Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation

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Abstract: Motivated by the critical dissipative quasi-geostrophic equation, we prove that drift-diffusion equations with L^2 initial data and minimal assumptions on the drift are locally Holder continuous. As an application we show that solutions of the quasi-geostrophic equation with initial L^2 data and critical diffusion $(-\Delta)^{1/2}$, are locally smooth for any space dimension.

1 Introduction

Non linear evolution equations with fractional diffusion arise in many contexts: In the quasi-geostrophic flow model (Constantin [3]), in boundary control problems (Duvaut-Lions [8]), in surface flame propagation and in financial mathematics. In this paper, motivated by the quasi-geostrophic model, we study the equation:

$$\begin{aligned} &\partial_t\theta + v\cdot\nabla\theta = -\Lambda\theta, \qquad x\in\mathbb{R}^N,\\ &\mathrm{div} v = 0, \end{aligned} \tag{1}$$

where $\Lambda\theta = (-\Delta)^{1/2}\theta$. The main two theorems are roughly the following a priori estimates:

Theorem 1 *(From L^2 to L^∞). Let $\theta(t,x)$ be a function in $L^\infty(0,T;L^2(\mathbb{R}^N))\cap L^2(0,T;H^{1/2}(\mathbb{R}^N))$. For every $\lambda>0$ we define:*

$$\theta_\lambda = (\theta-\lambda)_+.$$

If θ (and $-\theta$) satisfies for every $\lambda>0$ the level set energy inequalities:

$$\begin{aligned} &\int_{\mathbb{R}^N}\theta_\lambda^2(t_2,x)\,dx + 2\int_{t_1}^{t_2}\int_{\mathbb{R}^N}|\Lambda^{1/2}\theta_\lambda|^2\,dx\,dt\\ &\qquad\le \int_{\mathbb{R}^N}\theta_\lambda^2(t_1,x)\,dx, \qquad 0<t_1<t_2, \end{aligned}$$

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then:

$$\sup_{x\in\mathbb{R}^N} |\theta(T,x)| \le C^* \frac{\|\theta_0\|_{L^2}}{T^{N/2}}.$$

Remark: That solutions to equation (1) are expected to satisfy the energy inequality follows from writing Λ as the normal derivative of the harmonic extension of θ to the upper half space. Existence theory is sketching in appendix C. In the case of Quasi-geostrophic equation it can also be seen as a corollary of Cordoba and Cordoba [6].

Those energy inequalities are reminiscent of the notion of entropic solutions for scalar conservation laws. Consider a weak solution of (1) lying in $L^2(H^{1/2})$ and for which we can define the equality (in the sense of distribution for example):

$$\phi'(\theta) v \cdot \nabla\theta = \mathrm{div}(v\phi(\theta)),$$

for any Lipschitz function ϕ. Then θ verifies the level set energy inequalities. In the case of the Quasi-geostrophic equation, $v \in L^2(H^{1/2})$ and we can give a sense to:

$$v \cdot \nabla\phi(\theta).$$

Indeed, using the harmonic extension, it can be shown that if θ lies in $L^2(H^{1/2})$ so does $\phi(\theta)$. and so $\nabla\phi(\theta)$ lies in $L^2(H^{-1/2})$.

For the second theorem, (from L^∞ to C^α), we need better control of v:

Theorem 2 *(From L^∞ to C^α). We define $Q_r = [-r,0]\times[-r,r]^N$, for $r>0$. Assume now that $\theta(t,x)$ is bounded in $[-1,0]\times\mathbb{R}^N$ and $v|_{Q_1} \in L^\infty(-1,0;BMO)$, then θ is C^α in $Q_{1/2}$.*

Remark1: The global bound of θ is not really necessary, only local L^∞ and integrability at infinity against the Poisson kernel, as we will see later.

Remark2: Note that both theorems depend only on the resulting energy inequality and not on the special form of Λ.

From these two theorems, the regularity of solutions to the quasi-geostrophic equation follows.

Theorem 3 *Let θ be a solution to an equation*

$$\begin{aligned} &\partial_t\theta + u\cdot\nabla\theta = -\Lambda\theta, \qquad x\in\mathbb{R}^N, \\ &\mathrm{div} u = 0, \end{aligned} \tag{2}$$

with

$$u_j = \overline{R}_j[\theta], \tag{3}$$

$\overline{R}_j$ a singular integral operator. Assume also that θ verifies the level set energy inequalities stated in Theorem 1. Then, for every $t_0>0$ there exists α such that θ is bounded in $C^\alpha([t_0,\infty[\times\mathbb{R}^N)$.

Indeed, Theorem 1 gives that θ is uniformly bounded on $[t_0, \infty[$ for every $t_0 > 0$. Singular integral operators are bounded from L^∞ to BMO. This gives that $u \in L^\infty(t_0, \infty; BMO(\mathbb{R}^N))$ and, after proper scaling, Theorem 2 gives the result of Theorem 3.

Remark 1: Higher regularity then follows from standard potential theory, by noticing that the fundamental solution of the operator:

$$\partial_t + \Lambda\theta = 0$$

is the Poisson kernel and that in the non linear term we can substract a constant both θ_0 from θ and u_0 from u, this last one by a change of coordinates:

$$x^* = x - tu_0,$$

doubling its Holder decay (see appendix).

Strictly speaking, the dissipative quasi-geostrophic flow model in the critical case corresponds to the case $N = 2$ and

$$u_1 = -R_2\theta,$$
$$u_2 = R_1\theta,$$

where R_i is the usual Riesz transform defined from the Fourier transform: $\widehat{R_i\theta} = \frac{i\xi_i}{|\xi|}\widehat{\theta}$. This model was introduced by some authors as a toy model to investigate the global regularity of solutions to 3D fluid mechanics (see for instance [3]). When replacing the diffusion term $-\Lambda$ by $-\Lambda^\beta$, $0 \leq \beta \leq 2$, the situation is classically decomposed into 3 cases: The subcritical case for $\beta > 1$, the critical case for $\beta = 1$ and the supercritical case for $\beta < 1$.

Weak solutions has been constructed by Resnick in [11]. Constantin and Wu showed in [5] that in the subcritical case any solution with smooth initial value is smooth for all time. Constantin Cordoba and Wu showed in [4] that the regularity is conserved for all time in the critical case provided that the initial value is small in L^∞. In both the critical case and supercritical cases, Chae and Lee considered in [2] the well-posedness of solutions with initial conditions small in Besov spaces (see also Wu [15]).

Notice that our case corresponds to the critical case and global regularity in $C^{1,\beta}$, $\beta < 1$ is showed for any initial value in the energy space without hypothesis of smallness. This ensures that the solutions are classical.

Let us also cite a result of maximum principle due to Cordoba and Cordoba [6], results of behavior in large time due to Schonbeck and Schonbeck [13], [12], and a criteria for blow-up in Chae [1].

Remark 2: In a recently posted preprint in arXiv, Kiselev, Nazarov, and Volberg present a very elegant proof of the fact that in 2D, solutions with periodic C^∞ data for the quasi-geostrophic equation remain C^∞ for all time ([9]).

We conclude our introduction by pointing out that our techniques also can be seen as a parabolic De Giorgi Nash Moser method to treat "boundary parabolic problems" of the type:

$$
\begin{aligned}
\operatorname{div}(a\nabla\theta) = 0, &\qquad \text{in } \ \Omega\times[0,T]\\
[f(\theta)]_t = \theta_\nu &\qquad \text{on } \ \partial\Omega\times[0,T],
\end{aligned}
$$

that arise in boundary control (see Duvaut Lions [8]). Note also that similar results to Theorem 1 can be obtained even for systems (See Vasseur [14] and Mellet, Vasseur [10] for applications of the method in fluid mechanics).

2 L^∞ bounds

This section is devoted to the proof of Theorem 1. We use the level set energy inequality for:

$$\lambda = C_k = M(1-2^{-k}),$$

where M will be chosen later. This leads to the following energy inequality for the level set function $\theta_k = (\theta - C_k)_+$:

$$\partial_t \int_{\mathbb{R}^N} {\theta_k}^2\,dx + 2\int_{\mathbb{R}^N} |\Lambda^{1/2}\theta_k|^2\,dx \le 0. \tag{4}$$

Let us fix a $t_0 > 0$, we want to show that θ is bounded for $t > t_0$. We introduce $T_k = t_0(1-2^{-k})$, and the level set of energy/dissipation of energy:

$$U_k = \sup_{t\ge T_k}\left(\int_{\mathbb{R}^N} {\theta_k}^2\,dx\right) + 2\int_{T_k}^{\infty}\int_{\mathbb{R}^N} |\Lambda^{1/2}\theta_k|^2\,dx\,dt.$$

integrating (4) in time between s, $T_{k-1} < s < T_k$, and $t > T_k$ and between s and $+\infty$ we find:

$$U_k \le 2\int_{\mathbb{R}^N} {\theta_k}^2(s)\,dx.$$

Taking the mean value in s on $[T_{k-1}, T_k]$ we find:

$$U_k \le \frac{2^{k+1}}{t_0}\int_{T_{k-1}}^{\infty}\int_{\mathbb{R}^N} {\theta_k}^2\,dx\,dt. \tag{5}$$

We want to control the right-hand side by U_{k-1} in a non linear way. Sobolev and Holder inequalities give:

$$U_{k-1} \ge C\|\theta_{k-1}\|^2_{L^{\frac{2(N+1)}{N}}(]T_{k-1},\infty[\times\mathbb{R}^N)}.$$

Note that if $\theta_k > 0$ then $\theta_{k-1} \ge 2^{-k}M$. So

$$\mathbf{1}_{\{\theta_k>0\}} \le \left(\frac{2^k}{M}\theta_{k-1}\right)^{2/N}.$$

Hence:

$$\begin{aligned} U_k &\le \frac{2^{k+1}}{t_0}\int_{T_{k-1}}^{\infty}\int_{\mathbb{R}^N}\theta_{k-1}^2\mathbf{1}_{\{\theta_k>0\}}\,dx\,dt\\ &\le 2\frac{2^{\frac{N+2}{N}k}}{t_0M^{2/N}}\int_{T_{k-1}}^{\infty}\int_{\mathbb{R}^N}\theta_{k-1}^{2\frac{N+1}{N}}\,dx\,dt\\ &\le 2C\frac{2^{\frac{N+2}{N}k}}{t_0M^{2/N}}U_{k-1}^{\frac{N+1}{N}}. \end{aligned}$$

For M such that $M/t_0^{N/2}$ is big enough (depending on U_0) we have U_k which converges to 0. This gives $\theta \le M$ for $t \ge t_0$. The same proof on $-\theta$ gives the same bound for$|\theta|$. Note that $U_0 \le \|\theta_0\|_{L^2}^2$. The scaling invariance $\theta_\varepsilon(s,y) = \theta(\varepsilon s, \varepsilon y)$ gives the final dependence with respect to $\|\theta_0\|_{L^2}$. □

This theorem leads to the following corollary.

Corollary 4 *There exists a constant $C^* > 0$ such that any solution θ of (2) (3) verifies:*

$$\sup_{x\in\mathbb{R}^N}|\theta(T,x)| \le C^*\frac{\|\theta_0\|_{L^2(\mathbb{R}^N)}}{T^{N/2}},$$

$$\|u(T,\cdot)\|_{BMO(\mathbb{R}^N)} \le C^*\frac{\|\theta_0\|_{L^2(\mathbb{R}^N)}}{T^{N/2}}.$$

Proof. First note that the property on u follows directly from the property on θ and the imbedding of the Riesz function from L^∞ to BMO. We make use of the following result of Cordoba and Cordoba (see [6]): for any convex function ϕ we have the pointwise inequality:

$$-\phi'(\theta)\Lambda\theta \le -\Lambda(\phi(\theta)).$$

Making use of this inequality with:

$$\phi_k(\theta) = (\theta - C_k)_+ = \theta_k$$

leads to:

$$\partial_t\theta_k + u\cdot\nabla\theta_k \le -\Lambda\theta_k.$$

Multiplying by θ_k and integrating in x gives (4), using that u is divergence free. □

Remark: We point out that the level set energy inequalities we assume in Theorem 1 is heuristically a general fact (See appendix C).

3 Local energy inequality:

To get the C^α regularity we need to get an energy inequality which is local in time and space. Due to the non locality of the diffusion operator this cannot be obtained directly. It relies on the following classical representation of the operator Λ. We introduce first the harmonic extension L defined from $C_0^\infty(\mathbb{R}^N)$ to $C_0^\infty(\mathbb{R}^N \times \mathbb{R}^+)$ by:

$$\begin{aligned} -\Delta L(\theta) = 0 \qquad & \text{in } \ \mathbb{R}^N \times (0,\infty), \\ L(\theta)(x,0) = \theta(x) \qquad & \text{for } \ x \in \mathbb{R}^N. \end{aligned}$$

Then the following result holds true: consider θ defined on $\mathbb{R}^N$. then:

$$\Lambda\theta(x) = \partial_\nu[L\theta](x), \tag{6}$$

where we denote $\partial_\nu[L\theta]$ the normal derivative of $L\theta$ on the boundary $\{(x,0)|x \in \mathbb{R}^N\}$.

In the following, we will denote:

$$\theta^*(t,x,z) = L(\theta(t,\cdot))(x,z). \tag{7}$$

We denote $B_r = [-r,r]^N$, $B_r^* = B_r \times (0,r) \in \mathbb{R}^N \times (0,\infty)$, and $[y]_+ = \sup(0,y)$.

The rest of this section is devoted to the proof of the following proposition:

Proposition 5 *Let t_1, t_2 be such that $t_1 < t_2$ and let $\theta \in L^\infty(t_1,t_2;L^2(\mathbb{R}^N))$ with $\Lambda^{1/2}\theta \in L^2((t_1,t_2)\times\mathbb{R}^N)$, be solution to (1) with a velocity v satisfying:*

$$\|v\|_{L^\infty(t_1,t_2;BMO(\mathbb{R}^N))} + \sup_{t_1\le t\le t_2}\left|\int_{B_2} v(t,x)\,dx\right| \le C. \tag{8}$$

Then there exists a constant Φ (depending only on C) such that for every $t_1 \le t \le t_2$ and cut-off function η compactly supported in B_2^*:

$$\begin{aligned} &\int_{t_1}^{t_2}\int_{B_2^*} |\nabla(\eta[\theta^*]_+)|^2\,dx\,dz\,dt + \int_{B_2}(\eta[\theta]_+)^2(t_2,x)\,dx \\ &\le \int_{B_2}(\eta[\theta]_+)^2(t_1,x)\,dx + \Phi\int_{t_1}^{t_2}\int_{B_2}([\nabla\eta][\theta]_+)^2\,dx\,dt \\ &\qquad + \int_{t_1}^{t_2}\int_{B_2^*}([\nabla\eta][\theta^*]_+)^2\,dx\,dz\,dt. \end{aligned} \tag{9}$$

Proof. We have for every $t_1 < t < t_2$:

$$\begin{aligned} 0 &= \int_0^\infty\int_{\mathbb{R}^N} \eta^2[\theta^*]_+\Delta\theta^*\,dx\,dz \\ &= -\int_0^\infty\int_{\mathbb{R}^N}|\nabla(\eta[\theta^*]_+)|^2\,dx\,dz + \int_0^\infty\int_{\mathbb{R}^N}|\nabla\eta|^2[\theta^*]_+^2\,dx\,dz \\ &\qquad + \int_{\mathbb{R}^N}\eta^2[\theta]_+\Lambda\theta\,dx. \end{aligned}$$

Using equation (1), we find that:

$$
\begin{aligned}
&-\int_{\mathbb{R}^N} \eta^2 [\theta]_+ \Lambda\theta \, dx \\
&= \frac{\partial}{\partial t}\left(\int_{\mathbb{R}^N} \eta^2 \frac{[\theta]_+^2}{2}\, dx\right) - \int_{\mathbb{R}^N} \nabla\eta^2 \cdot v \frac{[\theta]_+^2}{2}\, dx.
\end{aligned}
$$

This leads to:

$$
\begin{aligned}
&\int_{t_1}^{t_2}\int_0^\infty\int_{\mathbb{R}^N} |\nabla(\eta[\theta^*]_+)|^2 \, dx\, dz\, ds + \int_{\mathbb{R}^N} \eta^2 \frac{[\theta]_+^2(t_2)}{2}\, dx \\
&\leq \int_{\mathbb{R}^N} \eta^2 \frac{[\theta]_+^2(t_1)}{2}\, dx + \int_{t_1}^{t_2}\int_0^\infty\int_{\mathbb{R}^N} |\nabla\eta|^2 [\theta^*]_+^2 \, dx\, dz\, ds \\
&\quad + \left|\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \eta\nabla\eta \cdot v[\theta]_+^2 \, dx\, ds\right|.
\end{aligned}
$$

To dominate the last term, we first use the Trace theorem and Sobolev imbedding to find:

$$
\begin{aligned}
\|\eta\theta_+\|^2_{L^{\frac{2N}{N-1}}(\mathbb{R}^N)} &\leq C \quad \|\eta\theta_+\|^2_{H^{1/2}(\mathbb{R}^N)} = C\int_{B_2} (\eta\theta_+^*)\Lambda(\eta\theta_+^*)\, dx \\
&= \quad C\int_{B_2^*} |\nabla L(\eta(\theta^*)_+)|^2 \, dx\, dz \\
&\leq \quad C\int_{B_2^*} |\nabla[\eta(\theta^*)_+]|^2 \, dx\, dz.
\end{aligned}
$$

In the last inequality we have used the fact that $L(\eta(\theta^*)_+)$ is harmonic and have the same trace than $\eta(\theta^*)_+$ at $z = 0$. Therefore we split:

$$
\begin{aligned}
&\left|\int_{t_1}^{t_2}\int_{\mathbb{R}^N} \nabla\eta^2 \cdot v\frac{[\theta]_+^2}{2}\, dx\, ds\right| \\
&\leq \varepsilon \int_{t_1}^{t_2} \|\eta\theta_+\|^2_{L^{\frac{2N}{N-1}}(\mathbb{R}^N)}\, ds + \frac{1}{\varepsilon}\int_{t_1}^{t_2} \|[\nabla\eta]v[\theta]_+\|^2_{L^{\frac{2N}{N+1}}}\, ds.
\end{aligned}
$$

The first term is absorbed by the left. The second can be bounded, using Holder inequality, by:

$$
\frac{1}{\varepsilon}\|v\|^2_{L^\infty(t_1,t_2;L^{2N}(B_2))}\int_{t_1}^{t_2}\int_{\mathbb{R}^N} |[\nabla\eta][\theta]_+|^2 \, dx\, ds,
$$

which gives the desired result. □

4 From L^2 to L^∞:

In these two sections (4 and 5) we follow De Giorgi's ideas in his "oscillation lemma" (see [7]) to prove Holder continuity: Suppose that θ oscillates in

$Q_1 = [-1,0]\times B_1$ between -2 and 2, but it is negative most of the time. In particular, if $\|\theta_+\|_{L^2}$ is very small, then we prove that $\theta_+|_{Q_{1/2}=[-1/2,0]\times B_{1/2}} \le 2-\lambda$, effectively reducing the oscillations of θ by λ (see section 4). Of course, we do not know a priori that this is the case. But we do know that in Q_1, θ is at least half of the time positive, or negative, say negative. Then we reproduce a version of De Giorgi's isoperimetric inequality that says that to go from zero to one θ needs "some room" (section 5). Therefore the set $\{\theta \le 1\}$ is "strictly larger" than the set $\{\theta \le 0\}$. Repeating this argument at truncation levels $\sigma_k = 2-2^{-k}$, we fall, after a finite number of steps, k_0, into the first case, effectively diminishing the oscillations of θ by $\lambda 2^{-k_0}$. This implies Holder continuity (section 6 and section 7).

This section is devoted to the proof of the following technical lemma. It says that, under suitable conditions on v, we can control the L^∞ norm of θ from the L^2 norm of both θ and θ^* locally.

Lemma 6 *There exists $\varepsilon_0 > 0$ and $\lambda > 0$ such that for every θ solution to (1) with a velocity v satisfying:*

$$\|v\|_{L^\infty(-4,0;BMO(\mathbb{R}^N))} + \sup_{-4\le t\le 0}\left|\int_{B_4} v(t,x)\,dx\right| \le C,$$

the following property holds true.

If we have:

$$\theta^* \le 2 \qquad \text{in } \ [-4,0]\times B_4^*,$$

and

$$\int_{-4}^0\int_{B_4^*}(\theta^*)_+^2\,dx\,dz\,ds + \int_{-4}^0\int_{B_4}(\theta)_+^2\,dx\,ds \le \varepsilon_0,$$

then:

$$(\theta)_+ \le 2-\lambda \qquad \text{on } \ [-1,0]\times B_1.$$

Proof. We split the proof of the lemma into several steps.

Step1. Useful barrier functions and setting of the constant λ: Consider the function b_1, defined by:

$$\begin{aligned} &\Delta b_1 = 0 && \text{in } \ B_4^* \\ &b_1 = 2 && \text{on the sides of the cube } B_4^* \text{ except for } z=0 \\ &b_1 = 0 && \text{for } z = 0. \end{aligned}$$

Then there exists $\lambda > 0$ such that:

$$b_1(x,z) \le 2-4\lambda \text{ on } B_2^*.$$

This result follows directly from the maximum principle. We consider now b_2 harmonic function defined by:

$$\begin{aligned} &\Delta b_2 = 0 \qquad \text{in } \ [0,\infty[\times[0,1], \\ &b_2(0,z) = 2 \qquad 0\le z\le 1, \\ &b_2(x,0) = b_2(x,1) = 0 \qquad 0 < x < \infty. \end{aligned}$$

Then there exists $\overline{C} > 0$ such that:

$$|b_2(x,z)| \leq \overline{C} e^{-\pi x}. \tag{10}$$

Notice that $\overline{C}$ is universal. Actually b_2 can be explicitly computed by the method of separation of variables:

$$b_2(x,z) = \sum_{p=0}^{\infty} \frac{4}{\pi(2p+1)} e^{-[(2p+1)\pi]x} \sin((2p+1)\pi z).$$

Step 2. Setting of constants: In this step we fix a set of constants. We make the choice to set them right away to convince the reader that there is no loop in the proof.

Lemma 7 *There exist* $0 < \delta < 1$ *and* $M > 1$ *such that for every* $k > 0$*:*

$$\begin{aligned} &N\overline{C} e^{-\pi \frac{2^{-k}}{\delta^k}} \leq \lambda 2^{-k-2}, \\ &\frac{M^{-k}}{\delta^{k+1}} \|P(1)\|_{L^2} \leq \lambda 2^{-k-2}, \\ &M^{-k} \geq C_0^k M^{-(1+1/N)(k-3)} \qquad k \geq 12N, \end{aligned}$$

where $\overline{C}$ *is defined from step 1,* $P(1)$ *is the the value at* $z = 1$ *of the Poisson kernel* $P(z)(x)$*, and* C_0 *is defined by (16).*

The proof is easy. We construct first δ to verify the first inequality in the following way. If $\delta < 1/4$, the inequality is true for $k > k_0$ due to the exponential decay. If necessary, we then choose δ smaller to make the inequality also valid for $k < k_0$. Now that δ has been fixed, we have to choose M large to satisfy the remaining inequalities. Note that the second inequality is equivalent to:

$$\left(\frac{2}{\delta M}\right)^k \leq \frac{\lambda \delta}{4\|P(1)\|_{L^2}}.$$

It is so sufficient to take:

$$M \geq \sup\left(\frac{2}{\delta}, \frac{8\|P(1)\|_{L^2}}{\lambda \delta^2}\right).$$

The third inequality is equivalent to:

$$\left(\frac{M}{C_0^N}\right)^{k/N} \geq M^{3(1+1/N)}.$$

For this case it is sufficient to take $M \geq \sup(1, C_0^{2N})$. Indeed, this ensures $M^2/C_0^{2N} \geq M$ and so:

$$\left(\frac{M}{C_0^N}\right)^{k/N} \geq M^{k/(2N)} \geq M^6,$$

for $k \geq 12N$. But $M^6 \geq M^{3(1+1/N)}$ for $M \geq 1$ and $N \geq 2$.

Therefore we can fix:

$$M = \sup\left(1, C_0^{2N}, \frac{2}{\delta}, \frac{8\|P(1)\|_{L^2}}{\lambda\delta^2}\right).$$

The constant λ, δ, and M are now fixed for the rest of the proof. The constant ε_0 will be constructed from those ones.

Step 3. Induction: We set:

$$\theta_k = (\theta - C_k)_+,$$

with $C_k = 2 - \lambda(1 + 2^{-k})$. We consider a cut-off function in x only such that:

$$\mathbf{1}_{\{B_{1+2^{-k-1}}\}} \leq \eta_k \leq \mathbf{1}_{\{B_{1+2^{-k}}\}},$$
$$|\nabla\eta_k| \leq C2^k,$$

and we denote:

$$A_k = 2\int_{-1-2^{-k}}^{0}\int_0^{\delta^k}\int_{\mathbb{R}^N} |\nabla(\eta_k\theta_k^*)|^2\,dx\,dz\,dt + \sup_{[-1-2^{-k},1]}\int_{\mathbb{R}^N}(\eta_k\theta_k)^2\,dx\,dt.$$

We want to prove simultaneously that for every $k \geq 0$:

$$A_k \leq M^{-k} \tag{11}$$
$$\eta_k\theta_k^* \text{ is supported in } 0 \leq z \leq \delta^k. \tag{12}$$

Step 4. Initial step: We prove in this step that (11), is verified for $0 \leq k \leq 12N$, and that (12) is verified for $k = 0$. We use the energy inequality (9) with cut-off function $\eta_k(x)\psi(z)$ where ψ is a fixed cut-off function in z only. Taking the mean value of (9) in t_1 between -4 and -2, we find that (11) is verified for $0 \leq k \leq 12N$ if ε_0 is taken such that:

$$C2^{24N}(1 + \Phi)\varepsilon_0 \leq M^{-12N}. \tag{13}$$

We have used that $|\nabla\eta_k|^2 \leq C2^{24N}$ for $0 \leq k \leq 12N$. Let us consider now the support property (12). By the maximum principle, we have:

$$\theta^* \leq (\theta_+\mathbf{1}_{B_4}) * P(z) + b_1(x, z),$$

in $\mathbb{R}^+ \times B_4^*$, where $P(z)$ is the Poisson kernel. Indeed, the right-hand side function is harmonic, positive and the trace on the boundary is bigger that the one of θ^*.

From step 1 we have: $b_1(x, z) \leq 2 - 4\lambda$. Moreover:

$$\|\theta_+\mathbf{1}_{B_4} * P(z)\|_{L^\infty(z\geq 1)} \leq C\|P(1)\|_{L^2}\sqrt{\varepsilon_0} \leq C\sqrt{\varepsilon_0}.$$

Choosing ε_0 small enough such that this constant is smaller that 2λ gives:

$$\theta^* \leq 2 - 2\lambda \qquad \text{for } \; z \geq 1, t \geq 0, x \in B,$$

so:

$$\theta_0^* = (\theta^* - (2 - 2\lambda))_+ \leq 0 \qquad \text{for } \; z \geq 1, t \geq 0, x \in B.$$

Hence $\eta_0 \theta_0^*$ is supported in $0 \leq z \leq \delta^0 = 1$.

Step 5. Propagation of the support property (12): Assume that (11) and (12) are verified at k. We want to show that (12) is verified at $(k+1)$. We will show also that the following is verified at k:

$$\eta_k \theta_{k+1}^* \leq [(\eta_k \theta_k) * P(z)] \eta_k. \tag{14}$$

We consider the set $\overline{B}_k^* = B_{1+2^{-k}} \times [0, \delta^k]$, and we want to control θ_k^* on this set by harmonic functions taking into account the contributions of the sides one by one. On $z = \delta^k$ we have no contribution thanks to the induction property (12) at k (the trace is equal to 0). The contribution of the side $z = 0$ can be controlled by: $\eta_k \theta_k * P(z)$ (It has the same trace than θ_k on $B_{1+2^{-k-1}}$).

On each of the other side we control the contribution by:

$$b_2((x_i - x^+)/\delta^k, z/\delta^k) + b_2((-x_i + x^-)/\delta^k, z/\delta^k),$$

where $x^+ = (1 + 2^{-k})$ and $x^- = -x^+$. Indeed, b_2 is harmonic, and on the side x_i^+ and x_i^- it is bigger than 2. Finally, by the maximum principle:

$$\theta_k^* \leq \sum_{i=1}^{N} \left[b_2((x_i - x^+)/\delta^k, z/\delta^k) + b_2((-x_i + x^-)/\delta^k, z/\delta^k) \right] + (\eta_k \theta_k) * P(z).$$

From Step 1, for $x \in B_{1+2^{-k}}$:

$$\begin{aligned} &\sum_{i=1}^{N} \left[b_2((x_i - x^+)/\delta^k, z/\delta^k) + b_2((-x_i + x^-)/\delta^k, z/\delta^k) \right] \\ &\qquad \leq N \overline{C} e^{-\frac{\pi 2^{-k}}{\delta^k}} \\ &\qquad \leq \lambda 2^{-k-2}, \end{aligned}$$

(thanks to Step 2). This gives (14) since:

$$\theta_{k+1}^* \leq (\theta_k^* - \lambda 2^{-k-1})_+.$$

More precisely, this gives:

$$\theta_{k+1}^* \leq ((\eta_k \theta_k) * P(z) - \lambda 2^{-k-2})_+.$$

So:

$$\eta_{k+1} \theta_{k+1}^* \leq ((\eta_k \theta_k) * P(z) - \lambda 2^{-k-2})_+.$$

From the second property of Step 2, we find for $z = \delta^{k+1}$:

$$\begin{aligned} |(\eta_k \theta_k) * P(z)| &\leq A_k \|P(z)\|_{L^2} \\ &\leq \frac{M^{-k}}{\delta^{k+1}} \|P(1)\|_{L^2} \leq \lambda 2^{-k-2}. \end{aligned}$$

The last inequality makes use of Step 2. Therefore:

$$\eta_{k+1}\theta^*_{k+1} \leq 0 \qquad \text{on } \ z = \delta^{k+1}.$$

Note, in particular, that with step 4 this gives that (12) is verified up to $k = 12N + 1$ and (14) up to $k = 12N$.

Step 6. Propagation of Property (11). We show in this step that if (12) is true for $k-3$ and (11) is true for $k-3$, $k-2$ and $k-1$ then (11) is true for k.

First notice that from Step 5, (12) is true at $k-2$, $k-1$, and k. We just need to show that:

$$A_k \leq C_0^k (A_{k-3})^{1+1/N} \qquad \text{for } k \geq 12N + 1, \tag{15}$$

with:

$$C_0 = C \frac{2^{1+2/N}}{\lambda^{2/N}}. \tag{16}$$

Indeed, the third inequality of Step 2 gives the result.

Step 7. Proof of (15): Since $\eta\theta^*_+$ has the same trace at $z = 0$ that $(\eta\theta_+)^*$ and the latter is harmonic we have:

$$\int |\nabla(\eta\theta^*_+)|^2 \geq \int |\nabla(\eta\theta_+)^*|^2 = \int |\Lambda^{1/2}(\eta\theta_+)|^2.$$

Sobolev and Holder inequalities give:

$$A_{k-3} \geq C \|\eta_{k-3}\theta_{k-3}\|^2_{L^{\frac{2(N+1)}{N}}([-1-2^{-k-3},0]\times\mathbb{R}^N)}.$$

From (14):

$$\|\eta_{k-3}\theta^*_{k-2}\|^2_{L^{\frac{2(N+1)}{N}}} \leq \|P(1)\|^2_{L^1} \|\eta_{k-3}\theta_{k-3}\|^2_{L^{\frac{2(N+1)}{N}}}.$$

So:

$$\begin{aligned} A_{k-3} &\geq C\|\eta_{k-3}\theta^*_{k-2}\|^2_{L^{\frac{2(N+1)}{N}}} + C\|\eta_{k-3}\theta_{k-3}\|^2_{L^{\frac{2(N+1)}{N}}} \\ &\geq C\left(\|\eta_{k-1}\theta^*_{k-1}\|^2_{L^{\frac{2(N+1)}{N}}} + \|\eta_{k-1}\theta_{k-1}\|^2_{L^{\frac{2(N+1)}{N}}} \right). \end{aligned}$$

Taking the mean value of (9) in t_1 between $-1-2^{-k-1}$ and $-1-2^{-k}$, we find:

$$A_k \leq C 2^k (\Phi + 2) \left(\int \eta^2_{k-1} {\theta_k}^2 + \int \eta^2_{k-1} {\theta^*_k}^2 \right).$$

Note that we have used here (12) since η_{k-1} is a cut-off function in x only. If $\theta_k > 0$ then $\theta_{k-1} \geq 2^{-k}\lambda$. So:

$$\mathbf{1}_{\{\theta_k>0\}} \leq \frac{C2^k}{\lambda}\theta_{k-1}.$$

Therefore:

$$A_k \leq \frac{C2^{k(1+2/N)}}{\lambda^{2/N}} A_{k-3}^{1+1/N}.$$

This gives (15). $\square$

5 The second technical lemma.

We set $Q_r = B_r \times [-r, 0]$ and $Q_r^* = B_r^* \times [-r, 0]$

Lemma 8 *For every $\varepsilon_1 > 0$, there exists a constant $\delta_1 > 0$ with the following property:*

For every solution θ to (1) with v verifying (8) and:

$$\theta^* \leq 2 \qquad \text{in } \; Q_4^*,$$
$$|\{(x,z,t) \in Q_4^*;\; \theta^*(x,z,t) \leq 0\}| \geq \frac{|Q_4^*|}{2},$$

we have the following implication:

$$|(x,z,t) \in Q_4^*;\; 0 < \{\theta^*(x,z,t) < 1\}| \leq \delta_1$$

implies:

$$\int_{Q_1} (\theta - 1)_+^2 \, dx\, dt + \int_{Q_1^*} (\theta^* - 1)_+^2 \, dx\, dx\, dt \leq C\sqrt{\varepsilon_1}.$$

Proof. Take $\varepsilon_1 \ll 1$. From the energy inequality (9), we get:

$$\int_{-4}^{0} \int_{B_1^*} |\nabla \theta_+^*|^2 \, dx\, dz\, dt \leq C.$$

Let:

$$K = \frac{4 \int |\nabla \theta_+^*|^2 \, dx\, dz\, dt}{\varepsilon_1}.$$

Then:

$$\left| \{t | \int_{B_1^*} |\nabla \theta_+^*|^2(t) \, dx\, dz \geq K\} \right| \leq \frac{\varepsilon_1}{4}. \tag{17}$$

For all $t \in \{t | \int_{B_1^*} |\nabla \theta_+^*|^2(t)\, dx\, dz \leq K\}$, the De Giorgi lemma (see appendix) gives that:

$$|\mathcal{A}(t)||\mathcal{B}(t)| \leq |\mathcal{C}(t)|^{1/2} K^{1/2},$$

where:

$$\begin{aligned}\mathcal{A}(t) &= \{(x,z) \in B_1^* \mid \theta^*(t,x,z) \leq 0\}\\ \mathcal{B}(t) &= \{(x,z) \in B_1^* \mid \theta^*(t,x,z) \geq 1\}\\ \mathcal{C}(t) &= \{(x,z) \in B_1^* \mid 0 < \theta^*(t,x,z) < 1\}.\end{aligned}$$

Let us set

$$\delta_1 = \varepsilon_1^8,$$

$$I = \{t \in [-4,0];\ |\mathcal{C}(t)|^{1/2} \leq \varepsilon_1^3 \ \text{ and } \int_{B_1^*} |\nabla \theta_+^*|^2(t)\,dx\,dz \leq K\}.$$

First we have, using Tchebichev inequality:

$$\begin{aligned}&\Big|\{t \in [-4,0];\ |\mathcal{C}(t)|^{1/2} \geq \varepsilon_1^3\}\Big|\\ &\qquad \leq \frac{|\{(t,x,z) \mid 0 < \theta^* < 1\}|}{\varepsilon_1^6}\\ &\qquad \leq \frac{\delta_1}{\varepsilon_1^6}\\ &\leq \varepsilon_1^2 \leq \varepsilon_1/4.\end{aligned}$$

Hence $|[-4,0]\backslash I| \leq \varepsilon_1/2$. Secondly we get for every $t \in I$ such that $|\mathcal{A}(t)| \geq 1/4$:

$$|\mathcal{B}(t)| \leq \frac{|\mathcal{C}(t)|^{1/2} K^{1/2}}{\mathcal{A}(t)} \leq 4\varepsilon_1^{5/2} \leq \varepsilon_1^2. \tag{18}$$

In particular:

$$\begin{aligned}\int {\theta_+^*}^2(t)\,dx\,dz &\leq 4(|\mathcal{B}(t)| + |\mathcal{C}(t)|)\\ &\leq 8\varepsilon_1^2.\end{aligned}$$

And so:

$$\int \theta_+^2(t)\,dx \leq \sqrt{K}\sqrt{\int {\theta_+^*}^2(t)\,dx\,dz} \leq C\sqrt{\varepsilon_1}.$$

We want to show that $|\mathcal{A}(t)| > 1/4$ for every $t \in I \cap [-1,0]$. First, since $|\{(t,x,z) \mid \theta^* \leq 0\}| \geq |Q_4^*|/2$, there exists $t_0 \leq -1$ such that $|\mathcal{A}(t_0)| \geq 1/4$. So for this t_0, $\int \theta_+^2(t_0)\,dx \leq C\sqrt{\varepsilon_1}$. Using the energy inequality (9), we have for every $t \geq t_0$:

$$\int \theta_+^2(t)\,dx \leq \int \theta_+^2(t_0)\,dx + C(t - t_0).$$

So for $t - t_0 \leq \delta^* = 1/(64C)$ we have:

$$\int \theta_+^2(t)\,dx \leq \frac{1}{64}.$$

(Note that δ^* do not depend on ε_1. Hence we can suppose $\varepsilon_1 \ll \delta^*$.) We have:

$$\begin{aligned}\theta_+^*(z) &\leq \theta_+ + \int_0^z \partial_z \theta_+^* \, dz \\ &\leq \theta_+ + \sqrt{z} \left(\int |\partial_z \theta_+^*|^2 \, dz \right)^{1/2}.\end{aligned}$$

So, for $t - t_0 \leq \delta^*$, $t \in I$ and $z \leq \varepsilon_1^2$ we have:

$$\theta_+^*(t,x,z) \leq \theta_+(t,x) + \left(\varepsilon_1^2 \int |\partial_z \theta_+^*|^2 \, dz \right)^{1/2}.$$

The integral in x of the right hand side term is less than $1/8 + \sqrt{\varepsilon_1} \leq 1/4$. So by Tchebichev:

$$|\{z \leq \varepsilon_1^2, \ x \in B_1, \ \theta_+^*(t) \geq 1\}| \leq \frac{\varepsilon_1^2}{4}.$$

Since $|\mathcal{C}(t)| \leq \varepsilon_1^6$, this gives

$$|\mathcal{A}(t)| \geq \varepsilon_1^2(1 - 1/4) - \varepsilon_1^6 \geq \varepsilon_1^2/2.$$

Then (18) gives:

$$|\mathcal{B}(t)| \leq 2\sqrt{\varepsilon_1},$$

and:

$$|\mathcal{A}(t)| \geq 1 - 2\sqrt{\varepsilon_1} - \varepsilon_1^6 \geq 1/4.$$

Hence, for every $t \in [t_0, t_0 + \delta^*] \cap I$ we have: $|\mathcal{A}(t)| \geq 1/4$. On $[t_0 + \delta^*/2, t_0\delta^*]$ there exists $t_1 \in I$ ($\delta^* \geq \varepsilon_1/4$). And so, we can construct an increasing sequence t_n, $0 \geq t_n \geq t_0 + n\delta^*/2$ such that $|\mathcal{A}(t)| \geq 1/4$ on $[t_n, t_n + \delta^*] \cap I \supset [t_n, t_{n+1}] \cap I$. Finally on $I \cap [-1, 0]$ we have $|\mathcal{A}(t)| \geq 1/4$. This gives from (18) that for every $t \in I \cap [-1, 0]$: $|\mathcal{B}(t)| \leq \varepsilon_1/16$. Hence:

$$|\{\theta^* \geq 1\}| \leq \varepsilon_1/16 + \varepsilon_1/2 \leq \varepsilon_1.$$

Since $(\theta^* - 1)_+ \leq 1$, this gives that:

$$\int_{Q_1^*} (\theta^* - 1)_+^2 \, dx \, dz \, dt \leq \varepsilon_1.$$

We have for every t, x fixed:

$$\theta - \theta^*(z) = \int_0^z \partial_z \theta^* \, dz.$$

So:

$$\begin{aligned}(\theta - 1)_+^2 &\leq 2 \left((\theta^*(z) - 1)_+^2 + \left(\int_0^z |\nabla \theta^*| \, dz \right)^2 \right) \\ &\leq \frac{2}{\sqrt{\varepsilon_1}} \int_0^{\sqrt{\varepsilon_1}} (\theta^* - 1)_+^2 \, dz + 2\sqrt{\varepsilon_1} \int_0^{\sqrt{\varepsilon_1}} |\nabla \theta^*|^2 \, dz.\end{aligned}$$

Therefore:

$$\int_{Q_1} (\theta - 1)_+^2 \, dx \, ds \leq C\sqrt{\varepsilon_1}.$$

□

6 Oscillation lemma

This section is dedicated to the proof of the following proposition:

Proposition 9 *There exits $\lambda^* > 0$ such that for every solution θ of (1) with v verifying (8), if:*

$$\begin{aligned} &\theta^* \leq 2 \qquad \text{in } \ Q_1^* \\ &|\{(t,x,z) \in Q_1^*; \ \theta^* \leq 0\}| \geq \frac{1}{2}, \end{aligned}$$

then:

$$\theta^* \leq 2 - \lambda^* \qquad \text{in } \ Q_{1/16}^*.$$

Proof. For every $k \in \mathbb{N}$, $k \leq K_+ = E(1/\delta_1 + 1)$ (where δ_1 is defined in Lemma 8 for ε_1 such that $4C\sqrt{\varepsilon_1} \leq \varepsilon_0$, ε_0 defined in Lemma 6), we define:

$$\overline{\theta}_k = 2(\overline{\theta}_{k-1} - 1) \qquad \text{with} \qquad \overline{\theta}_0 = \theta.$$

So we have: $\overline{\theta}_k = 2^k(\theta - 2) + 2$. Note that for every k, $\overline{\theta}_k$ verifies (1), $\overline{\theta}_k \leq 2$ and $|\{(t,x,z) \in Q_1^* \mid \overline{\theta}_k \leq 0\}| \geq \frac{1}{2}$. Assume that for all those k, $|\{0 < \overline{\theta}_k^* < 1\}| \geq \delta_1$. Then, for every k:

$$|\{\overline{\theta}_k^* < 0\}| = |\{\overline{\theta}_{k-1}^* < 1\}| \geq |\{\overline{\theta}_{k-1}^* < 0\}| + \delta_1.$$

Hence:

$$|\{\overline{\theta}_{K_+}^* \leq 0\}| \geq 1,$$

and $\overline{\theta}_{K_+}^* < 0$ almost everywhere, which means: $2^{K_+}(\theta^* - 2) + 2 < 0$ or

$$\theta^* < 2 - 2^{-K_+}.$$

And in this case we are done.

Else, there exists $0 \leq k_0 \leq K_+$ such that: $|\{0 < \overline{\theta}_{k_0}^* < 1\}| \leq \delta_1$. From Lemma 8 and Lemma 6 (applied on $\overline{\theta}_{k_0+1}$) we get $(\overline{\theta}_{k_0+1})_+ \leq 2 - \lambda$ which means:

$$\theta \leq 2 - 2^{-(k_0+1)}\lambda \leq 2 - 2^{-K_+}\lambda,$$

in $Q_{1/8}$.

Consider the function b_3 defined by:

$$\begin{aligned} &\Delta b_3 = 0 \qquad \text{in } \ B_{1/8}^*, \\ &b_3 = 2 \qquad \text{on the sides of the cube except for } z = 0 \\ &b_3 = 2 - 2^{-K_+} \inf(\lambda, 1) \qquad \text{on } z = 0. \end{aligned}$$

We have $b_3 < 2 - \lambda^*$ in $B_{1/16}^*$. And from the maximum principle we get $\theta^* \leq b_3$.

□

7 Proof of Theorem 2.

We fix $t_0 > 0$ and consider $t \in [t_0, \infty[\times\mathbb{R}^N$. We define:

$$F_0(s,y) = \theta(t + st_0/4, x + t_0/4(y - x_0(s))),$$

where $x_0(s)$ is solution to:

$$\begin{aligned} \dot{x}_0(s) &= \frac{1}{|B_4|}\int_{x_0(s)+B_4} v(t+st_0/4, x+yt_0/4)\,dy \\ x_0(0) &= 0. \end{aligned}$$

Note that $x_0(s)$ is uniquely defined from Cauchy Lipschitz theorem. We set:

$$\begin{aligned} \tilde{\theta}_0^*(s,y) &= \frac{4}{\sup_{Q_4^*} F_0^* - \inf_{Q_4^*} F_0^*}\left(F_0^* - \frac{\sup_{Q_4^*} F_0^* + \inf_{Q_4^*} F_0^*}{2}\right). \\ v_0(s,y) &= v(t+st_0/4, x+t_0/4(y-x_0(s))) - \dot{x}_0(s), \end{aligned}$$

and then for every $k > 0$:

$$\begin{aligned} F_k(s,y) &= F_{k-1}(\tilde{\mu}s, \tilde{\mu}(y - x_k(s))), \\ \tilde{\theta}_k^*(s,y) &= \frac{4}{\sup_{Q_4^*} F_k^* - \inf_{Q_4^*} F_k^*}\left(F_k^* - \frac{\sup_{Q_4^*} F_k^* + \inf_{Q_4^*} F_k^*}{2}\right), \\ \dot{x}_k(s) &= \frac{1}{|B_4|}\int_{x_k(s)+B_4} v_{k-1}(\tilde{\mu}s, \tilde{\mu}y)\,dy \\ x_k(0) &= 0 \\ v_k(s,y) &= v_{k-1}(\tilde{\mu}s, \tilde{\mu}(y - x_k(s))) - \dot{x}_k(s), \end{aligned}$$

where $\tilde{\mu}$ will be chosen later. We divide the proof in several steps.

Step 1. For k=0, $\tilde{\theta}_0$ is solution to (2) in $[-4,0]\times\mathbb{R}^N$, $\|v_0\|_{BMO} = \|v\|_{BMO}$, $\int v_0(s)\,dy = 0$ for every s and $|\tilde{\theta}_0| \le 2$. Assume that it is true at $k-1$. Then:

$$\partial_s F_k = \tilde{\mu}\partial_s\tilde{\theta}_{k-1}() - \tilde{\mu}\dot{x}_k(s)\cdot\nabla\tilde{\theta}_{k-1.}$$

So $\tilde{\theta}_k$ is solution of (2) and $|\tilde{\theta}_k| \le 2$. By construction, for every s we have $\int_{B_4} v_k(s,y)\,dy = 0$ and $\|v_k\|_{BMO} = \|v_{k-1}\|_{BMO} = \|v\|_{BMO}$. Moreover we have:

$$\begin{aligned} |\dot{x}_k(s)| &\le \int_{B_4} v_{k-1}(\tilde{\mu}(y - x_k(s)))\,dy \\ &\le C\|v_{k-1}(\tilde{\mu}y)\|_{L^p} \\ &\le C\tilde{\mu}^{-N/p}\|v_{k-1}\|_{L^p} \\ &\le C_p\tilde{\mu}^{-N/p}\|v_{k-1}\|_{BMO}. \end{aligned}$$

So, for $0 \le s \le 1$, $y \in B_4$ and $p > N$:

$$|\tilde{\mu}(y - x_k(s))| \le 4\tilde{\mu}(1 + C_p\tilde{\mu}^{-N/p}) \le C\tilde{\mu}^{1-N/p}.$$

For $\tilde{\mu}$ small enough this is smaller than 1.

Step 2. For every k we can use the oscillation lemma. If $|\{\tilde{\theta}_k^* \leq 0\}| \geq \frac{1}{2}|Q_4^*|$ then we have $\tilde{\theta}_k^* \leq 2 - \lambda^*$. Else we have $|\{-\tilde{\theta}_k^* \leq 0\}| \geq \frac{1}{2}|Q_4^*|$ and applying the oscillation lemma on $-\tilde{\theta}_k^*$ gives $\tilde{\theta}_k^* \geq -2 + \lambda^*$. In both cases this gives:

$$|\sup \tilde{\theta}_k^* - \inf \tilde{\theta}_k^*| \leq 2 - \lambda^*.$$

and so:

$$|\sup_{Q_1^*} F_k^* - \inf_{Q_1^*} F_k^*| \leq (1 - \lambda^*/2)^k |\sup_{Q_1^*} F_0^* - \inf_{Q_1^*} F_0^*|.$$

Step 3. For $s \leq \tilde{\mu}^{2n}$:

$$\sum_{k=0}^{n} \tilde{\mu}^{n-k} x_k(s) \leq \tilde{\mu}^{2n} \sum_{k=0}^{n} \frac{\tilde{\mu}^{n-k}}{\tilde{\mu}^{-N/p}} \leq \frac{\tilde{\mu}^n}{2},$$

for $\tilde{\mu}$ small enough. So

$$\left| \sup_{[-\tilde{\mu}^{2n},0]\times B^*_{\tilde{\mu}^n/2}} \theta^* - \inf_{[-\tilde{\mu}^{2n},0]\times B^*_{\tilde{\mu}^n/2}} \theta^* \right| \leq (1 - \lambda^*/2)^n.$$

This gives that θ^* is C^α at $(t, x, 0)$, and so θ is C^α at (t, x). □

A Proof of the De Giorgi isoperimetric lemma.

Let $\omega \in H^1([-1, 1]^{N+1})$. We denote:

$$\mathcal{A} = \{x;\ \omega(x) \leq 0)\}$$
$$\mathcal{B} = \{x;\ \omega(x) \geq 1)\}$$
$$\mathcal{C} = \{x;\ 0 < \omega(x) < 1)\},$$

and

$$\chi = \mathbf{1}_{\{y_1 + s(y_1 - y_2)/|y_1 - y_2| \in \mathcal{C}\}}.$$

We have:

$$
\begin{aligned}
|\mathcal{A}||\mathcal{B}| &\le \int_{\mathcal{A}}\int_{\mathcal{B}} (\omega(y_1)-\omega(y_2))\,dy_1\,dy_2 \\
&= \int_{\mathcal{A}}\int_{\mathcal{B}}\int_0^{|y_1-y_2|} \nabla\omega(y_1+s\frac{y_1-y_2}{|y_1-y_2|})\cdot\frac{y_1-y_2}{|y_1-y_2|}\,ds\,dy_1\,dy_2 \\
&= \int_{\mathcal{A}}\int_{\mathcal{B}}\int_0^{|y_1-y_2|} \chi\nabla\omega(y_1+s\frac{y_1-y_2}{|y_1-y_2|})\cdot\frac{y_1-y_2}{|y_1-y_2|}\,ds\,dy_1\,dy_2 \\
&\le \int_{\mathcal{A}}\int_{\mathcal{B}}\int_0^{\infty} \chi\left|\nabla\omega(y_1+s\frac{y_1-y_2}{|y_1-y_2|})\right|\,ds\,dy_1\,dy_2 \\
&\le \int_{B_1}\int_{B_1}\int_0^{\infty} \chi\left|\nabla\omega(y_1+s\frac{y_1-y_2}{|y_1-y_2|})\right|\,ds\,dy_1\,dy_2 \\
&\le \int_{S_{N-1}}\int_{B_1}\int_0^{\infty} \frac{|\nabla\omega(y_1+s\nu)|}{s^{N-1}}\mathbf{1}_{\{(y_1+s\nu)\in\mathcal{C}\}}s^{N-1}\,d\nu\,ds\,dy_1 \\
&\le C\int_{B_1}\int_{B_1} \frac{|\nabla\omega(y_1+y_2)|}{|y_2|^{N-1}}\mathbf{1}_{\{y_1+y_2\in\mathcal{C}\}}\,dy_2\,dy_1 \\
&\le C\|\nabla\omega\|_{L^2}|\mathcal{C}|^{1/2}.
\end{aligned}
$$

□

B Higher regularity

We give the proof of the following theorem.

Theorem 10 *Let θ be a solution of the quasi-geostrophic equation (2), (3) satisfying the regularity properties of Theorem 3:*

$$
\begin{aligned}
\theta &\in L^\infty(0,\infty;L^2)\cap L^2(0,\infty;H^{1/2}) \\
&\quad \cap L^\infty([t_0,\infty[\times\mathbb{R}^N)\cap C^\alpha([t_0,\infty[\times\mathbb{R}^N),
\end{aligned}
$$

for every $t_0>0$. Then θ belongs to $C^{1,\beta}([t_0,\infty[\times\mathbb{R}^N)$ for every $\beta<1$ and $t_0>0$ and is therefore a classical solution.

Proof: We want to show the regularity at a fixed point $y_0=(t_0,x_0)\in\,]0,\infty[\times\mathbb{R}^N\subset\mathbb{R}^m$ where $m=N+1$. Note that Changing $\theta(t,x)$ by $\theta(t,x-u(t_0,x_0)t)-\theta(t_0,x_0)$ if necessary, we can assume without loss of generality that $\theta(y_0)=0$ and $u(y_0)=0$. The fundamental solution of:

$$\partial_t\theta+\Lambda\theta=0$$

is the Poisson kernel:

$$P(t,x)=\frac{Ct}{(|x^2|+t^2)^{\frac{N+1}{2}}},$$

a homogeneous function of order $-N$ if extended for t negative. the solution θ of (2) can be represented as the sum of two terms.

$$\theta(t,x) = P(t,\cdot) * \theta_0 - g(t,x), \tag{19}$$

where:

$$\begin{aligned} g(t,x) &= \int_0^t \int_{\mathbb{R}^N} P(t-t_1, x-x_1) \mathrm{div}(u(t_1,x_1)\theta(t_1,x_1))\, dt_1\, dx_1 \\ &= \int_0^\infty \int_{\mathbb{R}^N} \nabla_x \tilde{P}(y-y_1)\cdot u(y_1)\theta(y_1)\, dy_1. \end{aligned}$$

In the last inequality, we denoted $y=(t,x)$, $\tilde{P}$ the extension of P for negative t with value 0, and we passed the divergence on $\tilde{P}$, which becomes a singular integral. The first term in (19) is smooth for $t>0$ and depends only on the initial data. We focus on the second one $g(y)$. We fix $e\in \mathbf{S}_m$, and estimate $g(y_0+he)-g(y_0)$ for $h>0$ in the standard way. We split the integral:

$$g(y_0)-g(y_0+he) = \int_0^\infty \int_{\mathbb{R}^N} Q_0(y_0-y_1, he) u(y_1)\theta(y_1)\, dy_1 \tag{20}$$

where:

$$Q_0(y,he) = \nabla_x\tilde{P}(y) - \nabla_x\tilde{P}(y+he),$$

into two parts, one on the ball B_{10h} centered to y_0 and radius $10h$, and the second on the complement. The first part has no cancelation so we separate the integrals:

$$\begin{aligned} &\int_{B_{10h}} \mathbf{1}_{\{t_1\geq 0\}} [\nabla_x\tilde{P}(y_0-y_1) - \nabla_x\tilde{P}(y_0+he-y_1)] u(y_1)\theta(y_1)\, dy_1 \\ =& \int_{B_{10h}} \mathbf{1}_{\{t_1\geq 0\}} \nabla_x\tilde{P}(y_0-y_1) u(y_1)\theta(y_1)\, dy_1 \\ &- \int_{B_{10h}} \mathbf{1}_{\{t\geq 0\}} \nabla_x\tilde{P}(y_0+he-y_1) u(y_1)\theta(y_1)\, dy_1. \end{aligned}$$

If θ is C^α, $\alpha>0$, from the Riesz transform u is also C^α, and since $\theta(y_0) = u(y_0) = 0$, we have:

$$|u(y_1)\theta(y_1)| \leq \inf(|y_1-y_0|^{2\alpha}, C). \tag{21}$$

So the first integral is convergent and bounded by $Ch^{2\alpha}$. To deal with the second one, notice that $\nabla_x\tilde{P}$ have mean value zero on any slice $t=C$ of B_{10h}, so we can add and substract $\theta(y_0+he)u(y_0+he)$. We have:

$$|\theta(y_1)u(y_1) - \theta(y_0+he)u(y_0+he)| \leq Ch^\alpha |y_0+he-y_1|^\alpha,$$

where we have used again that $u(y_0)=\theta(y_0)=0$. Hence the integral is also convergent and bounded by $Ch^{2\alpha}$. This gives that the contribution of B_{10h} on (20) is smaller that $Ch^{2\alpha}$.

Outside of a neighborhood of size $10h$ we use the cancelation of $\nabla_x \tilde{P}$. Up to Lipschitz regularity we just do:

$$\begin{aligned} &|\nabla_x[\tilde{P}(y_1 - y_0) - \tilde{P}(y_1 + he - y_0)]| \\ &\quad \leq \frac{h}{|y_1 - y_0|^{m+1}}, \end{aligned}$$

and integrate against $|u\theta|$ which verifies (21). This gives the bound:

$$\int_{|y_1 - y_0| \geq 10h} \frac{h}{|y_1 - y_0|^{m+1-2\alpha}} \, dy_1 \leq C h^{2\alpha},$$

provided that $2\alpha < 1$. Altogether, this gives that if $\theta \in C^\alpha$ with $2\alpha < 1$, then

$$|g(y_0) - g(y_0 + he)| \leq C h^{2\alpha}.$$

Bootstrapping the argument gives that θ is C^α for any $\alpha < 1$.

To go beyond Lipschitz we consider a second order increment quotient:

$$Q_1(y, he) = |\nabla[\tilde{P}(y + he) + \tilde{P}(y - he) - 2\tilde{P}(y)]|.$$

We have:

$$g(y_0 + he) + g(y_0 - he) - 2g(y_0) = \int_{\mathbb{R}^m} \mathbf{1}_{\{t_1 \geq 0\}} Q_1(y_0 - y_1, he) u(y_1) \theta(y_1) \, dy_1.$$

Note that $Q_1(y, he) = Q_0(y, he) - Q_0(y - he, he)$, so for $|y| < 20h$, the local estimate of the previous argument together with the C^α property of θ and u gives:

$$\int_{B_{20h}} |Q_1(y_0 - y_1) u(y_1) \theta(y_1)| \, dy_1 \leq C h^{2\alpha}.$$

For $|y| > 20h$ and y not in the strip $\mathcal{T}_h = [t_0 - h, t_0 + h] \times \mathbb{R}^N$, we have:

$$|Q_1(y_0 - y_1, he)| \leq C \frac{h^2}{|y_0 - y_1|^{m+2}}.$$

and the corresponding integral:

$$\begin{aligned} &\int_{|y_0 - y_1| \geq 20h} \mathbf{1}_{\{y_1 \notin \mathcal{T}_h\}} |Q_1(y_0 - y_1) u(y_1) \theta(y_1)| \, dy_1 \\ &\quad \leq C \int_{|y| \geq 20h} \frac{h^2}{|y|^{m+2}} (|y|^{2\alpha} \wedge 1) \, dy \\ &\quad \leq C h^{2\alpha}, \end{aligned}$$

whenever $2\alpha < 2$. It remains to control the contribution of the strip $\mathcal{T}_h \setminus B_{20h}$. The estimate on Q_0 gives that on this strip:

$$|Q_1(y_1 - y_0, he)| \leq C \frac{h}{|y_1 - y_0|^{N+2}} \leq C \frac{h}{|x_1 - x_0|^{N+2}}.$$

Not that on $\mathcal{T}_h \setminus B_{20h}$ we have $|x_1 - x_0| \geq h$. So the contribution of this strip is bounded by:

$$\begin{aligned}
&\int_{t_0-h}^{t_0+h} \int_{|x_1-x_0|\geq h} \frac{h}{|x_1 - x_0|^{N+2-2\alpha}}\, dx_1\, dt_1 \\
&\leq C\frac{h^{2\alpha}}{h} \int_{t_0-h}^{t_0+h} dt_1 \\
&\leq Ch^{2\alpha},
\end{aligned}$$

whenever $2\alpha < 2$. That goes all the way to $C^{1,\beta}$ for every $\beta < 1$. $\square$

C existence of solutions to (1)

In this appendix we sketch the existence theory of approximate solution of the equation (1) satisfying the truncated energy inequalities in the hypothesis of Theorem 1. We start by restricting the problem to $B_1 \times [0, \infty]$ and adding an artificial diffusion term $\varepsilon\Delta$. We will use the eigenfunctions σ_k and eigenvalues λ_k^2 of the Laplacian in B_1, that is:

$$\Delta\sigma_k + \lambda_k^2 \sigma_k = 0.$$

Note that $\sigma_k^*(x, z) = \sigma_k(x)e^{-\lambda_k z}$ is the harmonic extension of σ_k for the semi-infinite cylinder $Q_1 = B_1 \times [0, \infty]$ with data 0 in the lateral boundary, and:

$$\lambda_k \sigma_k(x) = \partial_\nu \sigma_k^*(x, 0),$$

where ∂_ν is the normal derivative. Also:

$$\int_{Q_1} \lambda_k \sigma_k^2 \, dx\, dz = \int_{\partial Q_1} \sigma_k^* \partial_\nu \sigma_k^* \, dx = \int_{Q_1} |\nabla \sigma_k^*|^2 \, dx\, dz,$$

and thus formula is also correct for any series

$$g(x) = \sum f_k \sigma_k(x),$$

provided that $\sum f_k^2 \lambda_k$ converges, i.e., $g \in H^{1/2}(B_1)$.

We want to solve then in $[0, \infty[\times B_1$ the equation:

$$\partial_t \theta + \mathrm{div}(v\theta) = \varepsilon\Delta\theta - (-\Delta^{1/2})\theta, \tag{22}$$

where $-\Delta^{1/2}\theta$ is understood as the operator that maps σ_k to $\lambda_k \sigma_k = \partial_\nu \sigma_k^*$.

For, say, v bounded and divergence free, this is straightforward using Galerkin method: Let us restrict (22) to σ_k, with $1 \leq k \leq k_0$, i.e. we seek a function:

$$\theta = \theta_{\varepsilon,k_0} = \sum_1^{k_0} f_k(t)\sigma_k(x)$$

that is a solution of the equation when tested against σ_k, $1 \le k \le k_0$. The functions f_k are solutions to the following system of ODEs:

$$f_k'(t) = -[\varepsilon\lambda_k^2 + \lambda_k]f_k(t) + \sum_{l=1}^{k_0} a_{kl} f_l(t), \qquad 1 \le k \le k_0,$$

with initial value:

$$f_k(0) = \int_{B_1} \theta_0(x)\sigma_k(x)\,dx,$$

where:

$$a_{kl} = \int_{B_1} v(t,x)\cdot\nabla\sigma_k(x)\sigma_l(x)\,dx.$$

Note that, since v is divergence free, the matrix a_{kl} is antisymmetric. This leads to the estimate:

$$\sum_{k=1}^{k_0} f_k^2(t_2) + \int_{t_1}^{t_2}\sum_{k=1}^{k_0}(\varepsilon\lambda_k^2+\lambda_k)f_k^2(s)\,ds$$
$$= \sum_{k=1}^{k_0} f_k^2(t_1).$$

In particular θ_{ε,k_0} satisfies the energy inequality:

$$\|\theta_{\varepsilon,k_0}(t_2)\|^2_{L^2(B_1)} + \int_{t_1}^{t_2}\left(\|\theta_{\varepsilon,k_0}(s)\|^2_{\dot H^{1/2}(B_1)} + \varepsilon\|\theta_{\varepsilon,k_0}(s)\|^2_{\dot H^1(B_1)}\right)ds$$
$$\le \|\theta_{\varepsilon,k_0}(t_1)\|^2_{L^2(B_1)}.$$

Notice also that what we call $H^{1/2}(B_1)$ corresponds to the extension of θ to the half cylinder, and such:

$$\|\theta\|_{\dot H^{1/2}(B_1)} \ge \|\theta\|_{\dot H^{1/2}(\mathbb{R}^N)}.$$

We now pass to the limit in k_0 and denote θ_ε the limit. If we test θ_{ε,k_0} with a function $\gamma \in L^\infty(0,T;L^2(B_1)) \cap L^2(0,T;H^1(B_1))$, there is no problem in passing to the limit in the term:

$$\int_{t_1}^{t_2}\int_{B_1}(\nabla\gamma)v\theta_{\varepsilon,k_0}\,dx\,ds,$$

since θ_{ε,k_0} converges strongly in $L^2([0,T]\times B_1)$. In particular, for $\gamma = (\theta_\varepsilon - \lambda)_+ = \theta_{\varepsilon,\lambda}$ the term converges to:

$$\int_{t_1}^{t_2}\int_{B_1}\nabla[\theta_{\varepsilon,\lambda}]^2 v\,dx\,ds = 0,$$

provided that v is divergence free. This leads to the following corollaries:

Corollary 11 *The function θ_ε satisfies the hypothesis of Lemma 6 independently of ε, and therefore:*

$$\|\theta_\varepsilon(T)\|_{L^\infty(B_1)} \leq \frac{C}{T^{N/2}} \|\theta_\varepsilon(0)\|_{L^2(B_1)}.$$

Corollary 12 *The same theorem is true for $v \in L^2([0,T] \times B_1)$ independently of the L^2 norm of v.*

Proof. We approximate v by a mollification v_δ. □

Corollary 13 *For θ_0 prescribed in $L^2(\mathbb{R}^N)$, the same result is true in $[0,T] \times \mathbb{R}^N$.*

Proof. We may rescale the previous theorems to the ball of radius M by applying them to $\overline{\theta}(t,x) = M^{N/2}\theta(Mt, Mx)$. This change preserves the L^2 norm, and so we get:

$$\sup_{B_M} M^{N/2}\theta(s,y) \leq \frac{C}{(s/M)^{N/2}},$$

or

$$\sup_{B_M} \theta(s,y) \leq \frac{C}{(s)^{N/2}},$$

provided that $v \in L^2(B_M)$ is divergence free. Then letting M go to infinity gives the result. □

Final remark: Since all the estimates are independent of ε we may let ε go to zero for the limit to be weak solution of the limiting equation, and satisfying the truncated energy inequalities.

Note also than the same approach can be taken for higher regularity. Indeed, the proof of higher regularity depends only on the truncated and localized energy inequality that is also satisfied by the ε-problem. We may then pass to the limit in ε and find a classical solution of the limiting problem.

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