Regularity theory for nonlinear integral operators

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Abstract: This article is dedicated to the proof of the existence of classical solutions for a class of non-linear integral variational problems. Those problems are involved in nonlocal image and signal processing.

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1 Introduction

The purpose of this work is to develop a regularity theory for non-local evolution equations of variational type with "measurable" kernels. More precisely, we consider solutions of the evolution equations of the type

$$w_t(t,x) = \int [w(t,y) - w(t,x)]K(t,x,y)\,dy, \tag{1.1}$$

where all that is required of the kernel K is that there exists $0 < s < 1$ and $0 < \Lambda$, such that

$$\begin{aligned} &\text{symmetry in } x, y: \qquad K(t,x,y) = K(t,y,x) \text{ for any } x \neq y, \\ &\mathbf{1}_{\{|x-y|\leq 3\}}\frac{(1-s/2)}{\Lambda}|x-y|^{-(N+s)} \leq K(t,x,y) \leq (1-s/2)\Lambda|x-y|^{-(N+s)}. \end{aligned} \tag{1.2}$$

The symmetry of the kernel K makes of the operator:

$$\int [w(y) - w(x)]K(x,y)\,dy$$

the Euler Lagrange equation of the energy integral

$$E(w) = \int\int [w(x) - w(y)]^2 K(x,y)\,dx\,dy.$$

It suggests a mathematical treatment based on the De Giorgi-Nash-Moser ideas [10, 15] from the calculus of variations. In fact, one of the immediate applications of our result is to nonlinear variational integrals

$$E_\phi(w) = \int\int \phi(w(x) - w(y))K(x-y)\,dx\,dy,$$

for ϕ a C^2 strictly convex functional. Indeed, the fact that $K(x,y)$ has the special form $K(x-y)$ makes the equation translation invariant, and as in the second order case, this implies that first derivatives of w satisfy an equation of the type (1.1). Our results are basically that solutions with initial data in L^2 become instantaneously bounded and Holder continuous. In these lines, see the work of Kassmann [13], Kassmann and Bass [2] (see also [3] and [1]), where the Moser approach for the stationary case is fully developped. For the non divergence case there is a recent work of Silvestre (see [16], [6], and references therein). We were motivated by our work on Navier-Stokes [18] and the quasigeostrophic equations [7]. In this work, the full regularity of the solutions to the surface quasi-geostrophic equation is shown in the critical case. It was followed by several works on the same subject in the super-critical case (see for instance [9]). Note also that the result was obtained, using completely different techniques by Kiselev, Nazarov and Volberg [14]. Our approach led to some progress in the supercritical case (see [17, 8]). It follows pretty much the lines of the De Giorgi's work [10]. Non linear equations of this form appear extensively in the phase transition literature (see Giacomin, Lebowitz, and Presutti [11]) and more recently on issues of image processing (Gilboa and Osher [12]).

2 Presentation of the results

Consider the variational integral

$$V(\theta) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi(\theta(y)-\theta(x))K(y-x)dydx,$$

for $\phi : \mathbb{R} \to [0,\infty)$ an even convex function of class $C^2(\mathbb{R})$ satisfying the conditions

$$\begin{aligned} &\phi(0) = 0, \\ &\Lambda^{-1/2} \le \phi''(x) \le \Lambda^{1/2}, \qquad x \in \mathbb{R}, \end{aligned} \tag{2.1}$$

for a given constant $1 < \Lambda < \infty$.

The kernel $K : \mathbb{R}^N - \{0\} \to (0,\infty)$ is supposed to satisfy the following conditions for a $0 < s < 2$.

$$\begin{aligned} &K(-x) = K(x), \qquad \text{for any } x \in \mathbb{R}^N - \{0\}, \\ &\mathbf{1}_{\{|x\le 3\}}(1-s/2)\frac{\Lambda^{-1/2}}{|x|^{N+s}} \le K(x) \le (1-s/2)\frac{\Lambda^{1/2}}{|x|^{N+s}}, \qquad \text{for any } x \in \mathbb{R}^N - \{0\}. \end{aligned} \tag{2.2}$$

With the above setting, the corresponding Euler-Lagrange equation for the variational integral $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi(\theta(y)-\theta(x))K(y-x)dydx$ is given by

$$-\int_{\mathbb{R}^N} \phi'(\theta(y)-\theta(x))K(y-x)dy = 0.$$

We are considering in this paper the associated time dependent problem:

$$\partial_t\theta(t,x) - \int_{\mathbb{R}^N} \phi'(\theta(t,y)-\theta(t,x))K(y-x)dy = 0. \tag{2.3}$$

The main goal of this paper is to address the regularity problem for solutions to the above parabolic-type equation and establish the following main theorem.

Theorem 2.1. *Consider an even convex function ϕ verifying the Hypothesis (2.1) and a kernel K verifying Hypothesis (2.2) for a $0 < s < 2$. Then, for any initial datum $\theta_0 \in H^1(\mathbb{R}^N)$, there exists a global classical solution to Equation 2.3 with $\theta(0,\cdot) = \theta_0$ in the $L^2(\mathbb{R}^N)$ sense. Moreover $\nabla_x\theta \in C^\alpha((t_0,\infty)\times\mathbb{R}^N)$ for any $t_0 > 0$.*

The existence of weak solutions with nonincreasing energy can be constructed following [4]. To address the regularity problem for solutions to Equation (2.3), we follow the classical idea of De Giorgi and look at the first derivative $D\theta$ of a solution θ to Equation (2.3). First, we use the change of variable $y = x + z$ to rewrite Equation (2.3) as follows

$$\partial_t \theta - \int_{\mathbb{R}^N} \phi'(\theta(x+z) - \theta(x)) K(z) dz = 0. \tag{2.4}$$

Now, we consider $w = D_e\theta$, the derivative in the direction e of θ. Derivating (formally) Equation (2.3) in the direction e we find

$$\partial_t w - \int_{\mathbb{R}^N} \phi''(\theta(x+z) - \theta(x))\{w(x+z) - w(x)\} K(z) dz = 0.$$

We then use the change of variable back to $y = x + z$ to rewrite the above equation in the following way

$$\partial_t w - \int_{\mathbb{R}^N} \phi''(\theta(y) - \theta(x))\{w(y) - w(x)\} K(y-x) dz = 0.$$

Consider the new kernel $K(t,x,y) = \phi''(\theta(t,y) - \theta(t,x))K(y-x)$ (with an obvious slight abuse for notation). Since ϕ is an even function, ϕ'' is also an even function, and hence the new kernel $K(t,x,y)$ is symmetric in x and y. Moreover, Hypothesis (2.2) and (2.1) implies that $K(t,x,y)$ satisfies the condition

$$(1-s/2)\mathbf{1}_{\{|x-y|\le 3\}} \frac{\Lambda^{-1}}{|x-y|^{N+s}} \le K(t,x,y) \le (1-s/2)\frac{\Lambda}{|x-y|^{N+s}}.$$

As a result, the function $w = D_e\theta$ satisfies Equation (1.1) with the kernel $K(t,x,y)$ verifying Hypothesis (1.2). Our goal is then to show that solutions to Equation (1.1) are in C^α.

To make the argument rigorous, we will consider the difference quotient $D_e^h\theta(\cdot) = \frac{1}{h}\{\theta(\cdot + he) - \theta(\cdot)\}$. We use again the version (2.4) of Equation (2.3). For any given $\eta \in C_c^\infty(\mathbb{R}^N)$, we use the difference quotient $D_e^{-h}\eta$ to test against it, and we get

$$\int_{\mathbb{R}^N} \partial_t\theta(t,x) D_e^{-h}\eta(x) dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi'(\theta(t,x+z) - \theta(t,x)) D_e^{-h}\eta(x) dx K(z) dz = 0.$$

Using the discrete integration by part $\int_{\mathbb{R}^N} f(x) D_e^{-h} g(x) dx = -\int_{\mathbb{R}^N} D_e^h f(x) g(x) dx$, we find

$$\int_{\mathbb{R}^N} \partial_t D_e^h\theta(t,x)\cdot\eta(x) dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} D_e^h[\phi'(\theta(\cdot+z) - \theta(\cdot))](x)\cdot\eta(x) dx K(z) dz = 0.$$

The change of variable $y = x + z$ leads to

$$\int_{\mathbb{R}^N} \partial_t D_e^h\theta(t,x)\cdot\eta(x) dx - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} D_e^h[\phi'(\theta(\cdot+y-x) - \theta(\cdot))](x)\cdot\eta(x) K(y-x) dx dy = 0.$$

Note that ϕ is an even function, so ϕ' is an odd function and consequently

$$D_e^h[\phi'(\theta(\cdot+y-x) - \theta(\cdot))](x) = -D_e^h[\phi'(\theta(\cdot+x-y) - \theta(\cdot))](y).$$

Using also the symmetry of K, we can symmetrize the operator to get

$$\int_{\mathbb{R}^N} \partial_t D_e^h\theta(t,x)\cdot\eta(x) dx - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} D_e^h[\phi'(\theta(\cdot+y-x) - \theta(\cdot))](x)\cdot[\eta(x) - \eta(y)] K(y-x) dx dy = 0. \tag{2.5}$$

Setting $Y = \theta(y+he) - \theta(x+he)$ and $X = \theta(y) - \theta(x)$, we get

$$\begin{aligned} D_e^h[\phi'(\theta(\cdot + y - x) - \theta(\cdot))](x) &= \frac{1}{h}\{\phi'(\theta(y+he_i) - \theta(x+he_i)) - \phi'(\theta(y) - \theta(x))\} \\ &= \frac{Y-X}{h}\int_0^1 \phi''(X + s(Y-X))\,ds \\ &= [D_e^h\theta(y) - D_e^h\theta(x)]\int_0^1 \phi''((1-s)[\theta(t,y) - \theta(t,x)] + s[\theta(t,y+he) - \theta(t,x+he)])\,ds. \end{aligned}$$

Hence, $w = D_e^h\theta$ solves the following equation

$$\int_{\mathbb{R}^N} \partial_t w(t,x)\eta(x)dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K^h(t,x,y)[\eta(x) - \eta(y)][w(t,x) - w(t,y)]dydx = 0.$$

where

$$K^h(t,x,y) = K(y-x)\int_0^1 \phi''((1-s)[\theta(t,y) - \theta(t,x)] + s[\theta(t,y+he) - \theta(t,x+he)])\,ds.$$

Note that this new kernels verified independently on h the properties (1.2) with the same Λ.

Theorem 2.1 is then a consequence of the following theorem.

Theorem 2.2. *Let w be a weak solution of (1.1) with a kernel verifying the properties (1.2), then for every $t_0 > 0$, $w \in C^\alpha((t_0,\infty)\times\mathbb{R}^N)$. The constant α and the norm of w depend only on t_0, N, $\|w^0\|_{L^2}$, and Λ.*

Passing into the limit $h \to 0$ gives the result of Theorem 2.1. The rest of the paper is dedicated to the proof of Theorem 2.2.

3 The first De-Giorgi's lemma

In this section and the next section, we focus on the differential equation stated in the sense of weak formulation in (1.1). We rewrite it in the following way.

$$\begin{aligned} \int_{\mathbb{R}^N} \partial_t w(t,x)\cdot\eta(x)dx + B[w(t,\cdot),\eta] = 0, \forall \eta \in C_c^\infty(\mathbb{R}^N), \\ B[u,v] = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(t,x,y)[u(x) - u(y)]\cdot[v(x) - v(y)]dxdy, \end{aligned} \tag{3.1}$$

where the kernel $K(t,x,y)$ is assumed to verify the Hypothesis (1.2). We first introduce the following function ψ:

$$\psi(x) = (|x|^{\frac{s}{2}} - 1)_+. \tag{3.2}$$

For any $L \geq 0$, we define

$$\psi_L(x) = L + \psi(x). \tag{3.3}$$

With the above setting, the first De Giorgi's lemma is as follows.

Lemma 3.1. *Let Λ be the given constant in condition (1.2). Then, there exists a constant $\epsilon_0 \in (0,1)$, depending only on N, s, and Λ, such that for any solution $w : [-2,0]\times\mathbb{R}^N \to \mathbb{R}$ to (3.1), the following implication for w holds true.*

If it is verified that

$$\int_{-2}^0\int_{\mathbb{R}^N} [w(t,x) - \psi(x)]_+^2 dxdt \leq \epsilon_0,$$

then we have

$$w(t,x) \le \frac{1}{2} + \psi(x)$$

for $(t,x) \in [-1,0] \times \mathbb{R}^N$. *(Hence, we have in particular that* $w \le 1/2$ *on* $[-1,0] \times B(1)$ *.)*

The main difficulty in our approach is due to the nonlocal operator. In [7], a localization of the problem was performed at the cost of adding one more variable to the problem. This was based on the "Dirichlet to Neuman" map. This approach still works for any fractional Laplacian (see Caffarelli and Silvestre [5]). However it breaks down for general kernels as (1.2). Instead, we keep track of the far away behavior of the solution via the function ψ.

Remark: All the computations on weak solutions in the proof can be justified by replacing the variable kernel in a neighborhood of the origin by the fractional Laplacian through a smooth cut off. Then the equation becomes a fractional heat equation with a bounded right hand side, thus C^2 in space. This makes the integrals involved uniformly convergent. Once the a priori Holder continuity is proven, we pass to the limit.

Proof. We split the proof in several steps.

First step: Energy estimates. Let $w : [-2,0] \times \mathbb{R}^N \to \mathbb{R}$ be a solution to equation (3.1). For $0 \le L \le 1$, we consider the truncated function $[w-\psi_L]_+$, where ψ_L is defined by (3.3). Then, we take the test function η to be $[w-\psi_L]_+$ in the weak formulation of equation (3.1), which gives

$$\begin{aligned} 0 &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} [w-\psi_L]_+^2 dx + B[w,(w-\psi_L)_+] \\ &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} [w-\psi_L]_+^2 dx + B[(w-\psi_L)_+,(w-\psi_L)_+] + B[(w-\psi_L)_-,(w-\psi_L)_+] \\ &\quad + B[\psi_L,(w-\psi_L)_+]. \end{aligned} \tag{3.4}$$

Now, due to the observation that $(w-\psi_L)_+ \cdot (w-\psi_L)_- = 0$ and the symmetry of K in x,y, we have

$$B[(w-\psi_L)_-,(w-\psi_L)_+] = 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(t,x,y)(w-\psi_L)_+(x)(w-\psi_L)_{neg}(y)dxdy,$$

where we denote $(w-\psi_L)_{neg} = -(w-\psi_L)_- \ge 0$. In particular

$$B[(w-\psi_L)_-,(w-\psi_L)_+] \ge 0.$$

This "good term" is not fully exploited in this section. It will be used in a crucial way in the next section. The remainder can be written as:

$$\begin{aligned} &B[\psi_L,(w-\psi_L)_+] \\ &= \frac{1}{2}\int\int_{|x-y|\ge 1} K(t,x,y)[\psi_L(x)-\psi_L(y)]\cdot\{(w-\psi_L)_+(x)-(w-\psi_L)_+(y)\}dxdy \\ &+ \frac{1}{2}\int\int_{|x-y|<1} K(t,x,y)[\psi_L(x)-\psi_L(y)]\cdot\{(w-\psi_L)_+(x)-(w-\psi_L)_+(y)\}dxdy. \end{aligned} \tag{3.5}$$

Using the inequality $|\psi(x)-\psi(y)| \le 2|y-x|^{\frac{s}{2}}$, for any x and y with $|y-x| \ge 1$, we get the following estimation of the "far-away" contribution.

$$\begin{aligned}
&\left|\int\int_{|x-y|\geq 1} K(t,x,y)[\psi_L(x)-\psi_L(y)]\cdot[w-\psi_L]_+(x)dxdy\right| \\
&=\left|\int\int_{|x-y|\geq 1} K(t,x,y)[\psi(x)-\psi(y)]\cdot[w-\psi_L]_+(x)dxdy\right| \\
&\leq \int_{\mathbb{R}^N}\int_{|y-x|\geq 1}\frac{2\Lambda}{|x-y|^{N+s}}2|y-x|^{\frac{s}{2}}dy\cdot(w-\psi_L)_+(x)dx \\
&=4\Lambda|S^{N-1}|\int_1^\infty r^{-\frac{s}{2}}dr\int_{\mathbb{R}^N}(w-\psi_L)_+(x)dx\leq C\int_{\mathbb{R}^N}(w-\psi_L)_+(x)dx.
\end{aligned}$$

By symmetry we end up to

$$\begin{aligned}
&\left|\int\int_{|x-y|\geq 1} K(t,x,y)[\psi_L(x)-\psi_L(y)]\cdot\{(w-\psi_L)_+(x)-(w-\psi_L)_+(y)\}dxdy\right| \\
&\leq C\int_{\mathbb{R}^N}(w-\psi_L)_+(x)dx.
\end{aligned} \tag{3.6}$$

The other part of the remainder ca be controlled in the following way:

$$\begin{aligned}
&\left|\int\int_{|x-y|<1} K(t,x,y)[\psi_L(x)-\psi_L(y)]\cdot\{(w-\psi_L)_+(x)-(w-\psi_L)_+(y)\}dxdy\right| \\
&\leq 2\int\int_{|x-y|<1} K(t,x,y)\chi_{\{[w-\psi_L](x)>0\}}|\psi_L(x)-\psi_L(y)||(w-\psi_L)_+(x)-(w-\psi_L)_+(y)|dxdy
\end{aligned} \tag{3.7}$$

where, in the above inequality, we have used the fact that

$$|(w-\psi_L)_+(x)-(w-\psi_L)_+(y)|\leq\{\chi_{\{[w-\psi_L](x)>0\}}+\chi_{\{[w-\psi_L](y)>0\}}\}|(w-\psi_L)_+(x)-(w-\psi_L)_+(y)|,$$

and the symmetry in x and y.

Now, by Holder's inequality, and using the elementary inequality $|\psi(y)-\psi(x)|<|y-x|$, for any x, y in $\mathbb{R}^N$, we can have the following estimation.

$$\begin{aligned}
&2\int\int_{|x-y|<1} K(t,x,y)\chi_{\{[w-\psi_L](x)>0\}}|\psi_L(x)-\psi_L(y)||(w-\psi_L)_+(x)-(w-\psi_L)_+(y)|dxdy \\
&\leq a\cdot\int\int_{|x-y|<1} K(t,x,y)\{(w-\psi_L)_+(x)-(w-\psi_L)_+(y)\}^2dydx \\
&+\frac{1}{a}\cdot\int\int_{|x-y|<1} K(t,x,y)|\psi(x)-\psi(y)|^2\cdot\chi_{\{[w-\psi_L](x)>0\}}dydx,
\end{aligned} \tag{3.8}$$

in which the arbitrary $a>0$ will be chosen later. Finally

$$\begin{aligned}
&\int\int_{|x-y|<1} K(t,x,y)|\psi(x)-\psi(y)|^2dy\cdot\chi_{\{[w-\psi_L](x)>0\}}dx \\
&\leq\int_{\mathbb{R}^N}\int_{|x-y|<1}\frac{2\Lambda}{|x-y|^{N+s}}|y-x|^2dy\cdot\chi_{\{[w-\psi_L](x)>0\}}dx=C_s\int_{\mathbb{R}^N}\chi_{\{[w-\psi_L](x)>0\}}dx.
\end{aligned}$$

Pulling this inequality in (3.7) with $a = 1/2$, and gathering it together with (3.5), (3.6), (3.7), we can rewrite the energy inequality as

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^N}[w-\psi_L]_+^2dx+\frac{1}{2}B[(w-\psi_L)_+,(w-\psi_L)_+]\\&\le C_{N,\Lambda,s}\{\int_{\mathbb{R}^N}(w-\psi_L)_+(x)dx+\int_{\mathbb{R}^N}\chi_{\{[w-\psi_L](x)>0\}}dx\},\end{aligned}\tag{3.9}$$

where $C_{N,\Lambda,s}$ is some universal constant depending only on N and Λ and s. Next, in order to employ the Sobolev embedding theorem, we need to compare $B[(w-\psi_L)_+,(w-\psi_L)_+]$ with $\|(w-\psi_L)_+\|^2_{H^{\frac{s}{2}}(\mathbb{R}^N)}$ as follow.

$$\begin{aligned}&\|(w-\psi_L)_+\|^2_{H^{\frac{s}{2}}(\mathbb{R}^N)}\\&=\int\int_{|x-y|\le 2}\frac{\{(w-\psi_L)_+(x)-(w-\psi_L)_+(y)\}^2}{|x-y|^{N+s}}+\int\int_{|x-y|>2}\frac{\{(w-\psi_L)_+(x)-(w-\psi_L)_+(y)\}^2}{|x-y|^{N+s}}\\&\le\Lambda\cdot B[(w-\psi_L)_+,(w-\psi_L)_+]+2\int\int_{|x-y|>2}\frac{1}{|x-y|^{N+s}}\{(w-\psi_L)_+^2(x)+(w-\psi_L)_+^2(y)\}dxdy\\&\le\Lambda\cdot B[(w-\psi_L)_+,(w-\psi_L)_+]+C\int_{\mathbb{R}^N}(w-\psi_L)_+^2dx.\end{aligned}$$

Hence,

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^N}[w-\psi_L]_+^2dx+\frac{1}{\Lambda}\|(w-\psi_L)_+\|^2_{H^{\frac{s}{2}}(\mathbb{R}^N)}\\&\le C_{N,\Lambda,s}\{\int_{\mathbb{R}^N}(w-\psi_L)_+dx+\int_{\mathbb{R}^N}\chi_{\{w-\psi_L>0\}}dx+|\int_{\mathbb{R}^N}(w-\psi_L)_+^2dx\}.\end{aligned}\tag{3.10}$$

Second step: Nonlinear recurrence. From this energy inequality, we establish a nonlinear recurrence relation to the following sequence of truncated energy.

$$U_k=\sup_{t\in[T_k,0]}\int_{\mathbb{R}^N}(w-\psi_{L_k})_+^2(t,x)dx+\int_{T_k}^0\|(w-\psi_{L_k})_+(t,\cdot)\|^2_{H^{\frac{s}{2}}(\mathbb{R}^N)}dt$$

where, in the above expression, $T_k=-1-\frac{1}{2^k}$, and $L_k=\frac{1}{2}(1-\frac{1}{2^k})$. Moreover, we will use the abbreviation $Q_k=[T_k,0]\times\mathbb{R}^N$.

Now, let us consider two variables σ, t which satisfies $T_{k-1}\le\sigma\le T_k\le t\le 0$. By taking the time integral over $[\sigma,t]$ in inequality 3.10, we yield

$$\begin{aligned}&\int_{\mathbb{R}^N}[w-\psi_{L_k}]_+^2(t,x)dx+\int_\sigma^t\|(w-\psi_{L_k})_+\|^2_{H^{\frac{s}{2}}(\mathbb{R}^N)}ds\\&\le\int_{\mathbb{R}^N}[w-\psi_{L_k}]_+^2(\sigma,x)dx+C_{N,\Lambda,s}\{\int_\sigma^t\int_{\mathbb{R}^N}(w-\psi_{L_k})_++\chi_{\{w-\psi_{L_k}>0\}}+(w-\psi_{L_k})_+^2dxds\}.\end{aligned}$$

Next, by first taking the average over $\sigma\in[T_{k-1},T_k]$, and then taking the sup over $t\in[T_k,0]$ in the above inequality, we deduce from the above inequality that

$$U_k\le 2^k(1+C_{N,s,\Lambda})\{\int_{Q_{k-1}}(w-\psi_{L_k})_++\chi_{\{w-\psi_{L_k}>0\}}+(w-\psi_{L_k})_+^2dxds\}.\tag{3.11}$$

Using the Sobolev embedding theorem $H^{\frac{s}{2}}(\mathbb{R}^N) \subset L^{\frac{2N}{N-s}}(\mathbb{R}^N)$ and interpolation we find

$$\|(w-\psi_{L_k})_+\|_{L^{2(1+\frac{s}{N})}(Q_k)} \leq C_N U_k^{\frac{1}{2}}.$$

Using Tchebychev inequality we get

$$\int_{Q_{k-1}} (w-\psi_{L_k})_+ \leq \int_{Q_{k-1}} (w-\psi_L)_+ \chi_{\{w-\psi_{L_{k-1}} > \frac{1}{2^{k+1}}\}} \leq (2^{k+1})^{1+\frac{2s}{N}} \int_{Q_{k-1}} (w-\psi_{L_{k-1}})_+^{2(1+\frac{s}{N})}$$
$$\leq (2^{k+1})^{1+\frac{2s}{N}} C_N^{2(1+\frac{s}{N})} U_{k-1}^{1+\frac{s}{N}}.$$

$$\int_{Q_{k-1}} \chi_{\{w-\psi_{L_k}>0\}} \leq (2^{k+1})^{2(1+\frac{s}{N})} \int_{Q_{k-1}} (w-\psi_{L_{k-1}})_+^{2(1+\frac{s}{N})} \leq (2^{k+1})^{2(1+\frac{s}{N})} C_N^{2(1+\frac{s}{N})} U_{k-1}^{1+\frac{s}{N}}.$$

$$\int_{Q_{k-1}} (w-\psi_{L_k})_+^2 \leq \int_{Q_{k-1}} (w-\psi_L)_+^2 \chi_{\{w-\psi_{L_{k-1}} > \frac{1}{2^{k+1}}\}} \leq (2^{k+1})^{\frac{2s}{N}} \int_{Q_{k-1}} (w-\psi_{L_{k-1}})_+^{2(1+\frac{s}{N})}$$
$$\leq (2^{k+1})^{\frac{2s}{N}} C_N^{2(1+\frac{s}{N})} U_{k-1}^{1+\frac{s}{N}}.$$

The above three inequalities, together with inequality (3.11), give

$$U_k \leq \{\overline{C}_{N,\Lambda,s}\}^k U_{k-1}^{1+\frac{s}{N}}, \forall k \geq 0, \tag{3.12}$$

for some universal constant $\overline{C}_{N,\Lambda,s}$ depending only on N, s, and Λ. Due to the nonlinear recurrence relation (3.12) for U_k, we know there exists some sufficiently small universal constant $\epsilon_0 = \epsilon_0(\overline{C}_{N,\Lambda,s})$, depending only on $\overline{C}_{N,\Lambda,s}$, such that the following implication is valid.

If $U_1 \leq \epsilon_0$, then it follows that $\lim_{k\to\infty} U_k = 0$.

Equation (3.11) with Tchebychev inequality gives that

$$U_1 \leq C \int_{-2}^{0} \int_{\mathbb{R}^N} |w-\psi|^2 \, dx \, dt,$$

and U_k converges to 0 implies that

$$w \leq \psi + \frac{1}{2} \qquad t \in [-1,0] \times \mathbb{R}^N.$$

□

We have the following corollary of Lemma 3.1. It shows that any solutions is indeed bounded for $t > 0$.

Corollary 3.2. *Any solution to (1.1) with initial value in $L^2(\mathbb{R}^N)$ is uniformly bounded on $(t_0, \infty) \times \mathbb{R}^N$ for any $0 < t_0 < 2$. Indeed:*

$$\sup_{t>t_0, x\in\mathbb{R}^N} |w(t,x)| \leq \frac{\|w^0\|_{L^2}}{2\sqrt{\varepsilon_0}(t_0/2)^{(N/s+1)/2}}.$$

Proof. Fix $0 < t_0 < 2$ and $x_0 \in \mathbb{R}^N$, for any $t > -2$, $x \in \mathbb{R}^N$ we consider

$$\bar{w}(t,x) = \frac{(t_0/2)^{(N/s+1)/2}\sqrt{\varepsilon_0}}{\|w^0\|_{L^2}} w(t_0 + t(t_0/2), x_0 + x(t_0/2)^{1/s}).$$

The function $\bar{w}$ still verifies the equation (3.1) with an other kernel verifying Hypothesis (1.2) with the same constant Λ. From the decreasing of energy, $\bar{w}$ verifies the assumptions of Lemma 3.1. Hence $\bar{w}(0,0) \leq 1/2$. Working with $-\bar{w}$ gives that $-\bar{w}(0,0) \leq 1/2$ too. □

We define $\psi_1(x) = (|x|^{s/4} - 1)_+$. We can rewrite the main lemma of this section in the following way. It will be useful for the next section.

Corollary 3.3. *Let Λ be the given constant in condition (1.2). Then, there exists a constant $\delta \in (0,1)$, depending only on N, s, and Λ, such that for any solution $w : [-2,0] \times \mathbb{R}^N \to \mathbb{R}$ to (3.1) satisfying*

$$w(t,x) \leq 1 + \psi_1(x) \qquad \text{on } [-2,0] \times \mathbb{R}^N,$$

and

$$|\{w > 0\} \cap ([-2,0] \times B_2)| \leq \delta,$$

we have

$$w(t,x) \leq \frac{1}{2}, \qquad (t,x) \in [-1,0] \times B_1.$$

Proof. Consider $R \geq 2$ such that $1 + \psi_1(R) \leq \psi(R)$. Note that R depends only on s. For any $(t_0, x_0) \in [-1,0] \times B_1$ we introduce w_R defined on $(-2,0) \times \mathbb{R}^N$ by

$$w_R(t,x) = w((t - t_0)/R^s, (x - x_0)/R).$$

Note that w_R verifies the equation (3.1) with an other kernel verifying Hypothesis (1.2) with the same constant Λ. Since ψ_1 increase with respect to $|x|$, for $|x| > 1$ we have

$$w_R(t,x) \leq 1 + \psi_1\left(\frac{x - x_0}{R}\right) \leq 1 + \psi_1\left(\frac{1 + |x|}{R}\right) \leq 1 + \psi_1\left((2/R)|x|\right) \leq 1 + \psi_1(x).$$

So, from the definition of R, for $|x| \geq R$ we have $w_R(t,x) \leq \psi(x)$. Hence, from the hypothesis we have

$$\begin{aligned}\int_{-2}^0 \int_{\mathbb{R}^N} (w_R(t,x) - \psi(x))_+^2 \, dx \, dt &= \int_{-2}^0 \int_{B_R} (w_R(t,x) - \psi(x))_+^2 \, dx \, dt \\ &\leq R^{N+s} \int_{-2}^0 \int_{B_2} (w(t,x))_+^2 \, dx \, dt \leq R^{N+s}(1 + \psi_1(2))^2 \delta.\end{aligned}$$

So, choosing $\delta = R^{-(N+s)}(1 + \psi_1(2))^{-2}\varepsilon_0$ gives that $w(t_0, x_0) \leq 1/2$ for $(t_0, x_0) \in (-1,0) \times B_1$. □

4 The second De-Giorgi's lemma

This section is dedicated to a lemma of local decrease of the oscillation of a solution to Equation (3.1). We define the following function

$$F(x) = \sup(-1, \inf(0, |x|^2 - 9)). \tag{4.1}$$

Note that F is Lipschitz, compactly supported in B_3, and equal to -1 in B_2.

For $\lambda < 1/3$, we define

$$\begin{aligned}\psi_\lambda(x) &= 0, \qquad \text{if } \ |x| \le \frac{1}{\lambda^{4/s}},\\ &= ((|x| - 1/\lambda^{4/s})^{s/4} - 1)_+, \qquad \text{if } \ |x| \ge \frac{1}{\lambda^{4/s}}.\end{aligned}$$

The normalized lemma will involve three consecutive cut-offs:

$$\varphi_0 = 1 + \psi_\lambda + F,$$

$$\varphi_1 = 1 + \psi_\lambda + \lambda F,$$

$$\varphi_2 = 1 + \psi_\lambda + \lambda^2 F.$$

We prove the following lemma:

Lemma 4.1. *Let Λ be the given constant in condition (1.2) and δ the constant defined in Corollary 3.3. Then, there exists $\mu > 0$, $\gamma > 0$, and $\lambda \in (0,1)$, depending only on N, Λ, and s, such that for any solution $w : [-3,0] \times \mathbb{R}^N \to \mathbb{R}$ to (3.1) satisfying*

$$w(t,x) \le 1 + \psi_\lambda(x) \qquad \text{on} [-3,0] \times \mathbb{R}^N,$$

$$|\{w < \varphi_0\} \cap ((-3,-2) \times B_1)| \ge \mu,$$

then we have either

$$|\{w > \varphi_2\} \cap ((-2,0) \times \mathbb{R}^N)| \le \delta,$$

or

$$|\{\varphi_0 < w < \varphi_2\} \cap ((-3,0) \times \mathbb{R}^N)| \ge \gamma.$$

The lemma says that in going from the φ_0 cut off to the φ_2 cut off, i.e., from the set $\{w > \varphi_0\}$ to $\{w > \varphi_2\}$ "some mass" is lost, i.e., if $|\{w > \varphi_2\}|$ is not yet subcritical (i.e., $\le \delta$) then

$$|\{w > \varphi_2\}| \le |\{w > \varphi_0\}| - \gamma.$$

Proof. In all the proof, we denote by C constants which depend only on s, N and Λ, but which can change from a line to another. We may fix any $0 < \mu < 1/8$. We will fix δ smaller than the one in Corollary 3.3 and such that the term $C\delta$ in (4.8) is smaller than $1/4$. The task consists now in showing that for $0 < \lambda < 1/3$ small enough, there exists a $\gamma > 0$ for which the lemma holds. The constraints on λ are (4.3), (4.5), (4.7), and (4.9). We split the proof into several steps.
First step: The energy inequality. We start again with the energy inequality (3.4), but use better the "good" term

$$B((w-\varphi)_+, (w-\varphi)_-) = \iint_{\mathbb{R}^{2N}} (w-\varphi)_+(x) K(t,x,y) (w-\varphi)_{\text{neg}}(y))\, dx\, dy$$

that we just neglected before.

We have, for φ_1 the intermediate cut off (see (3.4)):

$$\int ((w-\varphi_1)_+)^2\, dx \Big|_{T_1}^{T_2} + \int_{T_1}^{T_2} B((w-\varphi_1)_+, (w-\varphi_1)_+)\, dt =$$

$$- \int_{T_1}^{T_2} B((w-\varphi_1)_+, \varphi_1)\, dt - \int_{T_1}^{T_2} B((w-\varphi_1)_+, (w-\varphi_1)_-)\, dt.$$

The remainder term can be controlled in the following way.

$$B((w-\varphi_1)_+,\varphi_1) \leq \frac{1}{2}B((w-\varphi_1)_+,(w-\varphi_1)_+)+2\iint[\varphi_1(x)-\varphi_1(y)]K(x,y)[\varphi_1(x)-\varphi_1(y)][\chi_{B_3}(x)].$$

The first term $\frac{1}{2}B((w-\varphi_1)_+,(w-\varphi_1)_+)$ is absorbed on the left. The second one is smaller than

$$4\lambda^2\iint[F(x)-F(y)]K(x,y)[F(x)-F(y)]+4\iint[\psi_\lambda(x)-\psi_\lambda(y)]K(x,y)[\psi_\lambda(x)-\psi_\lambda(y)][\chi_{B_3}(x)],$$

which is smaller that $C\lambda^2$. This is obvious for the first term since F is Lipschitz and compactly supported. Since $\psi_\lambda(x)=0$ for $|x|<3$, the second term is equal to

$$\begin{aligned}
&4\iint \psi_\lambda(y)^2[\chi_{B_3}(x)]K(t,x,y)\,dx\,dy\\
&\leq 4(1-s/2)\Lambda|B_3|\int_{\{|y|>1/\lambda^{4/s}\}}\frac{(((|y|-1/\lambda^{4/s})^{s/4}-1)_+^2}{(|y|-3)^{N+s}}\,dy\\
&\leq 4(1-s/2)\Lambda|B_3|\lambda^2\int_{\{|z|>1\}}\frac{(((|z|-1)^{s/4}-\lambda)_+^2}{(|z|-3\lambda^{4/s})^{N+s}}\,dz\\
&\leq 4(1-s/2)\Lambda|B_3|\lambda^2\int_{\{|z|>1\}}\frac{(((|z|-1)^{s/4})_+^2}{(|z|-1/3)^{N+s}}\,dz\\
&\leq C\lambda^2,
\end{aligned}$$

since $\lambda<1/3$.

This leaves us with the inequality

$$\int(w-\varphi_1)_+^2dx\Big|_{T_1}^{T_2}+\frac{1}{2}\int_{T_1}^{T_2}B((w-\varphi_1)_+,(w-\varphi_1)_+)\,dt$$

$$+\int_{T_1}^{T_2}\int_{\mathbb{R}^{2N}}(w-\varphi_1)_+(x)K(x,y)(w-\varphi_1)_{\text{neg}}(y)\,dx\,dy\,dt\leq C\ \lambda^2(T_2-T_1).$$

In particular, since the second and third terms are positive, we get that for $-3<T_1<T_2<0$:

$$H(t)=\int_{\mathbb{R}^N}(w-\varphi_1)_+^2(t,x)dx$$

satisfies

$$H'(t)\leq C\ \lambda^2,$$

and

$$\int_{T_1}^{T_2}\int_{\mathbb{R}^{2N}}(w-\varphi_1)_+(x)K(x,y)(w-\varphi_1)_{\text{neg}}(y)\,dx\,dy\,dt\leq C\ \lambda^2[T_2-T_1]. \tag{4.2}$$

Note that, up to now, those estimates hold for any $0<\lambda<1/3$.

Second step: An estimate on those time slices where the "good" extra term helps. Remember that $\mu<1/8$ is fixed from the beginning of the proof. From our hypothesis

$$|\{w<\varphi_0\}\cap((-3,-2)\times B_1)|\geq\mu,$$

the set of times Σ in $(-3,-2)$ for which $|\{w(\cdot,T)<\varphi_0\}\cap B_1|\geq\mu/4$ has at least measure $\mu/(2|B_1|)$.

We estimate now that except for a few of those time slices, $\displaystyle\int_{\mathbb{R}^N}(w-\varphi_1)_+^2\,dx$ is very tiny:

Since $\quad \displaystyle\inf_{|x-y|\le 3} K(t,x,y) \ge C\Lambda^{-1} \quad$ we have that

$$C\,\lambda^2 \ge \int_{-3}^{-2} B((w-\varphi_1)_+,(w-\varphi_1)_-)dt \ge C\,\Lambda^{-1}\frac{\mu}{8}\int_\Sigma\int_{\mathbb{R}^N}(w-\varphi_1)_+\,dx\,dt$$

$$\ge C\,\Lambda^{-1}\frac{\mu}{8\lambda}\int_\Sigma\int_{\mathbb{R}^N}(w-\varphi_1)_+^2\,dx\,dt$$

since $(w-\varphi_1)_+ \le \lambda$.

In other words

$$\int_\Sigma\int_{\mathbb{R}^N}[(w-\varphi_1)_+(x)]^2\,dx\,dt \le \overline{C}\ \frac{\lambda^3}{\mu} \le \lambda^{3-1/8}$$

if λ is small enough such that

$$\lambda \le \left(\frac{\mu}{\overline{\overline{C}}}\right)^8. \tag{4.3}$$

In particular, from Tchebychev's inequality:

$$\int (w-\varphi_1)_+^2(t,x)\,dx \le \lambda^{3-\frac14} \tag{4.4}$$

for all $t\in\Sigma$, except for a very small subset F of t's of measure smaller than $\lambda^{1/8}$. We need it still much smaller than $\mu\sim|\Sigma|$. indeed, if λ is small enough such that

$$\lambda \le \left(\frac{\mu}{4|B_1|}\right)^8, \tag{4.5}$$

then, (4.4) holds on a set of ts in $[-3,-2]$ of measure bigger than $\mu/(4|B_1|)$.

Third step. In search of an intermediate set, where w is between φ_0 and φ_2. Let us go now to $(w-\varphi_2)_+$.

Assume that for at least one time $T_0 > -2$,

$$|\{x \mid (w-\varphi_2)_+(T_0,x)>0\}| > \delta/2,$$

i.e., goes over critical for the first lemma and let's go backwards in time until we reach a slice of time $T_1\in\Sigma$, where

$$\int_{\mathbb{R}^N}(w-\varphi_1)_+^2(T_1,x)\,dx \le \lambda^{3-\frac14}.$$

At T_0, for the intermediate cut off, φ_1, we have

$$\begin{aligned}\int_{\mathbb{R}^N}(w-\varphi_1)_+^2(T_0,x)\,dx &\ge \int(\varphi_1-\varphi_2)^2\chi_{\{(w-\varphi_2)_+>0\}}\\ &\ge \int(\lambda-\lambda^2)^2F^2(x)\chi_{\{(w-\varphi_2)_+>0\}} \ge C_F\frac{\lambda^2}{4}\delta^3,\end{aligned} \tag{4.6}$$

where the constant C_F depends only on the fixed function F. Indeed we have $\lambda < 1/2$, and F is increasing with respect to $|x|$ and smaller than $-C(3-|x|)$ for $|x| < 3$ closed to 3. Hence, the integral is minimum when all the mass $\{(w-\varphi_2)_+ > 0\}$ is concentrated on $3 - C\delta < |x| < 3$.

Now, at T_1,

$$\int_{\mathbb{R}^N} (w-\varphi_1)_+^2 (T_1, x)\, dx \le \lambda^{3-\frac{1}{4}} \ .$$

Thus, for λ small enough such that

$$\lambda^{1-1/4} \le C_F \frac{\delta^3}{64}, \tag{4.7}$$

in going from T_0 backwards to T_1, $H(t) = \int_{\mathbb{R}^N} (w-\varphi_1)_+^2 (t,x)\, dx$ has crossed a range between two multiples of $\delta^3\lambda^2$, from say $\lambda^2 \frac{\delta^3}{8}$, to $\lambda^2 \frac{\delta^3}{16}$. Since $H'(t) \le C\ \lambda^2$, in order to do so it needed in a range of times D, of at least length $\sim \delta^3$, where

$$D = \{t \in (T_1, T_0) : \ \lambda^{3-1/4} < H(t) < C_F \frac{\lambda^2}{4} \delta^3\}.$$

We want to show that in this range, we pick up an intermediate set, of nontrivial measure, where $(w-\varphi_0)_+ > 0$ and $(w-\varphi_2)_+ = 0$, implying that the measure

$$\mathcal{A}_2 = |\{(w-\varphi_2)_+ > 0\} \cap \{t \in (-3,0)\}|$$

effectively decreases some fixed amount from

$$\mathcal{A}_0 = |\{w-\varphi_0)_+ > 0\} \cap \{t \in (-3,0)\}| \ .$$

In these range of times D, given the gap between φ_1 and φ_2

$$|\{(w-\varphi_2)_+ > 0\}| \le C\ \delta \tag{4.8}$$

(if not $\int (w-\varphi_1)_+^2 > \delta^3\lambda^2$ as in the computation of (4.6)). As said in the beginning of the proof, we may consider a δ such that $C\delta < 1/4$. Moreover, those times of D for which

$$|\{(w-\varphi_0)_+ \le 0\} \cap B_2| \ge \mu$$

are in an exceptional subset $\mathcal{F}$ of very small size. Indeed

$$\begin{aligned} C\lambda^2 &\ge \int_{-3}^{0} \int_{\mathbb{R}^N} (w-\varphi_1)_+ K(t,x,y)(w-\varphi_1)_{\text{neg}} \\ &\ge C\mu \int_{\mathcal{F}} \int_{B_3} (w-\varphi_1)_+ \, dx\, dt \ge \frac{C\mu}{\lambda} \int_{\mathcal{F}} \int_{\mathbb{R}^N} (w-\varphi_1)_+^2 \, dx\, dt \\ &\ge \frac{C\mu|\mathcal{F}|(\lambda^2\delta^3/16)}{\lambda}. \end{aligned}$$

Hence

$$|\mathcal{F}| \le C\frac{\lambda}{\mu\delta^3}.$$

And so, for λ small enough such that

$$\lambda \le \mu\delta^3|D|/(2C), \tag{4.9}$$

we have

$$|\mathcal{F}| \leq \frac{|D|}{2}.$$

Note that the constraint (4.9) can be expressed depending only on s, N, Λ, δ, and, μ, since $|D| < C\delta^3$. For these times in D not in $\mathcal{F}$, we have:

$$A(t) = |\{\varphi_0 \leq w(t,\cdot) \leq \varphi_2\}| \geq 1/2.$$

That is

$$\begin{aligned}|\{\varphi_0 < w < \varphi_2\} \cap ((-3,0)\times\mathbb{R}^N)| &\geq \int_{-3}^{0} A(t)\,dt \\ &\geq \int_{D\setminus\mathcal{F}} A(t)\,dt \geq \frac{|D|}{4} \geq C\delta^3.\end{aligned}$$

□

5 Proof of the C^α regularity

we are now ready to show the following oscillation lemma. First, for λ as in the previous section, we define for any $\varepsilon > 0$

$$\begin{aligned}\psi_{\varepsilon,\lambda}(x) &= 0, \qquad \text{if } \ |x| \leq \frac{1}{\lambda^{4/s}}, \\ &= ((|x| - 1/\lambda^{4/s})^\varepsilon - 1)_+, \qquad \text{if } \ |x| \geq \frac{1}{\lambda^{4/s}}.\end{aligned}$$

Lemma 5.1. *there exists $\varepsilon > 0$ and λ^* such that for any solution to (3.1) in $[-3,0]\times\mathbb{R}^N$ such that*

$$-1 - \psi_{\varepsilon,\lambda} \leq w \leq 1 + \psi_{\varepsilon,\lambda},$$

we have

$$\sup_{[-1,0]\times B_1} w - \inf_{[-1,0]\times B_1} w \leq 2 - \lambda^*.$$

Proof. We may assume that

$$|\{w < \varphi_0\} \cap ((-3,-2)\times B_1)| > \mu.$$

Otherwise this is verified by $-w$, and we may work on this function.

Consider $k_0 = |(-3,0)\times B_3|/\gamma$. Then we fix ε small enough such that

$$\frac{(|x|^\varepsilon - 1)_+}{\lambda^{2k_0}} \leq (|x|^{s/4} - 1)_+,$$

for all x. We may take $\varepsilon = (s/4)\lambda^{2k_0}$ for instance. For $k \leq k_0$, we consider the sequence

$$w_{k+1} = \frac{1}{\lambda^2}(w_k - (1-\lambda^2)), \qquad w_1 = w.$$

By induction, we have that

$$(w_k)_+(t,x) \leq 1 + \frac{1}{\lambda^{2k}}\psi_{\varepsilon,\lambda}(x), \qquad t\in(-3,0), x\in\mathbb{R}^N.$$

So, for $k \leq k_0$ we have $w_k \leq 1 + \psi_\lambda$. By construction $|\{w_k < \varphi_0\} \cap (-3,-2) \times B_1|$ is increasing, so bigger than μ for any k. Hence, we can apply Lemma 4.1 on w_k. As long as $|\{w_k > \varphi_2\} \cap ((-2,0) \times \mathbb{R}^N)| \geq \delta$, we have

$$|\{w_{k+1} > \varphi_0\}| = |\{w_{k+1} > \varphi_2\}| + |\{\varphi_0 < w_{k+1} < \varphi_2\}|,$$

and

$$\begin{aligned} |\{w_{k+1} > \varphi_2\}| &\leq |\{w_{k+1} > \varphi_0\}| - \gamma \\ &\leq |\{w_k > \varphi_2\}| - \gamma \leq |(-3,0) \times B_3| - k\gamma. \end{aligned}$$

This cannot be true up to k_0. So there exists $k \leq k_0$ such that

$$|\{w_k > \varphi_2\} \cap ((-2,0) \times \mathbb{R}^N)| \leq \delta.$$

We can then apply the first De Giorgi lemma on w_{k+1}. Indeed

$$w_{k+1} \leq 1 + \psi_\lambda \leq 1 + \psi_1, \qquad \text{on } (-3,0) \times \mathbb{R}^N,$$

and

$$\begin{aligned} |\{w_{k+1} > 0\} \cap ((-2,0) \times B_2)| &\leq |\{w_{k+1} > \varphi_0\} \cap ((-2,0) \times B_2)| \\ &\leq |\{w_k > \varphi_2\} \cap ((-2,0) \times \mathbb{R}^N)| \leq \delta. \end{aligned}$$

Hence, from Corollary 3.3, we have

$$w_{k+1} \leq 1/2, \qquad \text{on}(-1,0) \times B_1.$$

This gives the result with

$$\lambda^* = \frac{\lambda^{2k_0}}{2}.$$

□

The C^α regularity follows in a standart way.

Proof. For any $(t_0, x_0) \in (0,\infty) \times \mathbb{R}^N$, consider first $K_0 = \inf(1, t_0/4)^{1/s}$, and

$$w_0(t,x) = w(t_0 + K_0^s t, x_0 + K_0 x).$$

This function still verifies an equation of the type of (3.1) in $(-4,0) \times \mathbb{R}^N$ with a kernel K verifying (1.2). From Corollary 3.2, It is bounded on $(-3,0) \times \mathbb{R}^N$. Consider $K < 1$ such that

$$\frac{1}{1 - (\lambda^*/2)} \psi_{\lambda,\varepsilon}(Kx) \leq \psi_{\lambda,\varepsilon}(x), \qquad \text{for } |x| \geq 1/K.$$

The coefficient K depends only on λ, λ^* and ε. Then we define by induction:

$$\begin{aligned} w_1(t,x) &= \frac{w_0(t,x)}{\|w_0\|_{L^\infty}}, \qquad (t,x) \in (-3,0) \times \mathbb{R}^N, \\ w_{k+1}(t,x) &= \frac{1}{1 - \lambda^*/4} \left(w_k(K^s t, Kx) - \bar{w}_k \right), \qquad (t,x) \in (-3,0) \times \mathbb{R}^N, \end{aligned}$$

where

$$\bar{w}_k = \frac{1}{|B_1|} \int_{-1}^0 \int_{B_1} w_k(t,x) \, dx \, dt.$$

By construction, w_k verifies the hypothesis of Lemma 5.1 for any k. Hence:

$$\sup_{(t_0+(-K^{ks},0))\times(x_0+B_{K^k})} w - \inf_{(t_0+(-K^{ks},0))\times(x_0+B_{K^k})} w \leq C(1-\lambda^*/4)^k.$$

So, w is C^α with

$$\alpha = \frac{\ln(1-\lambda^*/4)}{\ln(K^s)}.$$

□

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