THE DE GIORGI METHOD FOR ELLIPTIC AND PARABOLIC EQUATIONS AND SOME APPLICATIONS

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1. Introduction

This lecture notes are dedicated to the method introduced by De Giorgi to study the regularity of solutions to elliptic equations with rough coefficients. He introduced his technique in 1957 [5] to solve the 19th Hilbert problem. In this work, he showed the regularity of variational solutions to nonlinear elliptic problems. Independently, Nash introduced a similar techniques in 1958 [15]. Subsequently, Moser provided a new formulation of the proof in [14]. Those methods are now usually called De Giorgi-Nash-Moser techniques. The method has been extended to degenerated cases, like the p-Laplacian, first in the elliptic case by Ladyzhenskaya and Uralt'seva [13]. The degenerated parabolic cases were covered later by DiBenedetto [6] (see also DiBenedetto, Gianazza and Vespri [9, 7, 8]). More recently, the method has been extended to integral operators, like the fractional diffusion, in [3, 2] and Kassmann [12] and Kassmann and Felsinger [10]. Further application to fluid mechanics can be found in [1, 16, 11, 4].

In this lecture notes, we will focus on the original formulation of De Giorgi. We will stress on its beautiful geometric interpretation, and how it can be applied in different contexts. We do not try to show the more technical result available, but rather, try to get a simple description of the strength of the method, and the breadth of the possible applications.

The rest of the notes are as follows. in a first section, we introduce the original proof of De Giorgi for elliptic equations. In the second section we investigate the parabolic case. We include a non homogenous case, to illustrate how source terms can be handled. Finally we give an application in the context of compressible fluid mechanics.

2. The original result of De Giorgi

2.1. **The result.** The 19th Hilbert problem consists in showing the smoothness of local minimizers of convex energy functionals of the form

$$\mathcal{E}(w) = \int_{\Omega} F(\nabla w)\, dx, \tag{1}$$

where F is a smooth function from $\mathbb{R}^N$ to $\mathbb{R}$, and Ω is bounded open set of $\mathbb{R}^N$. This is a generalization of the Dirichlet integral

$$\int_\Omega |\nabla w|^2 \, dx.$$

Local minimizers of the Dirichlet integral verify the associate Euler-Lagrange equation which is nothing but the Laplace equation:

$$\Delta w = 0.$$

Solutions of the Laplace equation are known to be C^∞, as it can be easily shown, for instance, by using the mean value formula. We denote $L^2(\Omega)$ the space of square integrable functions on Ω.

Lemma 1. *Let $w \in L^2(\Omega)$ be a solution, in the sense of distributions, to the Laplace equation. For any radially symmetric compactly supported function η with integral 1, and any $x \in \Omega$, we have*

$$w(x) = \int_\Omega \eta(x-y) w(y) \, dy,$$

as long as the support of $\eta(\cdot - x)$ is included in Ω. Especially, for any $n \in \mathbb{N}$ there exists $C_n > 0$ such that for any $x \in \Omega$

$$|D^n w(x)| \le C_n \|w\|_{L^2(\Omega)} d(x, \partial\Omega)^{-(n+N/2)},$$

where $d(x, \partial\Omega)$ denotes the distance from x to the boundary Ω.

Proof. This is a very elementary proof. First, for any ball B included in Ω, from the the Stokes formula, any solution to the Laplace equation verifies

$$0 = \int_B \Delta w = \int_{\partial B} \partial_r w.$$

Let B_R be the support of η, and define for $0 < t < 1$

$$f(t) = \int_{B_R} \eta(y) w(x + ty) \, dy.$$

Note that, using the polar coordinates

$$f'(t) = \int_{B_R} \eta(y) y \cdot \nabla w(x+ty) \, dy = \int_0^R \left(\eta(r) r \int_{\partial B_r} \partial_r w \right) dr = 0.$$

Moreover, $f(0) = w(x)$, since the integral of η is equal to 1. So, for $t = 1$,

$$w(x) = \int_\Omega \eta(y) w(x+y) \, dy = \int_\Omega \eta(y-x) w(y) \, dy.$$

Consider now a fix function η compactly supported in B_1, and $d = d(x, \partial\Omega)$. The function $\eta_d = d^{-N} \eta(x/d)$ is such that the support of $\eta_d(\cdot - x)$in included in Ω. So

$$|D^n w(x)| = d^{-(N/2+n)} \left| \int d^{-N/2} D^n \eta((y-x)/d) w(y) \, dy \right| \le d^{-(N/2+n)} \|D^n \eta\|_{L^2} \|w\|_{L^2}.$$

□

The Dirichlet case is extremely simple. Surprisingly, the general case is far more difficult, and its resolution involves deep and beautiful mathematics.

First, let us precise the problem. By a local minimizer, it is meant that

$$\mathcal{E}(w) \leq \mathcal{E}(w+\varphi)$$

for any smooth φ compactly supported in Ω. As for the Dirichlet case, such a minimizer w satisfies an Euler-Lagrange equation. We denote $H^1(\Omega)$ the set of functions which are square integrable and such that their derivatives in the sense of distribution are themselves square integrable.

Lemma 2. *Consider F such that $D^2F(p) \leq \Lambda I$, for any $p \in \mathbb{R}^N$, for a fixed $\Lambda > 0$, where I is the identity $N \times N$ matrix. Then, any $w \in H^1(\Omega)$, local minimizer of (1), is solution in the sense of distribution to*

$$\operatorname{div}(DF(\nabla w)) = 0, \qquad x \in \Omega, \tag{2}$$

where DF is the gradient of the functional $p \to F(p)$.

Proof. This is, again, a very elementary result. For any $\varepsilon > 0$ and smooth function ϕ compactly supported, we have

$$\int_\Omega F(\nabla w + \varepsilon \nabla \phi)\, dx \geq \int_\Omega F(\nabla w)\, dx.$$

Those integrals are finite since F is at most quadratic, and $w \in H^1$. Using a Taylor expansion for F, we find

$$\int_\Omega DF(\nabla w) \cdot \nabla \phi \, dx \geq -\varepsilon \Lambda \int_\Omega |\nabla \phi|^2 \, dx.$$

This is true for any $\varepsilon > 0$, so at the limit ε goes to 0

$$\int_\Omega DF(\nabla w) \cdot \nabla \phi \, dx \geq 0.$$

The result is true also for ϕ replaced by $-\phi$. So, finally

$$\int_\Omega DF(\nabla w) \cdot \nabla \phi \, dx = 0.$$

This shows that w is solution to the Euler Lagrange equation is the sense of distribution. $\square$

From now on, we consider only function F which verifiy

$$\frac{1}{\Lambda} I \leq D^2F(p) \leq \Lambda I, \qquad \text{for any } p \in \mathbb{R}^N, \tag{3}$$

for a fixed $\Lambda > 1$. De Giorgi showed the following answer to the 19th problem of Hilbert:

Theorem 3. *Any local minimizer $w \in H^1(\Omega)$ of (1) lies in $C^\infty(\Omega)$.*

Note that the strict convexity is necessary for the result to be true. In one dimension, consider a functional F such that both 1 and -1 are minimum of F. Then $x \to |x|$ is a local minimizer of (1), but it is merely Lipschitz.

As long as we have enough regularity, (2) can be rewritten in the non divergence form as

$$D^2F(\nabla w) : D^2 w = 0.$$

For any open set O, we denote $C^\alpha(O)$ the space of function v such that

$$\|v\|_{C^\alpha(O)} = \|v\|_{L^\infty(O)} + \sup_{x,y\in O} \frac{|v(x) - v(y)|}{|x-y|^\alpha}$$

is finite. The strict convexity property of F translates into a strict ellipticity property for the equation on w. Note that if ∇w is C^α, then $D^2F(\nabla w)$ is also C^α. So (forgetting the dependence on w in $D^2F(\nabla w)$) the equation can be rewritten as

$$A(x) : D^2 w = 0,$$

for a C^α elliptic matrix A. Then the standard Schauder theory (which was known at the time) ensures that ∇w is $C^{1,\alpha}$. By differentiating the equation we get

$$D^2F(\nabla w) : D^2\partial_i w = -D^3F(\nabla w)\cdot \nabla\partial_i w : D^2 w.$$

Note that the second term is C^α. So, again, forgetting about the dependence of this term with respect to w we can use the Schauder theory on the linear problem

$$A(x) : D^2\partial_i w = f(x)$$

and find that $w \in C^{3,\alpha}$. Bootstrapping the argument, we get, finally, that $w \in C^\infty(\Omega)$. Note that at this point we have only $\nabla w \in L^2(\Omega)$. We need to fill the gap between L^2 to C^α.

The first idea of De Giorgi is to consider, for every $1 \le i \le N$, the derivative with respect to x_i of (2). Denote $u = \partial_i w$, this gives

$$\mathrm{div}(F''(\nabla w)\nabla u) = 0. \tag{4}$$

Thanks to (3), for every $x \in \Omega$

$$\frac{1}{\Lambda} I \le F''(\nabla w) \le \Lambda I.$$

Forgetting about the dependence of $A(x) = F''(\nabla w)$ on w, it can be rewritten as a classical linear elliptic equation in the divergence form:

$$\mathrm{div}(A(x)\nabla u) = 0, \tag{5}$$

with the elliptic condition on A:

$$\frac{1}{\Lambda} I \le A(x) \le \Lambda I, \qquad x \in \Omega. \tag{6}$$

De Giorgi showed the following theorem.

Theorem 4. *Let* $u \in H^1(\Omega)$ *be a weak solution to* (5) *with* A *verifying* (6)*. Then* $u \in C^\alpha(\tilde{\Omega})$ *for any* $\tilde{\Omega} \subset\subset \Omega$*, with*

$$\|u\|_{C^\alpha(\tilde{\Omega})} \leq C\|u\|_{L^2(\Omega)}.$$

The constant α *depends only on* Λ *and* N*. The constant* C *depends on* Λ*,* N*,* Ω*, and* $\tilde{\Omega}$*.*

To apply Theorem 4 to $u = \partial_i w$ and fill the gap by showing that ∇w is C^α, we would need first to show rigorously that indeed $u \in H^1$ (and so $\nabla w \in H^2$) and that it verifies (5) in the sense of distribution. This can be shown by standard method reminiscent of the proof that H^1 solutions to an elliptic equation are indeed H^2. For the sake of completeness, we give the proof of this fact at the end of this section.

Theorem 4 is significantly different in spirit with the results on elliptic regularity which existed before. Most of the previous results can be seen as perturbation of the Laplace operator. For instance, when the coefficients are C^α, we can use that locally, when zooming on small ball, the operator is almost constant, and so behaves like the Laplace operator. In the De Giorgi case this is not true anymore. Still, zooming is very important. It is worth noticing right away that the property (6) is preserved by the natural scaling of the equation.

From now on, we will denote L any operator $-\text{div}(A(x)\nabla\cdot)$, where A is uniformly elliptic, that is, verifies (6). Note that if a function u verifies $Lu = 0$, then, for any $\lambda \in \mathbb{R}$ and any $\varepsilon > 0$ the function

$$\tilde{u}(y) = \lambda u(x + \varepsilon y) \tag{7}$$

is solution of an other equation $\tilde{L}\tilde{u} = 0$ for an operator $\tilde{L}$ which verifies (6) for the same value of Λ. This property, based on the linearity of L, will be crucial in the proof.

2.2. **proof of Theorem 4.** The proof has two steps. The first step consists in showing that the energy control the supremum bound, and so that the function is indeed bounded. The second step consists in going from L^∞ to the modulus of continuity C^α. It uses a so called oscillation lemma.

2.2.1. *First lemma.* We denote $u_+ = \sup(0, u)$, and for any $r > 0$, B_r stands for the ball of center 0 and radius r. In this section, we show the following lemma.

Lemma 5. *There exists a constant* δ *depending only on* Λ *and* N *such that for any* u *solution to* $Lu = 0$ *in* B_1*, where* L *verifies* (6)*, we have the following property. If*

$$\|u_+\|_{L^2(B_1)} \leq \delta,$$

then

$$\|u_+\|_{L^\infty(B_{1/2})} \leq 1/2.$$

It will be very important later that C does not depend on u. The choice of the size of the balls is not consequential. Changing the size of the ball changes the constant C. But the size of the ball for the L^∞ norm has to be smaller than the ball for the L^2 norm. This is already a regularization result. The solution u can be unbounded on the boundary of B_1. It is regularized inside. This result will be applied later on rescaled functions $\tilde{u}$ as (7). As a first easy application of those rescaling techniques, we have the following corollary.

Corollary 6. *Let $u \in L^2(\Omega)$ be a solution to $Lu = 0$ where Ω is a bounded open set, and L verifies (6). Then, for any $\tilde{\Omega} \subset\subset \Omega$, $u \in L^\infty(\tilde{\Omega})$.*

Proof. let d be the distance from $\tilde{\Omega}$ to $\partial\Omega$. For any $x \in \tilde{\Omega}$, consider in B_1

$$\tilde{u}(y) = \delta \frac{d^{N/2}}{\|u\|_{L^2(\Omega)}} u(x + dy).$$

The function $\tilde{u}$ verifies $\tilde{L}\tilde{u} = 0$, where $\tilde{L}$ verifies (6), and $\|\tilde{u}\|_{L^2(B_1)} \leq \delta$. So, from Lemma 5,

$$\tilde{u}(x) \leq 1/2.$$

Applying the same result on $-\tilde{u}$ we obtain the bound by below, and

$$|\tilde{u}(x)| \leq 1/2.$$

Hence $\|u\|_{L^\infty(\tilde{\Omega})} \leq d^{-N/2}\|u\|_{L^2(\Omega)}/\delta$. ☐

We give now the proof of Lemma 5.

Proof. The proof is based on energy. We consider a sequence of level sets of energy on shrinking balls. As mentioned above, we have a regularization effect. So we expect to have a layer close to ∂B_1. In the recursive process, we want to escape from this layer. We consider the family of ball $\tilde{B}_k$ centered at 0 with radius $1/2(1 + 2^{-k})$. Note that $\tilde{B}_0 = B_1$ and $\tilde{B}_k$ converges to $B_{1/2}$ when k converges to infinity. the sequence of ball $\tilde{B}$ goes in a dyadic way from B_1 to $B_{1/2}$.

We consider in the same way, a family of "energy levels" C_k going in a dyadic way from 0 to 1/2:

$$C_k = 1/2(1 - 2^{-k}).$$

We define now the sequence of energy above the level set C_k in the ball $\tilde{B}_k$:

$$U_k = \int_{\tilde{B}_k} |u_k(x)|^2 \, dx,$$

where

$$u_k = (u - C_k)_+.$$

The aim is to derive an estimate of the form

$$U_{k+1} \leq C^k U_k^\beta, \tag{8}$$

for a suitable constant $C > 1$ and $\beta > 1$. It is worth noting that this inequality is *non linear.* This is fundamental. The term C^k converges to infinity when k converges to infinity. But, since $\beta > 1$, if the initial term U_0 (which is the L^2 norm of u_+ in B_1) is small enough, then the nonlinear term will be even smaller, and the sequence U_k will be shown to converge to 0. At the limit this gives

$$\int_{B_{1/2}} (u - 1/2)_+^2 \, dx = 0.$$

This means that in the ball of radius 1/2, the energy above the level 1/2 vanishes. And so, u is smaller than 1/2 in this ball.

The non linear inequality (8) is obtained via rather simple tools. We use linear tools as the energy estimate (which is linear since the equation itself is linear) and the Sobolev imbedding. The only nonlinear estimate that we use is the Tchebychev inequality (also known as Markov inequality) which is as follows:

$$|\{x, f(x) \geq R > 0\}| \leq \frac{\|f\|_{L^1}}{R}.$$

To gain the nonlinear estimate, we have to lose in term of level set on f. This explains why we introduce the level sets C_k. We split the proof in several steps.

Step 1. The two first ingredients are the Sobolev inequality, given by

$$\|v\|_{L^p(B_1)} \leq C \|\nabla v\|_{L^2(B_1)}$$

for $p(N) = \frac{2N}{N-2}$, whenever v is supported in B_1, and the energy inequality given by the following lemma.

Lemma 7. *(Energy inequality) If u is solution to $Lu = 0$ and $\varphi \in C_0^\infty(B_1)$ then*

$$\int_{B_1} (\nabla[\varphi u_+])^2 \, dx \leq C \|\nabla \varphi\|_{L^\infty}^2 \int_{B_1 \cap \operatorname{supp} \varphi} u_+^2 \, dx.$$

If A is symmetric then $C = \Lambda^2$.

Proof. We multiply $Lu = -\operatorname{div}(A(x)\nabla u)$ by $\varphi^2 u_+$ to get

$$\int \nabla^T(\varphi^2 u_+) A \nabla u_+ \, dx \leq 0.$$

We have to transfer a φ from the left ∇ to the right ∇. Indeed,

$$\begin{aligned}
&\int \nabla^T(\varphi^2 u_+)A\nabla u_+\,dx \\
&= \int \nabla^T(\varphi u_+)A\varphi\nabla u_+\,dx + \int (\varphi u_+)\nabla^T\varphi A\nabla u_+\,dx \\
&= \int \nabla^T(\varphi u_+)A\nabla(\varphi u_+)\,dx - \int \nabla^T(\varphi u_+)A(\nabla\varphi)u_+\,dx \\
&\qquad + \int (\varphi u_+)\nabla^T\varphi A\nabla u_+\,dx \\
&= \int \nabla^T(\varphi u_+)A\nabla(\varphi u_+)\,dx - \int \nabla^T(\varphi u_+)(A-A^T)(\nabla\varphi)u_+\,dx \\
&\qquad - \int u_+^2\nabla^T\varphi A\nabla\varphi\,dx.
\end{aligned}$$

In the symmetric case $A = A^T$, using (6) gives the result with $C = \Lambda^2$. Otherwise we use

$$\begin{aligned}
&\left|\int \nabla^T(\varphi u_+)(A-A^T)(\nabla\varphi)u_+\,dx\right| \\
&\le 2\Lambda\|\nabla(\varphi u_+)\|_{L^2}\|(\nabla\varphi)u_+\|_{L^2} \\
&\le 2\Lambda^{3/2}\left(\int \nabla^T(\varphi u_+)A\nabla(\varphi u_+)\,dx\right)^{1/2}\|(\nabla\varphi)u_+\|_{L^2} \\
&\le \frac{1}{2}\int \nabla^T(\varphi u_+)A\nabla(\varphi u_+)\,dx + 2\Lambda^3\int(\nabla\varphi)^2u_+^2\,dx.
\end{aligned}$$

And so

$$\begin{aligned}
\int |\nabla(\varphi u_+)|^2\,dx &\le \Lambda\int \nabla^T(\varphi u_+)A\nabla(\varphi u_+)\,dx \\
&\le C\int(\nabla\varphi)^2u_+^2\,dx.
\end{aligned}$$

□

Step 2. We introduce truncation functions ϕ_k verifying

$$\begin{aligned}
\phi_k &= 1 \quad \text{in } \tilde{B}_k, \\
&= 0 \text{ in } \tilde{B}_{k-1}^c,
\end{aligned}$$

with

$$|\nabla\phi_k| \le C2^k.$$

Note that

$$U_k \le \int \phi_k^2 u_k^2\,dx.$$

We have also $\mathbf{1}_{\tilde{B}_{k+1}} \leq \phi_k$, and $u_{k+1} \leq u_k$. The Sobolev inequality with $v = \phi_{k+1}u_{k+1}$, and the energy inequality with $\varphi = \phi_{k+1}$ give

$$\begin{aligned} \left(\int (\phi_{k+1}u_{k+1})^p \, dx \right)^{2/p} &\leq C \int |\nabla(\phi_{k+1}u_{k+1})|^2 \, dx \\ &\leq C2^{2k} \int |\phi_k u_{k+1}|^2 \, dx \\ &\leq C^k U_k. \end{aligned}$$

Using now the Tchebychev inequality, we get

$$\begin{aligned} U_{k+1} \leq \int (\phi_{k+1}u_{k+1})^2 \, dx &\leq \left(\int (\phi_{k+1}u_{k+1})^p \, dx \right)^{2/p} |\{\phi_{k+1}u_{k+1} > 0\}|^{2/N} \\ &\leq C^k U_k |\{\phi_k u_k > 2^{-2k}\}|^{2/N} \\ &\quad = C^k U_k |\{(\phi_k u_k)^2 > 2^{-4k}\}|^{2/N} \\ &\leq \frac{C^k}{2^{-8k/N}} U_k^{1+2/N} \leq (2^{8/N}C)^k U_k^{1+2/N}. \end{aligned}$$

This gives (8) with $\beta = 1 + 2/N$.

Step 3. To show that U_k converges to 0, we compare (8) with a geometric series. We want to show that taking U_0 small enough, we have for any k

$$C^k U_k^{\beta-1} \leq \frac{1}{(2C)^{\frac{1}{\beta-1}}}. \tag{9}$$

Consider k_0 such that

$$\frac{1}{2^{k_0}} \leq \frac{1}{(2C)^{\frac{1}{\beta-1}}},$$

and δ small enough such that (9) is valid for $k \leq k_0$. It is possible to find such that δ since k_0 is fixed. Let us show by induction that (9) is still true for any $k \geq k_0$. Fix a $k > k_0$. Assume that it is true for all $j \leq k$. Then

$$U_{k+1} \leq \frac{1}{(2C)^{\frac{k+1}{\beta-1}}},$$

and

$$C^{k+1} U_{k+1}^{\beta-1} \leq 2^{-(k+1)} \leq \frac{1}{(2C)^{\frac{1}{\beta-1}}}.$$

Hence we can compare to the geometric sequence and find that U_k converges to 0 when k goes to infinity. Passing into the limit, we find

$$\int_{B_{1/2}} (u - 1/2)_+^2 \, dx = 0.$$

This implies that $\|u_+\|_{L^\infty} \leq 1/2$. $\square$

2.3. **The Oscillation lemma.** The second step consists in obtaining a so-called oscillation lemma. We denote for any open set D: $\operatorname{osc}_D u = \sup_D u - \inf_D u$.

Lemma 8. *Let u be a solution of $Lu = 0$ in B_2 where A verifies (6). Then there exists $\lambda(\Lambda, N) < 1$ such that*

$$\operatorname{osc}_{B_{1/2}} u \leq \lambda \operatorname{osc}_{B_2} u \ .$$

This lemma implies C^α regularity of the solutions. Its strength is that it gives a definition that depends only on the L^∞ norm. The proof of C^α regularity follows that way. Consider $u \in H^1(\Omega)$ solution to $Lu = 0$ in Ω, and $\tilde{\Omega} \subset\subset \Omega$. Let $d = d(\tilde{\Omega}, \partial\Omega)$. Take any x_0 in $\tilde{\Omega}$. We introduce the rescaled functions

$$\begin{aligned} \bar{u}_1(y) &= u(x_0 + dy/2), \\ \bar{u}_n(y) &= \bar{u}_{n-1}(y/4). \end{aligned}$$

As before, $\bar{u}_n$ are solutions to (5) with diffusion matrices $A_n(y) = A(x_0 + y/4^n)$. Note that A_n verifies (6) for the same fixed Λ. We apply recursively Lemma 8 on $\bar{u}_n$. This gives

$$\sup_{|x_0 - x| \leq 4^{-n}} |u(x_0) - u(x)| \leq 2\|u\|_{L^\infty(B_1)}\lambda^n.$$

Note that this estimate does not depend on x_0. Hence u is in $C^\alpha(\tilde{\Omega})$ with

$$\alpha = -\frac{\ln \lambda}{2 \ln 2}.$$

We reformulate slightly the oscillation lemma in the following way.

Proposition 9. *Let $v \leq 1$, $Lv = 0$ in B_2. Assume that $|B_1 \cap \{v \leq 0\}| \geq \mu$ ($\mu > 0$). Then $\sup_{B_{1/2}} v \leq 1 - \lambda$, where λ depends only on μ, Λ, and N.*

In other words, if v is a solution of $Lv = 0$, smaller than one in B_2, and is "far from 1" in a set of non trivial measure, it cannot get too close to 1 in $B_{1/2}$.

Let us first show how this leads to Lemma 8. Consider the function

$$v(x) = \frac{2}{\operatorname{osc}_{B_2} u} \left(u(x) - \frac{\sup u + \inf u}{2} \right).$$

We have $-1 \leq v \leq 1$. Assume that v is half of the space smaller than 0 in B_1. Then we can apply Proposition 9 on v which gives that $\operatorname{osc}_{B_{1/2}} v \leq 2 - \lambda$. Hence $\operatorname{osc}_{B_{1/2}} u \leq (1 - \lambda/2) \operatorname{osc}_{B_2} u$. We get the same result if v is half of the space bigger than 0, working with $(-v)$.

To prove Proposition 9, we may first note that if the set

$$|\{v \leq 0\} \cap B_1| \geq |B_1| - \delta^2 \ ,$$

then

$$\|v_+\|_{L^2(B_1)} \leq \delta$$

and Lemma 5 imply that $v_+|_{B_{1/2}} \leq 1/2$.

So we must bridge the gap between knowing that $|\{v \le 0\}| \ge \frac{1}{2}|B_1|$ and knowing that $|\{v \le 0\}| \ge |B_1| - \delta^2$.

The main tool is the following De Giorgi isoperimetric inequality. It may be considered as a quantitative version of the fact that a function with a jump discontinuity cannot be in H^1.

Lemma 10. *There exists a constant $C_N > 0$ depending only on N such that the following holds true. Consider w such that $\int_{B_1} |\nabla w_+|^2\,dx \le C_0$. Set*

$$\begin{aligned}|A| &= |\{w \le 0\} \cap B_1|\\ |C| &= |\{w \ge 1/2\} \cap B_1|\\ |D| &= |\{0 < w < 1/2\} \cap B_1|.\end{aligned}$$

Then we have

$$C_0|D| \ge C_N(|C|\,|A|^{1-\frac{1}{N}})^2\ .$$

Proof. Consider $\bar{w} = \sup(0, \inf(w, 1/2))$. Note that $\nabla\bar{w} = \nabla w_+ \mathbf{1}_{\{0 \le w \le 1/2\}}$. For x in A and y in C, we have

$$\begin{aligned}1/2 = \bar{w}(y) - \bar{w}(x) &= \int_0^1 (y-x)\cdot\nabla\bar{w}(x + t(y-x))\,dt\\ &\le \int_0^{|y-x|} |\nabla\bar{w}|(x + se_\sigma)\,ds,\end{aligned}$$

where $e_\sigma = y - x/|y-x|$, and we made the change of variable $s = t|y-x|$. This quantity is smaller than the same integrating along the whole ray coming from x in the direction e_σ in B_1. With a slight abuse of notation (assigning the value 0 to $\nabla\bar{w}$ outside of B_1 for instance), we get

$$1/2 \le \int_0^\infty |\nabla\bar{w}|(x + se_\sigma)\,ds.$$

integrating this inequality for all $y \in C$, we get

$$\begin{aligned}|C|/2 &\le \int_C \left(\int_0^\infty |\nabla\bar{w}|(x + se_\sigma)\,ds\right)\,dy\\ &\le \int_{B_1} \left(\int_0^\infty |\nabla\bar{w}|(x + se_\sigma)\,ds\right)\,dy.\end{aligned}$$

Writing the first integral in polar coordinates for $(y-x)$, and noticing that the function does not depend on r we get

$$\begin{aligned}|C|/2 &\le \int_0^2 r^{N-1} \int_{\mathbb{S}_N} \left(\int_0^\infty |\nabla\bar{w}|(x + se_\sigma)\,ds\right)\,d\sigma\,dr\\ &\le \int_{\mathbb{S}_N} \int_0^\infty |\nabla\bar{w}|(x + se_\sigma)\,ds\,d\sigma\\ &= \int_{\mathbb{S}_N} \int_0^\infty s^{N-1} \frac{|\nabla\bar{w}|(x + se_\sigma)}{s^{N-1}}\,ds\,d\sigma.\end{aligned}$$

The last term is nothing but

$$\int_{B_1} \frac{|\nabla \bar{w}|(y)}{|x-y|^{N-1}}\, dy.$$

Integrating $x \in A$ we find

$$|A|\ |C|/2 \leq \int_{B_1} |\nabla \bar{w}|(y) \left(\int_A \frac{dx}{|x-y|^{N-1}} \right) dy.$$

Among all A with the same measure $|A|$ the integral in x is maximized by the ball of radius $|A|^{1/N}$, centered at y

$$\int_A \frac{dx}{|x-y|^{N-1}} \leq |A|^{1/N}\ .$$

So

$$|A|\ |C|/2 \leq |A|^{1/N} \left(\int_D |\nabla w_+|^2 \right)^{1/2} |D|^{1/2}\ .$$

Since $\int |\nabla w_+|^2\, dx \leq C_0$ the proof is complete. □

Proof of Proposition 9. We consider the new sequence of truncation

$$w_k = 2^k[v - (1 - 2^{-k})].$$

Note that for any k we have $w_k \leq 1$. So from the energy inequality, we have

$$\int_{B_1} |\nabla (w_k)_+|^2\, dx \leq C_0.$$

We have also $|\{w_k \leq 0\} \cap B_1| \geq \mu$. We apply Lemma 10 recursively on w_k as long as

$$\int_{B_1} (w_{k+1})_+^2\, dx \geq \delta^2.$$

We get

$$|\{w_{k+1} \geq 0\} \cap B_1| = |\{2w_k \geq 1\} \cap B_1| \geq \int_{B_1} (w_{k+1})_+^2\, dx \geq \delta^2.$$

So, from Lemma 10, there exists a constant α which does not depend on k such that

$$|\{0 < w_k < 1/2\} \cap B_1| \geq \alpha.$$

Then

$$|\{w_k \leq 0\} \cap B_1| \geq |\{w_{k-1} \leq 0\} \cap B_1| + \alpha \geq \mu + k\alpha.$$

This clearly fails after a finite number of k. At this k_0 we have for sure that

$$\int_{B_1} (w_{k_0+1})_+^2\, dx \leq \delta^2.$$

Lemma 5 then implies that $w_{k_0+1} \leq 1/2$ in $B_{1/2}$. Rescaling back to v gives the result with $\lambda = 2^{-(k_0+2)}$.

2.4. **Proof of Theorem 3.** To conclude the proof, we need to show that any $w \in H^1(\Omega)$ weak solution to (2) is such that $u = \partial_i w \in H^1$ and is a weak solution to (4). The proof use elementary techniques of elliptic theory and have nothing to do with the De Giorgi techniques. We present it here for the sake of completeness. It is related to the higher regularity proof of linear elliptic operators in the L^2 theory. In all the section we assume that (3) is verified. Recall that a weak solution to (2) verifies for every $\phi \in C_c^\infty(\Omega)$

$$\int_\Omega \nabla\phi DF(\nabla w)\, dx = 0.$$

By density, this is still valid for any $\phi \in H_0^1(\Omega)$.

By weak solution to (4), we mean that for every $\phi \in C_c^\infty(\Omega)$

$$\int_\Omega \nabla\phi^T D^2F(\nabla w)\nabla u\, dx = 0.$$

Again, by density, this is still true for any $\phi \in H^1(\Omega)$, as long as $u \in H^1(\Omega)$. We need the following proposition.

Proposition 11. *Let $w \in H^1(\Omega)$ be a weak solution to (2). Then, for any $\tilde\Omega \subset\subset \Omega$, $w \in H^2(\tilde\Omega)$, and for every i, $u = \partial_i w$ is a weak solution to (4).*

As, usual, we will first show a localized version in a fixed ball.

Lemma 12. *Let $w \in H^1(B_3)$ be a weak solution to (2). Then, $w \in H^2(B_1)$, and for every i, $u = \partial_i w$ is a weak solution to (4) in B_1.*

This lemma implies the proposition in the following way. Denote $d = d(\tilde\Omega, \partial\Omega)$, and for any $x \in \tilde\Omega$

$$\tilde w_x(y) = w(x + \frac{d}{3}y).$$

The function $\tilde w_x \in H^1(B_3)$ and verifies (2) in B_3 for $\tilde F(y) = F(3y/d)$. Since $\tilde\Omega$ is compact, there exists a finite number m of ball $B_{d/3}(x_i)$ such that $\tilde\Omega \subset \cup_{i=1}^m B_{d/3}(x_i)$, and $w \in H^2(\tilde\Omega)$. By the theorem of partition of the unity, any function $\phi \in C_c^\infty(\tilde\Omega)$ can be decomposed as

$$\phi = \sum_{i=1}^m \phi_i$$

where $\phi \in C_c^\infty(B_{d/3}(x_i))$. And so

$$\int_\Omega \nabla\phi^T D^2F(\nabla w)\nabla u\, dx = \sum_{i=1}^m \int_\Omega \nabla\tilde\phi_i^T D^2\tilde F(\nabla\tilde w_{x_i})\nabla\tilde u_{x_i}\, dx = 0.$$

We now turn to the proof of the lemma.

Proof. For any $-1 \le h \le 1$ we consider the operator defined on $C(\mathbb{R}^N)$

$$T_h\phi(x) = \frac{\phi(x + he_i) - \phi(x)}{h}.$$

For any cut-off function η, non negative, compactly supported in B_2 and equal to 1 in B_1, we can define two operators ηT_h and $T_h(\eta\cdot)$ well defined from $L^2(B_3)$ to $L^2(B_3)$, and from $H^1(B_3)$ to $H^1_0(B_3)$. We have the following properties.

Lemma 13. *For any $\tilde{\phi}, \tilde{\psi} \in L^2(B_3)$, $\phi \in H^1(B_3)$, $\Phi \in [L^2(B_3)]^N$, we have*

$$\int T_h(\eta\tilde{\phi})\tilde{\psi} = -\int \tilde{\phi}\eta T_{-h}\tilde{\psi}, \qquad \nabla[\eta^2 T_h\phi] = \eta^2 T_h[\nabla\phi] + 2\eta[T_h\phi]\nabla\eta \tag{10}$$

$$\eta T_h\{DF(\Phi)\}(x) = \eta T_h\Phi(x)\int_0^1 D^2F(\Phi(x) + t((\Phi(x+he_i) - (\Phi(x)))\,dt \tag{11}$$

$$\|\eta T_h\phi\|_{L^2(B_3)} \le C_\eta\|\nabla\phi\|_{L^2(B_3)}. \tag{12}$$

Moreover, if $\sup_{|h|<1}\|\eta T_h\tilde{\phi}\|_{L^2(B_3)}$ is bounded, then $\partial_i\tilde{\phi} \in L^2(B_1)$ and

$$\|\partial_i\tilde{\phi}\|_{L^2(B_1)} \le \sup_{|h|<1}\|\eta T_h\tilde{\phi}\|_{L^2(B_3)},$$

$$\partial_i\tilde{\phi} = \lim_{h\to 0}\eta T_h\tilde{\phi},$$

weakly in $L^2(B_1)$.

Proof. The first statement is obtained via a change of variable $y = x + he_i$.

$$\begin{aligned}\int T_h(\eta\tilde{\phi})\tilde{\psi} &= \int \frac{(\eta\tilde{\phi})(x+he_i)\tilde{\psi}(x)}{h}\,dx - \int \frac{(\eta\tilde{\phi})(x)\tilde{\psi}(x)}{h}\,dx \\ &= \int \frac{(\eta\tilde{\phi})(x)\tilde{\psi}(x-he_i)}{h}\,dx - \int \frac{(\eta\tilde{\phi})(x)\tilde{\psi}(x)}{h}\,dx \\ &= -\int \tilde{\phi}\eta T_{-h}\tilde{\psi}.\end{aligned}$$

This means that $-\eta T_{-h}$ is the dual operator of $T_h(\eta\cdot)$ in L^2.

The second equality is just the remark that ∇ and T_h are commuting.

After computation, we find

$$\begin{aligned}\eta T_h DF(\Phi)(x) &= \eta\frac{DF(\Phi(x+he_i)) - DF(\Phi(x))}{h} \\ &= \frac{\eta[\Phi(x+he_i) - \Phi(x)]}{h}\int_0^1 D^2F(\Phi(x) + t[\Phi(x+he_i) - \Phi(x)])\,dt,\end{aligned}$$

which gives the result.

We have

$$\eta T_h\phi = \eta\int_0^1 \partial_i\phi(x+te_i)\,dt.$$

Hence

$$\|\eta T_h\phi\|_{L^2(B_3)} \le \eta_{L^\infty}\|\partial_i\phi\|_{L^2(B_3)}.$$

Consider a sequence of mollifiers for $\varepsilon < 1/2$

$$\delta_\varepsilon(x) = \frac{1}{\varepsilon^N}\delta_1(x/\varepsilon),$$

where δ_1 is a smooth nonnegative function compactly supported in B_1 with integral 1. Consider

$$\tilde{\phi}_\varepsilon = \tilde{\phi} * \delta_\varepsilon.$$

Since $\tilde{\phi}_\varepsilon$ is smooth, for $x \in B_2$, $\eta T_h\tilde{\phi}_\varepsilon(x)$ converges to $\eta\partial_i\tilde{\phi}_\varepsilon(x)$ when h goes to 0. But $\eta T_h\tilde{\phi}_\varepsilon = \eta(T_h\tilde{\phi})_\varepsilon$ are uniformly bounded (both with respect to h and ε). Passing into the limit in h, we find that $\partial_i\tilde{\phi}_\varepsilon$ are uniformly bounded in $L^2(B_1)$. In turn, $\partial_i\tilde{\phi}$ is in $L^2(B_1)$ and $T_h\tilde{\phi}$ converges weakly to $\partial_i\tilde{\phi}$. $\square$

Let us fix a cut off function η. We consider the test function

$$\phi = T_{-h}(\eta^2 T_h w) \in H^1_0(B_3).$$

we got

$$\begin{aligned}
0 &= \int \nabla\phi DF(\nabla w)\,dx \\
&= -\int \eta\nabla(T_h w)\eta T_h DF(\nabla w)\,dx - 2\int \nabla\eta(T_h w)\eta T_h DF(\nabla w)\,dx \\
&= -\int \eta\nabla(T_h w)\left[\int_0^1 D^2F(\nabla w(x) + t(\nabla w(x+he_i) - \nabla w(x)))\,dt\right]\eta\nabla(T_h w)\,dx \\
&\quad -2\int (\nabla\eta)(T_h w)\left[\int_0^1 D^2F(\nabla w(x) + t(\nabla w(x+he_i) - \nabla w(x)))\,dt\right]\eta\nabla(T_h w)\,dx.
\end{aligned}$$

Using the property (3) we get

$$\frac{1}{\Lambda}\int \eta^2|\nabla(T_h w)|^2\,dx \le 2\Lambda\sqrt{\int |\nabla\eta|^2|T_h w|^2\,dx}\sqrt{\int \eta^2|\nabla(T_h w)|^2\,dx}$$

and so

$$\int \eta^2|\nabla(T_h w)|^2\,dx \le 4\Lambda^4\int |\nabla\eta|^2|T_h w|^2\,dx \le C\|w\|^2_{H^1(B_3)}.$$

This gives that $\partial_i w \in H^1(B_1)$. For any test function $\phi \in H^1_0(B_1)$, we consider the test function

$$T_{-h}\phi \in H^1(B_2).$$

We get

$$0 = -\int \nabla\phi T_h DF(\nabla w)\,dx.$$

Since $\partial_i w \in H^1(B_1)$, $T_h DF(\nabla w)$ converges to $\partial_i\{DF(\nabla w)\} = D^2F(\nabla w)\nabla\partial_i w$ weakly in $L^2(B_1)$, and at the limit

$$\int \nabla\phi DF^2(\nabla w)\nabla\partial_i w\,dx = 0.$$

which finishes the proof. $\square$

3. THE PARABOLIC CASE

The parabolic case was first introduced with the techniques of Nash. We present, here, a version with a "De Giorgi packaging". We begin with the homogenous case.

3.1. The homogenous case.

Consider the equation:

$$\partial_t u - \mathrm{div}(A(t,x)\nabla_x u) = 0 \qquad \text{in } (0,T)\times\Omega, \tag{13}$$

where A verifies

$$\frac{1}{\Lambda} \le A(t,x) \le \Lambda I, \qquad \text{for any } (t,x)\in(0,T)\times\Omega, \tag{14}$$

but without further assumption on the regularity of A. We have the following theorem.

Theorem 14. *Let $u\in L^\infty(0,T;L^2(\Omega))$, $\nabla u \in L^2((0,T)\times\Omega)$ be a weak solution to (13) in $(0,T)\times\Omega$. Then, there exists $\alpha>0$ such that for any $\tilde\Omega \subset\subset \Omega$, and any $0<s<T$*

$$u\in C^\alpha((s,T)\times\tilde\Omega).$$

Define $Q_r=(-r,0)\times B_r$. As in the previous section, we show the proof with going from (Q_2) to $Q_{1/2}$. Theorem 14 can easily be deduced from this case by adequate zooming as in the previous section. We consider the first De Giorgi lemma.

Lemma 15. *there exists $\delta>0$ such that for any u solution to (13) with (14) in $(-1,0)\times B_1$: If*

$$\int_{Q_1} |u_+|^2\,dx\,dt \le \delta,$$

then

$$u_+\le 1/2 \qquad \text{in } Q_{1/2}.$$

Proof. We have now to consider two layers in position and time. We consider the sequence of time

$$T_k = -1/2(1+2^{-k}),$$

and the sequence of cylinders

$$\tilde Q_k = (T_k,0)\times\tilde B_k,$$

where $\tilde B_k$ is defined as before as centered at 0 with radius $1/2(1+2^{-k})$. We define the sequence of energy

$$U_k=\int_{Q_k}|u_k|^2\,dx\,dt,$$

where as before

$$u_k=(u-C_k)_+,$$

with

$$C_k=1/2(1-2^{-k}).$$

We consider the exact same family of cutoff non negative functions ϕ_k compactly supported in $\tilde{B}_{k+1}$, equal to 1 in $\tilde{B}_k$ and such that $|\nabla \phi_k| \leq C 2^k$.

We obtain the energy inequality in the following way: We multiply the equation by $\phi_{k+1}^2 u_{k+1}$, and integrate in both time and space on $(s,t)\times B_1$ for

$$T_k \leq s \leq T_{k+1} \leq t \leq 0.$$

We find

$$\left(\int \phi_{k+1}^2 |u_{k+1}|^2\,dx\right)(t) \leq \left(\int \phi_{k+1}^2 |u_{k+1}|^2\,dx\right)(s) - \int_s^t \int \nabla(\phi_{k+1}^2 u_{k+1}) A \nabla u\,dx\,d\tau.$$

Using the energy inequality in x only from the previous section we get

$$\left(\int \phi_{k+1}^2 |u_{k+1}|^2\,dx\right)(t) + \frac{1}{\Lambda}\int_s^t \int |\nabla(\phi_{k+1}u_{k+1})|^2\,dx\,d\tau \leq \left(\int \phi_{k+1}^2 |u_{k+1}|^2\,dx\right)(s) + \Lambda \int_s^t \int |\nabla \phi_{k+1}|^2 |u_{k+1}|^2\,dx\,d\tau.$$

Using the range of s and t, we get

$$\left(\int \phi_{k+1}^2 |u_{k+1}|^2\,dx\right)(t) + \frac{1}{\Lambda}\int_{T_{k+1}}^t \int |\nabla(\phi_{k+1}u_{k+1})|^2\,dx\,d\tau \leq \left(\int \phi_{k+1}^2 |u_{k+1}|^2\,dx\right)(s) + \Lambda \int_{T_k}^0 \int |\nabla \phi_{k+1}|^2 |u_{k+1}|^2\,dx\,d\tau.$$

Taking the mean value in s between T_{k+1} and T_k we get

$$\left(\int \phi_{k+1}^2 |u_{k+1}|^2\,dx\right)(t) + \frac{1}{\Lambda}\int_{T_{k+1}}^t \int |\nabla(\phi_{k+1}u_{k+1})|^2\,dx\,d\tau \leq 2^{k+2}\int_{T_k}^{T_{k+1}} \int \phi_{k+1}^2 |u_{k+1}|^2\,dx\,ds + \Lambda \int_{T_k}^0 \int |\nabla \phi_{k+1}|^2 |u_{k+1}|^2\,dx\,d\tau.$$

Summing the estimate for the worst t, and $t=0$ we get

$$\mathcal{E}_{k+1} = \sup_{T_{k+1}\leq t\leq 0}\left(\int (\phi_{k+1}u_{k+1})^2\,dx\right) + \int_{T_{k+1}}^0 \int_{B_1} |\nabla \phi_{k+1} u_{k+1}|^2\,dx\,dt \leq C^k U_k.$$

The quantity $\mathcal{E}_{k+1}$ controls $\phi_{k+1}u_{k+1}$ in $L^\infty(T_{k+1},0;L^2(B_1))$ and, thanks to Sobolev, in the space $L^2(T_{k+1},0;L^{2N/(N-2)}(B_1))$. The coeficients $(\infty,2)$ and $(2,2N/(N-2))$ verifies both

$$\frac{2}{p}+\frac{N}{q}=\frac{N}{2}.$$

So, by interpolation, $\mathcal{E}_{k+1}$ control the $L^p(L^q)$ norm of any such (p,q) with $p\geq 2$. We consider the one verifying $p=q=2(2+N)/N$. So

$$\|\phi_{k+1}u_{k+1}\|^2_{L^{2(2+N)/N}(\tilde{Q}_{k+1})} \leq C\mathcal{E}_{k+1}.$$

And so,

$$\begin{aligned} U_{k+1} &= \int_{\tilde{Q}_{k+1}} |u_{k+1}|^2 \, dx \, dt \\ &= \int_{\tilde{Q}_{k+1}} |u_{k+1}|^2 \mathbf{1}_{\{u_{k+1}>0\}} \, dx \, dt \\ &= \int_{\tilde{Q}_{k+1}} |u_{k+1}|^2 \mathbf{1}_{\{u_k > 2^{-(k+2)}\}} \, dx \, dt \\ &\le \mathcal{E}_{k+1} |\{u_k^2 > 2^{-2(k+2)} \cap \tilde{Q}_{k+1}\}|^{\frac{2}{2+N}} \\ &\le \mathcal{E}_{k+1} \left(2^{2(k+2)} \int_{\tilde{Q}_{k+1}} |u_k|^2 \, dx \, dt \right)^{\frac{2}{2+N}} \\ &\le C^k U_k^{1+\frac{2}{2+N}}. \end{aligned}$$

So, a before, there exists δ small enough such that if $U_0 \le \delta$, we have

$$\lim_{k\to\infty} U_k = 0.$$

This ensures that

$$u_+ \le 1/2 \qquad \text{on } \tilde{Q}_\infty = (-1/2, 0) \times B_{1/2}.$$

□

We now turn to the second lemma of De Giorgi. We denote $\tilde{Q} = (-3/2, -1) \times B_1$. We then have the following theorem.

Theorem 16. *There exists a constant $0 < \lambda < 1$ such that for any solution u of (13) in Q_2 verifying (14), we have the following:*
If $-1 \le u \le 1$ on Q_2 and

$$|\{u \le 0\} \cap \tilde{Q}| \ge \frac{|\tilde{Q}|}{2},$$

we have

$$u \le 1 - \lambda \qquad \text{in } Q_{1/2}.$$

To show this theorem, we need a result similar to the isoperimetric inequality for the parabolic case. We show the following lemma.

Lemma 17. *There exists $\alpha > 0$ such that for any u solution to (13) in Q_2 verifying (14), with $u \le 1$ we have the following. Denote*

$$\begin{aligned} &|A| = |\{u \ge 1/2\} \cap Q_1| \\ &|C| = |\{u \le 0\} \cap \tilde{Q}| \\ &|D| = |\{0 < u < 1/2\} \cap (Q_1 \cup \tilde{Q})|. \end{aligned}$$

If $|A| \ge \delta$, $|C| \ge |\tilde{Q}|/2$ then

$$|D| \ge \alpha.$$

Proof. We assume that the lemma is wrong. Then we can find a sequence u_k such that the associated quantities

$$\begin{aligned}
&|A_k| = |\{u_k \geq 1/2\} \cap Q_1| \geq \delta\\
&|C_k| = |\{u_k \leq 0\} \cap \tilde{Q}| \geq |\tilde{Q}|/2\\
&|D_k| = |\{0 < u_k < 1/2\} \cap (Q_1 \cup \tilde{Q})| \leq 1/k.
\end{aligned}$$

Consider a cutoff function ϕ in x only, compactly supported in B_2 and equal to 1 in B_1. Consider $\phi^2 v_k$ where $v_k = (u_k)_+$. We have

$$\partial_t \phi^2 (v_k)^2 - \operatorname{div}(\phi^2 A \nabla (v_k)^2) + 2\phi \nabla \phi A \nabla (v_k)^2 + 2\phi^2 \nabla v_k A \nabla v_k = 0. \tag{15}$$

Then integrating in x, we have for $-1 < s < t < 0$

$$\left(\int \phi^2 (v_k)^2 \, dx\right)(t) + \frac{1}{\Lambda}\int_s^t |\nabla(\phi v_k)|^2 \leq \left(\int \phi^2 (v_k)^2 \, dx\right)(s) + C(t-s). \tag{16}$$

This energy inequality give that ϕv_k is uniformly bounded in $L^2(-2,0;H^1(B_2))$ (and so is $\phi|v_k|^2$). Hence, from (15), $\partial_t v_k$ is uniformly bounded in $L^1(Q_2)+L^\infty(-2,0;W^{-1}(Q_2))$. Using the Aubin lemma, we find that the sequence ϕv_k converges to ϕv up to a subsequence in L^2 when k goes to infinity. Using Tchebychev, for any $\varepsilon > 0$,

$$\lim_{k\to\infty} |\{|v_k - v| \geq \varepsilon\} \cap (-2,0) \times B_1| = 0.$$

Now, if $\varepsilon \leq v \leq 1/2 - \varepsilon$ then either $|v - v_k| \geq \varepsilon$ or $0 < v_k < 1/2$. So

$$|\{\varepsilon \leq v \leq 1/2-\varepsilon\} \cap Q_1 \cup \tilde{Q}| \leq |\{|v - v_k| \geq \varepsilon\} \cap Q_1 \cup \tilde{Q}| + |\{0 < v_k < 1/2\} \cap (Q_1 \cup \tilde{Q})|,$$

which converges to 0. So

$$|\{\varepsilon \leq v \leq 1/2 - \varepsilon\} \cap (Q_1 \cup \tilde{Q})| = 0.$$

This is true for any $\varepsilon > 0$ hence

$$|\{0 < v < 1/2\} \cap (Q_1 \cup \tilde{Q})| = 0.$$

For almost every $t \in (-2,0)$, $\nabla v(t,\cdot) \in L^2(B_1)$. So thanks to the isoperimetric lemma of De Giorgi in the stationary case, for almost every $t \in (-3/2, 0)$ we have either

$$v(t,\cdot) \leq 0 \quad \text{in } B_1, \qquad \text{or } v(t,\cdot) \geq 1/2 \quad \text{in } B_1.$$

In the same way, if $v_k \leq 0$, then either $|v - v_k| \geq \varepsilon$ or $v \leq \varepsilon$. So

$$\begin{aligned}
|\tilde{Q}|/2 &\leq |\{v_k \leq 0\} \cap \tilde{Q}| \leq |\{|v - v_k| \geq \varepsilon\} \cap \tilde{Q}| + |\{v \leq \varepsilon\} \cap \tilde{Q}|\\
&\leq |\{v \leq \varepsilon\} \cap \tilde{Q}|.
\end{aligned}$$

Passing into the limit $\varepsilon \to 0$, we find

$$|\{v \leq 0\} \cap \tilde{Q}| \geq |\tilde{Q}|/2.$$

So there is some times $-3/2 < s < -1$ such that $v(t,\cdot) \leq 0$ in B_1. Now, consider a nonnegative function in x, $\phi_1 \geq 0$ compactly supported in B_1 of integral 1. the energy inequality ensures that for those time s, and $t > s$

$$\left(\int \phi_1^2 (u_+)^2\, dx\right)(t) \leq \left(\int \phi_1^2 (u_+)^2\, dx\right)(s) + C_0(t-s) \leq C_0(t-s). \tag{17}$$

But $\int \phi_1^2 (u_+)^2\, dx$ in all time is either negative or bigger than 1/2. So it cannot reach the value 1/2, and finally

$$v(t,x) = 0 \qquad \text{on } Q_1.$$

But, for any $\varepsilon > 0$, if $v_k \geq 1/2$, then either $|v - v_k| \geq \varepsilon$ or $v \geq 1/2 - \varepsilon$. So

$$\delta \leq |\{v_k \geq 1/2\} \cap Q_1| \leq |\{|v - v_k| \geq \varepsilon\} \cap Q_1| + |\{v \geq 1/2 - \varepsilon\} \cap Q_1|.$$

So, passing into the limit first in $k \to \infty$, then $\varepsilon \to 0$, we get

$$|\{v \geq 1/2\} \cap Q_1| \geq \delta.$$

This leads to a contradiction. □

The proof of Theorem 16 and Theorem 14 now follows the lines of the previous section.

3.2. **The non homogenous case.** For a bounded set Ω, we consider the equation on u

$$\partial_t \rho + \mathrm{div}(\rho V_1) = 0, \tag{18}$$

$$\begin{aligned} &\partial_t(\rho u) + \mathrm{div}(\rho u (V_1 + V_2)) + \rho V_3 \cdot \nabla u \\ &\qquad -\mathrm{div}(A(t,x)\nabla_x u) = f + \mathrm{div} F \qquad \text{in } (0,T)\times\Omega, \end{aligned} \tag{19}$$

where A and ρ verify

$$\frac{1}{\Lambda} \leq \rho(t,x) \leq \Lambda, \tag{20}$$

$$\frac{1}{\Lambda} I \leq A(t,x) \leq \Lambda I, \qquad \text{for any } (t,x) \in (0,T)\times\Omega, \tag{21}$$

but without further assumption on the regularity of A and ρ. Note that if V_1 is divergence free, and ρ is bounded by below and by above at $t = 0$, then (20) is fulfilled for all time and space. The case include the classical parabolic case with drift in the divergence form ($V_1 = 0$, $\rho = 1$, and $V_3 = 0$)

$$\partial_t u + \mathrm{div}(\rho u V_2) + \mathrm{div}(A(t,x)\nabla_x u) = f + \mathrm{div} F,$$

Together with the case with the velocity in the non divergence form:

$$\partial_t u + V_3 \cdot \nabla u + \mathrm{div}(A(t,x)\nabla_x u) = f + \mathrm{div} F.$$

We assume that there exists $e > 0$ such that

$$f \in L^{p_f}(0,T;L^{q_f}(\Omega)), \qquad |V_i|^2 \in L^{p_i}(0,T;L^{q_i}(\Omega)), \qquad ||F|^2| \in L^{p_F}(0,T;L^{q_F}(\Omega))$$

where (p_f, q_f), (p_i, q_i) and (p_F, q_F) verify

$$1 < p < \infty, \qquad 1 < q < \infty, \qquad \frac{2}{p} + \frac{N}{q} = 2 - e. \tag{22}$$

Note that since Ω is bounded, it is not a restriction to consider the same e for all functions. We will denote

$$\|\mathcal{F}\|_{(0,T)\times\Omega} = \sum_{i=1}^{3} \||V_i|^2\|_{L^{p_i}(0,T;L^{q_i}(\Omega))} + \|f\|_{L^{p_F}(0,T;L^{q_f}(\Omega))} + \||F|^2\|_{L^{p_F}(0,T;L^{q_F}(\Omega))}.$$

We will consider only "suitable" solutions which verify for any $K > 0$

$$\begin{aligned} &\partial_t(\rho v_K^2) + \mathrm{div}(\rho v_K^2 V_1 + 2\rho u v_K V_2) + \rho V_3 \cdot \nabla v_K^2 + 2\nabla v_K^T A \nabla v_K \\ &\qquad \leq \mathrm{div}(A\nabla v_K^2) + 2(f + \mathrm{div} F) v_K + 2\rho u V_2 \cdot \nabla v_K, \end{aligned} \tag{23}$$

where $v_K = (u - K)_+$. This inequality is obtained formally by multiplying Equation (19) by $2v_K$ and using Equation (18). Indeed,

$$\begin{aligned} v_K\{\partial_t(\rho u) + \mathrm{div}(\rho V_1 u)\} &= \{\partial_t \rho + \mathrm{div}(\rho V_1)\}(u v_K) + \rho\{\partial_t u + V_1 \cdot \nabla u\} v_K \\ &= \rho\{\partial_t \frac{v_K^2}{2} + V_1 \cdot \nabla \frac{v_K^2}{2}\} \\ &= \partial_t(\rho \frac{v_K^2}{2}) + \mathrm{div}(\rho V_1 \frac{v_K^2}{2}). \end{aligned}$$

It is verified, for instance, for any weak solution $u \in L^\infty(0,T;L^2(\Omega))$, $\nabla u \in L^2((0,T)\times\Omega)$ of (19), as long as ρ and V_1 are smooth. Even this case is interesting, since the result does not depend on the amount of regularity on those quantities. We have the following theorem.

Theorem 18. *Let $u \in L^\infty(0,T;L^2(\Omega))$, $\nabla u \in L^2((0,T)\times\Omega)$ be a suitable solution to (19) in $(0,T)\times\Omega$, where (21), (20) and (22) are verified. Then, there exists $\alpha > 0$ such that for any $\tilde{\Omega} \subset\subset \Omega$, and any $0 < s < T$*

$$u \in C^\alpha((s,T)\times\tilde{\Omega}).$$

Note that for $\varepsilon < 1$,

$$\tilde{u}(s,y) = u(t + \varepsilon^2 s, x + \varepsilon y)$$

is solution to

$$\begin{aligned} &\partial_s(\tilde{\rho}\tilde{u}) + \mathrm{div}(\tilde{\rho}\tilde{u}(\tilde{V}_1 + \tilde{V}_2)) + \tilde{\rho}\tilde{V}_3 \cdot \nabla\tilde{u} - \mathrm{div}(\tilde{A}\nabla\tilde{u}) = \tilde{f} + \mathrm{div}\tilde{F}, \\ &\partial_t\tilde{\rho} + \mathrm{div}(\tilde{\rho}\tilde{V}_1) = 0, \end{aligned}$$

with

$$\begin{aligned} &\tilde{F} = \varepsilon F(t + \varepsilon^2 s, x + \varepsilon y), \\ &\tilde{V}_i = \varepsilon V_i(t + \varepsilon^2 s, x + \varepsilon y), \\ &\tilde{f} = \varepsilon^2 f(t + \varepsilon^2 s, x + \varepsilon y), \\ &\tilde{\rho} = \rho(t + \varepsilon^2 s, x + \varepsilon y), \\ &\tilde{A} = A(t + \varepsilon^2 s, x + \varepsilon y). \end{aligned}$$

And

$$\begin{aligned}
&\|\tilde f\|_{L^{p_1}(L^{q_1})}=\varepsilon^e\|f\|_{L^{p_1}(L^{q_1})},\\
&\||\tilde V_i|^2\|_{L^{p_2}(L^{q_2})}=\varepsilon^e\||V_i|^2\|_{L^{p_2}(L^{q_2})},\\
&\||\tilde F|^2\|_{L^{p_2}(L^{q_2})}=\varepsilon^e\||F|^2\|_{L^{p_2}(L^{q_2})}.
\end{aligned}$$

Note that $\tilde\rho$ and $\tilde A$ are verifying (20) and (21). Moreover, zooming if necessary, we can always assume that the norms of f and V are as small as we need on a fixed cylinder Q.

Note that the equation is not linear anymore. This introduces some new difficulties. The function $\tilde u(t,x)=\lambda u(t,x)$ is solution to

$$\partial_t(\rho\tilde u)+\mathrm{div}(\rho\tilde u(V_1+V_2))+\rho V_3\cdot\nabla\tilde u-\mathrm{div}(A(t,x)\nabla_x\tilde u)=\lambda f+\mathrm{div}(\lambda F),$$

where the right-hand side term becomes large when λ increases. This has to be dealt with when engineering the second lemma of De Giorgi.

We first derive the energy inequality in a form that will be useful for both lemmas.

Lemma 19. *Let u be a suitable solution to (13), and ϕ be a cut off function in x only, compactly supported in the domain of definition of u. We define $v=(u-K)_+$. Then for every $s<t$ we have*

$$\begin{aligned}
&\left(\int\rho\frac{v^2}{2}\phi^2\,dx\right)(t)-\left(\int\rho\frac{v^2}{2}\phi^2\,dx\right)(s)+\frac{1}{2\Lambda}\int_s^t\int|\nabla(\phi v)|^2\,dx\,dt\\
&\le C(1+K^2)(\|\mathcal F\|_{(t,s)\times\mathrm{supp}\phi}+1)\left(|\{v>0\}\cap\mathrm{supp}\phi|^{\frac{e}{N+2}}\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad|||(|\phi|+|\nabla\phi|)^2(v^2+\mathbf 1_{\{v>0\}})|||,
\end{aligned}$$

with

$$|||g|||=\sum_{(p,q)\in\mathcal I}\|g\|_{L^p(L^q)},$$

where the sum is taken on

$$\begin{aligned}
\mathcal I=\Bigg\{(p,q),\frac1p+\frac1{p'}+\frac{e}{N+2}=1,\frac1q+\frac1{q'}+\frac{e}{N+2}=1,\\
(p',q')\in\{(p_i,q_i),(p_F,q_F),(p_f,q_f)\}\Bigg\}.
\end{aligned}$$

For all the $(p,q)\in\mathcal I$ we have

$$\begin{aligned}
&0<\frac1{p'}+\frac{e}{N+2}=1-\frac{N}{2q'}-\frac{Ne}{N+2}<1,\\
&0<\frac1{q'}+\frac{e}{N+2}=1-\frac2N\left(1-\frac1{p'}\right)-\frac eN<1.
\end{aligned}$$

Hence

$$0 < \frac{1}{p} < 1,$$
$$0 < \frac{1}{q} < 1.$$

Moreover,

$$\frac{2}{p} + \frac{N}{q} = N. \tag{24}$$

Proof. Multiplying Equation (23) by ϕ^2, and integrating in x, and noticing that $vu = v^2 + Kv$, we get

$$\begin{aligned}
&\partial_t \int \rho \frac{v^2}{2}\phi^2\,dx + \frac{1}{\Lambda}\int |\nabla(\phi v)|^2\,dx \\
&\quad \le \int \Lambda |\nabla\phi|^2 v^2\,dx \\
&\qquad + \int \phi(\nabla\phi)\cdot(V_1 + 2V_2)\rho v^2\,dx + \int 2K\rho\phi\nabla\phi\cdot V_2 v\,dx \\
&\qquad + \int \rho\phi^2 v\nabla v\cdot(V_2 - V_3)\,dx + \int K\rho\phi^2\nabla v\cdot V_2\,dx \\
&\qquad + \int \phi^2 v f\,dx \\
&\qquad -2\int \phi v(\nabla\phi)\cdot F\,dx - \int \phi^2\nabla v\cdot F\,dx.
\end{aligned}$$

Using that $\phi\nabla v = \nabla(v\phi) - v\nabla\phi$, $|V_1| \le 1 + |V_1|^2$, $\rho \le \Lambda$, $v = v\mathbf{1}_{\{v>0\}}$, and $\nabla v = \nabla v\mathbf{1}_{\{v>0\}}$, we find:

$$\begin{aligned}
&\partial_t \int \rho \frac{v^2}{2}\phi^2\,dx + \frac{1}{\Lambda}\int |\nabla(\phi v)|^2\,dx \\
&\quad \le \frac{1}{2\Lambda}\int |\nabla(\phi v)|^2\,dx \\
&+C\int (\phi + |\nabla\phi|)^2(|V_1|^2 + (1+K^2)|V_2|^2 + |V_3|^2 + |F|^2 + |f|)(v^2 + \mathbf{1}_{\{v>0\}})\,dx.
\end{aligned}$$

Integrating in time, and using Holder gives the result. □

The first lemma is similar to the previous case. Except that it is easier to work with the energy/dissipation of energy instead of $\int |u|^2\,dx\,dt$.

Lemma 20. *there exists $\delta > 0$ such that for any u solution to (19) in $(-1,0)\times B_1$: If*

$$|||u_+^2||| \le \delta,$$

with

$$\|\mathcal{F}\|_{(-1,0)\times B_1} \le 1,$$

then

$$u_+ \le 1/2 \qquad \text{in} \;\; Q_{1/2}.$$

Proof. We use the same setting as for the homogenous case, Except we are working with

$$\mathcal{E}_k = \sup_{T_k \le t \le 0} \left(\int (\phi_k u_k)^2 \, dx \right) + \int_{T_k}^0 \int_{B_1} |\nabla(\phi_k u_k)|^2 \, dx \, dt.$$

Following the previous section we have now from the energy inequality

(25)

$$\mathcal{E}_{k+1} \le C^k \int_{\tilde{Q}_k} |u_k|^2 \, dx \, dt + C^k (1 + \|\mathcal{F}\|_{(-1,0)\times B_1}) |||u_k^2 \mathbf{1}_{\{\tilde{Q}_k\}}||| \left(|\{u_{k+1} > 0\} \cap \tilde{Q}_k|^{\frac{e}{N+2}} \right).$$

Thanks to Sobolev, $\mathcal{E}_k$ control the $L^\infty(L^1)$ norm and the $L^1(L^{N/(N-2)})$ norm of $(\phi_k u_k)^2$. By interpolation, it controls all the norms $L^p(L^q)$ with

$$\frac{2}{\bar{p}} + \frac{N}{\bar{q}} = N.$$

Especially all the norms involved in the definition of $|||\cdot|||$. Hence

$$|||\mathbf{1}_{\{\tilde{Q}_k\}} u_k^2||| \le C\mathcal{E}_k.$$

And

$$\begin{aligned} \mathcal{E}_{k+1} &\le C^k \mathcal{E}_k |\{u_{k+1} > 0\}|^{2/(N+2)} + C^k \mathcal{E}_k^{1+2/(N+2)} \\ &\le C^k \left(\mathcal{E}_k^{1+e/(N+2)} + \mathcal{E}_k^{1+2/(N+2)} \right). \end{aligned}$$

Note that $e < 2$. So as long as $\mathcal{E}_k \le 1$

$$\mathcal{E}_{k+1} \le 2C^k \mathcal{E}_k^{1+e/(N+2)}.$$

From (25), we can have $\mathcal{E}_1$ as small as we want, taking δ small enough. Then, for δ small enough, we have for all k, $\mathcal{E}_k < 1$ and $\mathcal{E}_k$ converges to 0 when $k \to \infty$. ☐

As before we denote $\tilde{Q} = (-3/2, -1) \times B_1$.

Theorem 21. *There exists* $0 < \lambda < 1$, *and* $\eta > 0$ *such that for any suitable solution* u *of (*19*) with*

$$|||\mathcal{F}|||_{Q_2} \le \eta :$$

If $u \le 1$ *on* Q_2 *and*

$$|\{u \le 0\} \cap \tilde{Q}| \ge \frac{|\tilde{Q}|}{2},$$

then $u \le 1 - \lambda$ *in* $Q_{1/2}$.

Because the equation is not linear any more, we need the smallness condition η. As before we first show the following lemma.

Lemma 22. *There exists* $\alpha > 0$ *such that for any* u *suitable solution to (*19*) with*

$$|||\mathcal{F}|||_{Q_2} \le 1 :$$

If $|A| \ge \delta$, $|C| > \frac{\tilde{Q}}{2}$ *then* $|D| \ge \alpha$.

Proof. of Theorem 21. The proof goes by contradiction. The energy gives now

$$\left(\int \phi^2 v_k^2\,dx\right)(t) + \frac{1}{2\Lambda}\int_s^t \int |\nabla(\phi v_k)|^2\,dx\,dt$$
$$\leq \left(\int \phi^2 v_k^2\,dx\right)(s) + C|t-s|^{e/(N+2)}.$$

The function ϕv_k is uniformly bounded in $L^2(-2,0;H^1(B_2))$, and so $\partial_t v_k$ is uniformly bounded in $L^1(Q_2)+L^\infty(-2,0;W^{-1}(B_2))$. Hence, up to a subsequence, ϕv_k converges in $L^2(Q_2)$ to ϕv. As before, we show that $|D|=0$. So for almost very time $t\in(-3/2,0)$, we have either $v(t,\cdot)\leq 0$, or $v(t,\cdot)\geq 1/2$ in B_1. But $|\{v\leq 0\}\cap\tilde{Q}|\geq|\tilde{Q}|/2|$. So, there exists $s_0\leq -1$ such that $\int \phi^2 v_k^2(s)\,dx=0$. Consider $s_0\leq s\leq 0$ the biggest s such that $\int \phi^2 v_k^2(\tau)\,dx=0$ for $s_0\leq\tau\leq s$. Assume that $s<0$. Using the energy inequality, we get that for every $s<t<0$

$$\begin{aligned}\int \phi^2 v_k^2(t)\,dx &\leq \int \phi^2 v_k^2(s)\,dx + C_0|t-s|^{e/(N+2)}\\ &\leq C_0|t-s|^{e/(N+2)}.\end{aligned}$$

For $|t-s|$ small enough, $\int \phi^2 v_k^2(t)\,dx\leq 1/8$. Hence we cannot have $v(t,\cdot)\geq 1/2$ for those values of t, and from the dichotomy, we have $v(t,\cdot)\leq 0$. Hence $\int\phi^2 v_k^2(t)\,dx=0$. This contradicts the definition of s. Hence $|A|=0$, which leads to a contradiction. □

Proof. Consider k^0 such that $k^0\alpha\geq|Q_1\cup\tilde{Q}|$, where α is defined in the previous lemma, and denote $\eta=\frac{2^{-k^0}}{1+\Lambda}$. We consider, as before, the sequence

$$w_k = 2^k(u-(1-2^{-k}))$$

which verifies for any k: $w_k\leq 1$ and $|\{w_k\leq 0\}\cap\tilde{Q}|\geq|\tilde{Q}|/2$, and

$$\partial_t(\rho w_k)-\mathrm{div}(A\nabla w_k)+\mathrm{div}(\rho w_k(V_1+V_2))+\rho V_3\cdot\nabla w_k = 2^k f+\mathrm{div}(2^kF-(2^k-1)\rho V_2).$$

and so for every $k\leq k^0$

$$\|\mathcal{F}_k\|_{Q_2}\leq(1+\lambda)2^k\|\mathcal{F}\|_{Q_2}\leq 1.$$

Then, for a fixed $k\leq k_0$,

$$\text{if}\quad |||(w_j)_+^2|||\geq\delta,\qquad \text{for any}\quad j\leq k+1,$$

we get

$$\delta\leq|\{w_{j+1}>0\}\cap Q_1| = |\{w_j>1/2\}\cap Q_1|.$$

So, from the previous lemma,

$$|\{0<w_j<1/2\}\cap(Q_1\cup\tilde{Q})|\geq\alpha,$$

and

$$|\{w_k<0\}\cap(Q_1\cup\tilde{Q})|\geq k\alpha.$$

But this fails for $k=k^0$. So, for sure, there exists a $k\leq k^0$ such that

$$|||(w_{k+1}^2)_+|||\leq\delta,$$

and

$$\|\mathcal{F}_k\|_{Q_2} \leq 1.$$

So from the first lemma, $w_{k+1} \leq 1/2$ in Q_1. This means that $u \leq 1 - 2^{-(2+k^0)}$ in $Q_{1/2}$. This gives the proof of the result with $\lambda = 2^{-(2+k^0)}$. □

This provides the following easy lemma.

Lemma 23. *If* $-1 \leq u \leq 1$ *in* Q_2 *and* $\|\mathcal{F}\|_{Q_2} \leq \eta$ *then*

$$\mathrm{osc}_{Q_{1/2}} \leq 1 - \lambda.$$

We can now prove the main theorem.

Proof. For $d > 0$, we define $\tilde{\Omega}$ the set of points of Ω whose distance from $\partial\Omega$ is greater to d. Consider $\varepsilon \leq \inf((1, 1/||\mathcal{F}||)^{1/e}, d, \sqrt{t_0}/2)$, and for every $(t,x) \in \tilde{\Omega} \times (t_0, T)$ and $s \leq t_0/(2\varepsilon^2)$

$$u_\varepsilon(s,y) = \varepsilon^{(N+1)/2} u(t + \varepsilon^2 s, x + \varepsilon y).$$

The function u_ε is a suitable solution in Q_1 to (19) with

$$\|\mathcal{F}\|_{Q_1} \leq 1.$$

Moreover

$$|||u_\varepsilon^2|||_{Q_1} \leq \varepsilon |||u^2|||_{\tilde{\Omega}}.$$

So, for ε small enough, this quantity will be smaller than δ. Thanks to Lemma 20, for all those $(t,x) \in \tilde{\Omega}$

$$u_\varepsilon \leq 1/2 \qquad \text{in } \; Q_{1/2}.$$

In the same way we find that u_ε is bounded by below on $Q_{1/2}$. This gives a uniform L^∞ bound of u on $\tilde{\Omega}$.

The C^α regularity is obtained in a similar way.

□

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