Regularity Analysis for Systems of Reaction-Diffusion Equations

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Abstract

This paper is devoted to the study of the regularity of solutions to some systems of Reaction–Diffusion equations. In particular, we show the global boundedness and regularity of the solutions in 1D and 2D, and we discuss the Hausdorff dimension of the set of singularities in higher dimensions. Our approach is inspired by De Giorgi's method for elliptic regularity with rough coefficients. The proof uses the specific structure of the system to be considered and is not a mere adaptation of scalar techniques; in particular the natural entropy of the system plays a crucial role in the analysis.

Key words. Reaction-Diffusion Systems. Regularity of solutions.

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1 Introduction

This paper is devoted to the analysis of the following system of Reaction-Diffusion equations

$$\begin{cases} \partial_t a_i - \nabla_x \cdot (D_i \nabla_x a_i) = Q_i(a), & i \in \{1, \dots, p\}, \\ Q_i(a) = (\mu_i - \nu_i) \left(k_\mathrm{f} \prod_{j=1}^{p} a_j^{\nu_j} - k_\mathrm{b} \prod_{j=1}^{p} a_j^{\mu_j} \right), & \\ a_{i\ |t=0} = a_i^0. & \end{cases} \tag{1.1}$$

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The equation holds for $t \geq 0$ and the space variable x lies in Ω where

- either $\Omega = \mathbb{R}^N$,

- or $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and the system is completed by imposing the Neumann boundary condition

$$D_i \nabla_x a_i \cdot \nu(x)_{|\partial\Omega} = 0,$$

where $\nu(x)$ stands for the outer normal vector at $x \in \partial\Omega$.
The matrices $D_i(x)$ are required to satisfy

$$\begin{cases} D_i \in \big(L^\infty(\Omega)\big)^{N\times N}, \\ D_i(x)\xi \cdot \xi \geq \alpha|\xi|^2, \qquad \alpha > 0 \qquad \text{for any } \xi \in \mathbb{R}^N,\ x \in \Omega. \end{cases} \tag{1.2}$$

Such a system is intended to describe e.g. the evolution of a chemical solution: the unknown a_i stands for the density of the species labelled by $i \in \{1, \ldots, p\}$ within the solution. Accordingly, the unknowns are naturally non negative quantities: $a_i \geq 0$. The right hand side of (1.1) follows from the mass action principle applied to the reversible reaction

$$\sum_{i=1}^p \nu_i A_i \leftrightarrow \sum_{i=1}^p \mu_i A_i,$$

where the μ_i and ν_i's — the so-called stoichiometric coefficients — are integers. The (positive) coefficients k_f and k_b are the rates corresponding to the forward and backward reactions, respectively.

The key feature relies on the conservation property

$$\begin{cases} \text{There exists } (m_1, \ldots, m_p) \in \mathbb{N}^p,\ m_i \neq 0, \text{ such that} \\ \displaystyle\sum_{i=1}^p m_i \mu_i = \sum_{i=1}^p m_i \nu_i \end{cases} \tag{1.3}$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^p \int_\Omega m_i \ a_i \,\mathrm{d}x = 0,$$

while the entropy is dissipated: we set $K = k_\mathrm{b}/k_\mathrm{f}$, then

$$\sum_{i=1}^p Q_i(a) \ln(a_i/K^{1/(p(\mu_i-\nu_i))}) = -k_\mathrm{f}\Big(\prod_{i=1}^p a_i^{\mu_i} - K\prod_{i=1}^p a_i^{\nu_i}\Big) \ln\left(\frac{\displaystyle\prod_{i=1}^p a_i^{\mu_i}}{K \displaystyle\prod_{i=1}^p a_i^{\nu_i}}\right) \leq 0. \tag{1.4}$$

Actually, we realize readily that, up to a change of unknowns, we can reduce the situation to

$$m_i = 1, \qquad k_\mathrm{f} = 1 = k_\mathrm{b}.$$

From now on, we adopt this framework and a crucial role will be played by the quantity

$$\overline{\mu} = \sum_{i=1}^{p} \mu_i = \sum_{i=1}^{p} \nu_i,$$

where the coefficients μ_i and ν_i are still integers.

As we shall see below, our analysis will be faced to restriction on the space dimension N and the parameter $\overline{\mu}$; one of the most interesting situation we are able to deal with is the following example corresponding to 4 species subject to the reactions

$$A_1 + A_3 \leftrightarrow A_2 + A_4,$$

which leads to

$$Q_i(a) = (-1)^{i+1}(a_2a_4 - a_1a_3), \tag{1.5}$$

We refer for a thorough introduction to the modeling issues and mathematical properties of such reaction diffusion systems to [12, 14, 15, 20, 21, 22, 24, 28, 31]. Information can also be found in the survey [7] with connection to coagulation-fragmentation models and in [25] for applications in biology. Let us also mention that (1.1) can be derived through hydrodynamic scaling from kinetic models, see [3].

In this contribution we are interested in the derivation of new L^∞ estimates and we investigate the regularity of the solutions of (1.1). Quite surprisingly, the question becomes trivial when the diffusion coefficients vanish: for $D_i = 0$, we are dealing with a system of ODEs and the conservation property (1.3) guarantees that solutions are globally defined. Conversely, certain reaction diffusion systems might exhibit blow up phenomena, see e.g. [23], as it is also well known when considering non linear heat equations [16, 35]. Therefore global well-posedness and discussion of smoothing effects is an issue.

While standard techniques can be applied to show the existence of a smooth solution locally in time, with, say, initial data in $L^1 \cap L^\infty(\Omega)$, extending the result on arbitrarily large time interval is a tough question. Roughly speaking, this is due to a lack of estimates since the only natural bounds are provided by the mass conservation (1.3) and the entropy dissipation (1.4). In turn this leads to an "L^1 framework" which is quite uncomfortable and with such a bound it is even not clear that the right hand side of (1.1) makes sense. Recently, by using the tricky techniques introduced in [27, 26], it has been shown in [11] that the solutions of (1.1) in the quadratic case (1.5) are a priori bounded in $L^2((0,T) \times \Omega)$ so that the nonlinear reaction term makes sense at least in L^1. Besides, [11] establishes the global existence of weak solutions of (1.1), (1.5). The dissipation property (1.4) is also the basis for studying the asymptotic trend to equilibrium [9, 10] in the spirit of the entropy/entropy dissipation techniques which are presented e.g. in [34] (we refer also to [2] for further investigation of the large time behavior of nonlinear evolution systems using the entropy dissipation).

Our approach is inspired by De Giorgi's methods for studying the regularity of solution of diffusion equations without requiring the regularity of the coefficients, see [8]. The crucial step consists in establishing a L^∞ estimate on the solution, and, then, we can deduce that hölderian regularity holds. This approach has been used in [33] to obtain an alternative proof to the regularity results for the Navier-Stokes equation [5, 18] and it also shares some features with the strategy introduced in [29, 30]. It has also been applied to study convection-diffusion equations [19] and regularity for the Quasi-geostrophic equation [6]. In the framework of Reaction-Diffusion systems, the De Giorgi's method was first introduced in [1]. In our situation, it is worth pointing out that the proof highly relies on the structure of the whole system and the argument is not a mere refinement of a scalar approach. Indeed, we use in a crucial way the properties of the underlying entropy functional. As we shall see however, restrictions appear between the space dimension N and the degree of nonlinearity of the reaction term measured by means of $\overline{\mu}$. For this reason, the L^∞ estimates can be proved in $2D$ for the quadratic operator (1.5) or in $1D$ considering cubic terms.

Theorem 1.1 *We consider the quadratic operator* (1.5) *(or assume $\overline{\mu} = 2$). Let $N = 2$ and suppose that the diffusion coefficients fulfill* (1.2). *Let $a_i^0 \geq 0$ satisfy*

$$\sum_{i=1}^{4} \int_\Omega a_i^0 \big(1 + |x| + |\ln(a_i^0)|\big)\,\mathrm{d}x = M_0 < \infty. \tag{1.6}$$

Then, (1.1) *admits a global solution such that for any $0 < T \leq T^\star < \infty$, a_i belongs to $L^\infty((T, T^\star) \times \Omega)$.*

Theorem 1.2 *Let $N = 1$ with $\overline{\mu} \leq 3$ and suppose that the diffusion coefficients fulfill* (1.2). *Let $a_i^0 \geq 0$ satisfy* (1.6). *Then, for any $0 < T \leq T^\star < \infty$, a_i belongs to $L^\infty((T, T^\star) \times \Omega)$.*

We point out that these statements do not require any regularity property on the diffusion coefficients D_i which are only supposed to be bounded. As a byproduct, we expect that such a result can be the starting point to apply the strategy developed in [9, 10] and then this would lead to the proof of the convergence to the equilibrium state for large time, with an exponential rate. Moreover, by using the new bound, a direct bootstrap argument combined to the standard local existence of smooth solutions shows the global regularity of the solution.

Corollary 1.1 *Let the assumptions of Theorem 1.1 or 1.2 be fulfilled. Suppose moreover that the D_i's are of class $C^k(\Omega)$ with bounded derivatives up to order k. Then the solution belongs to $L^\infty(T, T^\star; H^k(\Omega))$. Accordingly for C^∞ coefficients with bounded derivatives, the solution is C^∞ on $(T, T^\star) \times \Omega$.*

In higher dimensions, the same method provides information on the Hausdorff dimension (definitions are recalled in Section 4) of the set of the singular points of the solutions.

Theorem 1.3 *Let $N = 3$ or $N = 4$ and $\overline{\mu} = 2$. We suppose that the coefficients D_i are constant with respect to $x \in \Omega$. Let $a_i^0 \geq 0$ satisfy* (1.6). *We consider a solution of* (1.1) *on $(0, T) \times \Omega$. We call singular point any point (t, x) having a neighborhood on which one of the function a_i is not C^∞. Then, the Hausdorff dimension of the set of singular points of the solution a does not exceed $(N^2 - 4)/N$.*

The existence of a weak solution in this case has been proved in [26]. In the next section, we briefly recall the fundamental estimate that follows from (1.4). This bound is used in Section 3 where we adapt De Giorgi's approach to the system (1.1). There, the restriction on the space dimension and the degree of nonlinearity will appear clearly, through the application of the Gagliardo-Nirenberg inequality. Section 4 is devoted to the estimate of the Hausdorff dimension of the set of singularities in higher space dimensions.

2 Entropy Dissipation

In the following Sections, we adopt the viewpoint of discussing a priori estimates formally satisfied by the solutions of (1.1). As usual the derivation of such estimates relies on various manipulations as integrations by parts, inversion of integrals and so on. Then, they apply to the smooth solutions of the problem that can be shown to exist on a small enough time interval by using classical reasoning for non linear parabolic equations. Besides, these estimates also apply to solutions of suitable approximations of the problem (1.1). Such approximation should be defined so that the essential features of the system are preserved. Hence, let us reproduce the reasoning in [11]: by truncation and regularization we deal with an initial data

$$a_i^{0,\eta} \in C_c^\infty(\Omega), \qquad a_i^{0,\eta} \geq 0$$

which converges in $L^1(\Omega)$ to a_i^0 as $\eta > 0$ tends to 0 and such that

$$\sup_{\eta>0} \sum_{i=1}^{p} \int_\Omega a_i^{0,\eta}\big(1 + |x| + |\ln(a_i^{0,\eta})|\big)\,\mathrm{d}x \leq C_0\ \sum_{i=1}^{p} \int_\Omega a_i^{0}\big(1 + |x| + |\ln(a_i^{0})|\big)\,\mathrm{d}x = C_0\ M_0 < \infty.$$

Next, let us consider a cut-off function $\zeta \in C_c^\infty(\mathbb{R})$ such that $0 \leq \zeta(s) \leq 1$, $\mathrm{supp}(\zeta) \subset B(0,2)$ and $\zeta(s) = 1$ for $|s| \leq 1$. Then, in (1.1) we replace $Q_i(a)$ by

$$Q_i^\eta(a) = Q_i(a)\zeta(\eta|a|),$$

with $|a| = \sqrt{a_1^2 + ... + a_p^2}$. Accordingly, for any $\eta > 0$ fixed, and $a_i \in L^1(\Omega)$, $Q_i^\eta(a)$ belongs to $L^\infty(\Omega)$. We can show that the corresponding regularized problem admits a unique smooth solution, globally defined, see [17, 28]. Therefore, in what follows we discuss a priori estimates on solutions of (1.1): for the sake of simplicity we detail the arguments working directly on (1.1), but we keep in mind that the arguments apply to the regularized problem as well. In turn, we obtain bounds on the sequence a_i^η, which are uniform with respect to $\eta > 0$. Finally, existence of a global solution satisfying the estimates follows by performing the passage to the limit $\eta \to 0$; a detail that we skip here, referring for instance to [11].

As a warm up, let us discuss the a priori estimates that can be naturally deduced from (1.3) and (1.4). The results here apply in full generality, without assumptions on $p, N, \overline{\mu}$.

Proposition 2.1 *Let $a_i^0 \geq 0$ satisfy*

$$\sum_{i=1}^{p} \int_{\Omega} a_i^0 \big(1 + |x| + |\ln(a_i^0)|\big) \,\mathrm{d}x = M_0 < \infty. \tag{2.7}$$

We set

$$\mathfrak{D}(t,x) = \Big(\prod_{i=1}^{p} a_i^{\mu_i} - \prod_{i=1}^{p} a_i^{\nu_i}\Big) \ln\left(\frac{\prod_{i=1}^{p} a_i^{\mu_i}}{\prod_{i=1}^{p} a_i^{\nu_i}}\right)(t,x) \geq 0.$$

Then, for any $0 < T < \infty$, there exists $0 < C(T) < \infty$ such that

$$\sup_{0 \leq t \leq T} \Big\{ \sum_{i=1}^{p} \int_{\Omega} a_i \big(1 + |x| + |\ln(a_i)|\big)(t,x) \,\mathrm{d}x + \sum_{i=1}^{p} \int_0^t \int_{\Omega} \big|\nabla_x \sqrt{a_i}\big|^2 (s,x) \,\mathrm{d}x \,\mathrm{d}s + \sum_{i=1}^{p} \int_0^t \int_{\Omega} \mathfrak{D}(s,x) \,\mathrm{d}x \,\mathrm{d}s \Big\} \leq C(T).$$

If Ω is a bounded domain, this estimate holds for $T = +\infty$.

Proof. As a consequence of (1.3) and (1.4), we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{p} \int_{\Omega} a_i \big(1 + \ln(a_i)\big) \,\mathrm{d}x + \sum_{i=1}^{p} \int_{\Omega} D_i \nabla_x a_i \cdot \frac{\nabla_x a_i}{a_i} \,\mathrm{d}x + \int_{\Omega} \mathfrak{D} \,\mathrm{d}x = 0.$$

Then, (1.2) allows to bound from below

$$\sum_{i=1}^{p} \int_{\Omega} D_i \nabla_x a_i \cdot \frac{\nabla_x a_i}{a_i} \,\mathrm{d}x \geq \alpha \sum_{i=1}^{p} \int_{\Omega} \frac{|\nabla_x a_i|^2}{a_i} \,\mathrm{d}x = 4\alpha \sum_{i=1}^{p} \int_{\Omega} \big|\nabla_x \sqrt{a_i}\big|^2 \,\mathrm{d}x.$$

In a bounded domain this is enough to conclude since we have

$$\begin{aligned} \sum_{i=1}^{p} \int_{\Omega} a_i |\ln(a_i)| \,\mathrm{d}x &= \sum_{i=1}^{p} \int_{\Omega} a_i \ln(a_i) \,\mathrm{d}x - 2 \sum_{i=1}^{p} \int_{\Omega} a_i \ln(a_i) \mathbb{1}_{0 \leq a_i \leq 1} \,\mathrm{d}x \\ &\leq \sum_{i=1}^{p} \int_{\Omega} a_i \ln(a_i) \,\mathrm{d}x + p \frac{2}{e} |\Omega|. \end{aligned}$$

In the whole space, we compute, still by using (1.3) and denoting by M the sup norm of the diffusion coefficients,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{p} \int_{\Omega} a_i |x| \,\mathrm{d}x &= -\sum_{i=1}^{p} \int_{\Omega} D_i \nabla_x a_i \cdot \frac{x}{|x|} \,\mathrm{d}x \\ &\leq M \sum_{i=1}^{p} \int_{\Omega} |\nabla_x a_i| \,\mathrm{d}x = M \sum_{i=1}^{p} \int_{\Omega} \frac{|\nabla_x a_i|}{\sqrt{a_i}} \sqrt{a_i} \,\mathrm{d}x \\ &\leq \frac{\alpha}{2} \sum_{i=1}^{p} \int_{\Omega} \frac{|\nabla_x a_i|^2}{a_i} \,\mathrm{d}x + \frac{M^2}{2\alpha} \sum_{i=1}^{p} \int_{\Omega} a_i \,\mathrm{d}x. \end{aligned}$$

Hence, we arrive at

$$\sum_{i=1}^p \int_\Omega a_i\big(1+|x|+\ln(a_i)\big)\,\mathrm{d}x + \frac{\alpha}{2}\sum_{i=1}^p \int_0^t\int_\Omega \frac{|\nabla_x a_i|^2}{a_i}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_\Omega \mathfrak{D}\,\mathrm{d}x\,\mathrm{d}s$$
$$\le M_0 + \frac{M^2}{2\alpha}\sum_{i=1}^4 \int_0^t\int_\Omega a_i\,\mathrm{d}x\,\mathrm{d}s$$
$$\le \big(1+tM^2/(2\alpha)\big)M_0.$$

It remains to control the negative part of the $a_i\ln(a_i)$'s; to this end, we use the classical trick

$$\begin{aligned}\int_\Omega a_i|\ln(a_i)|\,\mathrm{d}x &= \int_\Omega a_i\ln(a_i)\,\mathrm{d}x - 2\int_\Omega a_i\ln(a_i)\big(\mathbb{1}_{0\le a_i\le e^{-|x|/2}} + \mathbb{1}_{e^{-|x|/2}\le a_i\le 1}\big)\,\mathrm{d}x\\ &\le \int_\Omega a_i\ln(a_i)\,\mathrm{d}x + \frac{4}{e}\int_\Omega e^{-|x|/4}\,\mathrm{d}x + \int_\Omega |x|a_i\,\mathrm{d}x\end{aligned}$$

since $-s\ln(s)\le \frac{2}{e}\sqrt{s}$ for any $0\le s\le 1$. We conclude readily by combining together all the pieces. ∎

3 L^∞ bounds

In the spirit of the Stampacchia cut-off method, L^∞ bounds of solutions of certain PDEs can be deduced from the behavior of suitable non linear functionals. Here, such a functional is constructed in a way that uses the dissipation property (1.4). Let us consider the non negative, C^1 and convex function

$$\Phi(z) = \begin{cases} (1+z)\ln(1+z) - z & \text{if } z\ge 0,\\ 0 & \text{if } z\le 0.\end{cases}$$

Then, for $k\ge 0$, we are interested in the evolution of

$$\sum_{i=1}^p \int_\Omega \Phi(a_i-k)\,\mathrm{d}x.$$

Lemma 3.1 *There exists a universal constant C, such that for every $a=(a_1,\ldots,a_p)$ solution of (1.1), for any $k\ge 0$, and for any $0\le s\le t<\infty$, we have*

$$\sum_{i=1}^p \int_\Omega \Phi(a_i-k)(t,x)\,\mathrm{d}x + 4\alpha\sum_{i=1}^p \int_s^t\int_\Omega \big|\nabla_x\sqrt{1+[a_i-k]_+}\big|^2(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\le \sum_{i=1}^p \int_\Omega \Phi(a_i-k)(s,x)\,\mathrm{d}x$$
$$+C\sum_{i=1}^p \int_s^t\int_\Omega \big(1+k^{\overline{\mu}} + (1+k)[a_i-k]_+^{\overline{\mu}-1}\big)\ \ln(1+[a_i-k]_+)(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau$$

where $[z]_+=\max(0,z)$ denotes the non negative part of z.

Remark 3.1 *Notice that the universal constant does not depend on the actual solution a nor on k. It is also worth noticing that, in order to make sense of this inequality we need only* $a_i^{\overline{\mu}-1}\ln(1+a_i)$ *to be integrable, although it is required to have* $a_i^{\overline{\mu}}$ *to be integrable to make sense of the equation (1.1). This point will be very important in the next section. We got this feature thanks to the similarity of the function* Φ *and the natural entropy of the system (1.1).*

Proof. Multiplying (1.1) by $\Phi'(a_i-k)$ and summing yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{p}\int_\Omega \Phi(a_i-k)\,\mathrm{d}x+\sum_{i=1}^{p}\int_\Omega D_i\nabla_x a_i\cdot\nabla_x a_i\ \Phi''(a_i-k)\,\mathrm{d}x=\sum_{i=1}^{p}\int_\Omega Q_i(a)\Phi'(a_i-k)\,\mathrm{d}x. \quad (3.8)$$

On the one hand, we observe that (1.2) leads to

$$\begin{aligned}\sum_{i=1}^{p}\int_\Omega D_i\nabla_x a_i\cdot\nabla_x a_i\ \Phi''(a_i-k)\,\mathrm{d}x&=\sum_{i=1}^{p}\int_\Omega D_i\nabla_x a_i\cdot\nabla_x a_i\ \frac{\mathbb{1}_{a_i\geq k}}{1+[a_i-k]_+}\,\mathrm{d}x\\ &=\sum_{i=1}^{p}\int_\Omega D_i\nabla_x(1+[a_i-k]_+)\cdot\nabla_x(1+[a_i-k]_+)\ \frac{\mathrm{d}x}{1+[a_i-k]_+}\\ &\geq\alpha\sum_{i=1}^{p}\int_\Omega\frac{\left|\nabla_x(1+[a_i-k]_+)\right|^2}{1+[a_i-k]_+}\,\mathrm{d}x\\ &\geq 4\alpha\sum_{i=1}^{p}\int_\Omega\left|\nabla_x\sqrt{1+[a_i-k]_+}\right|^2 dx.\end{aligned}$$

On the other hand, we rewrite the right hand side of (3.8) as

$$\begin{aligned}&\sum_{i=1}^{p}\int_\Omega Q_i(a)\ \ln(1+[a_i-k]_+)\,\mathrm{d}x\\ &=\sum_{i=1}^{p}\int_\Omega\big(Q_i(a)-Q_i(1+[a-k]_+)\big)\ \ln(1+[a_i-k]_+)\,\mathrm{d}x\\ &\qquad+\sum_{i=1}^{p}\int_\Omega Q_i(1+[a-k]_+)\ln(1+[a_i-k]_+)\,\mathrm{d}x,\end{aligned}$$

where (1.4) implies that the last term is ≤ 0. We are thus left with the task of estimating $\big(Q_i(a)-Q_i(1+[a-k]_+)\big)\ \ln(1+[a_i-k]_+)$.

To this end, let us consider the polynomial function $P:\mathbb{R}^p\to\mathbb{R}$ defined by $P(u)=\prod_{i=1}^{p}u_i^{\nu_i}$. Clearly, given $u,v\in\mathbb{R}^p$, we have

$$|P(u)-P(v)|=\left|\int_0^1\nabla P(u+s(v-u))\cdot(u-v)\,\mathrm{d}s\right|\leq C\|u-v\|\ \int_0^1\|\nabla P(u+s(v-u))\|\,\mathrm{d}s$$

where $\|\cdot\|$ stands for any norm on $\mathbb{R}^p$. As a matter of fact, since the μ_i's and ν_i's are non-zero integers, we have $\partial_j P(u)=\nu_j\prod_{i=1}^{p}u_i^{\nu'_{i,j}}$ where $\nu'_{i,j}=\nu_i$ if $i\neq j$ and $\nu'_{j,j}=\nu_j-1$. In particular, note that $\sum_{i=1}^p \nu'_{i,j} = \overline{\mu} - 1$. Therefore, working with the ℓ^1 norm, we get

$$\|\nabla P(u)\| \leq \sum_{j=1}^p \Big(\nu_j \prod_{i=1}^p |u_i|^{\nu'_{i,j}}\Big)$$

which yields, by using the convexity of the functions $z \mapsto z^{\nu'_{i,j}}$,

$$|P(u) - P(v)| \leq \sum_{\ell=1}^p |u_\ell - v_\ell| \times \sum_{j=1}^p \nu_j \Big(\prod_{i=1}^p |u_i|^{\nu'_{i,j}} + \prod_{i=1}^p |v_i|^{\nu'_{i,j}}\Big).$$

Clearly, we have $\prod_{i=1}^p |u_i|^{\nu'_{i,j}} \leq C\ \sum_{i=1}^p (1 + |u_i|^{\overline{\mu}-1})$ and finally we obtain

$$|P(u) - P(v)| \leq C\ \sum_{\ell=1}^p |u_\ell - v_\ell| \times \sum_{i,j=1}^p \nu_j (1 + |u_i|^{\overline{\mu}-1} + |v_i|^{\overline{\mu}-1}).$$

We apply this inequality with $u_i = a_i$ and $v_i = 1 + [a_i - k]_+$ and we make use of the following simple remarks

$$\begin{cases} 0 \leq (1 + [a_i - k]_+)^{\overline{\mu}-1} \leq C\ (1 + [a_i - k]_+^{\overline{\mu}-1}), \\ 0 \leq a_i \leq [a_i - k]_+ + k \text{ so that } 0 \leq a_i^{\overline{\mu}-1} \leq C([a_i - k]_+^{\overline{\mu}-1} + k^{\overline{\mu}-1}), \\ \big|a_i - (1 + [a_i - k]_+)\big| \leq 1 + \big|a_i - [a_i - k]_+\big| \leq 1 + k. \end{cases}$$

Applying the same reasoning with μ_i replacing ν_i, we arrive at

$$\big|Q_i(a) - Q_i(1 + [a - k]_+)\big| \leq C\ (1 + k)\ \sum_{j=1}^p \big(1 + k^{\overline{\mu}-1} + [a_j - k]_+^{\overline{\mu}-1}\big),$$

where the constant C depends on $\overline{\mu}$ and $p < \infty$. Then, we end the proof by using the simple inequality: for any $u, v \geq 0$, $u^{\overline{\mu}-1} \ln(1 + v) + v^{\overline{\mu}-1} \ln(1 + u) \leq 2\big(u^{\overline{\mu}-1} \ln(1 + u) + u^{\overline{\mu}-1} \ln(1 + u)\big)$. (As usual we have adopted the convention to keep the same notation C for a constant that does not depend on the solution, even when the value of the constant might change from one line to the other.) ∎

Remark 3.2 *We point out that the arguments above do not apply so easily to situations where the unknown a is an infinite sequence, like e.g. for coagulation-fragmentation models (the constant C involves a sum over the reactants).*

Let $0 < T < T^\star < \infty$ and $0 < K < \infty$ be fixed. Set

$$0 < t_n = T(1 - 1/2^n) < T < T^\star, \qquad 0 < k_n = K(1 - 1/2^n) < K.$$

Let us denote

$$\mathcal{U}_n = \sup_{t_n \leq t \leq T^\star} \sum_{i=1}^p \int_\Omega \Phi(a_i - k_n)(t, x)\,\mathrm{d}x + 4\alpha \sum_{i=1}^p \int_{t_n}^{T^\star} \int_\Omega \big|\nabla_x \sqrt{1 + [a_i - k_n]_+}\big|^2 (\tau, x)\,\mathrm{d}x\,\mathrm{d}\tau.$$

The aim is to show that, for a suitable choice of $K > 0$, $\mathcal{U}_n$ tends to 0 as $n \to \infty$ which will yield the L^∞ bound.

We start by making use of Lemma 3.1 with $0 \le t_{n-1} \le s \le t_n \le t \le T^\star$ and we average with respect to $s \in (t_{n-1}, t_n)$. After some manipulations, we obtain

$$\begin{aligned} \mathcal{U}_n \le \frac{2^n}{T} \sum_{i=1}^{p} \int_{t_{n-1}}^{t_n} \int_\Omega \Phi(a_i - k_n)(s,x)\,\mathrm{d}x\,\mathrm{d}s \\ +C \sum_{i=1}^{p} \int_{t_{n-1}}^{T^\star} \int_\Omega \big(1 + k_n^{\overline{\mu}} + (1+k_n)[a_i - k_n]_+^{\overline{\mu}-1}\big)\ \ln(1 + [a_i - k_n]_+)(\tau, x)\,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned} \tag{3.9}$$

The crucial step consists now in establishing the following non-linear estimate, where restriction on both the space dimension N and $\overline{\mu}$ appear.

Proposition 3.1 *Suppose $N \le 2$. There exists a constant $C > 0$ (which does not depend on the solution, nor on T, $T^\star$, K) such that*

$$\mathcal{U}_n \le C\ \mathcal{K}(n, K, T)\ \mathcal{U}_{n-1}^{(N+2)/N}$$

where

$$\mathcal{K}(n, K, T) = \frac{Q_K}{T}\ 2^{n(2N+4)/N} + (1 + K^{\overline{\mu}}) S_K\ 2^{n(N+4)/N} + (1 + K) R_K\ 2^{n((2N+4)/N - \overline{\mu})}$$

and $S_K = 2\max(1/K^{(N+4)/N}, 1/K^{(N+2)/N})$, $R_K = 2\max(1/K^{(2N+4)/N-\overline{\mu}}, 1/K^{2(N+1)/N-\overline{\mu}})$, $Q_K = S_K + 2\max(1/K^{4/N}, 1/K^{2/N})$.

Let us explain how the restrictions on N and $\overline{\mu}$ work. First of all, it will be crucial to remark that $\mathcal{K}(n, K, T)$ is bounded with respect to $K > 1$ provided $\overline{\mu} \le 2(N+1)/N - 1 = (N+2)/N$ which means $\overline{\mu} = 2$ in dimension $N = 2$ and $\overline{\mu} = 2$ or 3 in dimension $N = 1$. Second of all, we go back to Lemma 3.1 and we shall exploit the dissipation term that comes from the diffusion. Indeed, we expect an estimate of $\Phi(a_i - k)$ in $L^\infty(0, T^\star; L^1(\Omega))$ together with an estimate of $(1 + [a_i - k]_+)^{-1/2}\nabla_x(1 + [a_i - k]_+)$ in $L^2((0, T^\star) \times \Omega)$. Combining these information would lead to $\nabla_x Z([a_i - k]_+) \in L^2(0, T^\star; L^1(\Omega))$ where

$$Z(u) = \int_0^u \sqrt{\frac{\Phi(z)}{1+z}}\ \mathrm{d}z = \int_0^u \sqrt{\ln(1+z) + \frac{1}{1+z} - 1}\ \mathrm{d}z.$$

Let us consider a nonnegative function u defined on $[T, T^\star] \times \Omega$ such that $Z(u)$ belongs to $L^\infty(T, T^\star; L^1(\Omega))$ and $\nabla_x Z(u)$ belongs to $L^2(T, T^\star; L^1(\Omega))$ According to the Gagliardo-Nirenberg-Sobolev inequality (see [4], Th. IX.9, p. 162) the latter implies that

$$Z(u) \in L^2(T, T^\star; L^{N/(N-1)}(\Omega)).$$

we seek a homogeneous Lebesgue space with respect to the variables t, x. For $N \le 2$ we can obtain:

$$Z(u) \in L^{(N+2)/N}((T, T^\star) \times \Omega).$$

Indeed, if $N = 2$ we have $(N+2)/N = N/(N-1)$, and if $N = 1$:

$$\int_T^{T^\star}\int_\Omega |v|^3\,\mathrm{d}x\,\mathrm{d}t \le \int_T^{T^\star} \|v(t)\|_{L^1(\Omega)}\|v(t)\|^2_{L^\infty(\Omega)}\,dt \le \|v\|_{L^\infty(T,T^\star;L^1(\Omega))}\|v\|^2_{L^2(T,T^\star;L^\infty(\Omega))}.$$

Eventually, we aim at comparing $Z(u)^{(N+2)/N}$ to $\psi(u)\ln(1+u)$ where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ has a polynomial behavior. Specifying the behavior of ψ will induce restrictions on $\overline{\mu}$ that depend on the space dimension. Of course, it suffices to discuss the comparizon as $u \to 0$ and $u \to \infty$. Since $\ln(1+z) + 1/(1+z) - 1 \sim_{z\to 0} z^2/2$ we first obtain that $Z(u)^{(N+2)/N} \ge u^{(2N+4)/N}/8$ for $u \in [0,\delta]$, $\delta > 0$ small enough. It follows that $\psi(u)\ln(1+u)$ can indeed be dominated by $Z(u)$ for bounded u's provided $\psi(u) \sim_{u\to 0} u^{(N+4)/N}$. Next, there exists $A > 0$ such that for $z \ge A$ large enough, we have $\ln(1+z) + 1/(1+z) - 1 \ge \frac{1}{2}\ln(1+z)$. Thus, for $u \ge 2A$ we get

$$Z(u) \ge \frac{1}{\sqrt{2}}\int_{u/2}^u \sqrt{\ln(1+z)}\,\mathrm{d}z \ge \frac{1}{2\sqrt{2}}u\sqrt{\ln(1+u/2)} \ge C_1 u\sqrt{\ln(1+u)}.$$

Hence $Z(u)^{(N+2)/N}$ dominates $\psi(u)\ln(1+u)$ provided $N \le 2$ and $\psi(u) \sim_{u\to\infty} u^{(N+2)/N}$. Reasoning the same way, we also prove that there exists $C > 0$ such that $Z(u) \le C\ \Phi(u)$ holds for any $u \ge 0$. Let us summarize the properties that we need to justify Proposition 3.1.

Lemma 3.2 *Let us set*

$$\psi(u) = u^{(N+4)/N}\mathbb{1}_{0\le u\le 1} + u^{(N+2)/N}\mathbb{1}_{u\ge 1}.$$

There exists a constant $C > 0$ such that

$$\psi(u)\ln(1+u) \le C\ Z(u)^{(N+2)/N}, \qquad \textit{and} \qquad Z(u) \le C\ \Phi(u).$$

holds for any $u \ge 0$. Furthermore, for every nonnegative function u defined on $[T,T^\star]\times\Omega$ we have:

$$\int_T^{T^\star}\left|\int_\Omega |\nabla Z(u)|\,\mathrm{d}x\right|^2\mathrm{d}\tau \le \sup_{T\le\tau\le T^\star}\left(\int_\Omega \Phi(u)(\tau,x)\,\mathrm{d}x\right)\int_T^{T^\star}\int_\Omega \left|\nabla_x\sqrt{1+u}\right|^2(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau.$$

Proof of Proposition 3.1. The proof splits into two steps: in the former we modify (3.9) so that in the latter we can make the dissipation terms appear by appealing to the Gagliardo-Nirenberg inequality.
Step 1. The first step consists in showing the following inequality:

$$\mathcal{U}_n \le C\ \mathcal{K}(n,K,T)\ \sum_{i=1}^p \int_{t_{n-1}}^{T^\star}\int_\Omega \psi(a_i - k_{n-1})\ln(1+[a_i-k_{n-1}]_+)\,\mathrm{d}x\,\mathrm{d}\tau, \tag{3.10}$$

where the auxiliary function ψ has been introduced in Lemma 3.2. We start by noting that if $a_i \ge k_n \ge k_{n-1}$, then $(a_i - k_{n-1})/(k_n - k_{n-1}) \ge 1$. Therefore we can write for any $\alpha, \beta \ge 0$,

$$\begin{aligned}\mathbb{1}_{a_i\ge k_n} &\le \left(\frac{[a_i-k_{n-1}]_+}{k_n-k_{n-1}}\right)^\alpha \mathbb{1}_{\{k_n\le a_i\le 1+k_{n-1}\}} + \left(\frac{[a_i-k_{n-1}]_+}{k_n-k_{n-1}}\right)^\beta \mathbb{1}_{a_i\ge 1+k_{n-1}}\\ &\le \frac{2^{n\alpha}}{K^\alpha}[a_i-k_{n-1}]_+^\alpha \mathbb{1}_{0\le a_i-k_{n-1}\le 1} + \frac{2^{n\beta}}{K^\beta}[a_i-k_{n-1}]_+^\beta \mathbb{1}_{a_i-k_{n-1}\ge 1}.\end{aligned}$$

By using these simple estimates with $\alpha = (N+4)/N$, $\beta = (N+2)/N$ and $\alpha = (N+4)/N - \overline{\mu} + 1$, $\beta = (N+2)/N - \overline{\mu} + 1$ respectively (note that in both case $\alpha \geq \beta$), we are led to

$$(1 + k_n^{\overline{\mu}}) \ln(1 + [a_i - k_n]_+) \leq (1 + K^{\overline{\mu}})\ 2^{n(N+4)/N}\ S_K\ \psi(a_i - k_{n-1}) \ln(1 + [a_i - k_{n-1}]_+),$$

and

$$(1 + k_n)[a_i - k_n]_+^{\overline{\mu}-1} \ln(1 + [a_i - k_n]_+) \leq (1 + K)\ 2^{n((2N+4)/N - \overline{\mu})}\ R_K\ \psi(a_i - k_{n-1}) \ln(1 + [a_i - k_{n-1}]_+).$$

Coming back to (3.9) yields

$$\begin{aligned} \mathcal{U}_n \ \leq\ & \frac{2^n}{T} \sum_{i=1}^{p} \int_{t_{n-1}}^{t_n} \int_\Omega \Phi(a_i - k_{n-1})\, \mathrm{d}x\, \mathrm{d}s \\ & + C\Big((1 + K^{\overline{\mu}}) S_K 2^{n(N+4)/N} + (1+K) R_K 2^{n((2N+4)/N - \overline{\mu})}\Big) \\ & \qquad \times \sum_{i=1}^{p} \int_{t_{n-1}}^{T^\star} \int_\Omega \psi(a_i - k_{n-1}) \ln(1 + [a_i - k_{n-1}]_+)\, \mathrm{d}x\, \mathrm{d}\tau. \end{aligned}$$

The first integral in the right hand side can be dominated in a similar way (using $\alpha = 4/N$, $\beta = 2/N$); precisely, we have

$$\begin{aligned} & \frac{2^n}{T} \sum_{i=1}^{p} \int_{t_{n-1}}^{t_n} \int_\Omega \Phi(a_i - k_n)\, \mathrm{d}x\, \mathrm{d}s \\ & \quad \leq \frac{2^n}{T} \sum_{i=1}^{p} \int_{t_{n-1}}^{t_n} \int_\Omega (1 + [a_i - k_n]_+)\ \ln(1 + [a_i - k_n]_+)\, \mathrm{d}x\, \mathrm{d}s \\ & \quad \leq \frac{1}{T} 2^{n(2N+4)/N}\ Q_K\ \sum_{i=1}^{p} \int_{t_{n-1}}^{T^\star} \int_\Omega \psi(a_i - k_{n-1}) \ln(1 + [a_i - k_{n-1}]_+)\, \mathrm{d}x\, \mathrm{d}s. \end{aligned}$$

Therefore, we have proved from (3.9) that (3.10) holds.

<u>Step 2.</u> Now, we go back to Lemma 3.2 so that (3.10) becomes

$$\mathcal{U}_n \leq C\ \mathcal{K}(n, K, T)\ \sum_{i=1}^{p} \int_{t_{n-1}}^{T^\star} \int_\Omega \big| Z([a_i - k_{n-1}]_+) \big|^{(N+2)/N}\, \mathrm{d}x\, \mathrm{d}\tau. \tag{3.11}$$

Let us distinguish depending on the dimension $N = 1$ or $N = 2$ how we conclude by using the Gagliardo-Nirenberg-Sobolev inequality.

For $N = 2$, using the Gagliardo-Nirenberg-Sobolev inequality and Lemma 3.2, we obtain

$$\begin{aligned} \mathcal{U}_n \ \leq\ & C\ \mathcal{K}(n, K, T)\ \sum_{i=1}^{p} \Bigg[\int_{t_{n-1}}^{T^\star} \left(\int_\Omega \big| \nabla_x Z([a_i - k_{n-1}]_+) \big|\, \mathrm{d}x \right)^2 \mathrm{d}\tau \\ & \qquad + \int_{t_{n-1}}^{T^\star} \left(\int_\Omega \Phi([a_i - k_{n-1}]_+)\, dx \right)^2 ds \Bigg] \end{aligned}$$

Then, we use the second statement in Lemma 3.2 to obtain

$$
\begin{aligned}
\mathcal{U}_n \;&\leq\; C\ \mathcal{K}(n,K,T) \\
&\quad \times \sum_{i=1}^{p} \Bigg[\Bigg(\sup_{t_{n-1}\leq\tau\leq T^\star} \int_\Omega \Phi(a_i - k_{n-1})(\tau)\,\mathrm{d}x \int_{t_{n-1}}^{T^\star} \int_\Omega \left|\nabla_x \sqrt{1+[a_i-k_{n-1}]_+}\right|^2 \mathrm{d}x\,\mathrm{d}\tau \Bigg) \\
&\qquad\qquad + \int_{t_{n-1}}^{T^\star} \left(\int_\Omega \Phi([a_i-k_{n-1}]_+)\,dx \right)^2 ds \Bigg] \\
&\leq\; C(1+T^\star)\ \mathcal{K}(n,K,T)\ \mathcal{U}_{n-1}^2.
\end{aligned}
$$

For $N=1$, we proceed as follows

$$
\begin{aligned}
\mathcal{U}_n \;&\leq\; C\ \mathcal{K}(n,K,T) \sum_{i=1}^{p} \int_{t_{n-1}}^{T^\star} \left(\|Z(a_i-k_{n-1})(t,\cdot)\|^2_{L^\infty(\Omega)} \ \int_\Omega Z(a_i-k_{n-1})(t,x)\,\mathrm{d}x \right) \mathrm{d}t \\
&\leq\; C\ \mathcal{K}(n,K,T)\ \sum_{i=1}^{p} \Bigg[\sup_{t_{n-1}\leq t\leq T^\star} \int_\Omega Z(a_i-k_{n-1})\,\mathrm{d}x \\
&\qquad\qquad \times \int_{t_{n-1}}^{T^\star} \left(\int_\Omega \big(|Z(a_i-k_{n-1}| + |\nabla_x Z(a_i-k_{n-1}|\big)\,\mathrm{d}x \right)^2 \mathrm{d}t \Bigg] \\
&\leq\; C\ \mathcal{K}(n,K,T)\ \sum_{i=1}^{p} \Bigg[2T^\star \Big(\sup_{t_{n-1}\leq t\leq T^\star} \int_\Omega \Phi(a_i-k_{n-1})\,\mathrm{d}x \Big)^3 \\
&\qquad\qquad + \Big(\sup_{t_{n-1}\leq t\leq T^\star} \int_\Omega \Phi(a_i-k_{n-1})\,\mathrm{d}x \Big)^2 \int_{t_{n-1}}^{T^\star} \int_\Omega \left|\nabla_x \sqrt{1+[a_i-k_{n-1}]_+}\right|^2 \mathrm{d}x\,\mathrm{d}t \Bigg] \\
&\leq\; C\ \mathcal{K}(n,K,T)\ (1+T^\star)\ \mathcal{U}_{n-1}^3.
\end{aligned}
$$

This ends the proof of Proposition 3.1. ∎

Finishing the proof of the L^∞ bound needs the following elementary claim.

Lemma 3.3 *Let $\big(\mathcal{V}_n\big)_{n\in\mathbb{N}}$ be a sequence verifying*

$$\mathcal{V}_n \leq M^n \mathcal{V}_{n-1}^q$$

for some $M>0$, $q>1$. Then for any $n_0\in\mathbb{N}$, there exists ε, such that if $\mathcal{V}_{n_0}<\varepsilon$, then $\lim_{n\to\infty}\mathcal{V}_n = 0$.

Proof. Without loss of generality we suppose $n_0=0$. Let us set $\mathcal{W}_n = \ln(\mathcal{V}_n)$. We have

$$\mathcal{W}_n \leq n\ln(M) + q\mathcal{W}_{n-1}$$

which yields

$$\mathcal{W}_n \leq \ln(M)\sum_{j=0}^{n} q^{n-j} j + q^n \mathcal{W}_0 \leq q^n \ln(M^{1/(q(1-1/q)^2)}\ \mathcal{V}_0).$$

So, if $\mathcal{V}_0 < M^{-1/(q(1-1/q)^2)}$, $\mathcal{W}_n$ converges to $-\infty$, and $\mathcal{V}_n$ converges to 0. ∎

Hence, it remains to check that the first term of the iteration can be made small choosing K large enough. Indeed, let us go back to Proposition 3.1. Picking $K > 1$, we can summarize the obtained estimate as

$$\mathcal{U}_n \le C(1 + 1/T)\ 2^{n(2N+4)/N}\ \mathcal{U}_{n-1}^{(N+2)/N}.$$

The keypoint is to remark that Q_K, KR_K and $K^{\overline{\mu}}S_K$ remain bounded for large K's so that the constant C above does not depend on K. Hence, we apply Lemma 3.3 to $\mathcal{V}_n = \big(C(1+1/T)\big)^{N/2}\,\mathcal{U}_n$.

Now, let us specialize (3.11) to the case $n = 2$; we get (with C which still does not depend on K)

$$\mathcal{U}_2 \le C\left[\sum_{i=1}^{p}\int_0^{T^\star}\left(\int_\Omega |\nabla_x Z(a_i - K/2)_+|\,\mathrm{d}x\right)^2 \mathrm{d}t + \sum_{i=1}^{p}\int_0^{T^\star}\left(\int_\Omega |\Phi(a_i - K/2)_+\,\mathrm{d}x|\right)^2 \mathrm{d}t\right] \tag{3.12}$$

in dimension $N = 2$ and in dimension $N = 1$ the same expression is multiplied by the quantity $\sup_{0\le t\le T^\star}\sum_{i=1}^{p}\int_\Omega \Phi(a_i - K/2)\,\mathrm{d}x$. This allows to establish the following statement.

Lemma 3.4 *Let $\epsilon > 0$. Then, there exists $K_\epsilon \ge 1$ such that for any $K \ge K_\epsilon$ we have $\mathcal{U}_2 \le \epsilon$.*

Proof. The proof reduces to prove that the two integrals in the right hand side of (3.12) tend to 0 as $K \to +\infty$. As a matter of fact, there exists $C > 0$ such that for any $z \ge 0$ we have $(1 + z)\ln(1 + z) \le C\ z(1 + |\ln(z)|)$. Furthermore, there exists $C > 0$ such that for any $k > 1$ and $z \ge 0$, we have

$$[z - k]_+(1 + |\ln([z - k]_+)|) \le C\ z(1 + |\ln z|).$$

Accordingly, we deduce that $\Phi(a_i - K/2)$ converges to 0 for a.e $(t, x) \in (0, T) \times \Omega$ as K goes to infinity and is dominated by $a_i(1 + |\ln(a_i)|)$, which verifies

$$\sum_{i=1}^{p}\int_0^{T^\star}\left(\int_\Omega a_i(1 + |\ln(a_i)|)\,\mathrm{d}x|\right)^2 \mathrm{d}t < \infty$$

owing to Proposition 2.1. Applying the Lebesgue theorem then shows that

$$\lim_{K\to\infty}\left\{\sum_{i=1}^{p}\int_0^{T^\star}\left(\int_\Omega \Phi([a_i - K/2]_+)\,\mathrm{d}x\right)^2 \mathrm{d}t\right\} = 0.$$

Next, we simply write

$$\nabla_x Z(a_i - K/2) = \mathbb{1}_{a_i\ge K/2}\sqrt{\ln(1 + [a_i - K/2]_+) + \frac{1}{1 + [a_i - K/2]_+} - 1}\ \ \nabla_x a_i.$$

Then, we remark that $z \mapsto \ln(1 + z) + 1/(1 + z) - 1$ is non decreasing which allows to evaluate

$$|\nabla_x Z(a_i - K/2)| \le \mathbb{1}_{a_i\ge K/2}\sqrt{\ln(1 + a_i) + \frac{1}{1 + a_i} - 1}\ \ |\nabla_x a_i| = \mathbb{1}_{a_i\ge K/2}\,|\nabla_x Z(a_i)| \le |\nabla_x Z(a_i)|.$$

We readily conclude since the $\mathbb{1}_{a_i \geq K/2}\,|\nabla_x Z(u)|$ decreases to 0 as $K \to \infty$ for a.e $(t,x) \in (0,T) \times \Omega$ while Lemma 3.2 yields

$$\sum_{i=1}^{p} \int_0^{T^\star} \left(\int_\Omega |\nabla_x Z(a_i)|\,\mathrm{d}x \right)^2 \mathrm{d}t \leq \sum_{i=1}^{p} \sup_{0 \leq t \leq T^\star} \int_\Omega \Phi(a_i)\,\mathrm{d}x \int_0^{T^\star} \int_\Omega |\nabla_x \sqrt{a_i}|^2\,\mathrm{d}x\,\mathrm{d}t < \infty$$

by using the basic estimates in Proposition 2.1 again. We conclude by classical integration theory arguments. ∎

We can now finish the proof of Theorem 1.1. Let us emphasize the dependence with respect to K by denoting $\mathcal{U}_n^{(K)}$. We first fix K which makes $\mathcal{U}_2^{(K)}$ small enough (remark that K is more constrained as T is chosen small) so that we obtain by applying Lemma 3.3

$$\lim_{n\to\infty} \mathcal{U}_n^{(K)} = 0.$$

However, we clearly have

$$\mathcal{U}_n^{(K)} \geq \frac{1}{T^\star - t_n} \int_{t_n}^{T^\star} \int_\Omega \Phi(a_i - k_n)\,\mathrm{d}x\,\mathrm{d}t \geq 0.$$

Letting n go to infinity and applying the Fatou lemma, we deduce that

$$\frac{1}{T^\star - T} \int_T^{T^\star} \int_\Omega \Phi(a_i - K)\,\mathrm{d}x\,\mathrm{d}t = 0,$$

which implies that $0 \leq a_i(t,x) \leq K$ for a.e $(t,x) \in (T, T^\star) \times \Omega$.

4 Haussdorff Dimension of the Set of Singular Points

In this section we study the Hausdorff dimension of the blow-up points of the solutions of (1.1). The derivation of the necessary estimates remains close to the strategy described in the previous section; again restrictions on the space dimension and the degree of nonlinearity appear. It turns out that relevant results can be obtained by this method in dimension $N = 3$ and $N = 4$ with $\overline{\mu} = 2$, while we are not able to reach improvements in direction of higher nonlinearities for lower dimensions. For the sake of simplicity, in what follows we assume that the diffusion coefficients D_i are constant with respect to the space variable (but they still depend on i, otherwise the problem becomes trivial by remarking that $\rho(t,x) = \sum_{i=1}^p a_i(t,x)$ satisfies the heat equation $\partial_t \rho - D\Delta_x \rho = 0$, with D the common value of the diffusion coefficients). Then, we shall prove Theorem 1.3.

To begin with, let us recall a few definitions about Hausdorff dimension. For a given nonempty set $A \subset \mathbb{R}^d$, $s \geq 0$, $\delta > 0$, we set

$$\mathcal{H}^s_\delta(A) = \inf \left\{ \frac{\Gamma(1/2)^s}{2^s\ \Gamma(s/2+1)} \sum_i \big(\mathrm{diam}(A_i)\big)^s,\ A \subset \bigcup_i A_i,\ \mathrm{diam}(A_i) \leq \delta \right\},$$

and then $\mathcal{H}^s(A) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(A)$. The Hausdorff dimension of A is defined by

$$\dim_{\mathcal{H}}(A) = \inf\{s > 0,\ \mathcal{H}^s(A) = 0\} = \sup\{s > 0,\ \mathcal{H}^s(A) = +\infty\}.$$

We refer to [13] (p. 171) for more details.

The starting point of the proof is two-fold: first we identify the highest L^p norm available with the estimate deduced from mass conservation and entropy dissipation, and second we remark that the problem admits an invariant scaling. This is the purpose of the following claims.

Lemma 4.1 *Let $N > 2$. There exists $C > 0$ such that for any $u \in L^\infty(0,T;L^1(\Omega))$ verifying $\nabla\sqrt{u} \in L^2((0,T)\times\Omega)$, we have*

$$\left(\int_0^T \int_\Omega |u|^{(N+2)/N}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{N}{N+2}} \le C\Big((1+T)\|u\|_{L^\infty(0,T;L^1(\Omega))} + \|\nabla_x\sqrt{u}\|^2_{L^2((0,T)\times\Omega)}\Big).$$

Lemma 4.2 *Let a be a solution of* (1.1)*. Let $t_0 > 0$ and $x_0 \in \Omega$. Then, for any $0 < \varepsilon \ll 1$*

$$a_\varepsilon(t,x) = \varepsilon^{2/(\overline{\mu}-1)}\ a(t_0 + \varepsilon^2 t, x_0 + \varepsilon x)$$

satisfies (1.1)*.*

Keeping in mind Lemma 4.2 we consider now solutions of (1.1) that are defined for negative times. Let us set

$$k_n = 1 - 1/2^n, \qquad t_n = 1 + 1/2^n \qquad \mathcal{B}_n = B(0,t_n), \qquad \mathcal{Q}_n = (-t_n, 0)\times\mathcal{B}_n.$$

Note that $\mathcal{B}_n \subset \mathcal{B}_{n-1}$ and $\mathcal{Q}_n \subset \mathcal{Q}_{n-1}$. We introduce a cut-off function

$$\begin{cases} \zeta_n : \mathbb{R}^N \to \mathbb{R}, & 0 \le \zeta_n(x) \le 1, \\ \zeta_n(x) = 1 \text{ for } x \in \mathcal{B}_n, & \zeta_n(x) = 0 \text{ for } x \in \complement\mathcal{B}_{n-1}, \\ \displaystyle\sup_{i,j\in\{1,\dots,N\},\ x\in\mathbb{R}^N} |\partial^2_{ij}\zeta_n(x)| \le C\ 2^{2n}. & \end{cases}$$

We define

$$\mathcal{U}_n = \sup_{-t_n\le t\le 0} \sum_{i=1}^p \int_{\mathcal{B}_n} \Phi(a_i - k_n)\,\mathrm{d}x + \sum_{i=1}^p \iint_{\mathcal{Q}_n} \left|\nabla_x\sqrt{1 + [a_i - k_n]_+}\right|^2 \mathrm{d}x\,\mathrm{d}s.$$

Multiplying (1.1) by $\zeta_n(x)\Phi'(a_i - k_n)$ we obtain the following localized version of (3.8)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^p \int_\Omega \Phi(a_i - k_n)\ \zeta_n\,\mathrm{d}x + \sum_{i=1}^p \int_\Omega D_i\nabla_x a_i\cdot\nabla_x a_i\ \Phi''(a_i - k_n)\ \zeta_n\,\mathrm{d}x \\ = \sum_{i=1}^p \int_\Omega Q_i(a)\Phi'(a_i - k_n)\zeta_n\,\mathrm{d}x + \sum_{i=1}^p \int_\Omega D_i : D^2\zeta_n\ \Phi(a_i - k_n)\,\mathrm{d}x, \end{aligned} \tag{4.13}$$

where $D^2\zeta_n$ stands for the hessian matrix of ζ_n and $A : B = \sum_{k,l=1}^N A_{kl}B_{kl}$. Remark that $0 \leq \mathbb{1}_{\mathcal{B}_n}(x) \leq \zeta_n(x) \leq \zeta_{n-1}(x) \leq 1$ and $|\partial^2_{kl}\zeta_n(x)| \leq 2^{2n}\mathbb{1}_{\mathcal{B}_{n-1}}(x)$. Then, reproducing the proof of Lemma 3.1 and (3.9) we obtain

$$\begin{aligned} \mathcal{U}_n \leq C\Big(2^{2n} + \frac{2^n}{T}\Big) \sum_{i=1}^p \iint_{\mathcal{Q}_{n-1}} \Phi(a_i - k_n)(s,x)\,\mathrm{d}x\,\mathrm{d}s \\ +C\sum_{i=1}^p \iint_{\mathcal{Q}_{n-1}} \big(1 + k_n^{\overline{\mu}} + (1+k_n)[a_i - k_n]_+^{\overline{\mu}-1}\big)\ \ln(1 + [a_i - k_n]_+)(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned} \tag{4.14}$$

From this relation we shall be able to establish the following statements.

Proposition 4.1 *Let $N > 2$. The following relation holds*

$$\mathcal{U}_n \leq C\ 2^{n(\overline{\mu}+1)}\ \mathcal{U}_{n-1}^{1+2/N}$$

for any $n \geq 1$. Accordingly, if $\mathcal{U}_0$ is small enough then $\lim_{n\to\infty}\mathcal{U}_n = 0$.

Corollary 4.1 *There exists a universal constant $\eta_\star > 0$ such that any solution of (1.1) satisfying*

$$\sum_{i=1}^p \int_{-3}^0 \int_{B(0,3)} |a_i|^{(N+2)/N}\,\mathrm{d}x\,\mathrm{d}\tau \leq \eta_\star$$

is such that for any $i \in \{1,\ldots,p\}$ we have

$$0 \leq a_i(t,x) \leq 1 \qquad \textit{a. e. in } (-1,0)\times B(0,1).$$

Proof of Proposition 4.1. We start the proof by remarking that $0 \leq k_n \leq 1$, and $k_n \geq k_{n-1}$. Consequently we have $0 \leq [a_i - k_n]_+ \leq [a_i - k_{n-1}]_+$. Then, we make $a_i - k_{n-1}$ appear by using

$$\mathbb{1}_{a_i \geq k_n} \leq \left(\frac{a_i - k_{n-1}}{k_n - k_{n-1}}\right)^{\overline{\mu}-1} \mathbb{1}_{a_i \geq k_{n-1}} \leq 2^{n(\overline{\mu}-1)}\ [a_i - k_{n-1}]_+^{\overline{\mu}-1}.$$

It follows that the last integral in (4.14) can be dominated by

$$C(1 + 2^{n(\overline{\mu}-1)})\ \sum_{i=1}^p \iint_{\mathcal{Q}_{n-1}} [a_i - k_{n-1}]_+^{\overline{\mu}-1}\ \ln\big(1 + [a_i - k_{n-1}]_+\big)\,\mathrm{d}x\,\mathrm{d}\tau.$$

Similarly, we check that

$$\Phi(a_i - k_n) \leq (1+[a_i - k_n]_+)\ \ln\big(1+[a_i-k_n]_+\big) \leq (2^{n(\overline{\mu}-1)} + 2^{n(\overline{\mu}-2)})[a_i - k_{n-1}]_+^{\overline{\mu}-1}\ \ln\big(1+[a_i-k_{n-1}]_+\big).$$

Combining these inequalities, and identifying the highest power of 2^n involved we finally get from (4.14)

$$\mathcal{U}_n \leq C_T 2^{n(\overline{\mu}+1)} \sum_{i=1}^p \iint_{\mathcal{Q}_{n-1}} [a_i - k_{n-1}]_+^{\overline{\mu}-1}\ \ln\big(1 + [a_i - k_{n-1}]_+\big)\,\mathrm{d}x\,\mathrm{d}\tau. \tag{4.15}$$

The next step consists in evaluating $u^{\overline{\mu}-1}\ln(1+u)$ by functions having a polynomial behavior, so that the right hand side of (4.15) will be dominated by a power larger than 1 of $\mathcal{U}_{n-1}$, up to the use of the Gagliardo-Nirenberg-Sobolev estimate. To be more specific, we note that there exists $C>0$ such that for any $u\geq 0$, on the one hand

$$u^2 \mathbb{1}_{0\leq u\leq 1} + u\mathbb{1}_{u\geq 1} \leq C\ \Phi(u)$$

and, on the other hand

$$u^{\overline{\mu}-1}\ln(1+u) \leq Cu^{\overline{\mu}-1+\alpha_0}\mathbb{1}_{0\leq u\leq 1} + u^{\overline{\mu}-1+\alpha_\infty}\mathbb{1}_{u\geq 1}, \qquad 0\leq \alpha_0\leq 1, \qquad 0<\alpha_\infty.$$

The choice of the exponents α_0, α_∞ comes from information in Lemma 4.1 which then leads to the constraints $N\in\{3,4\}$, and $\overline{\mu}=2$. Hence, with these considerations (4.15) becomes

$$\begin{aligned}
\mathcal{U}_n &\leq C_T 2^{n(\overline{\mu}+1)} \sum_{i=1}^{p} \iint_{\mathcal{Q}_{n-1}} \Big([a_i-k_{n-1}]_+^{\overline{\mu}-1+\alpha_0}\mathbb{1}_{0\leq a_i-k_{n-1}\leq 1} \\
&\qquad\qquad + [a_i-k_{n-1}]_+^{\overline{\mu}-1+\alpha_\infty}\mathbb{1}_{a_i-k_{n-1}\geq 1}\Big)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\leq C_T 2^{n(\overline{\mu}+1)} \sum_{i=1}^{p} \left\{ \Big(\sup_{-t_n\leq t\leq 0} \int_{\mathcal{B}_{n-1}} \Phi(a_i-k_{n-1})\,\mathrm{d}x \Big)^{2/N} \iint_{\mathcal{Q}_{n-1}} \left|\nabla_x\sqrt{1+[a_i-k_{n-1}]_+}\right|^2 \mathrm{d}x\,\mathrm{d}\tau \right\} \\
&\leq C_T 2^{n(\overline{\mu}+1)} \mathcal{U}_{n-1}^{1+2/N}.
\end{aligned}$$

Coming back to Lemma 3.3 finishes the proof of Proposition 4.1. ∎

Proof of Corollary 4.1. We are thus now left with the task of discussing the smallness of

$$\mathcal{U}_0 = \sup_{-2\leq t\leq 0} \sum_{i=1}^{p} \int_{\mathcal{B}(0,2)} \Phi(a_i)\,\mathrm{d}x + \int_{-2}^{0}\int_{B(0,2)} \left|\nabla_x\sqrt{a_i}\right|^2 \mathrm{d}x\,\mathrm{d}\tau.$$

It turns out that this quantity can be evaluated by involving the norm $L^{(N+2)/N}$ on a slightly larger domain; this is precisely the purpose of Corollary 4.1. The mass conservation yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{p}\int_{\Omega} \zeta_0(x)a_i(t,x)\,\mathrm{d}x = \sum_{i=1}^{p}\int_{\Omega} D_i : D_x^2\zeta_0(x)\ a_i(t,x)\,\mathrm{d}x \leq C\sum_{i=1}^{p}\int_{B(0,3)} a_i(t,x)\,\mathrm{d}x,$$

since ζ_0 has bounded second order derivatives and is supported on $B(0,3)$. We integrate over the time interval (t,τ), and then we average over $\tau\in(-3,-2)$. Hence, we get

$$\begin{aligned}
\sup_{-2\leq t\leq 0}\sum_{i=1}^{p}\int_{B(0,2)} a_i(t,x)\,\mathrm{d}x &\leq C\sum_{i=1}^{p}\int_{-3}^{0}\int_{B(0,3)} a_i(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\leq C\sum_{i=1}^{p}\int_{-3}^{0}\int_{B(0,3)} a_i(\tau,x)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\leq \sum_{i=1}^{p}\left(\int_{-3}^{0}\int_{B(0,3)} |a_i(\tau,x)|^{(N+2)/N}\,\mathrm{d}x\,\mathrm{d}\tau\right)^{N/(N+2)}.
\end{aligned}$$

Similarly, the entropy dissipation yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{p}\int_{\Omega}\zeta_0(x)\ a_i\ln(a_i)\,\mathrm{d}x+\int_{\Omega}\zeta_0(x)\ \zeta_0(x)\frac{D_i\nabla_x a_i\cdot\nabla_x(a_i)}{a_i}\,\mathrm{d}x\\
&\quad=\sum_{i=1}^{p}\int_{\Omega}D_i:D_x^2\zeta_0(x)\ a_i\ln(a_i)\,\mathrm{d}x\\
&\quad\le C\sum_{i=1}^{p}\int_{B(0,3)}a_i|\ln(a_i)|\,\mathrm{d}x.
\end{aligned}$$

Again we integrate with respect to the time variable. We shall also use the trick

$$u|\ln(u)|=u\ln(u)-2u\ln(u)\mathbb{1}_{0\le u\le 1}\le u\ln(u)+C\sqrt{u}.$$

It follows that

$$\begin{aligned}
&\sup_{-2\le t\le 0}\sum_{i=1}^{p}\int_{B(0,2)}a_i|\ln(a_i)|\,\mathrm{d}x+\alpha\sum_{i=1}^{p}\int_{-2}^{0}\int_{B(0,2)}\left|\nabla_x\sqrt{a_i}\right|^2\mathrm{d}x\,\mathrm{d}\tau\\
&\qquad\le C\sum_{i=1}^{p}\int_{-3}^{0}\int_{B(0,3)}\left(a_i|\ln(a_i)|+\sqrt{a_i}\right)\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}$$

Furthermore, we have $u|\ln(u)|=u\ln(u)\mathbb{1}_{u\ge 1}-u\ln(u)\mathbb{1}_{0\le u\le 1}\le C(|u|^{(N+2)/N}+\sqrt{u})$. Therefore, using the Holder inequality, we end up with

$$\sup_{-2\le t\le 0}\sum_{i=1}^{p}\int_{B(0,2)}a_i|\ln(a_i)|\,\mathrm{d}x\le C\sum_{i=1}^{p}\left(\|a_i\|^{(N+2)/N}_{L^{(N+2)/N}((-3,0)\times B(0,3))}+\|a_i\|^{1/2}_{L^{(N+2)/N}((-3,0)\times B(0,3))}\right).$$

Coming back to the definition of $\mathcal{U}_0$ finishes the proof. ∎

Now, these statements allows us to deduce some property of the solution of the original Cauchy problem by using the scaling argument in Lemma 4.2. Indeed, we notice that

$$\int_{-3}^{0}\int_{B(0,3)}|a_\varepsilon(\tau,x)|^{(N+2)/N}\,\mathrm{d}x\,\mathrm{d}\tau=\varepsilon^{2(N+2)/N-(N+2)}\int_{t_0-3\varepsilon^2}^{t_0+3\varepsilon^2}\int_{|y-x_0|\le 3\varepsilon}|a(s,y)|^{(N+2)/N}\,\mathrm{d}y\,\mathrm{d}s$$

holds (recall that we are dealing with the case $\overline{\mu}=2$ only). We deduce the following statement.

Lemma 4.3 *Let $N=3$ or $N=4$ and $\overline{\mu}=2$. Then there exists a universal constant $\eta_\star>0$ such that for any a solution of (1.1), any $t_0>0$, $x_0\in\Omega$ and $0<\varepsilon\ll 1$, we have the following property. If:*

$$\sum_{i=1}^{p}\frac{1}{\varepsilon^{N+2}}\int_{t_0-3\varepsilon^2}^{t_0+3\varepsilon^2}\int_{|y-x_0|\le 3\varepsilon}|a(s,y)|^{(N+2)/N}\,\mathrm{d}y\,\mathrm{d}s\le\eta_\star\ \varepsilon^{-2(N+2)/N}$$

then a_i satisfies $0\le a_i(t,x)\le 1/\varepsilon^2$ on $|t-t_0|\le\varepsilon^2$, $|x-x_0|\le\varepsilon$ and a_i is C^∞ on this set.

Notice that it is enough to show the boundedness of the a_i. Then the full regularity on the (possibly smaller) neighborhood is obtained by induction, using classical theory of parabolic equations.

We start by localizing: namely, we consider $(0,T)\times B(0,R)$, $0<T,R<\infty$. We set

$$\mathcal{S}=\{(t,x)\in(0,T)\times B(0,R),\ u \text{ is not } C^\infty \text{ on a neighborhood of } (t,x)\}.$$

We cover $\mathcal{S}$ by rectangles with step size ε^2 in the time direction and ε in the space directions, centered at points $(t,x)\in\mathcal{S}$. By the Vitali covering lemma (see [32], p. 9) we actually consider a countable covering of disjoints such rectangles (at the price of multiplying ε by a suitable constant which does not matter). We denote $\{C_j,\ j\in\mathbb{N}\}$ the set of these rectangles, centered at $(t_j,x_j)\in\mathcal{S}$. Since (t_j,x_j) denies the conclusion of Lemma 4.3, we have

$$\sum_{i=1}^{p}\frac{1}{\varepsilon^{N+2}}\iint_{\widetilde{C_j}}|a_i(s,y)|^{(N+2)/N}\,\mathrm{d}y\,\mathrm{d}s\geq\eta_\star\ \varepsilon^{-2(N+2)/N}$$

where $\widetilde{C_j}$ stands for the rectangle centered at (t_j,x_j) with step size $3\varepsilon^2$ in the time direction and 3ε in the space directions. We introduce the function

$$F_{\mathcal{S}}(t,x)=\mathbb{1}_{\widetilde{C_j}}(t,x)\ \sum_{i=1}^{p}\frac{1}{\varepsilon^{N+2}}\iint_{\widetilde{C_j}}|a_i(s,y)|^{(N+2)/N}\,\mathrm{d}y\,\mathrm{d}s.$$

Hence, denoting by $\mathcal{L}$ the Lebesgue measure, we have the following estimate

$$\begin{aligned}\mathcal{L}\Big(\bigcup_{j\in\mathbb{N}}C_j\Big) &\leq \mathcal{L}\big((t,x)\in(0,T)\times B(0,R),\ F_{\mathcal{S}}(t,x)\geq\eta_\star/\varepsilon^{(N+2)/N}\big)\\ &\leq \frac{\varepsilon^{(N+2)/N}}{\eta_\star}\int_0^T\int_\Omega F_{\mathcal{S}}(t,x)\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

as a consequence of the Tchebyschev inequality. It yields by direct evaluation

$$\begin{aligned}\mathcal{L}\Big(\bigcup_{j\in\mathbb{N}}C_j\Big) &\leq \frac{\varepsilon^{2(N+2)/N}}{\eta_\star}\sum_{i=1}^{p}\sum_{j\in\mathbb{N}}\left(\iint_{\widetilde{C_j}}|a_i|^{(N+2)/N}\,\mathrm{d}y\,\mathrm{d}s\ \frac{\int_0^T\int_\Omega\mathbb{1}_{\widetilde{C_j}}(t,x)\,\mathrm{d}x\,\mathrm{d}t}{\mathcal{L}(\widetilde{C_j})}\right)\\ &= \frac{\varepsilon^{2(N+2)/N}}{\eta_\star}\sum_{i=1}^{p}\|a_i\|^{(N+2)/N}_{L^{(N+2)/N}((0,T)\times\Omega)}.\end{aligned}$$

Since the Lebesgue measure of the C_j's is proportional to ε^{N+2}, we deduce that the cardinal of the covering is of order $\mathcal{O}(\varepsilon^{2(N+2)/N-(N+2)}=\varepsilon^{-(N^2-4)/N})$. Furthermore, the C_j's realize a covering of $\mathcal{S}$ with sets of diameter ε; we conclude that the Hausdorff dimension of $\mathcal{S}$ is dominated by $(N^2-4)/N$.

∎

Remark 4.1 *It is not obvious that we can improve this estimate, which is in the spirit of [29, 30] for the Navier-Stokes equations, up to a sharp result as in [5, 18]. A difficulty is related to the fact that we are dealing with diffusion coefficients that depend on the component of the system, which prevents from using regularizations by a common heat kernel.*

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