Positive lower bound for the numerical solution of a convection-diffusion equation

Claire Chainais-Hillairet, Benoît Merlet and Alexis F. Vasseur

Abstract In this work, we apply a method due to De Giorgi [3] in order to establish a positive lower bound for the numerical solution of a stationary convection-diffusion equation.

Key words: finite volume scheme, isoperimetric inequality
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1 Introduction

The continuous problem

Let Ω be an open bounded polygonal domain of $\mathbb{R}^2$. Let $f \in L^\infty(\Omega)$, we consider the following system of equations :

$$-\Delta\Psi = f \qquad \text{in } \Omega, \tag{1a}$$

$$\operatorname{div}(-\nabla v + \nabla\Psi v) = 0 \qquad \text{in } \Omega. \tag{1b}$$

This system is supplemented with mixed Dirichlet-Neumann boundary conditions. We assume that $\partial\Omega = \Gamma^D \cup \Gamma^N$ with $\Gamma^D \cap \Gamma^N = \emptyset$ and $\mathrm{m}(\Gamma^D) > 0$ and we consider $v^D \in L^\infty(\Gamma^D)$, $\Psi^D \in L^\infty(\Gamma^D)$. The boundary conditions write:

$$\Psi = \Psi^D,\ v = v^D \text{on } \Gamma^D \text{ and } \nabla\Psi \cdot \mathbf{n} = 0 = \nabla v \cdot \mathbf{n} \text{ on } \Gamma^N \tag{2}$$

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and v^D is bounded from below by a positive constant. Up to a rescaling, we assume:

$$v^D \geq 1 \text{ a.e. on } \Gamma^D. \tag{3}$$

With this hypothesis, one can prove, using De Giorgi's method (see [3, 5]), that the weak solution v to (1), (2) has a positive lower bound:

$$\exists \alpha > 0 \text{ such that } v \geq \alpha \text{ in } \Omega.$$

Our aim is to adapt De Giorgi's method in a discrete setting. We establish that the approximation $v_{\mathscr{T}}$ of v obtained by one of the finite volume discretizations described below satisfies (4) with some $\alpha > 0$ only depending on the continuous data and on the regularity of the mesh. The method could be extended to finite volume approximations of evolutive convection-diffusion equations and drift-diffusion systems.

The numerical scheme

The mesh of the domain Ω, $\mathscr{M} = (\mathscr{T}, \mathscr{E}, \mathscr{P})$, is classically given by: $\mathscr{T}$, a family of open polygonal control volumes, $\mathscr{E}$, a family of edges, $\mathscr{P} = (x_K)_{K\in\mathscr{T}}$ a family of points. As we deal with a two-point flux approximation of convection-diffusion equations, we assume that the mesh is admissible in the sense of [4] (Definition 9.1).

We distinguish in $\mathscr{E}$ the interior edges, $\sigma = K|L$, from the exterior edges: $\mathscr{E} = \mathscr{E}_{int} \cup \mathscr{E}_{ext}$. Among the exterior edges, we distinguish the edges included in Γ^D from the edges included in Γ^N: $\mathscr{E}_{ext} = \mathscr{E}^D_{ext} \cup \mathscr{E}^N_{ext}$. For a given control volume $K \in \mathscr{T}$, we define $\mathscr{E}_K$ the set of its edges, which is also split into $\mathscr{E}_K = \mathscr{E}_{K,int} \cup \mathscr{E}^D_{K,ext} \cup \mathscr{E}^N_{K,ext}$. For each edge $\sigma \in \mathscr{E}$, we pick one cell in the non empty set $\{K : \sigma \in \mathscr{E}_K\}$ and denote it by K_σ. In the case of an interior edge $\sigma = K|L$, K_σ is either K or L.

For all edges $\sigma \in \mathscr{E}$, we set $\mathrm{d}_\sigma = \mathrm{d}(x_K, x_L)$ if $\sigma = K|L \in \mathscr{E}_{int}$ and $\mathrm{d}_\sigma = \mathrm{d}(x_K, \sigma)$ if $\sigma \in \mathscr{E}_{ext}$ with $\sigma \in \mathscr{E}_K$ and the transmissibility coefficient is defined by $\tau_\sigma = \mathrm{m}(\sigma)/\mathrm{d}_\sigma$, for all $\sigma \in \mathscr{E}$. We assume that the mesh satisfies the regularity constraint:

$$\exists \xi > 0 \text{ such that } \mathrm{d}(x_K, \sigma) \geq \xi\, \mathrm{d}_\sigma, \quad \forall K \in \mathscr{T}, \forall \sigma \in \mathscr{E}_K. \tag{4}$$

The size of the mesh is defined by $h = \max\{\text{diam }(K) : K \in \mathscr{T}\}$.

Per se, a finite volume scheme for a conservation law with unknown u provides a vector $u_{\mathscr{T}} = (u_K)_{K\in\mathscr{T}}$ of approximate values and the associated piecewise constant function. However, since there are Dirichlet boundary conditions on a part of the boundary, we need to define approximate values for u at the corresponding boundary edges: $u_{\mathscr{E}^D} = (u_\sigma)_{\sigma\in\mathscr{E}^D_{ext}}$. Then, we set $u_{\mathscr{M}} = (u_{\mathscr{T}}, u_{\mathscr{E}^D})$ and we define

$$Du_{K,\sigma} = u_{K,\sigma} - u_K \quad \text{and} \quad D_\sigma u = |Du_{K,\sigma}|, \quad \forall K \in \mathscr{T}, \forall \sigma \in \mathscr{E}_K, \tag{5}$$

where $u_{K,\sigma}$ is either u_L ($\sigma = K|L$), u_σ ($\sigma \in \mathscr{E}^D_{K,ext}$) or u_K ($\sigma \in \mathscr{E}^N_{K,ext}$).

For all $K \in \mathscr{T}$, f_K denotes the mean value of f over K. For all $\sigma \in \mathscr{E}^D_{ext}$, Ψ^D_σ and v^D_σ are respectively the mean values of Ψ^D and v^D over σ. We set:

$$v_\sigma = v^D_\sigma, \quad \Psi_\sigma = \Psi^D_\sigma, \quad \forall \sigma \in \mathscr{E}^D_{ext}. \tag{6}$$

We are now in the position to define the finite volume scheme for (1):

$$-\sum_{\sigma\in\mathscr{E}_K}\tau_\sigma D\Psi_{K,\sigma}=\mathrm{m}(K)f_K,\quad \forall K\in\mathscr{T}, \tag{7a}$$

$$\sum_{\sigma\in\mathscr{E}_K}\mathscr{F}_{K,\sigma}=0,\quad \forall K\in\mathscr{T}. \tag{7b}$$

where the numerical convection-diffusion fluxes $\mathscr{F}_{K,\sigma}$ are given by

$$\mathscr{F}_{K,\sigma}=\tau_\sigma\Big(B(-D\Psi_{K,\sigma})v_K-B(D\Psi_{K,\sigma})v_{K,\sigma}\Big),\qquad \forall K\in\mathscr{T},\forall\sigma\in\mathscr{E}_K, \tag{8}$$

and B is a Lipschitz-continuous function on $\mathbb{R}$ satisfying

$$B(0)=1,\quad B(s)>0\quad\text{and}\quad B(s)-B(-s)=-s\quad\forall s\in\mathbb{R}. \tag{9}$$

The upwind scheme corresponds to the case $B(s)=1+s^-$ and the Scharfetter-Gummel scheme to the case $B(s)=s/(e^s-1)$. They both satisfy (9).

Main result

The scheme (6)-(7)-(8) can be written as two linear systems of equations on the unknowns $\Psi_{\mathscr{T}}$ and $\mathbf{v}_{\mathscr{T}}$: $\mathbb{A}\Psi_{\mathscr{T}}=\mathbf{B}_\Psi$ and $\mathbb{M}v_{\mathscr{T}}=\mathbf{B}_v$. It is well-known that $\mathbb{A}$ is a positive-definite symmetric matrix and that $\mathbb{M}$ is an M-matrix. Therefore, existence and uniqueness of a solution to the scheme is ensured. Moreover, the non negativity of the boundary condition v^D implies $\mathbf{B}_v\geq 0$ and, since $\mathbb{M}$ is an M-matrix, we get $v_{\mathscr{T}}\geq 0$. We establish a positive lower bound for $v_{\mathscr{T}}$.

Theorem 1. *There exists $h_0>0$ and $\alpha>0$ only depending on Ω, f,Ψ^D, v^D, ξ and B such that if $0<h<h_0$, then*

$$v_{\mathscr{T}}\geq\alpha\qquad in\ \Omega. \tag{10}$$

The proof of Theorem 1 follows the lines of the proof at the continuous level. We introduce a sequence $(w^j_{\mathscr{M}})_{j\geq 0}$ defined recursively by: $w^0_{\mathscr{M}}=1/2-v_{\mathscr{M}}$ and $w^{j+1}_{\mathscr{M}}=2w^j_{\mathscr{M}}-1/2$ for all $j\geq 0$. A direct computation shows:

$$w^j_{\mathscr{M}}=2^j w_{\mathscr{M}}-(2^j-1)/2=1/2-2^j v_{\mathscr{M}},\qquad \forall j\geq 0. \tag{11}$$

It is clear that $w^j_{\mathscr{T}}\leq 1/2$ and that $w^j_{\mathscr{E}^D}\leq -2^j+1/2<0$ for all $j\geq 0$. Moreover, if $w^{j_0}_{\mathscr{T}}\leq 1/4$ for some $j_0\geq 0$, we can conclude that $w^0_{\mathscr{T}}\leq 1/2-2^{-j_0-2}$ and (10) is established with $\alpha=2^{-j_0-2}$.

The proof then splits into three steps. In Section 2, we establish some discrete *a priori* H^1-estimates on $w^j_{\mathscr{M}}$ for $j\geq 0$. Then, we prove that the desired estimate $w^{j_0}_{\mathscr{T}}\leq 1/4$ can be obtained under a smallness assumption on $\|w^{j_0}_{\mathscr{T}}\|_{L^2}$. We conclude in Section 4 by proving that this smallness assumption is verified for some j_0. This is a consequence of the discrete H^1-estimate and of an isoperimetric inequality established in Section 3.

2 Estimates on the approximate solution

L^∞-estimate on the potential Ψ

As $f \in L^\infty(\Omega)$, the solution $\Psi_{\mathscr{T}}$ to the scheme (7a) with (5) and (6) satisfies L^∞-estimates, see for instance Lemma 6 in [2] and the references therein. These L^∞-estimates imply a uniform bound on the $D_\sigma\Psi$ and, as B is Lipschitz-continuous,

$$\exists\beta > 0 \text{ such that } B(D_\sigma\Psi) \geq \beta \quad \forall\sigma \in \mathscr{E}. \tag{12}$$

Let us note that $\beta = 1$ for the upwind scheme ($B(s) = 1 + s^-$).

Energy estimates

Thanks to (11), we have $v_K = 2^{-j-1} - 2^{-j} w_K^j$ for all $K \in \mathscr{T}$ and $j \geq 0$. Using (9), the numerical fluxes $\mathscr{F}_{K,\sigma}$ defined by (8) rewrite

$$\mathscr{F}_{K,\sigma} = 2^{-j-1}\tau_\sigma D\Psi_{K,\sigma} - 2^{-j}\mathscr{G}_{K,\sigma}^j \quad \forall j \geq 0,$$

where

$$\mathscr{G}_{K,\sigma}^j = \tau_\sigma\Big(B(-D\Psi_{K,\sigma})w_K^j - B(D\Psi_{K,\sigma})w_{K,\sigma}^j\Big), \forall K \in \mathscr{T},\ \forall\sigma \in \mathscr{E}_K,\ \forall j \geq 0. \tag{13}$$

Therefore, the scheme (7) implies:

$$\sum_{\sigma\in\mathscr{E}_K} \mathscr{G}_{K,\sigma}^j = \frac{1}{2}\sum_{\sigma\in\mathscr{E}_K} \tau_\sigma D\Psi_{K,\sigma} = -\frac{1}{2}\mathrm{m}(K) f_K, \quad \forall K \in \mathscr{T},\ \forall j \geq 0. \tag{14}$$

Lemma 1. *There exists C_0 only depending on Ω, ξ, Ψ^D, f and B such that for every $j \geq 0$,*

$$\sum_{\sigma\in\mathscr{E}} \tau_\sigma (D_\sigma w^{j,+})^2 \leq C_0. \tag{15}$$

Proof. Let us fix $j \geq 0$ and let us drop the superscript j. We use that $w_{\mathscr{M}}$ solves (13)-(14) with $w_{\mathscr{E}^D}^+ = 0$ to prove that $w_{\mathscr{T}}^+$ satisfies (15).

Let us multiply (14) by w_K^+ and sum over $K \in \mathscr{T}$. Rearranging the sum as a sum over edges and taking into account that $w_\sigma^+ = 0$ for $\sigma \in \mathscr{E}_{ext}^D$ we get

$$\sum_{\sigma\in\mathscr{E};K=K_\sigma} \mathscr{G}_{K,\sigma} D(w^+)_{K,\sigma} = \frac{1}{2}\sum_{K\in\mathscr{T}} \mathrm{m}(K) f_K w_K^+. \tag{16}$$

But, thanks to (9), the numerical fluxes defined by (13) can be rewritten either as

$$\begin{aligned} &\mathscr{G}_{K,\sigma} = \tau_\sigma\left[D\Psi_{K,\sigma} w_K - B(D\Psi_{K,\sigma}) Dw_{K,\sigma}\right] \\ \text{or as}\quad &\mathscr{G}_{K,\sigma} = \tau_\sigma\left[D\Psi_{K,\sigma} w_{K,\sigma} - B(-D\Psi_{K,\sigma}) Dw_{K,\sigma}\right]. \end{aligned}$$

We use the first expression when $D\Psi_{K,\sigma} \geq 0$ and the second one when $D\Psi_{K,\sigma} < 0$. Combined with the inequalities $x(y^+ - x^+) \leq \frac{1}{2}[(y^+)^2 - (x^+)^2]$ and $(x - y)(x^+ -$

$y^+) \geq (x^+ - y^+)^2$, we get

$$\mathscr{G}_{K,\sigma} D(w^+)_{K,\sigma} \leq \frac{1}{2}\tau_\sigma D\Psi_{K,\sigma}(D(w^+)^2)_{K,\sigma} - \tau_\sigma B(D_\sigma \Psi)(D_\sigma w^+)^2. \qquad (17)$$

Therefore, we deduce from (16), (17) and (7a):

$$\sum_{\sigma\in\mathscr{E}} \tau_\sigma B(D_\sigma\Psi)(D_\sigma w^+)^2 \leq \frac{1}{2}\sum_{K\in\mathscr{T}} \mathrm{m}(K) f_K (w_K^+)^2 - \frac{1}{2}\sum_{K\in\mathscr{T}} \mathrm{m}(K) f_K w_K^+.$$

Eventually, since $B(D_\sigma\Psi) \geq \beta$, $w_{\mathscr{T}}^+ \leq \dfrac{1}{2}$ and $f \in L^\infty$, we get (15). □

From L^2-bound to L^∞-bound for $w_{\mathscr{T}}^{j,+}$

We establish the desired bound $w_{\mathscr{T}}^j \leq \frac{1}{4}$ under a smallness assumption on $\|w_{\mathscr{T}}^{j,+}\|_{L^2}$.

Lemma 2. *There exists $\delta > 0$ only depending on Ω, f, Ψ^D and ξ such that*

$$\sum_{K\in\mathscr{T}} \mathrm{m}(K)(w_K^{j,+})^2 \leq \delta^2 \quad \Longrightarrow \quad w_{\mathscr{T}}^{j,+} \leq \frac{1}{4}. \qquad (18)$$

Proof. As above, we drop the superscript j in the proof. For all $i \geq 0$, we set

$$C_i = 1/4 - 2^{-i-2}, \quad z_{\mathscr{M}}^i = (w_{\mathscr{M}} - C_i)^+, \quad E_i = \int_\Omega (z_{\mathscr{T}}^i)^2 = \sum_{K\in\mathscr{T}} \mathrm{m}(K)([w_K - C_i]^+)^2.$$

We note that $C_i - C_{i-1} = 2^{-i-2}$ and that $\{z_{\mathscr{T}}^i > 0\} \subset \{z_{\mathscr{T}}^{i-1} > 0\}$. Moreover,

$$\begin{aligned} & z_{\mathscr{T}}^{i-1} = z_{\mathscr{T}}^i + 2^{-i-2} \quad \text{on } \{z_{\mathscr{T}}^i > 0\} \\ \text{and} \quad & \int_\Omega (z_{\mathscr{T}}^{i-1})^2 \geq \int_\Omega (z_{\mathscr{T}}^i)^2 + 2^{-2(i+2)}\mathrm{m}(\{z_{\mathscr{T}}^i > 0\}). \end{aligned}$$

It yields

$$E_i \leq E_{i-1} \quad \text{and} \quad \mathrm{m}(\{z_{\mathscr{T}}^i > 0\}) \leq 2^{2(i+2)} E_{i-1}. \qquad (19)$$

Moreover, thanks to Young's inequality, we get

$$\int_\Omega z_{\mathscr{T}}^i \leq \frac{1}{2}E_i + \frac{1}{2}\mathrm{m}(\{z_{\mathscr{T}}^i > 0\}) \leq \frac{1}{2}\left(1 + 2^{2(i+2)}\right) E_{i-1}. \qquad (20)$$

Applying now Hölder inequality with $q \in (1, +\infty)$ and $q^* = q/(q-1)$, we get

$$E_i \leq \left(\int_\Omega |z_{\mathscr{T}}^i|^{2q}\right)^{1/q} \mathrm{m}(\{z_{\mathscr{T}}^i > 0\})^{1/q^*}.$$

Since $z_{\mathscr{E}^D}^i = 0$, we apply a discrete Poincaré-Sobolev inequality to bound the integral (see Theorem 4.3 in [1]) and using (19) it yields

$$E_i \le \frac{C}{\xi}\Big(\sum_{\sigma\in\mathscr{E}} \tau_\sigma (D_\sigma z^i)^2\Big)\Big(2^{2(i+2)}E_{i-1}\Big)^{1/q^*} \tag{21}$$

with C depending on Ω and q.

Let us now remark that the numerical fluxes $\mathscr{G}_{K,\sigma}$ defined by (13) rewrite as

$$\mathscr{G}_{K,\sigma} = \tau_\sigma\Big(B(-D\Psi_{K,\sigma})(w_K - C_i) - B(D\Psi_{K,\sigma})(w_{K,\sigma} - C_i)\Big) + \tau_\sigma D\Psi_{K,\sigma}C_i.$$

Then, following the proof of Lemma 1, we obtain

$$\begin{aligned}\sum_{\sigma\in\mathscr{E}} \tau_\sigma B(D_\sigma\Psi)(D_\sigma z^i)^2 &\le \frac{1}{2}\sum_{K\in\mathscr{T}} \mathrm{m}(K) f_K (z^i_K)^2 - \Big(\frac{1}{2} - C_i\Big)\sum_{K\in\mathscr{T}} \mathrm{m}(K) f_K z^i_K,\\ &\le \frac{1}{2}\|f\|_\infty E_i + \frac{1}{2}\|f\|_\infty \int_\Omega z^i_{\mathscr{T}}.\end{aligned}$$

Combined with (12), (19) and (20), this yields

$$\sum_{\sigma\in\mathscr{E}} \tau_\sigma (D_\sigma z^i)^2 \le \frac{\|f\|_\infty}{\beta}\left(1 + 2^{2i+3}\right)E_{i-1}.$$

From (21), we deduce

$$E_i \le \frac{C}{\xi}\frac{\|f\|_\infty}{\beta}(1 + 2^{2i+3})2^{2(i+2)/q}E_{i-1}^{1+1/q^*}.$$

Thus the sequence $(E_i)_{i\ge 0}$ satisfies a relation $E_i \le KD^iE_i^\gamma$ with $D > 1$ and $\gamma > 1$. As shown for instance in [5], there exists $\delta > 0$ such that for an initial condition small enough, $E_0 \le \delta^2$, the sequence converges to 0. This proves the lemma. □

3 A discrete counterpart of De Giorgi's isoperimetric inequality

Lemma 3. *Let* $w_{\mathscr{M}} = (w_{\mathscr{T}}, w_{\mathscr{E}^D})$ *satisfying* $w_{\mathscr{E}^D} \le 0$. *Let us define:*

$$\mathscr{A} = \{x \in \Omega; w_{\mathscr{T}}(x) \ge 1/4\} \quad \text{and} \quad \mathscr{B} = \{x \in \Omega; 0 < w_{\mathscr{T}}(x) < 1/4\}.$$

We assume that there exists $\mu > 0$ *and* $C_0 > 0$ *such that:*

$$\mathrm{m}(\mathscr{A}) \ge \mu \quad \text{and} \quad \sum_{\sigma\in\mathscr{E}} \tau_\sigma (D_\sigma w^+)^2 \le C_0. \tag{22}$$

There exists $h_0 > 0$ *and* $\varepsilon > 0$ *only depending on* ξ, Ω, Γ^D, C_0 *and* μ *such that if* $h \le h_0$ *then* $\mathrm{m}(\mathscr{B}) \ge \varepsilon$.

Proof. Let us first recall Lemma 4.1 in [1] (see also Theorem 5.11.1 in [6]). There exists C depending only on Ω and Γ^D such that, for all $u_{\mathscr{T}} \in X(\mathscr{T})$:

$$\|u_{\mathscr{T}}\|_{L^{N/(N-1)}(\Omega)} \leq C\Big(\sum_{\sigma\in\mathscr{E}_{int}} \mathrm{m}(\sigma)|D_\sigma u| + \sum_{\sigma\in\mathscr{E}_{ext}^D} \mathrm{m}(\sigma)|u_K|\Big). \tag{23}$$

For all $\zeta \in (0, \frac{1}{4})$, we define the set $E_\zeta = \{x \in \Omega;\ w_{\mathscr{T}}(x) > \zeta\}$ which contains $\mathscr{A}$. The characteristic function of E_ζ belongs to $X(\mathscr{T})$ and therefore verifies (23). As $N = 2$ and using (22), we get:

$$\sqrt{\mu} \leq \big(\mathrm{m}(E_\zeta)\big)^{1/2} \leq C\Big(\sum_{\substack{\sigma=K|L\in\mathscr{E}_{int};\\ w_K\leq\zeta<w_L}} \mathrm{m}(\sigma) + \sum_{\substack{\sigma\in\mathscr{E}_{ext}^D;\\ w_K>\zeta}} \mathrm{m}(\sigma)\Big). \tag{24}$$

Let us introduce $\mathscr{C} = \{x \in \Omega; w_{\mathscr{T}}(x) \leq 0\}$. For a Dirichlet boundary edge $\sigma \subset \Gamma^D$ with $\sigma \in \mathscr{E}_K$, we write that $\sigma \in \mathscr{E}_{ext,\mathscr{A}}^D$ if $K \subset \mathscr{A}$, $\sigma \in \mathscr{E}_{ext,\mathscr{B}}^D$ if $K \subset \mathscr{B}$ and $\sigma \in \mathscr{E}_{ext,\mathscr{C}}^D$ if $K \subset \mathscr{C}$. Integrating (24) over $\zeta \in (0, \frac{1}{4})$, we get :

$$\begin{aligned}
\frac{1}{4}\sqrt{\mu} \leq C\Big(&\sum_{\substack{\sigma=K|L\\ K\subset\mathscr{B},\ L\subset\mathscr{B}\cup\mathscr{C}}} \mathrm{m}(\sigma)|w_K^+ - w_L^+| + \sum_{\substack{\sigma=K|L\\ K\subset\mathscr{B},\ L\subset\mathscr{A}}} \mathrm{m}(\sigma)|w_K^+ - \frac{1}{4}| \\
&+ \sum_{\substack{\sigma=K|L\\ K\subset\mathscr{A},\ L\subset\mathscr{C}}} \frac{1}{4}\mathrm{m}(\sigma) + \sum_{\sigma\in\mathscr{E}_{ext,\mathscr{B}}^D} \mathrm{m}(\sigma)|w_K^+| + \sum_{\sigma\in\mathscr{E}_{ext,\mathscr{A}}^D} \mathrm{m}(\sigma)\frac{1}{4}\Big) \\
\leq C\Big(&\sum_{\substack{\sigma=K|L\\ K\subset\mathscr{B},\ L\subset\mathscr{A}\cup\mathscr{B}\cup\mathscr{C}}} \mathrm{m}(\sigma)|w_K^+ - w_L^+| + \sum_{\sigma\in\mathscr{E}_{ext,\mathscr{B}}^D} \mathrm{m}(\sigma)|w_K^+| \\
&+ \sum_{\substack{\sigma=K|L\\ K\subset\mathscr{A},\ L\subset\mathscr{C}}} \mathrm{m}(\sigma)|w_K^+ - w_L^+| + \sum_{\sigma\in\mathscr{E}_{ext,\mathscr{A}}^D} \mathrm{m}(\sigma)|w_K^+|\Big).
\end{aligned}$$

We apply now Cauchy-Schwarz' inequality to the right-hand side, which yields:

$$\frac{\mu}{16} \leq CC_0\Big(\sum_{\sigma\subset\bar{B}} \mathrm{m}(\sigma)\mathrm{d}_\sigma + \sum_{\substack{\sigma=K|L\\ K\subset\mathscr{A},\ L\subset\mathscr{C}}} \mathrm{m}(\sigma)\mathrm{d}_\sigma + \sum_{\sigma\in\mathscr{E}_{ext,\mathscr{A}}^D} \mathrm{m}(\sigma)\mathrm{d}_\sigma\Big).$$

On the one hand, the hypothesis (4) ensures that $\sum_{\sigma\subset\bar{B}} \mathrm{m}(\sigma)\mathrm{d}_\sigma \leq \mathrm{m}(B)/\xi$. On the other hand, if $\sigma = K|L$ with $K \subset \mathscr{A}$ and $L \subset \mathscr{C}$ or if $\sigma \in \mathscr{E}_{ext,\mathscr{A}}^D$, we have $|D_\sigma w^+| \geq 1/4$ and

$$\sum_{\substack{\sigma=K|L\\ K\subset\mathscr{A},\ L\subset\mathscr{C}}} \mathrm{m}(\sigma)\mathrm{d}_\sigma + \sum_{\sigma\in\mathscr{E}_{ext,\mathscr{A}}^D} \mathrm{m}(\sigma)\mathrm{d}_\sigma \leq \frac{16C_0}{\xi^2}h^2.$$

Hence, $\dfrac{\mu}{16} \leq CC_0\left(\dfrac{\mathrm{m}(B)}{\xi} + \dfrac{16C_0}{\xi^2}h^2\right)$, so that $\mathrm{m}(B) \geq \dfrac{\xi\mu}{16CC_0} - \dfrac{16C_0}{\xi}h^2$. $\square$

4 Proof of Theorem 1

Let $\delta > 0$, given by Lemma 2, and let us assume that we have some $j_0 \geq 0$ such that

$$\sum_{K\in\mathscr{T}} \mathrm{m}(K)((w_K^{j+1})^+)^2 \geq \delta^2, \qquad \text{for } 0 \leq j \leq j_0 - 1. \tag{25}$$

Since $w_{\mathscr{T}}^j \leq \frac{1}{2}$ for all $j \geq 0$, we have

$$\sum_{K\in\mathscr{T}} \mathrm{m}(K)((w_K^{j+1})^+)^2 \leq \frac{1}{4}\mathrm{m}\left(\left\{w_{\mathscr{T}}^{j+1} \geq 0\right\}\right) \leq \frac{1}{4}\mathrm{m}\left(\left\{w_{\mathscr{T}}^{j} \geq \frac{1}{4}\right\}\right).$$

Hence, $\mathrm{m}(\{w_{\mathscr{T}}^j \geq 1/4\}) \geq 4\delta^2$ for $0 \leq j \leq j_0$. Applying Lemma 3 with $\mu = 4\delta^2$ and C_0 given by Lemma 1, there exists $\varepsilon = \varepsilon(C_0,\delta) > 0$ such that if $h < h_0 = h_0(C_0,\delta,\xi)$ we have $\mathrm{m}(\{0 < w_{\mathscr{T}}^j < \frac{1}{4}\}) \geq \varepsilon$. Since $w_{\mathscr{T}}^{j+1} \leq 0$ if and only if $w_{\mathscr{T}}^j \leq \frac{1}{4}$, we deduce

$$\mathrm{m}(\{w_{\mathscr{T}}^{j+1} \leq 0\}) \geq \mathrm{m}(\{w_{\mathscr{T}}^{j} \leq 0\}) + \mathrm{m}(\{0 < w_{\mathscr{T}}^{j} < \frac{1}{4}\}) \geq \mathrm{m}(\{w_{\mathscr{T}}^{j} \leq 0\}) + \varepsilon.$$

By induction, we get $\mathrm{m}(\{w_{\mathscr{T}}^{j_0} \leq 0\}) \geq \mathrm{m}(\{w_{\mathscr{T}}^{0} \leq 0\}) + j\varepsilon$ and since Ω is bounded,

$$j_0 < \mathrm{m}(\Omega)/\varepsilon.$$

We conclude that (25) is wrong for $j_0 = j_0^\star := \lceil \mathrm{m}(\Omega)/\varepsilon \rceil$. From Lemma 2, we get $w_{\mathscr{T}}^{j_0^\star} \leq 1/4$, that is $v_{\mathscr{T}} \geq 2^{-j_0^\star - 2}$. This concludes the proof of Theorem 1.

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