Relative entropy and the stability of shocks and contact discontinuities for systems of conservation laws with non BV perturbations

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Abstract: We develop a theory based on relative entropy to show the uniqueness and L^2 stability (up to a translation) of extremal entropic Rankine-Hugoniot discontinuities for systems of conservation laws (typically 1-shocks, n-shocks, 1-contact discontinuities and n-contact discontinuities of large amplitude) among bounded entropic weak solutions having an additional trace property. The existence of a convex entropy is needed. No BV estimate is needed on the weak solutions considered. The theory holds without smallness condition. The assumptions are quite general. For instance, strict hyperbolicity is not needed globally. For fluid mechanics, the theory handles solutions with vacuum.

Keywords: System of conservation laws, compressible Euler equation, Rankine-Hugoniot discontinuity, shock, contact discontinuity, relative entropy, stability, uniqueness.

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1 Introduction

In this article, we develop a theory for uniqueness and global L^2 stability of extremal entropy-admissible Rankine-Hugoniot discontinuities (typically 1-shocks, n-shocks, 1-contact discontinuities and n-contact discontinuities) for a wide class of systems of conservation laws endowed with a convex entropy. The uniqueness and stability is shown in the class of bounded weak entropic solutions verifying the following trace property.

Definition 1. *Let $U \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$. We say that U verifies the strong trace property if for any Lipschitzian curve $t \to X(t)$, there exists two bounded functions $U_-, U_+ \in L^\infty(\mathbb{R}^+)$ such that for any $T > 0$*

$$\lim_{n\to\infty} \int_0^T \sup_{y\in(0,1/n)} |U(t, x(t) + y) - U_+(t)|\, dt = \lim_{n\to\infty} \int_0^T \sup_{y\in(-1/n,0)} |U(t, x(t) + y) - U_-(t)|\, dt = 0.$$

Note that, for each fixed curve, this is equivalent to the convergence for almost every time t. Obviously, any BV function verifies this strong trace property. But this requirement is weaker than the BV property. Let us emphasize that this notion of trace is more restrictive than the strong trace introduced in [42], which is known to be verified for bounded solutions of scalar conservation laws. This has been shown in the multidimensional case, first with a non-degeneracy property, in [42]. In the one-dimensional case, a different proof based on compensated compactness was proposed by Chen and Rascle [14]. For a general flux function the strong trace problem has been solved in the 1D case in [22]. The general multidimensional case has been obtained by Panov [34, 33] (see also Kwon [21], De Lellis, Otto, and Westdickenberg [17] for interesting generalizations). In the case of systems, this is mainly an open problem. This has been shown only for the particular case of isentropic gas dynamics with $\gamma = 3$ for traces in time (traces in space can be shown the same way) in [40]. Unfortunately, there are no such results for the strong trace property of Definition 1 outside of the usual BV theory.

Stability of shocks in the class of BV solutions has been investigated by a number of authors. In the case of small perturbations in $L^\infty \cap BV$, Bressan, Crasta, and Piccoli [7] developed a powerful theory of L^1 stability for entropy solutions obtained by either the Glimm scheme [19] or the wave front-tracking method. A simplified approach has been proposed by Bressan, Liu, and Yang [8] and Liu and Yang [30]. (See also [5].) The theory also works in some cases for small perturbations in $L^\infty \cap BV$ of large shocks. See, for instance, Lewicka and Trivisa [25] or Bressan and Colombo [6].

However, our stability result goes beyond the known results valid in the class of BV solutions, with perturbations small in BV. Our approach is based on the relative entropy method first used by Dafermos and DiPerna to show L^2 stability and uniqueness of Lipschitzian solutions to conservation laws [15, 16, 18]. Note that in [18], uniqueness of small shocks for strictly hyperbolic 2×2 systems is shown in a class of admissible weak solutions with small oscillation in $L^\infty \cap BV$. The analysis in [18] also implies the uniqueness of shocks for 2×2 systems in the Smoller-Johnson class [37]. In each case genuine nonlinearity is assumed. The ideas of DiPerna were developed further by Chen and Frid in the papers [10, 11]. In subsequent work, they established, together with Li [12], the uniqueness of solutions to the Riemann problem in a large class of entropy solutions (locally BV without smallness conditions) for the 3×3 Euler system in Lagrangian coordinates. They also establish a large-time stability result in this context. See also Chen and Li [13] for an extension to the relativistic Euler equations. However, no stability in L^2 for all time is included in those results.

Our approach is based on fairly mild assumptions on the system and the Rankine-Hugoniot discontinuity. Basically, we need the discontinuity to be extremal (1-shock or n-shock and well separated from the other Hugoniot discontinuities), to be either a contact discontinuity or verify the Liu condition. In the case of a Liu shock, the corresponding shock curve should satisfy the Liu property everywhere (so that the shock speed varies monotonically along the curve), and we need also a property of growth of the strength of the shock along the shock curve, where the strength is measured via the entropy. Very little constraint is needed on the other shock families. Lax properties are typically enough. But we may even relax it to cases where the system is neither genuinely nonlinear nor strictly hyperbolic, and even to cases where the shock curves are not well-defined. The theory works fine even for large shocks. Note that the present study is another step in the program described in [43]. A first step was achieved for scalar conservation laws in [24].

We will give a precise description of our hypotheses and main results in the next section. First let us mention a few particular cases in which our theory applies. Our first examples include the isentropic Euler system and the full Euler system for a polytropic gas. Both systems are treated in Eulerian coordinates. The isentropic Euler system is the following.

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + P(\rho)) = 0. \end{cases} \tag{1}$$

We assume a smooth pressure law $P : \mathbb{R}^+ \to \mathbb{R}$ with the following properties

$$P'(\rho) > 0, \qquad [\rho P(\rho)]'' \geq 0. \tag{2}$$

As usual, we consider only entropic solutions of this system, namely, those verifying additionally the entropy inequality:

$$\partial_t \eta(\rho, \rho u) + \partial_x G(\rho, \rho u) \leq 0,$$

with

$$\eta(\rho, \rho u) = \frac{(\rho u)^2}{2\rho} + S(\rho), \qquad G(\rho, \rho u) = \frac{(\rho u)^3}{2\rho^2} + \rho u\, S'(\rho),$$

and with $S''(\rho) = \rho^{-1} P'(\rho) > 0$. Note that we need only a single convex entropy, even if in this case there exists an entire family of convex entropies.

The full Euler system reads

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + P) = 0 \\ \partial_t(\rho E) + \partial_x(\rho u E + uP) = 0, \end{cases} \tag{3}$$

where $E = \frac{1}{2}u^2 + e$. The equation of state for a polytropic gas is given by

$$P = (\gamma - 1)\rho e \tag{4}$$

where $\gamma > 1$. In that case, we consider the entropy/entropy-flux pair

$$\eta(\rho, \rho u, \rho E) = (\gamma - 1)\rho \ln \rho - \rho \ln e, \qquad G(\rho, \rho u, \rho E) = (\gamma - 1)\rho u \ln \rho - \rho u \ln e, \tag{5}$$

where, in conservative variables, we have $e = \dfrac{\rho E}{\rho} - \dfrac{(\rho u)^2}{2\rho^2}$.

For the Euler systems (1) and (3), we have the following theorem.

Theorem 1.1. *Consider a shock $(U_L, U_R) = ((\rho_L, u_L), (\rho_R, u_R))$ with velocity σ associated to the system (1)-(2), (resp. $(U_L, U_R) = ((\rho_L, u_L, E_L), (\rho_R, u_R, E_R))$ associated to the system (3)-(4)). For any $K > 0$, there exists $C_K > 0$ with the following property. For any $0 < \varepsilon < 1$ we have the two following cases:*

- *If (U_L, U_R) is a 1-shock, then for any weak entropic solution $U = (\rho, u) \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ of (1) (resp. $U = (\rho, u, E) \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ of (3)) verifying the strong trace property of Definition 1 and such that $\|(\rho, u)\|_{L^\infty} \leq K$, (resp. $\|(\rho, u, E)\|_{L^\infty} \leq K$) and*

$$\int_0^\infty |U_0(x) - U_R|^2\, dx \leq \varepsilon, \qquad \int_{-\infty}^0 |U_0(x) - U_L|^2\, dx \leq \varepsilon^4,$$

then there exists a Lipschtiz path $x(t)$ such that for any $T > 0$ we have both

$$\int_0^\infty |U(T, x + x(T)) - U_R|^2\, dx \leq C\varepsilon(1 + T), \qquad \int_{-\infty}^0 |U(T, x + x(T)) - U_L|^2\, dx \leq \varepsilon^4.$$

- *If (U_L, U_R) is a n-shock, then for any weak entropic solution $U \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ of (1) (resp. of (3)) verifying the strong trace property of Definition 1 and such that $\|(\rho, u)\|_{L^\infty} \le K$ (resp. $\|(\rho, u, E)\|_{L^\infty} \le K$), and*

  $$\int_0^\infty |U_0(x) - U_R|^2\,dx \le \varepsilon^4, \qquad \int_{-\infty}^0 |U_0(x) - U_L|^2\,dx \le \varepsilon,$$

  then there exists a Lipschtiz path $x(t)$ such that for any $T > 0$ we have

  $$\int_0^\infty |U(T, x + x(T) - U_R|^2\,dx \le \varepsilon^4, \qquad \int_{-\infty}^0 |U(T, x + x(T)) - U_L|^2\,dx \le C\varepsilon(1 + T).$$

In both case we have

$$|x(t) - \sigma t| \le C\sqrt{\varepsilon t(1 + t)}.$$

Note that this implies both the uniqueness and the stability of the shocks. The solutions are not assumed to be away from vacuum. This gives a L^2 stability results up to the translation $x(t)$. It is worth noting that the treatment of the entropy on the left and the right of the shock is not the same. For example, in the case of a 1-shock, we will show that the total relative entropy on the left is strictly decreasing. Hence, if at $t = 0$, $U^0(x) = U_L$ for $x < 0$, then it stays that way on the left of $x(t)$ (while keeping the relative entropy on the right under control). The idea here comes from [41], where a similar stability appeared in the study of a semi-discrete shock for an isentropic gas with $\gamma = 3$. For 2×2 systems, all Rankine-Hugoniot discontinuities are extremal, hence Theorem 1.1 applies to all admissible shocks of (1). (The theorem also holds for contact discontinuities in this case, although the pressure laws are usually nonphysical.) For the full Euler system, all shocks are 1-shocks or n-shocks (3-shocks in this case), so again the result applies to any entropy admissible shock. However, our result does not provide the stability of contact discontinuities for this system. The problem is that the contact discontinuities correspond to 2-waves (with a middle eigenvalue). Note that in the isentropic case with $P(\rho) = \rho^\gamma$ ($\gamma > 1$), it is enough to assume that the initial values are bounded since solutions can be constructed conserving this property (see Chen [9], or Lions Perthame Tadmor [27], for instance).

We now show an application of our method in the general setting of strictly hyperbolic conservation laws with either linearly degenerate or genuinely nonlinear characteristic fields. We consider an $n \times n$ system of conservation laws

$$\partial_t U + \partial_x A(U) = 0, \tag{6}$$

which has a strictly convex entropy η. Assume that A and η are of class C^2 on an open state domain $\mathcal{V} \subset \mathbb{R}^n$. We have the following result.

Theorem 1.2. *Assume that the smallest (resp. largest) eigenvalue of $\nabla A(V)$ is simple for all $V \in \mathcal{V}$, and that the corresponding 1-characteristic family (resp. n-characteristic family) of (6) is either genuinely nonlinear or linearly degenerate. Then, for any $V_0 \in \mathcal{V}$, there exists $K > 0$ and $C > 0$ such that, for any entropy-admissible 1-shock or 1-contact discontinuity (resp. n-shock or n-contact discontinuity) with speed σ and endstates (U_L, U_R) verifying $U_L \in B_K(V_0)$ and $U_R \in B_K(V_0)$, the following is true. For $\varepsilon > 0$ small enough, and for any weak entropic solution U bounded in $B_K(V_0)$ on $(0, T)$ such that*

$$\int_0^\infty |U_0(x) - U_R|^2\,dx \le \varepsilon, \qquad \int_{-\infty}^0 |U_0(x) - U_L|^2\,dx \le \varepsilon^4,$$

(resp. $\int_0^\infty |U_0(x) - U_R|^2\, dx \le \varepsilon^4$ *and* $\int_{-\infty}^0 |U_0(x) - U_L|^2\, dx \le \varepsilon$*), there exists a Lipschitzian curve* $x(t)$ *such that, for any* $0 < t < T$*, we have both*

$$\int_0^\infty |U(t, x + x(t)) - U_R|^2\, dx \le C\varepsilon(1 + t), \qquad \int_{-\infty}^0 |U(t, x + x(t)) - U_L|^2\, dx \le \varepsilon^4,$$

(resp. $\int_0^\infty |U(t, x + x(t)) - U_R|^2\, dx \le \varepsilon^4$ *and* $\int_{-\infty}^0 |U(t, x + x(t)) - U_L|^2\, dx \le C\varepsilon(1 + t)$*). In both cases we have*

$$|x(t) - \sigma t| \le C\sqrt{\varepsilon t(1 + t)}.$$

In particular, this provides L^2 stability, up to a drift, for suitably weak shocks and contact discontinuities in a class of perturbations without BV conditions. Note that the assumption of genuinely nonlinearity or linear degeneracy applies only to the wave family associated to the extremal eigenvalue. No such assumptions are needed on the other wave families.

The theorems above highlight only a few applications of our theory. In the next section, we develop our methods in a more general framework. The assumptions on the Hugoniot curves are quite natural and we require no smallness condition on the discontinuities at play. We can even relax the strict hyperbolicity condition and consider cases in which the middle eigenvalues degenerate and possibly cross each other.

To conclude this introduction, let us mention that the relative entropy method is also an important tool in the study of asymptotic limits to conservation laws. Applications of the relative entropy method in this context began with the work of Yau [44] and have been studied by many others. For incompressible limits, see Bardos, Golse, Levermore [1, 2], Lions and Masmoudi [26], Saint Raymond et al. [20, 36, 31, 35]. For the compressible limit, see Tzavaras [39] in the context of relaxation and [4, 3, 32] in the context of hydrodynamical limits. Up to now, this method works as long as the limit solution is Lipschitz. It would be of significant interest to extend the method to shocks (see [43]).

2 Presentation of the results

2.1 General framework

We want to study a system of m equations of the form

$$\partial_t U + \partial_x A(U) = 0, \tag{7}$$

where the flux function A is defined on an open, bounded, convex set $\mathcal{V} \subset \mathbb{R}^m$.

$$A : \mathcal{V} \subset \mathbb{R}^m \longrightarrow \mathbb{R}^m.$$

We assume that $A \in C^2(\mathcal{V})$. Additionally, we assume the existence of a strictly convex entropy

$$\eta : \mathcal{V} \subset \mathbb{R}^m \longrightarrow \mathbb{R},$$

of class C^2, and an associated entropy flux

$$G : \mathcal{V} \subset \mathbb{R}^m \longrightarrow \mathbb{R},$$

of class C^2, such that the following compatibility relation holds on $\mathcal{V}$.

$$\partial_j G = \sum_{i=1}^m \partial_i \eta\, \partial_j A_i \qquad \text{for any } 1 \le j \le m. \tag{8}$$

If we want to apply our theory to the systems of gas dynamics, we have to define these functions on a suitable subset of the boundary of $\mathcal{V}$, namely the points corresponding to vacuum states. For this reason, we introduce

$$\mathcal{U} = \overline{\mathcal{V}},$$

the closure of $\mathcal{V}$. We assume, for simplicity, that A, η, and G are continuous on $\mathcal{U}$ (but with no additional regularity up to the boundary). We denote by $\mathcal{U}^0$ the subset of $\mathcal{U}$ where at least one of the functions η, A, G is not C^1 (typically the vacuum states).

Remark. It is possible to consider a more general situation in which η is unbounded on $\mathcal{V}$. In that case, we can add the "vacuum points" in the following way, as in [43].

$$\mathcal{U} = \{V \in \mathbb{R}^m \mid \exists V_k \in \mathcal{V}, \lim_{k\to\infty} V_k = V, \limsup_{k\to\infty} \eta(V_k) < \infty\}.$$

We consider here a slightly less general framework, which is enough to treat the Euler systems on a state domain with bounded velocities.

Next, we define, for any $V \in \mathcal{V}$, $U \in \mathcal{U}$, the relative entropy function

$$\eta(U \mid V) = \eta(U) - \eta(V) - \nabla\eta(V) \cdot (U - V).$$

Since η is continuous (in fact, convex) on $\mathcal{U}$ and strictly convex in $\mathcal{V}$, we have

$$\eta(U \mid V) \geq 0, \qquad U \in \mathcal{U},\ V \in \mathcal{V},$$

and

$$\eta(U \mid V) = 0 \quad \text{if and only if} \quad U = V.$$

Indeed, the following lemma shows that the relative entropy is comparable to the square of the L^2 norm.

Lemma 1. *For any compact set $\Omega \subset \mathcal{V}$, there exist $C_1, C_2 > 0$ such that*

$$C_1|U - V|^2 \leq \eta(U \mid V) \leq C_2|U - V|^2,$$

for any $U \in \mathcal{U}$ and $V \in \Omega$.

We give a proof of this well-known estimate in the appendix.

For a pair of states $U_L \neq U_R$ in $\mathcal{V}$, we say that (U_L, U_R) is an entropic Rankine-Hugoniot discontinuity if there exists $\sigma \in \mathbb{R}$ such that

$$\begin{aligned} &A(U_R) - A(U_L) = \sigma(U_R - U_L), \\ &G(U_R) - G(U_L) \leq \sigma(\eta(U_R) - \eta(U_L)). \end{aligned} \tag{9}$$

Equivalently, this means that the discontinuous function U defined by

$$U(t,x) = \begin{cases} U_L, & \text{if } x < \sigma t, \\ U_R, & \text{if } x > \sigma t, \end{cases}$$

is a weak solution to (7) verifying also, in the sense of distributions, the entropy inequality

$$\partial_t \eta(U) + \partial_x G(U) \leq 0. \tag{10}$$

Finally, we denote by $\lambda^-(U)$ and $\lambda^+(U)$ the smallest and largest eigenvalues, respectively, of $\nabla A(U)$. Hereafter, we assume that $\lambda^\pm(U)$ are simple eigenvalues for all $U \in \mathcal{V}$ and that $U \to \lambda^\pm(U)$ lie in $L^\infty(\mathcal{U})$. (They may be undefined on $\mathcal{U}^0$.)

2.2 Hypotheses on the system

First, we assume that for any (U_-, U_+) entropic Rankine-Hugoniot discontinuity with $U_- \neq U_+$ we have both $U_- \notin \mathcal{U}^0$ and $U_+ \notin \mathcal{U}^0$. (Typically, there is no shock connecting the vacuum.)

We will consider two sets of assumptions. One set will imply the result on the 1-shock (or 1-contact discontinuity), the second set (dual from the first one) will imply the result on the n-shock (or n-contact discontinuity). A system satisfying both set of hypotheses, verifies both results.

First set of hypotheses

The first set of hypotheses, related to some $U_L \in \mathcal{V}$, is the following ((H1) to (H3)).

(H1) (Family of 1-contact discontinuities or 1-shocks verifying the Liu condition)
There exists a neighborhood $B \subset \mathcal{V}$ of U_L such that for any $U \in B$, there is a one parameter family of states $S_U(s) \in \mathcal{U}$ defined on an interval $[0, s_U]$, such that $S_U(0) = U$, and

$$A(S_U(s)) - A(U) = \sigma_U(s)(S_U(s) - U), \qquad s \in [0, s_U],$$

(which means that $(U, S_U(s))$ is a Rankine-Hugoniot discontinuity with velocity $\sigma_U(s)$). We assume that $U \to s_U$ is Lipschitz on B and both $(s, U) \to S_U(s)$ and $(s, U) \to \sigma_U(s)$ are C^1 on $\{(s, U) | U \in B, 0 \leq s \leq s_U\}$. We assume also that the following properties hold for $U \in B$.

- (a) $\sigma'_U(s) \leq 0$ for $0 \leq s \leq s_U$ (the speed of the shock decreases with s), and $\sigma_U(0) = \lambda^-(U)$.
- (b) (1-shock) If $\sigma'_U \not\equiv 0$, then $\dfrac{d}{ds}\eta(U|S_U(s)) \geq 0$ (the shock "strengthens" with s) for all s.

(H2) If (U, V) is an entropic Rankine-Hugoniot discontinuity with velocity σ such that $V \in B$, then $\sigma \geq \lambda^-(V)$.

(H3) If (U, V) is an entropic Rankine-Hugoniot discontinuity with velocity σ such that $U \in B$ and $\sigma < \lambda^-(U)$, then (U, V) is a 1-shock. In particular, $V = S_U(s)$ for some $0 \leq s \leq s_U$.

Second set of hypotheses

The second set of hypotheses, related to some $U_R \in \mathcal{V}$, is the following ((H′1) to (H′3)).

(H′1) (Family of n-contact discontinuities or n-shocks verifying the Liu condition)
There exists a neighborhood $B \subset \mathcal{V}$ of U_R such that for every $U \in B$ there is a one parameter family of states $S_U(s) \in \mathcal{U}$ defined on an interval $[0, s_U]$, such that $S_U(0) = U$, and

$$A(S_U(s)) - A(U) = \sigma_U(s)(S_U(s) - U), \qquad s \in [0, s_U],$$

(which means that $(S_U(s), U)$ is a Rankine-Hugoniot discontinuity with velocity $\sigma_U(s)$). We assume that $U \to s_U$ is Lipschitz and both $(s, U) \to S_U(s)$ and $(s, U) \to \sigma_U(s)$ are C^1 on $\{(s, U) | U \in B, s \in [0, s_U]\}$. We assume also that the following properties hold for $U \in B$.

- (a) $\sigma'_U(s) \geq 0$ for $0 \leq s \leq s_U$ (the speed of the shock increases with s), and $\sigma_U(0) = \lambda^+(U)$.
- (b) (n-shock) If $\sigma'_U \not\equiv 0$, then $\dfrac{d}{ds}\eta(U|S_U(s)) \geq 0$ (the shock "strengthens" with s) for all s.

(H′2) If (V, U) is an entropic Rankine-Hugoniot discontinuity with velocity σ such that $V \in B$, then $\sigma \leq \lambda^+(V)$.

(H$'$3) If (V, U) is an entropic Rankine-Hugoniot discontinuity with velocity σ such that $U \in B$ and $\sigma > \lambda^+(U)$, then (V, U) is an n-shock. In particular, $V = S_U(s)$ for some $0 \leq s \leq s_U$.

Remarks

- Note that a given system (7) verifies Properties (H1) to (H3) relative to $U \in \mathcal{V}$ if and only if the system
$$\partial_t U - \partial_x A(U) = 0, \tag{11}$$
verifies Properties (H$'$1) to (H$'$3) relative to the same $U \in \mathcal{V}$. The properties are, in this way, dual.

- In the case $\sigma'_U(s) = 0$ for all s, Property (H1) just assumes the existence of a 1-family of contact discontinuities.

- In the case where $\sigma_U(s)$ is not constant in s, Property (H1) assumes the existence of a family a 1-shocks $(U, S_U(s))$ verifying the Liu entropy condition for all s (Property (a)). The only additional requirement is (b), which is a condition on the growth of the shock along $S_U(s)$, where the growth is measured through the pseudo-metric induced from the entropy. This condition arises naturally in the study of admissibility criteria for systems of conservation laws. In particular, it ensures that Liu admissible shocks are entropic even for moderate to strong shocks. Indeed, this fact follows from the important formula
$$G(S_{U_L}(s)) - G(U_L) = \sigma_{U_L}(s)\,(\eta(S_{U_L}(s)) - \eta(U_L)) + \int_0^s \sigma'_{U_L}(\tau)\eta(U_L \mid S_{U_L}(\tau))\,d\tau,$$
which is proved in Section 3. (See also [15, 23, 29].)

- Hypothesis (H2) is fulfilled under the very general assumption that all the entropic Rankine-Hugoniot discontinuities verify the Lax entropy conditions, that is
$$\lambda_i(U_-) \geq \sigma \geq \lambda_i(U_+),$$
for any i-shocks (U_-, U_+) with velocity σ and any $1 \leq i \leq n$. Indeed, we need only the second inequality, and the fact that $\lambda_i(U_+) \geq \lambda^-(U_+)$.

- Hypothesis (H3) is a requirement that the family of 1-discontinuities is well-separated from the other Rankine-Hugoniot discontinuities and do not interfere with them. In the case of strictly hyperbolic systems, it is, for instance, a consequence of the extended Lax admissibility condition
$$\lambda_{i+1}(U_+) \geq \sigma \geq \lambda_{i-1}(U_-),$$
for all i-shocks (U_-, U_+), $i > 1$, with velocity σ. Indeed, we use only the second inequality and the fact that $\lambda_{i-1}(U_-) \geq \lambda^-(U_-)$. Note that we need to separate only the 1-shocks issued from B, that is close to U_L.

- The existence of an entropy η implies that the system (7) is hyperbolic. Since $A \in C^2(\mathcal{V})$, the eigenvalues of $\nabla A(U)$ vary continuously on $\mathcal{V}$. In particular, since $\lambda^\pm(U)$ are simple for $U \in \mathcal{V}$, the implicit function theorem ensures that the maps $U \to \lambda^\pm(U)$ are in $C^1(\mathcal{V})$. Note, however, that those maps may be discontinuous on $\mathcal{U}$.

2.3 Statement of the result

Our main result is the following.

Theorem 2.1. *Consider a system of conservation laws (7), such that A is C^2 on an open convex subset $\mathcal{V}$ of $\mathbb{R}^m$. We assume that there exists a C^2 strictly convex entropy η on $\mathcal{V}$ verifying (8). We assume that η, A and G are continuous on $\mathcal{U} = \overline{\mathcal{V}}$. Let $U_L \in \mathcal{V}$. Assume that the system (7) verifies the Properties (H1)–(H3). Consider $U_R \in \mathcal{V}$ such that (U_L, U_R) is a 1-shock (or 1-contact discontinuity) with velocity σ. (This means that there exists $s > 0$ such that $U_R = S_{U_L}(s)$ and $\sigma = \sigma_{U_L}(s)$). Then there exist constants $C > 0$, $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and any weak entropic solution U of (7) with values in $\mathcal{U}$ on $(0, T)$ verifying the strong trace property of Definition 1, and*

$$\int_0^\infty |U_0(x) - U_R|^2\, dx \le \varepsilon, \qquad \int_{-\infty}^0 |U_0(x) - U_L|^2\, dx \le \varepsilon^4,$$

there exists a Lipschitzian map $x(t)$ such that for any $0 < t < T$:

$$\int_0^\infty |U(t, x + x(t)) - U_R|^2\, dx \le C\varepsilon(1 + t), \qquad \int_{-\infty}^0 |U(t, x + x(t)) - U_L|^2\, dx \le \varepsilon^4.$$

Moreover,

$$|x(t) - \sigma t| \le C\sqrt{\varepsilon t(1 + t)}.$$

This shows that the theory is essentially an L^2 theory. The correction of the position of the approximated shock $x(t)$ is fundamental, since the result is trivially wrong without it, even for Burgers' equation in the scalar case (see [24]). Part of the difficulty of the proof is to find this correct position. Indeed, we will construct it so that $\int_{-\infty}^0 \eta(U(t, x+x(t))|U_L)\, dx$ is decreasing and remains smaller than ε^4 for all time.

Applying the main theorem on the $\tilde{U}(t, x) = U(t, -x)$, which is an entropic solution to (11), we obtain the following corollary.

Corollary 1. *Consider a system of conservation laws (7), such that A is C^2 on a open convex subset $\mathcal{V}$ of $\mathbb{R}^m$. We assume that there exists a C^2 strictly convex entropy η on $\mathcal{V}$ verifying (8). We assume that η, A and G are continuous on $\mathcal{U} = \overline{\mathcal{V}}$. Let $U_R \in \mathcal{V}$. Assume that (7) verifies the properties (H′1)–(H′3). Let $U_L \in \mathcal{V}$ such that (U_L, U_R) is a n-shock (or n-contact discontinuity) with velocity σ. (This means that there exists $s > 0$ such that $U_L = S_{U_R}(s)$ and $\sigma = \sigma_{U_R}(s)$). Then there exist constants $C > 0$, $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and any weak entropic solution U of (7) with values in $\mathcal{U}$ on $(0, T)$ verifying the strong trace property of Definition 1, and*

$$\int_0^\infty |U_0(x) - U_R|^2\, dx \le \varepsilon^4, \qquad \int_{-\infty}^0 |U_0(x) - U_L|^2\, dx \le \varepsilon,$$

there exists a Lipschitzian map $x(t)$ such that for any $0 < t < T$:

$$\int_0^\infty |U(t, x + x(t)) - U_R|^2\, dx \le \varepsilon^4, \qquad \int_{-\infty}^0 |U(t, x + x(t)) - U_L|^2\, dx \le C\varepsilon(1 + t).$$

Moreover,

$$|x(t) - \sigma t| \le C\sqrt{\varepsilon t(1 + t)}.$$

In particular, if we consider a 2×2 system which verifies both (H1)–(H3) for any $U_L \in \mathcal{V}$ and (H$'$1)–(H$'$3) for any $U_R \in \mathcal{V}$, then all shocks (and contact discontinuities) are unique and stable in L^2.

Note that the assumptions on the system are quite minimal. There is an assumption only on the wave coming from U_L (or going to U_R in the case of the corollary). There are absolutely no assumptions on the other waves (which may not even exist or may be neither genuinely non-linear nor linearly degenerate). The extremal property that the shock curve under consideration corresponds to the smallest eigenvalue of $\nabla A(U)$ (resp. the largest eigenvalue of $\nabla A(U)$) is only prescribed on a small neighborhood of U_L (resp. of U_R). Finally the theory allows us (via the extended set $\mathcal{U}$) to consider weak solutions which may take values U corresponding to points of non-differentiability of A and η. This includes, for example, the vacuum states in fluid mechanics.

2.4 Main ideas of the proof

The main idea of the proof is to find a path $t \to x(t)$ for which the total relative entropy of $U(t,x)$ with respect to U_L on the left of $x(t)$, given by

$$\int_{-\infty}^{0} \eta(U(t, x + x(t)) \,|\, U_L)\, dx,$$

is strictly decreasing. The following estimate underlies most of our analysis.

Lemma 2. *If $V \in \mathcal{V}$ and U is any weak entropic solution of (7), then $\eta(U \,|\, V)$ is a solution in the sense of distributions to*

$$\partial_t \eta(U \,|\, V) + \partial_x F(U, V) \leq 0,$$

where

$$F(U, V) = G(U) - G(V) - \nabla\eta(V) \cdot (A(U) - A(V)).$$

The proof of this lemma is direct from the definition of the relative entropy (Note that V is constant with respect to t and x). From this lemma, and the strong trace property of Definition 1, we will show that

$$\frac{d}{dt} \int_{-\infty}^{0} \eta(U(t, x + x(t)) \,|\, U_L)\, dx \leq x'(t)\eta(U(t, x(t)-) \,|\, U_L) - F(U(t, x(t)-), U_L),$$

for almost every t. Let us fix a small drift velocity $v_0 > 0$ of order ε. Ideally, we would like to have

$$x'(t) = \frac{F(U(t, x(t)-), U_L)}{\eta(U(t, x(t)-) \,|\, U_L)} - v_0. \tag{12}$$

This would preserve the estimate

$$\int_{-\infty}^{0} \eta(U(t, x + x(t)) \,|\, U_L)\, dx \leq \varepsilon^4,$$

which is assumed at the initial time, and ensure that

$$\frac{d}{dt} \int_{-\infty}^{0} \eta(U(t, x + x(t)) \,|\, U_L)\, dx \leq -v_0\varepsilon^2, \qquad \text{whenever } \eta(U(t, x(t)-) \,|\, U_L) \geq \varepsilon^2.$$

If we denote by $\mathcal{I}$ the set of time for which $\eta(U(t, x(t)-) \,|\, U_L) \geq \varepsilon^2$, then we must have

$$|\mathcal{I}| \leq \frac{\varepsilon^4}{v_0 \varepsilon^2} \sim \varepsilon.$$

Before going on, let us note that U has little regularity, and (12) cannot be solved in the classical sense. Hence we can define $x(t)$ only in the Filippov way. (This is roughly the idea. For technical reasons, the construction is even more complicated than that.) We will have to check carefully that we can do it using only the strong trace property of Definition 1. Even so, we cannot ensure that (12) holds almost everywhere. However, we will use the fact that for almost every time t, especially when $U(t,x(t)+) \neq U(t,x(t)-)$, the following Rankine–Hugoniot relation holds:

$$A(U(t,x(t)+) - A(U(t,x(t)-) = x'(t)(U(t,x(t)+) - U(t,x(t)-)).$$

We will make these claims more precise in the sections to follow.

At this point we need to check that the total relative entropy of $U(t,x)$ with respect to U_R on the right of $x(t)$ remains under control. A similar estimate to the one above gives

$$\int_0^\infty \eta(U(t,x+x(t))\,|\,U_R)\,dx = \int_0^\infty \eta(U_0(x)\,|\,U_R)\,dx \\ + \int_0^t \{F(U(t,x(t)+),U_R) - x'(t)\eta(U(t,x(t)+)\,|\,U_R)\}\;dt.$$

Since U is bounded, we have the easy estimate

$$\int_{\mathcal{I}} \{F(U(t,x(t)+),U_R) - x'(t)\eta(U(t,x(t)+)\,|\,U_R)\}\;dt \leq C\varepsilon.$$

In the remaining case, we have to control $F(U(t,x(t)+),U_R) - x'(t)\eta(U(t,x(t)+)\,|\,U_R)$ for t such that $\eta(U(t,x(t)-)\,|\,U_L) \leq \varepsilon^2$, that is

$$|U(t,x(t)-) - U_L| \sim \varepsilon.$$

The extremality property of (U_L,U_R) ensures that the worst scenario corresponds to

$$U(t,x(t)+) = S_{U(t,x(t)-)}(s), \qquad x'(t) = \sigma_{U(t,x(t)-)}(s) \qquad \text{for given } s > 0.$$

But, $S_{U(t,x(t)-)}(s) = S_{U_L}(s) + \mathcal{O}(\varepsilon)$, and the key structural lemma will show that

$$F(S_{U_L}(s),U_R) - \sigma_{U_L}(s)\eta(S_{U_L}(s)\,|\,U_R) \leq 0, \qquad \text{for } s > 0.$$

Hence, for $t \notin \mathcal{I}$,

$$F(U(t,x(t)+),U_R) - x'(t)\eta(U(t,x(t)+)\,|\,U_R) \sim \varepsilon.$$

The rest of the paper is organized as follows. In the next section, we prove the main structural lemmas. In the following section we construct the path $t \to x(t)$. The next one is dedicated to the proof of the main theorem. In the last section, we prove theorems on the example stated in the introduction.

3 Structural lemmas

The first lemma of this section gives an explicit formula for the entropy lost at a Rankine–Hugoniot discontinuity (U_-,U_+), where $U_+ = S_{U_-}(s)$ for some $s > 0$. The estimate can be traced back to the work of Lax [23].

Lemma 3. *Assume $(U_-, U_+) \in \mathcal{V}^2$ is an entropic Rankine-Hugoniot discontinuity with velocity σ; that is, (U_-, U_+) verifies (9). Then, for any $V \in \mathcal{U}$*

$$F(U_+, V) - \sigma\eta(U_+ \,|\, V) \leq F(U_-, V) - \sigma\eta(U_- \,|\, V),$$

where F is defined as in Lemma 2. Furthermore, if $U_- \in B$, as in Hypothesis (H1), and there exists $s > 0$ such that $U_+ = S_{U_-}(s)$ and $\sigma = \sigma_{U_-}(s)$ (that is, (U_-, U_+) is a 1-discontinuity), then

$$F(U_+, V) - \sigma\eta(U_+ \,|\, V) = F(U_-, V) - \sigma\eta(U_- \,|\, V) + \int_0^s \sigma'_{U_-}(\tau)\eta(U_- \,|\, S_{U_-}(\tau))\, d\tau.$$

Proof. Since $(U_-, U_+) \in \mathcal{V}^2$ is an entropic Rankine-Hugoniot discontinuity with velocity σ we have

$$-\nabla\eta(V) \cdot (A(U_+) - A(U_-)) = -\sigma\nabla\eta(V) \cdot (U_+ - U_-),$$

and

$$G(U_+) - G(U_-) \leq \sigma(\eta(U_+) - \eta(U_-)).$$

Summing those two estimates gives the first result.

Assume now that it is a 1-discontinuity. Then, define

$$\begin{aligned} \mathcal{F}_1(s) &= F(S_{U_-}(s), V) - F(U_-, V), \\ \mathcal{F}_2(s) &= \sigma(s)(\eta(S_{U_-}(s) \,|\, V) - \eta(U_- \,|\, V)) + \int_0^s \sigma'_{U_-}(\tau)\eta(U_- \,|\, S_{U_-}(\tau))\, d\tau. \end{aligned}$$

We want to show that $\mathcal{F}_1(s) = \mathcal{F}_2(s)$ for all s. Since $S_{U_-}(0) = U_-$, the equality is true for $s = 0$. Next we have

$$\begin{aligned} \mathcal{F}_1'(s) &= \frac{d}{ds}G(S_{U_-}(s)) - \nabla\eta(V) \cdot \frac{d}{ds}A(S_{U_-}(s)) \\ &= [\nabla\eta(S_{U_-}(s)) - \nabla\eta(V)] \cdot \frac{d}{ds}[A(S_{U_-}(s)) - A(U_L)], \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}_2'(s) &= \sigma'_{U_-}(s)[\nabla\eta(V) \cdot ((S_{U_-}(s) - V) - (U_- - V)) - \nabla\eta(S_{U_-}(s)) \cdot (S_{U_-} - U_-)] \\ &\qquad + \sigma_{U_-}(s)[\nabla\eta(S_{U_-}(s)) - \nabla\eta(V)] \cdot S'_{U_-}(s) \\ &= [\nabla\eta(S_{U_-}(s)) - \nabla\eta(V)] \cdot \frac{d}{ds}[\sigma_{U_-}(s)(S_{U_-}(s) - U_L)]. \end{aligned}$$

Using the fact that $(U_-, S_{U_-}(s))$ with velocity $\sigma_{U_-}(s)$ verifies the Rankine-Hugoniot conditions, we get

$$\mathcal{F}_1'(s) = \mathcal{F}_2'(s) \qquad \text{for } s > 0.$$

□

The next lemma is fundamental to the theory. It is a slight variation on a crucial lemma of DiPerna [18].

Lemma 4. *For any $U \in B$ and any $s > 0$, $s_0 > 0$, we have*

$$F(S_U(s), S_U(s_0)) - \sigma_U(s)\eta(S_U(s) \,|\, S_U(s_0)) = \int_{s_0}^s \sigma'_U(\tau)(\eta(U|S_U(\tau)) - \eta(U \,|\, S_U(s_0)))\, d\tau \leq 0.$$

Proof. We use the estimate of Lemma 3 twice with $V = S_U(s_0)$ and $U_- = U$. The first time we take $U_+ = S_U(s)$, and the second time $U_+ = S_U(s_0)$. The difference of the two results gives the lemma. □

4 Construction of $t \to x(t)$

In this section, we contruct the path $t \to x(t)$ and study its properties. Unlike the construction in [24], we do not appeal directly to the theory of Filippov. Instead, we build $x(t)$ as the limit of approximate solutions, and then show that the Filippov properties are satisfied. The procedure is interesting in its own right and has similarities with the method of Hermes (see [**?**]). We will focus on the family of 1-discontinuities. The corresponding results are easily obtained for the family of n-discontinuities.

Throughout this section, we assume that $(U_L, U_R) \in \mathcal{V}^2$ is a fixed 1-discontinuity with velocity σ, and that U is a fixed weak entropic solution of (7) verifying the strong trace property of Definition 1. We also fix $\varepsilon > 0$. As before, $\lambda^-(U)$ denotes the smallest eigenvalue of $\nabla A(U)$ and is defined for all $U \in \mathcal{U} \setminus \mathcal{U}^0$. We define on $\mathcal{U}$ the velocity function

$$V(U) = \begin{cases} \min\left\{\dfrac{F(U,U_L)}{\eta(U\,|\,U_L)} - \varepsilon,\ \lambda^-(U) - \varepsilon\right\}, & \text{if } U \in \mathcal{U} \setminus (\mathcal{U}^0 \cup \{U_L\}), \\ \lambda^-(U_L) - \varepsilon, & \text{if } U = U_L, \\ \dfrac{F(U,U_L)}{\eta(U\,|\,U_L)} - \varepsilon, & \text{if } U \in \mathcal{U}^0. \end{cases} \tag{13}$$

We have the following lemma.

Lemma 5. *The function $U \to V(U)$ is continuous on $\mathcal{U} \setminus \mathcal{U}^0$, and is bounded and upper semi-continuous on $\mathcal{U}$. Moreover, there exists a constant C (independent of ε) such that if $U \in B$ and $\eta(U\,|\,U_L) \le \varepsilon^2$, then*

$$V(U) \ge \lambda^-(U_L) - C\varepsilon.$$

Note that this map may not be continuous on $\mathcal{U}^0$.

Proof. First note that the function $U \to \frac{F(U,U_L)}{\eta(U\,|\,U_L)}$ is continuous on $\mathcal{U} \setminus \{U_L\}$. Also, by assumption, $\lambda^-(U)$ is continuous on $\mathcal{U} \setminus \mathcal{U}^0$ and bounded on $\mathcal{U}$. So $U \to V(U)$ is continuous on $\mathcal{U} \setminus (\{U_L\} \cup \mathcal{U}^0)$, and upper semi-continuous on $\mathcal{U} \setminus \{U_L\}$. At the point U_L, we have

$$\begin{aligned} \nabla_U F(U,U_L)\big|_{U=U_L} &= 0, \qquad & \nabla_U^2 F(U,U_L)\big|_{U=U_L} &= D^2\eta(U_L)\nabla A(U_L), \\ \nabla_U \eta(U\,|\,U_L)\big|_{U=U_L} &= 0, \qquad & \nabla_U^2 \eta(U\,|\,U_L)\big|_{U=U_L} &= D^2\eta(U_L). \end{aligned}$$

Owing to the strict convexity of η at U_L, an expansion near U_L gives

$$\frac{F(U,U_L)}{\eta(U\,|\,U_L)} = \frac{(U-U_L)^T D^2\eta(U_L)\nabla A(U_L)(U-U_L)}{(U-U_L)^T D^2\eta(U_L)(U-U_L)} + \mathcal{O}(|U-U_L|). \tag{14}$$

Since $D^2\eta(U_L)$ is symmetric positive definite and $D^2\eta(U_L)\nabla A(U_L)$ is symmetric, those two matrices are diagonalizable in the same basis. This gives

$$\lambda^-(U_L)D^2\eta(U_L) \le D^2\eta(U_L)\nabla A(U_L) \le \lambda^+(U_L)D^2\eta(U_L). \tag{15}$$

Therefore,

$$\liminf_{U\to U_L} \frac{F(U,U_L)}{\eta(U\,|\,U_L)} - \varepsilon \ge \lambda^-(U_L) - \varepsilon = V(U_L),$$

which implies

$$\lim_{U\to U_L} V(U) = \lambda^-(U_L) - \varepsilon.$$

So, V is continuous at $U = U_L$ and bounded on $\mathcal{U}$.

More precisely, (14) and (15) give that

$$\frac{F(U, U_L)}{\eta(U \mid U_L)} \geq \lambda^-(U_L) - \mathcal{O}(|U - U_L|).$$

Since $U \to \lambda^-(U)$ lies in $C^1(B)$, we have also that

$$\lambda^-(U) \geq \lambda^-(U_L) - \mathcal{O}(|U - U_L|).$$

This provides the last statement of the lemma. □

For any Lipschitzian path $t \to x(t)$ we define

$$\begin{aligned} V_{\max}(t) &= \max\{V(U(t, x(t)-)),\, V(U(t, x(t)+))\}, \\ V_{\min}(t) &= \begin{cases} \min\{V(U(t, x(t)-)),\, V(U(t, x(t)+))\}, & \text{if both } U(t, x(t)-),\ U(t, x(t)+) \notin \mathcal{U}^0, \\ -\infty, & \text{otherwise.} \end{cases} \end{aligned}$$

This section is dedicated to the following proposition.

Proposition 1. *For any $(U_L, U_R) \in \mathcal{V}^2$ 1-discontinuity with velocity σ, $\varepsilon > 0$, and U a weak entropic solution of (7) verifying the strong trace property of Definition 1, there exists a Lipschitzian path $t \to x(t)$ such that for almost every $t > 0$*

$$V_{\min}(t) \leq x'(t) \leq V_{\max}(t).$$

Proof. Consider the function

$$v_n(t, x) = \int_0^1 V(U(t, x + \frac{y}{n}))\, dy.$$

By virtue of Lemma 5, v_n is bounded, measurable in t, and Lipschitz in x. We denote by x_n the unique solution to

$$\begin{cases} x_n'(t) = v_n(t, x_n(t)), & t > 0, \\ x_n(0) = 0, \end{cases}$$

in the sense of Carathéodory. Since v_n is uniformly bounded, x_n is uniformly Lipschitzian (in time) with respect to n. Hence, there exists a Lipschitzian path $t \to x(t)$ such that (up to a subsequence) x_n converges to x in $C^0(0, T)$ for every $T > 0$. We construct $V_{\max}(t)$ and $V_{\min}(t)$ as above for this particular fixed path $t \to x(t)$. Let us show that for almost every $t > 0$

$$\lim_{n \to \infty} [x_n'(t) - V_{\max}(t)]_+ = 0, \tag{16}$$

$$\lim_{n \to \infty} [V_{\min}(t) - x_n'(t)]_+ = 0. \tag{17}$$

Both limits can be proved the same way. Let us focus on the first one. We have

$$
\begin{aligned}
[x_n'(t) - V_{\max}(t)]_+ &= \left[\int_0^1 V(U(t, x_n(t) + \frac{y}{n}))\, dy - V_{\max}(t)\right]_+ \\
&= \left[\int_0^1 [V(U(t, x_n(t) + \frac{y}{n})) - V_{\max}(t)]\, dy\right]_+ \\
&\leq \int_0^1 \left[V(U(t, x_n(t) + \frac{y}{n})) - V_{\max}(t)\right]_+ dy \\
&\leq \operatorname*{ess\,sup}_{y\in(0,1)} \left[V(U(t, x_n(t) + \frac{y}{n})) - V_{\max}(t)\right]_+ \\
&\leq \operatorname*{ess\,sup}_{z\in(-\varepsilon_n,\varepsilon_n)} [V(U(t, x(t) + z)) - V_{\max}(t)]_+ ,
\end{aligned}
$$

where, for a given $t > 0$, $\varepsilon_n \to 0$ is chosen so that $x_n(t) - x(t) \in (-\varepsilon_n, \varepsilon_n - \frac{1}{n})$. We claim that for almost every $t > 0$, the last term above goes to zero as $n \to \infty$. Indeed, fix $t > 0$ for which $U(t, x(t) + \cdot)$ has a left and right trace in the sense of Definition 1. That is,

$$
\lim_{\varepsilon\to 0} \left\{ \operatorname*{ess\,sup}_{y\in(0,\varepsilon)} |U(t, x(t) + y) - U_+(t)| \right\} = \lim_{\varepsilon\to 0} \left\{ \operatorname*{ess\,sup}_{y\in(0,\varepsilon)} |U(t, x(t) - y) - U_-(t)| \right\} = 0,
$$

Since V is upper semi-continuous on $\mathcal{U}$, we have that for all $r > 0$ there exists $\delta > 0$ such that

$$
|U - U_\pm(t)| < \delta \quad \Rightarrow \quad [V(U) - V(U_\pm(t))]_+ < r.
$$

Therefore,

$$
\lim_{\varepsilon\to 0} \left\{ \operatorname*{ess\,sup}_{y\in(0,\varepsilon)} [V(U(t, x(t) \pm y)) - V(U_\pm(t))]_+ \right\} = 0,
$$

and it follows easily that

$$
\lim_{\varepsilon\to 0} \left\{ \operatorname*{ess\,sup}_{z\in(-\varepsilon,\varepsilon)} [\, V(U(t, x(t) + z)) - V_{\max}(t)\,]_+ \right\} = 0.
$$

This verifies the claim above and finishes the proof of (16). The proof of (17) is similar; we just use the continuity of V on $\mathcal{U} \setminus \mathcal{U}^0$ and the definition of $V_{\min}$ for $U \in \mathcal{U}^0$.

Finally, the sequence x_n' converges to x' in the sense of distributions. Also, the function $[\cdot]_+$ is convex. Therefore, integrating (16) and (17) on $[0, T]$ and passing to the limit, we obtain

$$
\int_0^T [V_{\min}(t) - x'(t)]_+\, dt = \int_0^T [x'(t) - V_{\max}(t)]_+\, dt = 0.
$$

In particular, for almost every $t > 0$ we have

$$
V_{\min}(t) \leq x'(t) \leq V_{\max}(t).
$$

$\square$

We end this section with an elegant formulation of the Rankine-Hugoniot condition and related entropy estimates, as originally presented by Dafermos in the BV case. We show that the estimates remain true for solutions having the strong trace property (in fact, the strong trace property defined in [42] suffices). The proof is given in the appendix.

Lemma 6. *Consider $t \to x(t)$ a Lipschitzian path, and U an entropic weak solution to (7) verifying the strong trace property. Then, for almost every $t > 0$ we have*

$$\begin{aligned} &A(U(t,x(t)+)) - A(U(t,x(t)-)) = x'(t)(U(t,x(t)+) - U(t,x(t)-)), \\ &G(U(t,x(t)+)) - G(U(t,x(t)-)) \leq x'(t)(\eta(U(t,x(t)+)) - \eta(U(t,x(t)-))). \end{aligned}$$

Moreover, for almost every $t > 0$ and $V \in \mathcal{V}$

$$\begin{aligned} &\frac{d}{dt}\int_{-\infty}^{0} \eta(U(t,y+x(t))\,|\,V)\,dy \leq -F(U(t,x(t)-),V) + x'(t)\eta(U(t,x(t)-)\,|\,V), \\ &\frac{d}{dt}\int_{0}^{\infty} \eta(U(t,y+x(t))\,|\,V)\,dy \leq F(U(t,x(t)+),V) - x'(t)\eta(U(t,x(t)+)\,|\,V). \end{aligned}$$

5 Proof of Theorem 2.1

This section is dedicated to the proof of our main result, Theorem 2.1. We consider a system of conservation laws (7) such that A is C^2 on a open convex subset $\mathcal{V}$ of $\mathbb{R}^m$. We assume that there exists a strictly convex entropy η of class C^2 on $\mathcal{V}$ verifying (8). We assume that η, A and G are continuous on $\mathcal{U} = \overline{\mathcal{V}}$. Let $(U_L, U_R) \in \mathcal{V}^2$ be an entropic 1-discontinuity. For the neighborhood $B \ni U_L$ given by (H1) we define

$$\varepsilon_0^2 = \frac{1}{2}\inf_{U \notin B} \eta(U\,|\,U_L).$$

Consider U weak entropic solution of (7) with values in $\mathcal{U}$ on $(0,T)$ verifying the strong trace property of Definition 1, and

$$\int_0^\infty \eta(U_0(x)\,|\,U_R)\,dx \leq \varepsilon, \qquad \int_{-\infty}^0 \eta(U_0(x)\,|\,U_L)\,dx \leq \varepsilon^4,$$

for some $\varepsilon < \varepsilon_0$. Consider the path $t \to x(t)$ constructed in Proposition 1. We first prove a pair of lemmas which provide estimates for the rate of change of the total relative entropy with respect to $x(t)$.

Lemma 7. *For almost every time $0 < t < T$, we have*

$$\frac{d}{dt}\int_{-\infty}^{0} \eta(U(t,y+x(t))\,|\,U_L)\,dy \leq 0.$$

Furthermore, if we let $\mathcal{I} \subset (0,T)$ denote the set of time such that $\eta(U(t,y+x(t)-)\,|\,U_L) \geq \varepsilon^2$, then we have

$$|\mathcal{I}| \leq \varepsilon.$$

Proof. From Lemma 6, we have for almost every $t > 0$

$$\frac{d}{dt}\int_{-\infty}^{0} \eta(U(t,y+x(t))\,|\,U_L)\,dy \leq -F(U(t,x(t)-),U_L) + x'(t)\eta(U(t,x(t)-)\,|\,U_L).$$

At times t for which $U(t, x(t)-) = U_L$, the right-hand side vanishes (and obviously $t \notin \mathcal{I}$). For the other times t, Proposition 1 ensures that either

$$x'(t) \leq V(U(t, x(t)-)) \leq \frac{F(U(t, x(t)-), U_L)}{\eta(U(t, x(t)-) \,|\, U_L)} - \varepsilon, \qquad \text{or} \qquad x'(t) \leq V(U(t, x(t)+)).$$

For the times corresponding to the first case, we have

$$\frac{d}{dt} \int_{-\infty}^{0} \eta(U(t, y + x(t)) \,|\, U_L)\, dy \leq -\varepsilon \eta(U(t, x(t)-) \,|\, U_L) \leq 0.$$

Especially, for every such time in $\mathcal{I}$, we have

$$\frac{d}{dt} \int_{-\infty}^{0} \eta(U(t, y + x(t)) \,|\, U_L)\, dy \leq -\varepsilon^3. \tag{18}$$

For the second case, we appeal to Lemma 6. For almost every time we have either $U(t, x(t)-) = U(t, x(t)+)$ (which is covered by the first case) or $x'(t)$ is the velocity associated to an entropic Rankine–Hugoniot discontinuity $(U(t, x(t)-), U(t, x(t)+))$. If $U(t, x(t)+) \in B \subset \mathcal{V}$, we have

$$x'(t) \leq V(U(t, x(t)+)) \leq \lambda^-(U(t, x(t)+)) - \varepsilon,$$

which is in contradiction with Hypothesis (H2). Hence, for almost every time t not covered by the first case, $U(t, x(t)+) \notin B$, and so $\eta(U(t, x(t)+) \,|\, U_L) \geq \varepsilon_0^2$. In that case, we deduce from Lemmas 3 and 6 that

$$\begin{aligned}
\frac{d}{dt} \int_{-\infty}^{0} \eta(U(t, y + x(t)) \,|\, U_L)\, dy &\leq -F(U(t, x(t)-), U_L) + x'(t)\eta(U(t, x(t)-) \,|\, U_L) \\
&\leq -F(U(t, x(t)+), U_L) + x'(t)\eta(U(t, x(t)+) \,|\, U_L) \\
&\leq -\varepsilon \eta(U(t, x(t)+) \,|\, U_L) \leq -\varepsilon^3.
\end{aligned}$$

In particular, for every $t \in \mathcal{I}$, we have (18). Integrating in time, we find

$$|\mathcal{I}|\varepsilon^3 \leq \int_{-\infty}^{0} \eta(U_0(x) \,|\, U_L)\, dx \leq \varepsilon^4,$$

which gives the desired result. ☐

Before we prove the second lemma, we mention the following basic fact.

Remark. If $f : \mathcal{U} \to \mathbb{R}$ is C^1 on $\mathcal{V}$ and bounded on $\mathcal{U}$, then for any fixed $V \in \mathcal{V}$, there exists $C_V > 0$ such that for all $U \in \mathcal{U}$

$$|f(U) - f(V)| \leq C_V |U - V|.$$

The proof of this fact follows the same outline as the proof of Lemma 1.

Lemma 8. *There exists a constant $C > 0$ such that for every $0 < \varepsilon < \varepsilon_0$*

$$F(U(t, x(t)+), U_R) - x'(t)\eta(U(t, x(t)+) \,|\, U_R) \leq C\varepsilon$$

for almost every $t \notin \mathcal{I}$.

Proof. First note that $t \notin \mathcal{I}$ implies $U(t, x(t)-) \in B$. Also, from Lemma 1, we have

$$|U(t, x(t)-) - U_L| \leq C\varepsilon.$$

We consider three cases.

1. (Points of continuity.) For almost every $t \notin \mathcal{I}$ such that

   $$U(t, x(t)+) = U(t, x(t)-),$$

   we have, from Proposition 1 and Lemma 5,

   $$x'(t) \geq V(U(t, x(t)+) \geq \lambda^-(U_L) - C\varepsilon.$$

   Hence, changing the constant C from line to line if necessary,

   $$\begin{aligned} &F(U(t, x(t)+), U_R) - x'(t)\eta(U(t, x(t)+) \,|\, U_R) \\ &\quad \leq F(U(t, x(t)-), U_R) - \lambda^-(U_L)\eta(U(t, x(t)-) \,|\, U_R) + C\varepsilon \\ &\quad \leq F(U_L, U_R) - \lambda^-(U_L)\eta(U_L \,|\, U_R) + C\varepsilon \\ &\quad \leq F(S_{U_L}(0), S_{U_L}(s_0)) - \sigma_{U_L}(0)\eta(S_{U_L}(0) \,|\, S_{U_L}(s_0)) + C\varepsilon \leq C\varepsilon, \end{aligned}$$

   where, in the last inequality, we used Lemma 4 with $S_{U_L}(s_0) = U_R$.

2. (Discontinuities excluding 1-shocks.) If $(U(t, x(t)-), U(t, x(t)+))$ is an entropic Rankine–Hugoniot discontinuity with velocity $x'(t) \geq \lambda^-(U(t, x(t)-))$, then

   $$\begin{aligned} &F(U(t, x(t)+), U_R) - x'(t)\eta(U(t, x(t)+) \,|\, U_R) \\ &\quad \leq F(U(t, x(t)-), U_R) - x'(t)\eta(U(t, x(t)-) \,|\, U_R) \\ &\quad \leq F(U(t, x(t)-), U_R) - \lambda^-(U_-)\eta(U(t, x(t)-) \,|\, U_R) \\ &\quad \leq F(U_L, U_R) - \lambda^-(U_L)\eta(U_L \,|\, U_R) + C\varepsilon \leq C\varepsilon. \end{aligned}$$

   In the second line we used Lemma 3, and in the fourth line Lemma 4.

3. (1-shocks.) On the other hand, if $(U(t, x(t)-), U(t, x(t)+))$ is an entropic discontinuity with velocity $x'(t) < \lambda^-(U(t, x(t)-))$, then, by Hypothesis (H3), it is a 1-shock and, since $U \to s_U$ is Lipschitz on B, there exists $s' \in [0, s_{U_L}]$ such that we have both $U(t, x(t)+) = S_{U(t,x(t)-)}(s) = S_{U_L}(s') + \mathcal{O}(\varepsilon)$ and $x'(t) = \sigma_{U(t,x(t)-)}(s) = \sigma_{U_L}(s') + \mathcal{O}(\varepsilon)$. Note that $B' = \overline{\{U \in B | \eta(U|U_L) \leq \varepsilon_0^2\}}$ is closed and included in B. Moreover $U \to s_U$ is continuous on B' and there is no state $U_+ \in \mathcal{U}^0$ such that (U, U_+) is an admissible discontinuity. Hence the set $\{(S_U(s) | U \in B', s \in [0, s_U]\}$ is at a positive distance from $\mathcal{U}^0$ (where η and A may be not C^1). The functions $F(\cdot, U_R)$ and $\eta(\cdot|U_R)$ are then uniformly Lipschitz on this set. Hence, by Lemma 4,

   $$\begin{aligned} &F(U(t, x(t)+), U_R) - x'(t)\eta(U(t, x(t)+) \,|\, U_R) \\ &\quad \leq F(S_{U_L}(s'), U_R) - \sigma_{U_L}(s')\eta(S_{U_L}(s') \,|\, U_R) + \mathcal{O}(\varepsilon) \leq \mathcal{O}(\varepsilon). \end{aligned}$$

□

We can now finish the proof of the theorem. Lemma 7 implies that for all $0 < t < T$

$$\int_{-\infty}^0 \eta(U(t, y + x(t)) \,|\, U_L)\, dy \leq \int_{-\infty}^0 \eta(U_0(y) \,|\, U_L)\, dy \leq \varepsilon^4. \tag{19}$$

On the other hand,

$$\int_0^\infty \eta(U(t,y+x(t))\,|\,U_R)\,dy$$
$$= \int_0^\infty \eta(U_0(y)\,|\,U_R)\,dy + \int_{(0,t)\cap\mathcal{I}} \frac{d}{dr}\left\{\int_0^\infty \eta(U(r,y+x(r))\,|\,U_R)\,dy\right\} dr$$
$$+ \int_{(0,t)\cap\mathcal{I}^c} \frac{d}{dr}\left\{\int_0^\infty \eta(U(r,y+x(r))\,|\,U_R)\,dy\right\} dr.$$

Since F, x', and η are bounded and $|\mathcal{I}| \le \varepsilon$, we have

$$\left|\int_{\mathcal{I}} \frac{d}{dr}\left\{\int_0^\infty \eta(U(r,y+x(r))\,|\,U_R)\,dy\right\} dr\right|$$
$$= \int_{\mathcal{I}} |F(U(r,x(r)+),U_R) - x'(r)\eta(U(r,x(r)+)\,|\,U_R)|\,dr \le C|\mathcal{I}| \le C\varepsilon.$$

Applying this estimate together with Lemmas 1 and 8 above, we obtain

$$\int_0^\infty \eta(U(t,y+x(t))\,|\,U_R)\,dy \le C\varepsilon(1+t). \tag{20}$$

It remains to show that $|x(t)-\sigma t| \le C\sqrt{\varepsilon t(1+t)}$. We denote by $U_\pm$ the function defined by

$$U_\pm(x) = \begin{cases} U_L, & \text{if } x<0,\\ U_R, & \text{if } x>0.\end{cases}$$

Note that the exact shock solution is $U_\pm(x-\sigma t)$. By virtue of (19) and (20), we have for every $0<t<T$

$$\|U(t,\cdot) - U_\pm(\cdot - x(t))\|_{L^2} \le C\sqrt{(1+t)\varepsilon}.$$

for some $C > \sqrt{\frac{\varepsilon_0^3}{C_1}}$, where C_1 is given by Lemma 1. Let $M = \|x'(t)\|_{L^\infty}$. Then for $T>0$, we consider an even cutoff function $\phi \in C^\infty(\mathbb{R})$ such that

$$\begin{cases} \phi(x) = 1, & \text{if } |x| \le MT,\\ \phi(x) = 0, & \text{if } |x| \ge 2MT,\\ \phi'(x) \le 0, & \text{if } x \ge 0,\\ |\phi'(x)| \le 2(MT)^{-1}, & \text{if } x\in\mathbb{R}.\end{cases}$$

Then, for almost every $0<t<T$, we have

$$\begin{aligned} 0 &= \int_0^t \int_{\mathbb{R}} \phi(x)\left[\partial_t U + \partial_x A(U)\right]\,dx\,dt \\ &= \int_{\mathbb{R}} \phi(x)\left[U_\pm(x-x(t)) - U_\pm(x)\right]\,dx - \int_0^t\int_{\mathbb{R}} A(U_\pm(x-x(s)))\phi'(x)\,dx\,ds \\ &\quad + \int_{\mathbb{R}} \phi(x)\left[U(t,x) - U_\pm(x-x(t))\right]\,dx + \int_{\mathbb{R}} \phi(x)\left[U_\pm(x) - U^0(x)\right]\,dx \\ &\quad - \int_0^t\int_{\mathbb{R}} \left[A(U(s,x)) - A(U_\pm(x-x(s)))\right]\phi'(x)\,dx\,ds. \end{aligned}$$

The terms on the second line above to reduce to

$$x(t)(U_L - U_R) - t(A(U_L) - A(U_R)) = (x(t) - \sigma t)(U_L - U_R).$$

The third line can be controlled by

$$\|\phi\|_{L^2(\mathbb{R})} \left(\|U(t,\cdot) - U_\pm(\cdot - x(t))\|_{L^2(\mathbb{R})} + \|U^0 - U_\pm\|_{L^2(\mathbb{R})} \right) \leq C\sqrt{4MT}\sqrt{(1+T)\varepsilon}.$$

Finally, since A has a suitable Lipschitz property at the points $U_L, U_R \in \mathcal{V}$ (see the remark preceding Lemma 8), the last term has the following bound:

$$\begin{aligned}
&\left| \int_0^t \int_{\mathbb{R}} \left[A(U(s,x)) - A(U_\pm(x - x(s))) \right] \phi'(x)\, dx\, ds \right| \\
&\qquad\qquad \leq C\|\phi'\|_{L^\infty(\mathbb{R})} \int_0^t \int_{-2MT}^{2MT} |U(s,x) - U_\pm(x - x(s))| \; dx\, ds \\
&\qquad\qquad \leq \frac{2C}{MT} \int_0^t \sqrt{4MT}\, \|U(s,\cdot) - U_\pm(\cdot - x(s))\|_{L^2(\mathbb{R})}\, dx\, ds \leq C\sqrt{T(1+T)\varepsilon}.
\end{aligned}$$

Combining the estimates above we obtain

$$|x(t) - \sigma t| \leq \frac{C\sqrt{T(1+T)\varepsilon}}{|U_L - U_R|} \leq C\sqrt{T(1+T)\varepsilon}.$$

This concludes the proof of the theorem.

6 Proof of the theorems stated in the introduction

We can now prove Theorems 1.1 and 1.2 which were stated the introduction. In each case we want to show that the hypotheses of Theorem 2.1 are fulfilled.

6.1 Isentropic Euler system

To illustrate the foregoing ideas, we apply our results to the isentropic Euler system (1). We will show that our hypotheses are satisfied for a fairly large class of pressure laws $P(\rho)$.

The eigenvalues for this system are

$$\lambda_1(\rho, \rho u) = u - \sqrt{P'(\rho)}, \qquad \lambda_2(\rho, \rho u) = u + \sqrt{P'(\rho)}. \tag{21}$$

To ensure hyperbolicity we assume $P'(\rho) > 0$. If $\{\sigma, (\rho_L, u_L), (\rho_R, u_R)\}$ satisfies the Rankine-Hugoniot condition, then the following relations hold

$$u_L \pm \sqrt{\frac{\rho_R}{\rho_L}\left[\frac{P_R - P_L}{\rho_R - \rho_L}\right]} \;=\; \sigma \;=\; u_R \pm \sqrt{\frac{\rho_L}{\rho_R}\left[\frac{P_R - P_L}{\rho_R - \rho_L}\right]}, \tag{22}$$

$$(u_L - u_R)^2 = \frac{(P_R - P_L)(\rho_R - \rho_L)}{\rho_R \rho_L}. \tag{23}$$

In view of (21), the plus and minus signs in (22) distinguish the 2-shocks and 1-shocks, respectively. It will not be necessary to distinguish between shocks and contact discontinuities here, so we will call all such discontinuities shocks. Let us assume that the entropy admissible shocks are characterized by the relation $u_L > u_R$. The opposite case is treated similarly. In this case, for a fixed $(\rho_L, u_L) \in (\mathbb{R}^+ \times \mathbb{R})$, the admissible right endstates have the form $(\rho_R, u_R(\rho_R))$ where $\rho_R \in \mathbb{R}^+$ and

$$u_R(\rho_R) = u_L - \sqrt{\frac{(P(\rho_R) - P(\rho_L))(\rho_R - \rho_L)}{\rho_R \rho_L}}$$

It follows from the observations above that 1-shocks correspond to the states $\rho_R > \rho_L$ and 2-shocks correspond to $\rho_R < \rho_L$. Note that in either case $\operatorname{sgn}(\rho_R - \rho_L)\, u_R'(\rho_R) < 0$. Therefore, u_R is strictly decreasing along each shock curve. Note that this property depends only on the hyperbolicity assumption. In fact, no further assumptions are necessary to show that condition (H1b) is satisfied for the family of 1-shocks. Indeed, the relative entropy in this case is given by

$$\eta((\rho_L, \rho_L u_L) \,|\, (\rho_R, \rho_R u_R)) = \frac{\rho_L}{2}(u_L - u_R)^2 + S(\rho_L \,|\, \rho_R).$$

The observation above shows that $|u_L - u_R(\rho_R)|$ increases as $|\rho_R - \rho_L|$ increases. Also, since S is strictly convex, $S(\rho_L \,|\, \rho_R)$ increases as $|\rho_R - \rho_L|$ increases. This verifies our claim. The proof of (H$'$1b) for the family of 3-shocks is similar.

Now let us investigate the natural condition on $P(\rho)$ which ensures that the Liu condition holds globally along each shock curve. Thanks to (22), we can parametrize the shock speeds as follows

$$\sigma_1(\rho_R) = u_L - \sqrt{\frac{\rho_R}{\rho_L}\left[\frac{P(\rho_R) - P(\rho_L)}{\rho_R - \rho_L}\right]} \qquad \rho_R > \rho_L, \qquad \text{(1-shock)}$$

$$\sigma_2(\rho_L) = u_R + \sqrt{\frac{\rho_L}{\rho_R}\left[\frac{P(\rho_L) - P(\rho_R)}{\rho_L - \rho_R}\right]} \qquad \rho_R < \rho_L. \qquad \text{(2-shock)}$$

Hypotheses (H1a) and (H$'$1a) are satisfied for all 1-shocks and 2-shocks, respectively, if and only if the function

$$\phi(\rho) = \frac{\rho}{\rho_0}\left[\frac{P(\rho) - P(\rho_0)}{\rho - \rho_0}\right]$$

is nondecreasing. The derivative of ϕ can be expressed as follows.

$$\phi'(\rho) = \begin{cases} \dfrac{1}{\rho_0(\rho - \rho_0)^2} \displaystyle\int_{\rho_0}^{\rho} (q - \rho_0)[qP(q)]''\, dq & \text{if } \rho \neq \rho_0; \\[2ex] \dfrac{1}{2\rho_0}[\rho P(\rho)]''\Big|_{\rho = \rho_0} & \text{if } \rho = \rho_0, \end{cases}$$

Therefore, the global Liu-admissibility hypothesis holds if and only if

$$[\rho P(\rho)]'' \geq 0.$$

This may include, for instance, systems which fail to be genuinely nonlinear. On the other hand, it is interesting to note that the genuine nonlinearity condition, $[\rho P(\rho)]'' \neq 0$, holds if and only if $\sigma' \neq 0$; that is, if and only if the shock speed is strictly monotone along each shock curve. In fact, the equivalence of genuine nonlinearity and the global *strict* Liu-condition can be shown for a wide class of conservation laws, as suggested by Liu [28]. Finally, it follows from (22) and the analysis above that the Lax admissibility conditions also hold globally when $[\rho P(\rho)]'' \geq 0$. This verifies (H2)-(H3) and the corresponding dual conditions. Therefore, Theorem 2.1 applies and the proof is complete.

6.2 Full Euler system

Now let us show that our theorem applies to the full Euler system for a polytropic gas. To simplify the presentation we work with the nonconservative variables (ρ, u, e). The sound speed in the case of a polytropic pressure law (4) is given by

$$c = \sqrt{\partial_\rho P + \rho^{-2} P \partial_e P} = \sqrt{\gamma(\gamma-1)e}.$$

Hence the eigenvalues of the system are

$$\lambda_1(\rho,u,e) = u - \sqrt{\gamma(\gamma-1)e}, \qquad \lambda_2(\rho,u,e) = u, \qquad \lambda_3(\rho,u,e) = u + \sqrt{\gamma(\gamma-1)e}. \tag{24}$$

We consider a discontinuity $\{\sigma, (\rho_L, u_L, e_L), (\rho_R, u_R, e_R)\}$ verifying the Rankine-Hugoniot condition. Excluding contact discontinuities ($u_L = u_R$), the following relations hold.

$$u_L \pm \sqrt{\frac{\gamma P_L}{\rho_L}} \cdot \sqrt{\frac{\gamma-1}{2\gamma} + \frac{\gamma+1}{2\gamma}\left[\frac{P_R}{P_L}\right]} = \sigma = u_R \pm \sqrt{\frac{\gamma P_R}{\rho_R}} \cdot \sqrt{\frac{\gamma-1}{2\gamma} + \frac{\gamma+1}{2\gamma}\left[\frac{P_L}{P_R}\right]}, \tag{25}$$

$$\frac{P_R}{P_L} = \frac{\left[\frac{\gamma+1}{\gamma-1}\right]\frac{\rho_R}{\rho_L} - 1}{\left[\frac{\gamma+1}{\gamma-1}\right] - \frac{\rho_R}{\rho_L}}, \qquad \frac{e_R}{e_L} = \frac{P_R}{P_L}\cdot\frac{\rho_L}{\rho_R}, \qquad \frac{\gamma-1}{\gamma+1} < \frac{\rho_R}{\rho_L} < \frac{\gamma+1}{\gamma-1}. \tag{26}$$

We refer the reader to [38] for a proof of these relations. Note that the plus and minus signs in (25) distinguish the 3-shocks and 1-shocks, respectively. Let us fix a left endstate $(\rho_L, u_L, e_L) \in (\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+)$. The entropy admissible shocks can be parametrized by any of the three quantities P_R, ρ_R, and e_R, which are related by (26). The following conditions are equivalent and characterize the family of 1-shocks.

$$P_R > P_L, \qquad \rho_R > \rho_L, \qquad e_R > e_L. \tag{1-shock}$$

Similarly, the 3-shocks are characterized by

$$P_R < P_L, \qquad \rho_R < \rho_L, \qquad e_R < e_L. \tag{3-shock}$$

Note carefully that, for ρ_L fixed, ρ_R is bounded above and bounded away from zero, whereas P_R and e_R range over all positive real numbers. Furthermore, each of the quantities is an increasing function of the others as indicated above. In particular, (24)-(26) easily imply that both shock families verify the Lax and Liu admissibility conditions globally.

It remains only to check that the relative entropy increases along the shock curves. We will prove that (H1b) holds for the family of 1-shocks. The proof of (H′1b) for the family of 3-shocks is essentially the same. A somewhat tedious calculation shows that

$$\eta((\rho_L, \rho_L u_L, \rho_L E_L)\,|\,(\rho_R, \rho_R u_R, \rho_R E_R)) = (\gamma-1)h(\rho_L\,|\,\rho_R) - \rho_L \ln(e_L\,|\,e_R) + \frac{\rho_L}{2e_R}(u_L - u_R)^2,$$

where $h(x) = x\ln x$. Observe first that $h(x)$ and $-\ln x$ are both strictly convex functions. Since $|\rho_R - \rho_L|$ and $|e_R - e_L|$ are increasing along the shock curve, the first two terms in the formula above are also increasing. To deal with the last term, we recall (23) which also holds for the full Euler system and which we rewrite as follows.

$$(u_L - u_R)^2 = \frac{P_R}{\rho_R}\left[1 - \frac{P_L}{P_R}\right]\left[\frac{\rho_R}{\rho_L} - 1\right].$$

Hence we have

$$\frac{\rho_L}{2e_R}(u_L-u_R)^2 = \frac{(\gamma-1)\rho_L\rho_R}{2P_R}(u_L-u_R)^2 = \frac{1}{2}(\gamma-1)\rho_L\left[1-\frac{P_L}{P_R}\right]\left[\frac{\rho_R}{\rho_L}-1\right].$$

Finally, we note that the bracketed terms are both positive and increasing along the 1-shock curves, hence the product is increasing. This establishes hypothesis (H1b), and the theorem follows.

6.3 Strictly Hyperbolic system with small amplitude

The proof of Theorem 1.2 follows the same outline as the proof of Theorem 2.1. We simply need to check that our structural hypotheses are verified locally. In particular, we will show that both the Liu and Lax admissibility conditions apply. Again, we restrict our attention to 1-shocks and 1-contact discontinuities. The argument for n-shocks and n-contact discontinuites is similar.

By assumption, the state space $\mathcal{V}$ is open and $\lambda^-(V)$ is a simple eigenvalue for all $V\in\mathcal{V}$. Therefore, in a neighborhood of each state $V\in\mathcal{V}$, there exists a 1-shock curve $S_V(s)$ verifying

$$\sigma_V'(0) = \frac{1}{2}\frac{d}{ds}\bigg|_{s=0}\lambda_1(S_V(s)). \tag{27}$$

If the 1-characteristic family is genuinely nonlinear, the parameterization of $S_V(s)$ is chosen so that the right-hand side of (27) is negative. For linearly degenerate fields, the right-hand side vanishes and $\sigma_V'(s)$ is identically zero for all s along the shock curve $S_V(s)$. In either case, a straightforward continuity argument shows that for each $V_0\in\mathcal{V}$ there exists $K>0$ (sufficiently small) so that for all $V\in B_K(V_0)$, the 1-shock curve $S_V(s)$ exists in a neighborhood of V which contains $B_K(V_0)$ and such that $\sigma_V'(s)\le 0$ for $S_V(s)\in B_K(V_0)$. Therefore, any $U_L\in B_K(V_0)$ verifies a local (with respect to the parameter s) version of (H1a). To be more precise, all 1-shocks appearing in the solution U (which has values in $B_K(V_0)$) verify the Liu condition.

Now, with respect to property (H1b), an easy calculation shows that

$$\frac{d}{ds}\,\eta(U\,|\,S_U(s)) = S_U'(s)^T\cdot D^2\eta(S_U(s))\cdot[S_U(s)-U].$$

Since $D^2\eta$ is positive definite, the previous quantity is nonnegative for $s\ge 0$ sufficiently small (since $S_U(s)-U$ and $S_U'(s)^T$ point in nearly the same direction). Hence, (H1b) holds (locally in s) for all $U_L\in B_K(V_0)$ provided K is small enough.

Finally, K can be chosen so that all shocks (U_-,U_+) appearing in the solution U, excluding the family of 1-shocks, verify hypotheses (H2) and (H3), Indeed,

$$\sigma(U_-,U_+) = \lambda(U_\pm) + \mathcal{O}(|U_+-U_-|),$$

where $\lambda(U_\pm)$ is some intermediate eigenvalue of $\nabla A(U_\pm)$. Since ∇A is continuous and $\lambda_1=\lambda^-$ is a simple eigenvalue, there exists $\delta>0$ such that $\lambda^-(V)<\lambda(V)-\delta$ for all $V\in B_K(V_0)$. Then, for K small enough, it follows that $\sigma(U_-,U_+)\ge\lambda^-(U_\pm)$ for all (intermediate) Rankine-Hugoniot discontinuities with endstates $U_-,U_+\in B_K(V_0)$. This completes the proof.

A Proof of Lemmas 1 and 6

We first give the proof of Lemma 1.

Proof. On $\mathcal{V}^2$, we have

$$\eta(U|V) = \int_0^1\int_0^1 (U-V)^T\cdot D^2\eta(V+st(U-V))\cdot(U-V)\,t\,ds\,dt.$$

Consider a convex, compact set $\tilde{\Omega}$ such that $\Omega \subset \tilde{\Omega} \subset \mathcal{V}$ and $\overline{\tilde{\Omega}^c} \cap \Omega = \emptyset$, where $\tilde{\Omega}^c = \mathcal{V} \setminus \tilde{\Omega}$. Also, let $0 < \Lambda^-_{\tilde{\Omega}} \leq \Lambda^+_{\tilde{\Omega}} < \infty$ denote, respectively, the smallest and largest eigenvalues of $D^2\eta$ on $\tilde{\Omega}$. Then, for any $U \in \tilde{\Omega}$ and $V \in \Omega$, we have

$$\frac{\Lambda^-_{\tilde{\Omega}}}{2} |U - V|^2 \leq \eta(U \,|\, V) \leq \frac{\Lambda^+_{\tilde{\Omega}}}{2} |U - V|^2.$$

On the other hand, there exists a constant $C > 0$ such that

$$\inf_{U \in \tilde{\Omega}^c, V \in \Omega} \eta(U \,|\, V) \geq \frac{1}{C}, \qquad \sup_{U \in \tilde{\Omega}^c, V \in \Omega} \eta(U \,|\, V) \leq C.$$

Indeed, $\eta(\cdot \,|\, \cdot)$ is continuous on the compact set $\overline{\tilde{\Omega}^c} \times \Omega$. So it attains its maximum and minimum. The maximum is bounded. The mininum is attained at some point (U_0, V_0) with $U_0 \neq V_0$, and so $\eta(U_0 \,|\, V_0) \neq 0$. Hence the results holds true with

$$C_1 = \min \left\{ \frac{\Lambda^-_{\tilde{\Omega}}}{2}, \ \frac{1}{C \displaystyle\sup_{U \in \tilde{\Omega}^c, V \in \Omega} |U - V|^2} \right\}, \qquad C_2 = \max \left\{ \frac{\Lambda^+_{\tilde{\Omega}}}{2}, \ \frac{C}{\displaystyle\inf_{U \in \tilde{\Omega}^c, V \in \Omega} |U - V|^2} \right\}.$$

□

We now give the proof of Lemma 6.

Proof. Since A, η, and G are continuous on $\mathcal{U}$, the functions of (t, x) $A(U)$, $\eta(U \,|\, U_L)$, $\eta(U \,|\, U_R)$, $F(U, U_L)$, and $F(U, U_R)$ also verify the strong trace property of Definition 1.

Consider the family of mollifyer functions

$$\phi_\varepsilon(y) = \frac{1}{\varepsilon} \phi_1(\frac{y}{\varepsilon}),$$

where $\phi_1 \in C^\infty(\mathbb{R})$ is nonnegative, compactly supported in $(-1, 0)$, and such that $\int \phi_1(y)\, dy = 1$. We define

$$\Phi_\varepsilon(x) = \int_x^\infty \phi_\varepsilon(y)\, dy.$$

Note that Φ_ε is bounded by 1. It is equal to 1 for $x < -\varepsilon$ and equal to 0 for $x > 0$. Especially, it converges to $1_{\{x<0\}}$ when ε goes to 0.

We consider also a test function in time only $\psi \in C^\infty(\mathbb{R}^+)$, $\psi \geq 0$, compactly supported in $(0, T) \subset (0, \infty)$.

For any function V verifying the strong trace property of Definition 1, we have

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}} \psi(t) \phi_\varepsilon(y - x(t)) V(t, y)\, dy\, dt \\ &= \int_0^\infty \psi(t) V(t, x(t)-)\, dt + \int_0^\infty \int_{\mathbb{R}} \psi(t) \phi_\varepsilon(y - x(t)) [V(t, y) - V(t, x(t)-)]\, dy\, dt. \end{aligned}$$

The last term is smaller than

$$\|\psi\|_{L^\infty}\int_0^T\int_0^1 \phi_1(y)|V(t,x(t)-\varepsilon y)-V(t,x(t)-)|\,dy\,dt$$
$$\leq \|\psi\|_{L^\infty}\|\phi_1\|_{L^\infty}\sup_{y\in(0,\varepsilon)}\int_0^T |V(t,x(t)-y)-V(t,x(t)-)|\,dt.$$

This converges to 0 for ε going to 0, thanks to the strong trace property. Hence

$$\lim_{\varepsilon\to 0}\int_0^\infty\int_{\mathbb{R}}\psi(t)\phi_\varepsilon(y-x(t))V(t,y)\,dy\,dt=\int_0^\infty \psi(t)V(t,x(t)-)\,dt. \tag{28}$$

In the same way, we show that

$$\lim_{\varepsilon\to 0}\int_0^\infty\int_{\mathbb{R}}\psi(t)\phi_\varepsilon(x(t)-y)V(t,y)\,dy\,dt=\int_0^\infty \psi(t)V(t,x(t)+)\,dt. \tag{29}$$

And so

$$\lim_{\varepsilon\to 0}\int_0^\infty\int_{\mathbb{R}}\psi(t)(\phi_\varepsilon(y-x(t))-\phi_\varepsilon(x(t)-y))V(t,y)\,dy\,dt=\int_0^\infty \psi(t)[V(t,x(t)-)-V(t,x(t)+)]\,dt. \tag{30}$$

We consider the test function in time and space

$$\psi(t)\Phi_\varepsilon(x-x(t)).$$

For the equation of Lemma 2 with $V\in\mathcal{V}$, we get:

$$-\int_0^\infty\int_{\mathbb{R}}\psi(t)\Phi'_\varepsilon(x-x(t))[x'(t)\eta(U(t,x)\,|\,V)-F(U(t,x),V)]\,dx\,dt$$
$$\geq -\int_0^\infty\int_{\mathbb{R}}\psi'(t)\Phi_\varepsilon(x-x(t))\eta(U(t,x)\,|\,V)\,dx\,dt.$$

Passing to the limit and using (28), we find

$$-\int_0^\infty\psi'(t)\int_{-\infty}^0\eta(U(t,x(t)+x)\,|\,V)\,dx\,dt\leq\int_0^\infty\psi(t)[x'(t)\eta(U(t,x(t)-)\,|\,V)-F(U(t,x(t)-),V)]\,dt.$$

This is the desired result in the sense of distribution.

We consider now the test function in time and space

$$\psi(t)\Phi_\varepsilon(x(t)-x).$$

For the equation of Lemma 2 with $V\in\mathcal{V}$, we get in the same way, using (29),

$$-\int_0^\infty\psi'(t)\int_0^\infty\eta(U(t,x(t)+x)\,|\,V)\,dx\,dt\leq\int_0^\infty\psi(t)[-x'(t)\eta(U(t,x(t)+)\,|\,V)+F(U(t,x(t)+),V)]\,dt.$$

We take, now, as a test function

$$\psi(t)[\Phi_\varepsilon(x-x(t))+\Phi_\varepsilon(x(t)-x)-1].$$

Note that this function converges to 0 in L^1 when ε converges to 0. For equation (7), this gives

$$\int_0^\infty \int_{\mathbb{R}} \psi(t)[\Phi'_\varepsilon(x - x(t)) - \Phi'_\varepsilon(x(t) - x)][x'(t)U(t,x) - A(U(t,x))]\,dx\,dt$$
$$= \int_0^\infty \int_{\mathbb{R}} \psi'(t)[\Phi_\varepsilon(x - x(t)) + \Phi_\varepsilon(x(t) - x) - 1]U(t,x)\,dx\,dt.$$

The right hand side term converges to 0. Thanks to (30), the left hand side term converges to

$$\int_0^\infty \psi(t)[x'(t)(U(t,x(t)+) - U(t,x(t)-)) - (A(U(t,x(t)+)) - A(U(t,x(t)-)))]\,dt = 0.$$

This provides the first equality in the sense of distribution. The second one can be proven the same way from (10). □

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