

# UNIQUENESS AND STABILITY OF ENTROPY SHOCKS TO THE ISENTROPIC EULER SYSTEM IN A CLASS OF INVISCID LIMITS FROM A LARGE FAMILY OF NAVIER-STOKES SYSTEMS

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**ABSTRACT.** We prove the uniqueness and stability of entropy shocks to the isentropic Euler systems among all vanishing viscosity limits of solutions to associated Navier-Stokes systems. To take into account the vanishing viscosity limit, we show a contraction property for any large perturbations of viscous shocks to the Navier-Stokes system. The contraction estimate does not depend on the strength of the viscosity. This provides a good control on the inviscid limit process. We prove that, for any initial value, there exist a vanishing viscosity limit to solutions of the Navier-Stokes system. The convergence holds in a weak topology. However, this limit satisfies some stability estimates measured by the relative entropy with respect to an entropy shock. In particular, our result provides the uniqueness of entropy shocks to the shallow water equation in a class of inviscid limits of solutions to the viscous shallow water equations.

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## 1. INTRODUCTION

We consider the vanishing viscosity limit ( $\nu \rightarrow 0$ ) of the one-dimensional barotropic Navier-Stokes system in the Lagrangian coordinates:

$$(1.1) \quad \begin{cases} v_t^\nu - u_x^\nu = 0, \\ u_t^\nu + p(v^\nu)_x = \nu \left( \frac{\mu(v^\nu)}{v^\nu} u_x^\nu \right)_x, \end{cases}$$

where  $v$  denotes the specific volume,  $u$  is the fluid velocity, and  $p(v)$  is the pressure law. We consider the case of a polytropic perfect gas where the pressure is given by

$$(1.2) \quad p(v) = v^{-\gamma}, \quad \gamma > 1,$$

with  $\gamma$  the adiabatic constant. Here,  $\mu$  denotes the viscosity coefficient given by

$$(1.3) \quad \mu(v) = bv^{-\alpha}.$$

Notice that if  $\alpha > 0$ ,  $\mu(v)$  degenerates near the vacuum, i.e., near  $v = +\infty$ . Very often, the viscosity coefficient is assumed to be constant, i.e.,  $\alpha = 0$ . However, in the physical context the viscosity of a gas depends on the temperature (see Chapman and Cowling [6]). In the barotropic case, the temperature depends directly on the density ( $\rho = 1/v$ ). The viscosity is expected to degenerate near the vacuum as a power of the density, which is translated into  $\mu(v) = bv^{-\alpha}$  in terms of  $v$  with  $\alpha > 0$ .

At least formally, as  $\nu \rightarrow 0$ , the limit system of (1.1) is given by the isentropic Euler system:

$$(1.4) \quad \begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = 0. \end{cases}$$

The idea of approximating inviscid gases by viscous gases with vanishing viscosity is due to the seminal paper by Stokes [44]. The vanishing viscosity limit has been used later to construct entropy solutions to the isentropic Euler system, see DiPerna [20, 21], Hoff-Liu [29], Goodman-Xin [24], Yu [50]. Recently, Chen and Perepelitsa [8] proved the convergence of solutions to the Navier-Stokes system with  $\alpha = 0$  towards an entropy solution of the isentropic Euler system with finite energy initial data. Some results exist for the inviscid limit of the Navier-Stokes-Fourier system in very special cases, see for instance Feireisl [22] and [48], and the references cited therein.

In this article, we prove the existence of vanishing viscosity limits of solutions to (1.1) in some weak sense, and obtain a stability estimate of those limits. We then present the class of inviscid limits, in which the entropy shocks to (1.4) are unique.

Our result provides an answer, in the case of a shock, to the conjecture: The compressible Euler system admits a unique entropy solution in the class of vanishing viscosity solutions to the associated compressible Navier-Stokes system. As a breakthrough result related to this conjecture, Bianchini-Bressan [2] constructed a globally-in-time unique entropy solution to a strictly hyperbolic  $n \times n$  system with small BV initial datum, which is obtained from

vanishing “artificial” viscosity limit of the associate parabolic system. However, to the best of our knowledge, there is no result on uniqueness of discontinuous entropy solutions in the class of vanishing physical viscosity solutions to the Navier-Stokes systems.

Previous uniqueness results for special discontinuous solutions (as solutions to the Riemann problem) required suitable regularity like locally  $BV$  or strong trace properties (see Chen-Frid-Li [7] and Vasseur et al. [33, 37, 47]). Unfortunately, the global-in-time propagation of those regularities is unknown in general (except for the system with  $\gamma = 3$  see [45]). In the multi-D case, De Lellis-Székelyhidi[18] and Chiodaroli et al. [9, 10, 11, 12] showed non-uniqueness of entropy solutions. They showed that entropy solutions to the isentropic Euler systems in more than one space dimension are not unique, by constructing infinitely many entropy solutions based on the convex integration method [17, 18].

It is well known that the system (1.1) admits viscous shock waves connecting two end states  $(v_-, u_-)$  and  $(v_+, u_+)$ , provided the two end states satisfy the Rankine-Hugoniot condition and the Lax entropy condition (see Matsumura and Wang [38]):

$$(1.5) \quad \begin{aligned} \exists \sigma \quad \text{s.t.} \quad & \begin{cases} -\sigma(v_+ - v_-) - (u_+ - u_-) = 0, \\ -\sigma(u_+ - u_-) + p(v_+) - p(v_-) = 0, \end{cases} \\ & \text{and either } v_- > v_+ \text{ and } u_- > u_+ \text{ or } v_- < v_+ \text{ and } u_- > u_+ \text{ holds.} \end{aligned}$$

In other words, for given constant states  $(v_-, u_-)$  and  $(v_+, u_+)$  satisfying (1.5), there exists a viscous shock wave  $(\tilde{v}^\nu, \tilde{u}^\nu)(x - \sigma t)$  as a solution of

$$(1.6) \quad \begin{cases} -\sigma(\tilde{v}^\nu)' - (\tilde{u}^\nu)' = 0, \\ -\sigma(\tilde{u}^\nu)' + p(\tilde{v}^\nu)' = \nu \left( \frac{\mu(\tilde{v}^\nu)}{\tilde{v}^\nu} (\tilde{u}^\nu)' \right)' \\ \lim_{\xi \rightarrow \pm\infty} (\tilde{v}^\nu, \tilde{u}^\nu)(\xi) = (v_\pm, u_\pm). \end{cases}$$

Here, if  $v_- > v_+$ ,  $(\tilde{v}, \tilde{u})(x - \sigma t)$  is a 1-shock wave with velocity  $\sigma = -\sqrt{-\frac{p(v_+) - p(v_-)}{v_+ - v_-}}$ , whereas if  $v_- < v_+$ , that is a 2-shock wave with  $\sigma = \sqrt{-\frac{p(v_+) - p(v_-)}{v_+ - v_-}}$ .

Let  $(\bar{v}, \bar{u})$  be an associated entropy (inviscid) shock wave connecting the two end states  $(v_-, u_-)$  and  $(v_+, u_+)$  satisfying (1.5) as follows:

$$(1.7) \quad (\bar{v}, \bar{u})(x - \sigma t) = \begin{cases} (v_-, u_-) & \text{if } x - \sigma t < 0, \\ (v_+, u_+) & \text{if } x - \sigma t > 0. \end{cases}$$

As mentioned above, our goal is to show the uniqueness of the entropy shock to (1.4) in a suitable class, based on a generalization of our recent result [32] on the contraction property of viscous shocks to (1.1). More precisely, we prove the contraction of any large perturbations of viscous shocks to (1.1) in the case of  $0 < \alpha \leq \gamma \leq \alpha + 1$  and  $\gamma > 1$ , which improves the special case  $\gamma = \alpha$  in [32]. The contraction holds up to a shift, and is measured by a weighted relative entropy. Notice that since the relative entropy is locally quadratic, the contraction measured by the relative entropy can be regarded as  $L^2$ -type contraction. To prove the contraction, we employ the new approach introduced by the authors [32], which basically uses the relative entropy method. The relative entropy method has been extensively used in studying the contraction (or stability) of viscous (or inviscid) shock waves (see [13, 14, 30, 31, 32, 33, 34, 35, 36, 40, 41, 42, 47, 48]).

**1.1. Main results.** To handle the stability and uniqueness of the entropy shocks, we use the relative entropy associated to the entropy of (1.4) as follows: For any functions  $v_1, u_1, v_2, u_2$ ,

$$(1.8) \quad \eta((v_1, u_1)|(v_2, u_2)) := \frac{|u_1 - u_2|^2}{2} + Q(v_1|v_2),$$

where  $Q(v_1|v_2)$  is the relative functional associated with the strictly convex function

$$Q(v) := \frac{v^{-\gamma+1}}{\gamma-1}.$$

For the global-in-time existence of solutions to (1.1), we introduce the function space:

$$\begin{aligned} \mathcal{X}_T := \{ & (v, u) \in \mathbb{R}^+ \times \mathbb{R} \mid v - \underline{v}, u - \underline{u} \in L^\infty(0, T; H^1(\mathbb{R})), \\ & u - \underline{u} \in L^2(0, T; H^2(\mathbb{R})), v^{-1} \in L^\infty((0, T) \times \mathbb{R}) \}, \end{aligned}$$

where  $\underline{v}$  and  $\underline{u}$  are smooth monotone functions such that

$$(1.9) \quad \underline{v}(x) = v_\pm \quad \text{and} \quad \underline{u}(x) = u_\pm \quad \text{for } \pm x \geq 1.$$

The first theorem is on stability and uniqueness of the entropy shocks to (1.4):

**Theorem 1.1.** *Let  $\gamma > 1$  and  $\alpha, b > 0$  be any constants satisfying  $\alpha \leq \gamma \leq \alpha + 1$ . For each  $\nu > 0$ , consider the system (1.1)-(1.3). For a given constant state  $(v_-, u_-) \in \mathbb{R}^+ \times \mathbb{R}$ , there exists a constant  $\varepsilon_0 > 0$  such that for any  $\varepsilon < \varepsilon_0$  and any  $(v_+, u_+) \in \mathbb{R}^+ \times \mathbb{R}$  satisfying (1.5) with  $|p(v_-) - p(v_+)| = \varepsilon$ , the following holds.*

*Let  $(\tilde{v}^\nu, \tilde{u}^\nu)$  be a viscous shock connecting the two end states  $(v_-, u_-)$  and  $(v_+, u_+)$  as a solution of (1.6).*

*Then for a given initial datum  $(v^0, u^0)$  of (1.4) satisfying*

$$(1.10) \quad \int_{\mathbb{R}} \frac{|u^0 - \bar{u}|^2}{2} + Q(v^0|\bar{v}) dx < \infty,$$

*the following is true.*

*(i) (Well-prepared initial data) There exists a sequence of smooth functions  $\{(v_0^\nu, u_0^\nu)\}_{\nu>0}$  such that*

$$(1.11) \quad \begin{aligned} & \lim_{\nu \rightarrow 0} v_0^\nu = v^0, \quad \lim_{\nu \rightarrow 0} u_0^\nu = u^0 \quad \text{a.e.}, \quad v_0^\nu > 0, \\ & \lim_{\nu \rightarrow 0} \int_{\mathbb{R}} \left( \frac{1}{2} \left( u_0^\nu + \nu \left( p(v_0^\nu)^{\frac{\alpha}{\gamma}} \right)_x - \tilde{u}^\nu - \nu \left( p(\tilde{v}^\nu)^{\frac{\alpha}{\gamma}} \right)_x \right)^2 + Q(v_0^\nu|\tilde{v}^\nu) \right) dx \\ & = \int_{-\infty}^{\infty} \left( \frac{|u^0 - \bar{u}|^2}{2} + Q(v^0|\bar{v}) \right) dx. \end{aligned}$$

*(ii) For a given  $T > 0$ , let  $\{(v^\nu, u^\nu)\}_{\nu>0}$  be a sequence of solutions in  $\mathcal{X}_T$  to (1.1) with the initial datum  $(v_0^\nu, u_0^\nu)$  as above. Then there exist functions  $v_\infty$  and  $u_\infty$  such that as  $\nu \rightarrow 0$  (up to a subsequence),*

$$(1.12) \quad v^\nu \rightharpoonup v_\infty, \quad u^\nu \rightharpoonup u_\infty \quad \text{in } \mathcal{M}((0, T) \times \mathbb{R}) \text{ (space of Radon measures)}.$$

*In addition, there exist shift  $X_\infty \in BV((0, T))$  and constant  $C > 0$  such that the limit functions  $v_\infty, u_\infty$  satisfy that for a.e.  $t \in (0, T)$ ,*

$$(1.13) \quad \int_{-\infty}^{\infty} \eta((v_\infty, u_\infty)(t, x)|(\bar{v}, \bar{u})(x - X_\infty(t))) dx \leq C \int_{-\infty}^{\infty} \eta((v^0, u^0)|(\bar{v}, \bar{u})) dx.$$

Moreover, the shift  $X_\infty$  satisfies

$$(1.14) \quad |X_\infty(t) - \sigma t| \leq \frac{C}{|v_- - v_+|} \left( \mathcal{E}_0 + (1+t)\sqrt{\mathcal{E}_0} \right),$$

where

$$\mathcal{E}_0 := \int_{-\infty}^{\infty} \eta((v^0, u^0)|(\bar{v}, \bar{u})) dx.$$

Therefore, entropy shocks (1.7) (with small amplitude) of the isentropic Euler system (1.4) are stable and unique in the class of weak inviscid limits of solutions to the Navier-Stokes system (1.1).

**Remark 1.1.** 1. By (1.13), the limits  $v_\infty, u_\infty$  satisfy  $v_\infty \in \bar{v} + L^\infty(0, T; L_{loc}^1(\mathbb{R}))$  and  $u_\infty \in \bar{u} + L^\infty(0, T; L^2(\mathbb{R}))$ .

2. Theorem 1.1 provides the stability and uniqueness of weak Euler shocks in the wide class of weak inviscid limits to solutions to the Navier-Stokes system.

3. In fact, the smallness of amplitude of shocks is not needed in the proof of Theorem 1.1. The constraint is due to Theorem 1.2.

4. It is worth emphasizing from the assumption on  $\alpha$  and  $\gamma$  that Theorem 1.1 also holds in the case of the shallow water equations (i.e.,  $\gamma = 2$ ) in a class of inviscid limits of solutions to the viscous shallow water equations (i.e.,  $\gamma = 2, \alpha = 1$ ). We refer to Gerbeau-Perthame [23] for a derivation of the viscous shallow water equations from the incompressible Navier-Stokes equations with free boundary.

**Remark 1.2.** For the global-in-time existence (and uniqueness) of any large solutions to (1.1) in  $\mathcal{X}_T$ , we refer to [39] in the case of  $\alpha < 1/2$  and  $\gamma > 1$ . More precisely, they proved that  $\rho = 1/v$  and  $u$  satisfy

$$\rho - \underline{\rho}, u - \underline{u} \in L^\infty(0, T; H^1(\mathbb{R})), \quad u - \underline{u} \in L^2(0, T; H^2(\mathbb{R})), \quad \rho^{-1} \in L^\infty((0, T) \times \mathbb{R}).$$

This implies that there exists a solution in  $\mathcal{X}_T$  to the system (1.1) with  $\alpha < 1/2$  and  $\gamma > 1$ , since the system (1.1) is equivalent to the one in the Eulerian coordinates for such strong solutions. The result of [39] was extended by Haspot [27] to the case of  $\alpha \in (1/2, 1]$ . Recently, Constantin-Drivas-Nguyen-Pasqualotto [15, Theorem 1.5] extended it to the case of  $\alpha > 1/2$  and  $\gamma \in [\alpha, \alpha + 1]$ , but they handled it on the periodic domain. It would be possible to extend their result to the case where solutions have different limits at the infinity on the whole space as a perturbation of a shock, by combining the two methods: one is the relative entropy method with BD entropy as in [39]; the other is the analysis of the evolution of the active potential as in [15]. We leave it as a future work.

The starting point of the proof of Theorem 1.1 is to derive a uniform-in- $\nu$  estimate for any large perturbations of viscous shocks to (1.1). It is equivalent to obtain the contraction property of any LARGE perturbations of viscous shocks to (1.1) with a fixed  $\nu = 1$ :

$$(1.15) \quad \begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = \left( \frac{\mu(v)}{v} u_x \right)_x. \end{cases}$$

As in [32], we first introduce the following relative functional  $E(\cdot|\cdot)$  to measure the contraction:

$$(1.16) \quad \begin{aligned} &\text{for any functions } v_1, u_1, v_2, u_2, \\ E((v_1, u_1)|(v_2, u_2)) &:= \frac{1}{2} \left( u_1 + \left( p(v_1)^{\frac{\alpha}{\gamma}} \right)_x - u_2 - \left( p(v_2)^{\frac{\alpha}{\gamma}} \right)_x \right)^2 + Q(v_1|v_2), \end{aligned}$$

where the constants  $\gamma, \alpha$  are in (1.2) and (1.3). The functional  $E$  is associated to the BD entropy (see Bresch-Desjardins [3, 4, 5]). Since  $Q(v_1|v_2)$  is positive definite, (1.16) is also positive definite, that is, for any functions  $(v_1, u_1)$  and  $(v_2, u_2)$  we have  $E((v_1, u_1)|(v_2, u_2)) \geq 0$ , and

$$E((v_1, u_1)|(v_2, u_2)) = 0 \text{ a.e.} \Leftrightarrow (v_1, u_1) = (v_2, u_2) \text{ a.e.}$$

The following result provides a contraction property measured by the relative functional (1.16).

**Theorem 1.2.** *For any  $\gamma > 1$  and  $\alpha, b > 0$  satisfying  $\alpha \leq \gamma \leq \alpha + 1$ , consider the system (1.15) with (1.2)-(1.3). For a given constant state  $(v_-, u_-) \in \mathbb{R}^+ \times \mathbb{R}$ , there exist constants  $\varepsilon_0, \delta_0 > 0$  such that the following is true.*

*For any  $\varepsilon < \varepsilon_0$ ,  $\delta_0^{-1}\varepsilon < \lambda < \delta_0$ , and any  $(v_+, u_+) \in \mathbb{R}^+ \times \mathbb{R}$  satisfying (1.5) with  $|p(v_-) - p(v_+)| = \varepsilon$ , there exists a smooth monotone function  $a : \mathbb{R} \rightarrow \mathbb{R}^+$  with  $\lim_{x \rightarrow \pm\infty} a(x) = 1 + a_\pm$  for some constants  $a_-, a_+$  with  $|a_+ - a_-| = \lambda$  such that the following holds.*

*Let  $\tilde{U} := (\tilde{v}, \tilde{u})$  be the viscous shock connecting  $(v_-, u_-)$  and  $(v_+, u_+)$  as a solution of (1.6) with  $\nu = 1$ . For a given  $T > 0$ , let  $U := (v, u)$  be a solution in  $\mathcal{X}_T$  to (1.15) with a initial datum  $U_0 := (v_0, u_0)$  satisfying  $\int_{-\infty}^{\infty} E(U_0|\tilde{U})dx < \infty$ . Then there exists a shift  $X \in W^{1,1}((0, T))$  such that*

$$\begin{aligned} & \int_{-\infty}^{\infty} a(x)E(U(t, x + X(t))|\tilde{U}(x))dx \\ & + \delta_0 \frac{\varepsilon}{\lambda} \int_0^T \int_{-\infty}^{\infty} |\sigma a'(x)|Q(v(t, x + X(t))|\tilde{v}(x)) dx dt \\ & + \delta_0 \int_0^T \int_{-\infty}^{\infty} a(x)v^{\gamma-\alpha}(t, x + X(t)) \left| \partial_x(p(v(t, x + X(t))) - p(\tilde{v}(x))) \right|^2 dx dt \\ & \leq \int_{-\infty}^{\infty} a(x)E(U_0(x)|\tilde{U}(x))dx, \end{aligned} \tag{1.17}$$

and

$$\begin{aligned} |\dot{X}(t)| & \leq \frac{1}{\varepsilon^2}(1 + f(t)), \\ & \text{for some positive function } f \text{ satisfying } \|f\|_{L^1(0, T)} \leq \frac{2\lambda}{\delta_0\varepsilon} \int_{-\infty}^{\infty} E(U_0|\tilde{U})dx. \end{aligned} \tag{1.18}$$

**Remark 1.3.** 1. Theorem 1.2 provides a contraction property for viscous shocks with suitably small amplitude parametrized by  $\varepsilon = |p(v_-) - p(v_+)|$ . This smallness together with (1.5) implies  $|v_- - v_+| = \mathcal{O}(\varepsilon)$  and  $|u_- - u_+| = \mathcal{O}(\varepsilon)$ . For such a fixed small shock, the contraction holds for any large solutions to (1.15), without any smallness condition imposed on  $U_0$ . This implies that the contraction still holds for any large solutions to (1.1), which provides a weak compactness to prove Theorem 1.1 as the inviscid limit problem ( $\nu \rightarrow 0$ ).  
2. In (1.17), the dissipation terms will be used to show the convergence of  $\{u^\nu\}_{\nu>0}$  in (1.12).

**Remark 1.4.** The contraction property is non-homogenous in  $x$ , as measured by the function  $x \rightarrow a(x)$ . This is consistant with the hyperbolic case (with  $\nu = 0$ ). In the hyperbolic case, it was shown in [40] that a homogenous contraction cannot hold for the full Euler system. However, the contraction property is true if we consider a non-homogenous pseudo-distance [47] providing the so-called  $a$ -contraction [33]. Our main result shows that the non-homogeneity of the pseudo-distance can be chosen of a similar size as the strength of the shock (as measured by the quantity  $\lambda$ ).

The rest of the paper is as follows. We explain main ideas of proofs of the main results in Section 2. In Section 3, we provide a transformation of the system (1.15), and an equivalent version of Theorem 1.2, and useful inequalities. Section 4 is dedicated to the proof of Theorem 3.1. Finally, Section 5 is dedicated to the proof of the main Theorem.

## 2. IDEAS OF THE PROOF.

We describe in this section the methodology and main ideas of our results.

**Uniform estimates with respect to the viscosity.** The main results of this paper boil down to the proof of stability of the viscous shocks to the Navier-Stokes equations UNIFORMLY with respect to the strength of the viscosity. This can be obtained by considering only the case of the viscosity  $\nu = 1$ , replacing the notion of stability by the notion of contraction, valid even for large perturbations (Theorem 1.2).

Indeed, if  $(v^\nu, u^\nu)$  is a solution of (1.1), then

$$v(t, x) = v^\nu(\nu t, \nu x), \quad u(t, x) = u^\nu(\nu t, \nu x)$$

is a solution to (1.15), i.e., the Navier-Stokes equations with  $\nu = 1$ . Note that, even if the initial perturbation  $(v_0^\nu - \tilde{v}^\nu, u_0^\nu - \tilde{u}^\nu)$  is small, let say of order  $\mathcal{E}$ , then the perturbation  $(v_0 - \tilde{v}, u_0 - \tilde{u})$  is big (of order  $\mathcal{E}/\nu$ ):

$$\int_{\mathbb{R}} E(U_0(\xi)) |\tilde{U}(\xi)| d\xi = \frac{1}{\nu} \int_{\mathbb{R}} E_\nu(U_0^\nu(\xi)) |\tilde{U}^\nu(\xi)| d\xi = \frac{\mathcal{E}}{\nu},$$

where  $E(\cdot)$  is defined in (1.16), and the rescaled  $\nu$ -dependent functional  $E_\nu(\cdot)$  is defined in (5.4).

However, a contraction independent of the size of the perturbation in the case  $\nu = 1$ , as in Theorem 1.2, provides, after rescaling, a similar contraction for any values  $\nu$ :

$$\int_{\mathbb{R}} a(\xi/\nu) E_\nu(U^\nu(t, \xi + \nu X(t/\nu)) |\tilde{U}^\nu(\xi)| d\xi \leq \int_{\mathbb{R}} a(\xi/\nu) E_\nu(U_0^\nu) |\tilde{U}^\nu| d\xi.$$

This gives a uniform stability result with respect to  $\nu$  provided that the weight function  $a$  is uniformly bounded from below and from above, that we have a control on the shift which is independent of the transformation

$$X(t) \longrightarrow \nu X\left(\frac{t}{\nu}\right),$$

and that we have a uniform bound of  $\int_{\mathbb{R}} E_\nu(U_0^\nu) |\tilde{U}_0| d\xi$ . The first two conditions are verified by Theorem 1.2 thanks to (1.18), and considering  $\lambda < 1$ , and the last one is verified thanks to the well-prepared initial data (1.11).

**The contraction when  $\nu = 1$ : Theorem 3.1.** This result is a generalization of the result in [32] where only the case  $\alpha = \gamma$  was considered. The extension introduces severe technical difficulties. A key to the extension is the local minimization explained below.

*Step one: Considering a new velocity variable.* We need to control the growth of the perturbation due to the hyperbolic terms (flux functionals). Thanks to the relative entropy method, the linear fluxes are easier to handle (the relative functional of linear quantity vanishes). Therefore, the main hyperbolic quantities to control are the pressure terms depending only on the specific volume  $v$ . At the core of the method, we are using a generalized Poincaré inequality Proposition A.1, first proved in [32]. The Navier-Stokes system can be

seen as a degenerate parabolic system. But the diffusion is in the other variable, the velocity variable  $u$ . Bresch and Desjardins (see [5, 3, 4]) showed that compressible Navier-Stokes systems have a natural perturbed velocity quantity associated to the viscosity:

$$h^\nu = u^\nu + \nu \left( p(v^\nu)^{\frac{\alpha}{\gamma}} \right)_x.$$

Remarkably, the system in the variables  $(v^\nu, h^\nu)$  exhibits a diffusion in the  $v$  variable (the Smoluchowski equation), rather than in the velocity variable. For this reason, we are working with the natural relative entropy of this system, which corresponds to the usual relative entropy of the associated p-system in the  $U_h^\nu = (v^\nu, h^\nu)$  variable:

$$\eta(U_h^\nu | \tilde{U}_h^\nu) = E_\nu(U^\nu | \tilde{U}^\nu).$$

For the rest of the proof of this theorem, we consider only  $\nu = 1$  and work only in the new set of variable  $(v, h)$ . To simplify the notation, we denote  $U = (v, h)$  from now on.

*Step 2: Evolution of the relative entropy.* Computing the evolution of the relative entropy in Lemma 4.2, we get

$$\begin{aligned} & \frac{d}{dt} \int_{-\infty}^{\infty} a(\xi) \eta(U(t, \xi + X(t)) | \tilde{U}_\varepsilon(\xi)) d\xi \\ &= \dot{X}(t) Y(U(t, \cdot + X(t))) + \mathcal{J}^{bad}(U(t, \cdot + X(t))) - \mathcal{J}^{good}(U(t, \cdot + X(t))). \end{aligned}$$

The functional  $\mathcal{J}^{good}(U)$  is non-negative (good term) and can be split into three terms (see (4.10), (4.18)):

$$\mathcal{J}^{good}(U) = \mathcal{J}_1^{good}(U) + \mathcal{G}_2(U) + \mathcal{D}(U),$$

where only  $\mathcal{J}_1^{good}(U)$  depends on  $h$  (and actually does not depend on  $v$ ). The term  $\mathcal{D}(U)$  corresponds to the diffusive term (which depends on  $v$  only, thanks to the transformation of the system).

*Step 3: Construction of the shift.* The shift  $X(t)$  produces the term  $\dot{X}(t)Y(U)$ . The key idea of the technique is to take advantage of this term when  $Y(U(t, \cdot))$  is not too small, by compensating all the other terms via the choice of the velocity of the shift (see (4.20)). Specifically, we algebraically ensure that the contraction holds as long as  $|Y(U(t))| \geq \varepsilon^2$ . The rest of the analysis is to ensure that when  $|Y(U(t))| \leq \varepsilon^2$ , the contraction still holds.

The condition  $|Y(U(t))| \leq \varepsilon^2$  ensures a smallness condition that we want to fully exploit. This is where the non-homogeneity of the semi-norm is crucial. In the case where the function  $a$  is constant,  $Y(U)$  is a linear functional in  $U$ . The smallness of  $Y(U)$  gives only that a certain weighted mean value of  $U$  is almost null. However, when  $a$  is decreasing,  $Y(U)$  becomes convex. The smallness  $Y(U(t)) \leq \varepsilon^2$  implies, for this fixed time  $t$  (See Lemma 4.4 with (4.14) and (4.1)):

$$(2.1) \quad \int_{\mathbb{R}} \varepsilon e^{-C\varepsilon|\xi|} Q(v(t, \xi + X(t)) | \tilde{v}_\varepsilon(\xi)) d\xi \leq C \left( \frac{\varepsilon}{\lambda} \right)^2.$$

This gives a control in  $L^2$  for moderate values of  $v$ , and in  $L^1$  for big values of  $v$ , in the layer region ( $|\xi - X(t)| \lesssim 1/\varepsilon$ ).

The problem now looks, at first glance, as a typical problem of stability with a smallness condition. There are, however, three major difficulties: The bad term  $\mathcal{J}^{bad}(U)$  has some terms depending on the variable  $h$  for which we do not have diffusion, we have some smallness in  $v$ , only for a very weak norm, and only localized in the layer region. More importantly, the smallness is measured with respect to the smallness of the shock. It basically



says that, considering only the moderate values of  $v$ : the perturbation is not bigger than  $\varepsilon/\lambda$  (which is still very big with respect to the size of the shock  $\varepsilon$ ). Actually, as we will see later, it is not possible to consider only the linearized problem: Third order terms appear in the expansion using the smallness condition (the energy method involving the linearization would have only second order term in  $\varepsilon$ ).

In the argument, for the values of  $t$  such that  $|Y(U(t))| \leq \varepsilon^2$ , we construct the shift as a solution to the ODE:  $\dot{X}(t) = -Y(U(t, \cdot + X(t)))/\varepsilon^4$ . From this point, we forget that  $U = U(t, \xi)$  is a solution to (3.3) and  $X(t)$  is the shift. That is, we leave out  $X(t)$  and the  $t$ -variable of  $U$ . Then we show that for any function  $U$  satisfying  $|Y(U)| \leq \varepsilon^2$ , we have

$$(2.2) \quad -\frac{1}{\varepsilon^4}Y^2(U) + \mathcal{J}^{bad}(U) - \mathcal{J}^{good}(U) \leq 0.$$

This is the main Proposition 4.1 (actually, the proposition is slightly stronger to ensure the control of the shift). This implies clearly the contraction. From now on, we are focusing on the proof of this statement.

*Step 4: Maximization in  $h$  for  $v$  fixed.* We need to get rid of the dependence on the  $h$  variable from the bad parts  $\mathcal{J}^{bad}(U)$ . The idea in [32] (for  $\gamma = \alpha$ ) is to maximize the bad term with respect to  $h$  for  $v$  fixed:

$$\mathcal{B}(v) = \sup_h \left( \mathcal{J}^{bad}(v, h) - \mathcal{J}_1^{good}(h) \right).$$

We then had an inequality depending only on  $v$  and  $\partial_x v$  (through  $\mathcal{D}(U)$ ) for which we can apply a generalized Poincaré inequality. This does not work anymore when  $\gamma \neq \alpha$ . This is because  $\mathcal{B}(v)$  involves powers of  $p(v)$  which cannot be controlled by the good terms due to big values of  $p(v)$ . The new idea is to maximize in  $h$  ONLY for the fixed values of  $v$  such that  $p(v) - p(\tilde{v}_\varepsilon) \leq \delta_3$  for a constant  $\delta_3$  to be determined (and depending on the Poincaré inequality). This leads to the decomposition (4.17), (4.18). The bad terms involving values  $p(v) - p(\tilde{v}_\varepsilon) > \delta_3$  can now be controlled using additional information from the unconditional estimate  $|Y(U)| \leq \varepsilon^2$  (see (4.35), (4.68)).

*Step 5: Expansion in  $\varepsilon$ .* Although we have no control on the supremum of  $|p(v) - p(\tilde{v}_\varepsilon)|$ , we can control independently the contribution of the values  $|p(v) - p(\tilde{v}_\varepsilon)| \geq \delta_3$  in Proposition 4.3 (for the same  $\delta_3$  related to the maximization process above. The coefficient  $\delta_3$  can be chosen very small, but independent of  $\varepsilon$  and of  $\varepsilon/\lambda$ ). The last step is to perform an expansion in the size of the shock  $\varepsilon$  every small, uniformly in  $v$  (but for a fixed small value of  $\delta$ ). As in [32], the expansion has to be done up to the third order. It leads to the exact same generic expression (A.5). The generalized nonlinear Poincaré inequality, Proposition A.1 concludes the proof.

### The inviscid limit: Theorem 1.1.

We have now a stability result uniform with respect to the viscosity. It is natural to expect a stability result on the corresponding inviscid limit. The result, however, is not immediate. Several difficulties have to be overcome. First, due to the BD representation as above, the stability result for  $\nu$  fixed is on the quantities:

$$U_h^\nu = (v^\nu, h^\nu), \quad h^\nu = u^\nu + \nu \left( p(v^\nu)^{\frac{\alpha}{\gamma}} \right)_x.$$

This is the reason we need a compatibility condition on the family of initial values  $U_0^\nu$ . This also leads to a very weak convergence (in measure in  $(t, x)$  only). The next difficulty is

that for small values of  $v$ , the relative entropy control only the  $L^1$  norm of  $Q(v) = 1/v^{\gamma-1}$ . Therefore the pressure  $p(v) = 1/v^\gamma$  cannot be controlled at all. Therefore, we do not control the time derivative of  $u$  in any distributional sense in  $x$ . Moreover, we have to study carefully the effect of small and big values of  $v$  together with big values of  $|u|$  through truncations (see (5.9) and (5.13)). This is particularly important to pass to the limit on the shift in the contraction inequality (note that the shift converges only in  $L^p_{\text{loc}}(\mathbb{R}^+)$ , for  $1 \leq p < \infty$ ). Finally, it has to be shown that the shift converges to  $\sigma t$  when the perturbation converges to 0. This can be obtained, thanks to the convergence of  $v$  in  $C^0(\mathbb{R}^+, W_{\text{loc}}^{-s,1}(\mathbb{R}))$ . It is interesting to note that the continuity (in time) of  $v$  is enough. We do not obtain any such control on  $u$  (nor  $h$ ).

### 3. PRELIMINARIES

**3.1. Transformation of the system (1.15).** We here provides an equivalent version of Theorem 1.2 as in [32]. First of all, since the strength of the coefficient  $b$  in  $\mu(v)$  does not affect our analysis, as in [32], we set  $b = \gamma$  (for simplification) and introduce a new effective velocity

$$h := u + \left( p(v)^{\frac{\alpha}{\gamma}} \right)_x.$$

The system (1.15) with  $\mu(v) = \gamma v^{-\alpha}$  is then transformed into

$$(3.1) \quad \begin{cases} v_t - h_x = - \left( v^\beta p(v)_x \right)_x \\ h_t + p(v)_x = 0, \end{cases}$$

where  $\beta := \gamma - \alpha$ . Notice that the above system has a parabolic regularization on the specific volume, contrary to the regularization on the velocity for the original system (1.15). This is better for our analysis, since the hyperbolic part of the system is linear in  $u$  (or  $h$ ) but nonlinear in  $v$  (via the pressure). This effective velocity was first introduced by Shelukhin [43] for  $\alpha = 0$ , and in the general case (in Eulerian coordinates) by Bresch-Desjardins [3, 4, 5], and Haspot [26, 25, 28]. It was also used in [49].

As mentioned in Theorem 1.2, we consider shock waves with suitably small amplitude  $\varepsilon$ . For that, let  $(\tilde{v}_\varepsilon, \tilde{u}_\varepsilon)(x - \sigma_\varepsilon t)$  denote a shock wave with amplitude  $|p(v_-) - p(v_+)| = \varepsilon$  as a solution of (1.6) with  $\mu(v) = \gamma v^{-\gamma}$ . Then, setting  $\tilde{h}_\varepsilon := \tilde{u}_\varepsilon + \left( p(\tilde{v}_\varepsilon)^{\frac{\alpha}{\gamma}} \right)_x$ , the shock wave  $(\tilde{v}_\varepsilon, \tilde{h}_\varepsilon)(x - \sigma_\varepsilon t)$  satisfies

$$(3.2) \quad \begin{cases} -\sigma_\varepsilon \tilde{v}'_\varepsilon - \tilde{h}'_\varepsilon = - \left( \tilde{v}_\varepsilon^\beta p(\tilde{v}_\varepsilon)' \right)' \\ -\sigma_\varepsilon \tilde{h}'_\varepsilon + p(\tilde{v}_\varepsilon)' = 0 \\ \lim_{\xi \rightarrow \pm\infty} (\tilde{v}_\varepsilon, \tilde{h}_\varepsilon)(\xi) = (v_\pm, u_\pm). \end{cases}$$

For simplification of our analysis, we rewrite (3.1) into the following system, based on the change of variable  $(t, x) \mapsto (t, \xi = x - \sigma_\varepsilon t)$ :

$$(3.3) \quad \begin{cases} v_t - \sigma_\varepsilon v_\xi - h_\xi = - \left( v^\beta p(v)_\xi \right)_\xi \\ h_t - \sigma_\varepsilon h_\xi + p(v)_\xi = 0 \\ v|_{t=0} = v_0, \quad h|_{t=0} = u_0. \end{cases}$$

For the global-in-time existence of solutions to (3.3), we consider the function space:

$$(3.4) \quad \mathcal{H}_T := \{(v, h) \in \mathbb{R}^+ \times \mathbb{R} \mid v - \underline{v} \in C(0, T; H^1(\mathbb{R})), \\ h - \underline{u} \in C(0, T; L^2(\mathbb{R})), v^{-1} \in L^\infty((0, T) \times \mathbb{R})\},$$

where  $\underline{v}$  and  $\underline{u}$  are as in (1.9).

Theorem 1.2 is a direct consequence of the following theorem on the contraction of shocks to the system (3.3).

**Theorem 3.1.** *Assume  $\gamma > 1$  and  $\alpha > 0$  satisfying  $\alpha \leq \gamma \leq \alpha + 1$ . For a given constant state  $(v_-, u_-) \in \mathbb{R}^+ \times \mathbb{R}$ , there exist constants  $\varepsilon_0, \delta_0 > 0$  such that the following holds.*

*For any  $\varepsilon < \varepsilon_0$ ,  $\delta_0^{-1}\varepsilon < \lambda < \delta_0$ , and any  $(v_+, u_+) \in \mathbb{R}^+ \times \mathbb{R}$  satisfying (1.5) with  $|p(v_-) - p(v_+)| = \varepsilon$ , there exists a smooth monotone function  $a : \mathbb{R} \rightarrow \mathbb{R}^+$  with  $\lim_{x \rightarrow \pm\infty} a(x) = 1 + a_\pm$  for some constants  $a_-, a_+$  with  $|a_- - a_+| = \lambda$  such that the following holds.*

*Let  $\tilde{U}_\varepsilon := (\tilde{v}_\varepsilon, \tilde{h}_\varepsilon)$  be a viscous shock connecting  $(v_-, u_-)$  and  $(v_+, u_+)$  as a solution of (3.2). For a given  $T > 0$ , let  $U := (v, h)$  be a solution in  $\mathcal{H}_T$  to (3.3) with a initial datum  $U_0 := (v_0, u_0)$  satisfying  $\int_{-\infty}^{\infty} \eta(U_0 | \tilde{U}_\varepsilon) dx < \infty$ , there exists a shift function  $X \in W^{1,1}((0, T))$  such that*

$$(3.5) \quad \begin{aligned} & \int_{\mathbb{R}} a(\xi) \eta(U(t, \xi + X(t)) | \tilde{U}_\varepsilon(\xi)) d\xi \\ & + \delta_0 \frac{\varepsilon}{\lambda} \int_0^T \int_{-\infty}^{\infty} \sigma a'(\xi) Q(v(t, \xi + X(t)) | \tilde{v}_\varepsilon(\xi)) d\xi dt \\ & + \delta_0 \int_0^T \int_{-\infty}^{\infty} a(\xi) v^{\gamma-\alpha}(t, \xi + X(t)) \left| \partial_x (p(v(t, \xi + X(t))) - p(\tilde{v}_\varepsilon(\xi))) \right|^2 d\xi dt \\ & \leq \int_{\mathbb{R}} a(\xi) \eta(U_0 | \tilde{U}_\varepsilon) d\xi, \end{aligned}$$

and

$$(3.6) \quad |\dot{X}(t)| \leq \frac{1}{\varepsilon^2} (1 + f(t)),$$

for some positive function  $f$  satisfying  $\|f\|_{L^1(0, T)} \leq \frac{2\lambda}{\delta_0 \varepsilon} \int_{-\infty}^{\infty} \eta(U_0 | \tilde{U}_\varepsilon) d\xi$ .

**Remark 3.1.** 1. In [32], the authors proved Theorem 1.2 in the case of  $\alpha = \gamma$ . Therefore, it suffices to prove the remaining cases where  $0 < \alpha < \gamma \leq \alpha + 1$ . That is,  $\beta = \gamma - \alpha \in (0, 1]$ .

2. Notice that it is enough to prove Theorem 3.1 for 1-shocks. Indeed, the result for 2-shocks is obtained by the change of variables  $x \rightarrow -x$ ,  $u \rightarrow -u$ ,  $\sigma_\varepsilon \rightarrow -\sigma_\varepsilon$ .

Therefore, from now on, we only consider a 1-shock  $(\tilde{v}_\varepsilon, \tilde{h}_\varepsilon)$ , i.e.,  $v_- > v_+$ ,  $u_- > u_+$ , and

$$(3.7) \quad \sigma_\varepsilon = -\sqrt{\frac{p(v_+) - p(v_-)}{v_+ - v_-}}.$$

**Remark 3.2.** As mentioned in Remark 1.2, we consider the solution  $(v, u) \in \mathcal{X}_T$  to (1.15). Then, (3.3) admits the solution  $(v, h)$  in  $\mathcal{H}_T$ . Indeed, since  $v_t = u_x \in L^2(0, T; H^1(\mathbb{R}))$  by (1.15)<sub>1</sub>, we have  $v - \underline{v} \in C(0, T; H^1(\mathbb{R}))$ . To show  $h - \underline{u} \in C(0, T; L^2(\mathbb{R}))$ , we first find that for  $(v, u) \in \mathcal{X}_T$ ,

$$h - \underline{u} = u - \underline{u} + \frac{\alpha}{\gamma} p(v)^{\frac{\alpha}{\gamma}-1} v_x \in L^\infty(0, T; L^2(\mathbb{R})).$$

Moreover, together with the fact that  $v \in L^\infty((0, T) \times \mathbb{R})$  by Sobolev embedding, we find that

$$\begin{aligned} u_t &= -p'(v)v_x + \frac{d}{dv}\left(\frac{\mu(v)}{v}\right)v_x u_x + \frac{\mu(v)}{v}u_{xx} \in L^2(0, T; L^2(\mathbb{R})), \\ \left(p(v)^{\frac{\alpha}{\gamma}-1}v_x\right)_t &= \left(\frac{\alpha}{\gamma} - 1\right)p(v)^{\frac{\alpha}{\gamma}-2}v_t v_x + p(v)^{\frac{\alpha}{\gamma}-1}v_{xt} \in L^2(0, T; L^2(\mathbb{R})), \end{aligned}$$

which implies  $h_t \in L^2(0, T; L^2(\mathbb{R}))$ , and therefore  $h - \underline{u} \in C(0, T; L^2(\mathbb{R}))$ .

**3.2. Global and local estimates on the relative quantities.** We here present useful inequalities on  $Q$  and  $p$  that are crucially used for the proofs of main results. First, the following lemma provides some global inequalities on the relative function  $Q(\cdot)$  corresponding to the convex function  $Q(v) = \frac{v^{-\gamma+1}}{\gamma-1}$ ,  $v > 0$ ,  $\gamma > 1$ .

**Lemma 3.1.** *For given constants  $\gamma > 1$ , and  $v_- > 0$ , there exist constants  $c_1, c_2 > 0$  such that the following inequalities hold.*

1) For any  $w \in (0, v_-)$ ,

$$(3.8) \quad \begin{aligned} Q(v|w) &\geq c_1|v - w|^2, \quad \text{for all } 0 < v \leq 3v_-, \\ Q(v|w) &\geq c_2|v - w|, \quad \text{for all } v \geq 3v_-. \end{aligned}$$

2) Moreover if  $0 < w \leq u \leq v$  or  $0 < v \leq u \leq w$  then

$$(3.9) \quad Q(v|w) \geq Q(u|w),$$

and for any  $\delta_* > 0$  there exists a constant  $C > 0$  such that if, in addition,  $v_- > w > v_- - \delta_*/2$  and  $|w - u| > \delta_*$ , we have

$$(3.10) \quad Q(v|w) - Q(u|w) \geq C|u - v|.$$

3) For any  $w \in (v_-/4, v_-)$ ,

$$(3.11) \quad |p(v) - p(w)| \leq c_5|v - w|, \quad \text{for all } v \geq v_-/2,$$

*Proof.* We refer to [32, Lemma 2.4, 2.5]. □

Next, we use (3.8) and (3.9) above to prove the following lemma, which is used for the proof of Theorem 1.1.

**Lemma 3.2.** *For given constants  $\gamma, M > 1$ , there exist constants  $C > 0$  and  $k_0 > 1$  such that the following inequalities hold.*

1) For any  $k \geq 3M$  and  $w \in (M^{-1}, M)$ ,

$$(3.12) \quad \max\{(k^{-1} - v)_+, (v - k)_+\} \leq CQ(v|w), \quad \text{for any } v > 0.$$

2) For any  $w_1, w_2 \in (M^{-1}, M)$ ,

$$(3.13) \quad Q(v|w_1) \leq CQ(v|w_2), \quad \text{for any } v \in (0, k_0^{-1}) \cup (k_0, \infty).$$

*Proof.* • *proof of (3.12) :* i) If  $k^{-1} \leq v \leq k$ , then  $\max\{(k^{-1} - v)_+, (v - k)_+\} = 0 \leq CQ(v|w)$ .  
ii) If  $0 < v < k^{-1}$ , we have

$$(3.14) \quad \max\{(k^{-1} - v)_+, (v - k)_+\} = k^{-1} - v < k^{-1}.$$

Since  $v < k^{-1} \leq M^{-1}/3$ , we use (3.8) and (3.9) to have

$$Q(v|w) \geq Q(M^{-1}/3|w) \geq c_1|M^{-1}/3 - w|^2.$$

Moreover, since  $w > M^{-1}$ , we have

$$Q(v|w) \geq c_1 4M^{-2}/9 \geq (c_1 4M^{-1}/3)k^{-1},$$

which together with (3.14) implies the desired inequality.

iii) If  $v > k$ , we have

$$\max\{(k^{-1} - v)_+, (v - k)_+\} = v - k \leq v - 3M.$$

Likewise, using (3.8), we have

$$Q(v|w) \geq c_2|v - w|.$$

Since  $v > 3M > w$ , we have

$$Q(v|w) \geq c_2(v - 3M),$$

which completes the desired inequality.

• *proof of (3.13)* : We set  $C := 2 \max\{\frac{Q'(M)}{Q'(M^{-1})}, 1\}$ . Since  $Q' < 0$ , there exists  $k_1 > 1$  such that for all  $v > k_1$ ,

$$-Q'(M)v \geq (1 - C)Q(v) + (Q'(M)M - Q(M)) - C(Q'(M^{-1})M^{-1} - Q(M^{-1})).$$

Moreover, since  $Q'$  is increasing and  $\frac{d}{dv}(Q'(v)v - Q(v)) > 0$ , we have

$$\begin{aligned} (-CQ'(w_2) + Q'(w_1))v &\geq (-CQ'(M^{-1}) + Q'(M))v \geq -Q'(M)v \\ &\geq (1 - C)Q(v) + (Q'(M)M - Q(M)) - C(Q'(M^{-1})M^{-1} - Q(M^{-1})) \\ &\geq (1 - C)Q(v) + (Q'(w_1)w_1 - Q(w_1)) - C(Q'(w_2)w_2 - Q(w_2)), \end{aligned}$$

which together with the definition of  $Q(\cdot|\cdot)$  yields that

$$Q(v|w_1) \leq CQ(v|w_2), \quad \text{for all } v > k_1.$$

On the other hand, since  $Q(v) \rightarrow +\infty$  as  $v \rightarrow 0+$ , there exists  $k_0 > k_1$  such that for all  $v < k_0^{-1}$ ,

$$Q(v) \geq (CQ'(M^{-1}) - Q'(M))v + (Q'(M)M - Q(M)) - C(Q'(M^{-1})M^{-1} - Q(M^{-1})).$$

Then we have

$$\begin{aligned} (C - 1)Q(v) &\geq Q(v) \\ &\geq (CQ'(M^{-1}) - Q'(M))v + (Q'(M)M - Q(M)) - C(Q'(M^{-1})M^{-1} - Q(M^{-1})) \\ &\geq (CQ'(w_2) - Q'(w_1))v + (Q'(w_1)w_1 - Q(w_1)) - C(Q'(w_2)w_2 - Q(w_2)), \end{aligned}$$

which yields that  $Q(v|w_1) \leq CQ(v|w_2)$  for all  $v < k_0^{-1}$ .  $\square$

We present now some local estimates on  $p(v|w)$  and  $Q(v|w)$ .

**Lemma 3.3.** *For given constants  $\gamma > 1$  and  $v_- > 0$  there exist positive constants  $C$  and  $\delta_*$  such that for any  $0 < \delta < \delta_*$ , the following is true.*

1) *For any  $(v, w) \in \mathbb{R}_+^2$  satisfying  $|p(v) - p(w)| < \delta$  and  $|p(w) - p(v_-)| < \delta$ ,*

$$(3.15) \quad p(v|w) \leq \left( \frac{\gamma + 1}{2\gamma} \frac{1}{p(w)} + C\delta \right) |p(v) - p(w)|^2.$$

2) *For any  $(v, w) \in \mathbb{R}_+^2$  such that  $|p(w) - p(v_-)| \leq \delta$ , and satisfying either  $Q(v|w) < \delta$  or  $|p(v) - p(w)| < \delta$ ,*

$$(3.16) \quad |p(v) - p(w)|^2 \leq CQ(v|w).$$

*Proof.* We refer to [32, Lemma 2.6].  $\square$

## 4. PROOF OF THEOREM 3.1

Throughout this section,  $C$  denotes a positive constant which may change from line to line, but which stays independent on  $\varepsilon$  (the shock strength) and  $\lambda$  (the total variation of the function  $a$ ). We will consider two smallness conditions, one on  $\varepsilon$ , and the other on  $\varepsilon/\lambda$ . In the argument,  $\varepsilon$  will be far smaller than  $\varepsilon/\lambda$ .

**4.1. Properties of small shock waves.** In this subsection, we present useful properties of the 1-shock waves  $(\tilde{v}_\varepsilon, \tilde{h}_\varepsilon)$  with small amplitude  $\varepsilon$ . In the sequel, without loss of generality, we consider the 1-shock wave  $(\tilde{v}_\varepsilon, \tilde{h}_\varepsilon)$  satisfying  $\tilde{v}_\varepsilon(0) = \frac{v_- + v_+}{2}$ . Notice that the estimates in the following lemma also hold for  $\tilde{h}_\varepsilon$  since we have  $\tilde{h}'_\varepsilon = \frac{p'(\tilde{v}_\varepsilon)}{\sigma_\varepsilon} \tilde{v}'_\varepsilon$  and  $C^{-1} \leq \frac{p'(\tilde{v}_\varepsilon)}{\sigma_\varepsilon} \leq C$ . But, since the below estimates for  $\tilde{v}_\varepsilon$  are enough in our analysis, we give the estimates only for  $\tilde{v}_\varepsilon$ .

**Lemma 4.1.** *We fix  $v_- > 0$  and  $h_- \in \mathbb{R}$ . Then there exists  $\varepsilon_0 > 0$  such that for any  $0 < \varepsilon < \varepsilon_0$  the following is true. Let  $\tilde{v}_\varepsilon$  be the 1-shock wave with amplitude  $|p(v_-) - p(v_+)| = \varepsilon$  and such that  $\tilde{v}_\varepsilon(0) = \frac{v_- + v_+}{2}$ . Then, there exist constants  $C, C_1, C_2 > 0$  such that*

$$(4.1) \quad -C^{-1}\varepsilon^2 e^{-C_1\varepsilon|\xi|} \leq \tilde{v}'_\varepsilon(\xi) \leq -C\varepsilon^2 e^{-C_2\varepsilon|\xi|}, \quad \forall \xi \in \mathbb{R}.$$

Therefore, as a consequence, we have

$$(4.2) \quad \inf_{[-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}]} |\tilde{v}'_\varepsilon| \geq C\varepsilon^2.$$

*Proof.* Using  $v_-/2 < \tilde{v}_\varepsilon < v_-$ , the proof follows the same arguments as in [32, Lemma 2.1]. Therefore, we omit its details.  $\square$

**4.2. Relative entropy method.** Our analysis is based on the relative entropy. The method is purely nonlinear, and allows to handle rough and large perturbations. The relative entropy method was first introduced by Dafermos [16] and Diperna [19] to prove the  $L^2$  stability and uniqueness of Lipschitz solutions to the hyperbolic conservation laws endowed with a convex entropy.

To use the relative entropy method, we rewrite (3.3) into the following general system of viscous conservation laws:

$$(4.3) \quad \partial_t U + \partial_\xi A(U) = \begin{pmatrix} -\partial_\xi(v^\beta \partial_\xi p(v)) \\ 0 \end{pmatrix},$$

where

$$U := \begin{pmatrix} v \\ h \end{pmatrix}, \quad A(U) := \begin{pmatrix} -\sigma_\varepsilon v - h \\ -\sigma_\varepsilon h + p(v) \end{pmatrix}.$$

The system (4.3) has a convex entropy  $\eta(U) := \frac{h^2}{2} + Q(v)$ , where  $Q(v) = \frac{v^{-\gamma+1}}{\gamma-1}$ , i.e.,  $Q'(v) = -p(v)$ .

Using the derivative of the entropy as

$$(4.4) \quad \nabla \eta(U) = \begin{pmatrix} -p(v) \\ h \end{pmatrix},$$

the above system (4.3) can be rewritten as

$$(4.5) \quad \partial_t U + \partial_\xi A(U) = \partial_\xi \left( M(U) \partial_\xi \nabla \eta(U) \right),$$

where  $M(U) = \begin{pmatrix} v^\beta & 0 \\ 0 & 0 \end{pmatrix}$ , and (3.2) can be rewritten as

$$(4.6) \quad \partial_\xi A(\tilde{U}_\varepsilon) = \partial_\xi \left( M(\tilde{U}_\varepsilon) \partial_\xi \nabla \eta(\tilde{U}_\varepsilon) \right).$$

Consider the relative entropy function defined by

$$\eta(U|V) = \eta(U) - \eta(V) - \nabla \eta(V)(U - V),$$

and the relative flux defined by

$$A(U|V) = A(U) - A(V) - \nabla A(V)(U - V).$$

Let  $G(\cdot; \cdot)$  be the flux of the relative entropy defined by

$$G(U; V) = G(U) - G(V) - \nabla \eta(V)(A(U) - A(V)),$$

where  $G$  is the entropy flux of  $\eta$ , i.e.,  $\partial_i G(U) = \sum_{k=1}^2 \partial_k \eta(U) \partial_i A_k(U)$ ,  $1 \leq i \leq 2$ . Then, for our system (4.3), we have

$$(4.7) \quad \begin{aligned} \eta(U|\tilde{U}_\varepsilon) &= \frac{|h - \tilde{h}_\varepsilon|^2}{2} + Q(v|\tilde{v}_\varepsilon), \\ A(U|\tilde{U}_\varepsilon) &= \begin{pmatrix} 0 \\ p(v|\tilde{v}_\varepsilon) \end{pmatrix}, \\ G(U; \tilde{U}_\varepsilon) &= (p(v) - p(\tilde{v}_\varepsilon))(h - \tilde{h}_\varepsilon) - \sigma_\varepsilon \eta(U|\tilde{U}_\varepsilon), \end{aligned}$$

where the relative pressure is defined as

$$(4.8) \quad p(v|w) = p(v) - p(w) - p'(w)(v - w).$$

We consider a weighted relative entropy between the solution  $U$  of (4.5) and the viscous shock  $\tilde{U}_\varepsilon := \begin{pmatrix} \tilde{v}_\varepsilon \\ \tilde{h}_\varepsilon \end{pmatrix}$  in (3.2) up to a shift  $X(t)$ :

$$a(\xi) \eta(U(t, \xi + X(t)) | \tilde{U}_\varepsilon(\xi)).$$

where  $a$  is a smooth weight function.

In Lemma 4.2, we will derive a quadratic structure on  $\frac{d}{dt} \int_{\mathbb{R}} a(\xi) \eta(U(t, \xi + X(t)) | \tilde{U}_\varepsilon(\xi)) d\xi$ . For that, we introduce a simple notation: for any function  $f : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$  and the shift  $X(t)$ ,

$$f^{\pm X}(t, \xi) := f(t, \xi \pm X(t)).$$

We also introduce the function space:

$$\mathcal{H} := \{(v, h) \in \mathbb{R}^+ \times \mathbb{R} \mid v^{-1}, v \in L^\infty(\mathbb{R}), h - \tilde{h}_\varepsilon \in L^2(\mathbb{R}), \partial_\xi(p(v) - p(\tilde{v}_\varepsilon)) \in L^2(\mathbb{R})\},$$

on which the functionals  $Y, \mathcal{J}^{bad}, \mathcal{J}^{good}$  in (4.10) are well-defined for all  $t \in (0, T)$ .

**Remark 4.1.** As mentioned in Remark 3.2, we consider the solution  $(v, h) \in \mathcal{H}_T$  to (3.3). Then, using the fact that  $v_\xi \in C(0, T; L^2(\mathbb{R}))$ ,  $\tilde{v}'_\varepsilon \in L^2(\mathbb{R})$ , and  $v^{-1}, v \in C(0, T; L^\infty(\mathbb{R}))$ , we find

$$\partial_\xi(p(v) - p(\tilde{v}_\varepsilon)) \in C(0, T; L^2(\mathbb{R})),$$

which implies  $(v, h) \in C(0, T; \mathcal{H})$ .

**Lemma 4.2.** *Let  $a : \mathbb{R} \rightarrow \mathbb{R}^+$  be any positive smooth bounded function whose derivative is bounded and integrable. Let  $\tilde{U}_\varepsilon := \begin{pmatrix} \tilde{v}_\varepsilon \\ \tilde{h}_\varepsilon \end{pmatrix}$  be the viscous shock in (3.2). For any solution  $U \in C(0, T; \mathcal{H})$  to (4.5), we have*

$$(4.9) \quad \frac{d}{dt} \int_{\mathbb{R}} a(\xi) \eta(U^X(t, \xi) | \tilde{U}_\varepsilon(\xi)) d\xi = \dot{X}(t) Y(U^X) + \mathcal{J}^{bad}(U^X) - \mathcal{J}^{good}(U^X),$$

where

$$(4.10) \quad \begin{aligned} Y(U) &:= - \int_{\mathbb{R}} a' \eta(U | \tilde{U}_\varepsilon) d\xi + \int_{\mathbb{R}} a \partial_\xi \nabla \eta(\tilde{U}_\varepsilon) (U - \tilde{U}_\varepsilon) d\xi, \\ \mathcal{J}^{bad}(U) &:= \int_{\mathbb{R}} a' (p(v) - p(\tilde{v}_\varepsilon)) (h - \tilde{h}_\varepsilon) d\xi + \sigma_\varepsilon \int_{\mathbb{R}} a \partial_\xi \tilde{v}_\varepsilon p(v | \tilde{v}_\varepsilon) d\xi \\ &\quad - \int_{\mathbb{R}} a' v^\beta (p(v) - p(\tilde{v}_\varepsilon)) \partial_\xi (p(v) - p(\tilde{v}_\varepsilon)) d\xi - \int_{\mathbb{R}} a' (p(v) - p(\tilde{v}_\varepsilon)) (v^\beta - \tilde{v}_\varepsilon^\beta) \partial_\xi p(\tilde{v}_\varepsilon) d\xi \\ &\quad - \int_{\mathbb{R}} a \partial_\xi (p(v) - p(\tilde{v}_\varepsilon)) (v^\beta - \tilde{v}_\varepsilon^\beta) \partial_\xi p(\tilde{v}_\varepsilon) d\xi, \\ \mathcal{J}^{good}(U) &:= \frac{\sigma_\varepsilon}{2} \int_{\mathbb{R}} a' |h - \tilde{h}_\varepsilon|^2 d\xi + \sigma_\varepsilon \int_{\mathbb{R}} a' Q(v | \tilde{v}_\varepsilon) d\xi + \int_{\mathbb{R}} a v^\beta |\partial_\xi (p(v) - p(\tilde{v}_\varepsilon))|^2 d\xi. \end{aligned}$$

**Remark 4.2.** *In what follows, we will define the weight function  $a$  such that  $\sigma_\varepsilon a' > 0$ . Therefore,  $-\mathcal{J}^{good}$  consists of three good terms, while  $\mathcal{J}^{bad}$  consists of bad terms.*

*Proof.* To derive the desired structure, we use here a change of variable  $\xi \mapsto \xi - X(t)$  as

$$(4.11) \quad \int_{\mathbb{R}} a(\xi) \eta(U^X(t, \xi) | \tilde{U}_\varepsilon(\xi)) d\xi = \int_{\mathbb{R}} a^{-X}(\xi) \eta(U(t, \xi) | \tilde{U}_\varepsilon^{-X}(\xi)) d\xi.$$

Then, using the same computation in [32, Lemma 2.3] (see also [46, Lemma 4]), we have

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}} a^{-X}(\xi) \eta(U(t, \xi) | \tilde{U}_\varepsilon^{-X}(\xi)) d\xi \\ &= -\dot{X} \int_{\mathbb{R}} a'^{-X} \eta(U | \tilde{U}_\varepsilon^{-X}) d\xi + \int_{\mathbb{R}} a^{-X} \left[ \left( \nabla \eta(U) - \nabla \eta(\tilde{U}_\varepsilon^{-X}) \right) \left( -\partial_\xi A(U) + \partial_\xi \left( M(U) \partial_\xi \nabla \eta(U) \right) \right) \right. \\ &\quad \left. - \nabla^2 \eta(\tilde{U}_\varepsilon^{-X}) (U - \tilde{U}_\varepsilon^{-X}) \left( -\dot{X} \partial_\xi \tilde{U}_\varepsilon^{-X} - \partial_\xi A(\tilde{U}_\varepsilon^{-X}) + \partial_\xi \left( M(\tilde{U}_\varepsilon^{-X}) \partial_\xi \nabla \eta(\tilde{U}_\varepsilon^{-X}) \right) \right) \right] d\xi \\ &= \dot{X} \left( - \int_{\mathbb{R}} a'^{-X} \eta(U | \tilde{U}_\varepsilon^{-X}) d\xi + \int_{\mathbb{R}} a^{-X} \partial_\xi \nabla \eta(\tilde{U}_\varepsilon^{-X}) (U - \tilde{U}_\varepsilon^{-X}) \right) + I_1 + I_2 + I_3 + I_4, \end{aligned}$$



where

$$\begin{aligned}
I_1 &:= - \int_{\mathbb{R}} a^{-X} \partial_{\xi} G(U; \tilde{U}_{\varepsilon}^{-X}) d\xi, \\
I_2 &:= - \int_{\mathbb{R}} a^{-X} \partial_{\xi} \nabla \eta(\tilde{U}_{\varepsilon}^{-X}) A(U | \tilde{U}_{\varepsilon}^{-X}) d\xi, \\
I_3 &:= \int_{\mathbb{R}} a^{-X} \left( \nabla \eta(U) - \nabla \eta(\tilde{U}_{\varepsilon}^{-X}) \right) \partial_{\xi} \left( M(U) \partial_{\xi} (\nabla \eta(U) - \nabla \eta(\tilde{U}_{\varepsilon}^{-X})) \right) d\xi, \\
I_4 &:= \int_{\mathbb{R}} a^{-X} \left( \nabla \eta(U) - \nabla \eta(\tilde{U}_{\varepsilon}^{-X}) \right) \partial_{\xi} \left( (M(U) - M(\tilde{U}_{\varepsilon}^{-X})) \partial_{\xi} \nabla \eta(\tilde{U}_{\varepsilon}^{-X}) \right) d\xi \\
I_5 &:= \int_{\mathbb{R}} a^{-X} (\nabla \eta)(U | \tilde{U}_{\varepsilon}^{-X}) \partial_{\xi} \left( M(\tilde{U}_{\varepsilon}^{-X}) \partial_{\xi} \nabla \eta(\tilde{U}_{\varepsilon}^{-X}) \right) d\xi.
\end{aligned}$$

Using (4.7) and (4.4), we have

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}} a'^{-X} G(U; \tilde{U}_{\varepsilon}^{-X}) d\xi = \int_{\mathbb{R}} a'^{-X} \left( (p(v) - p(\tilde{v}_{\varepsilon}^{-X})) (h - \tilde{h}_{\varepsilon}^{-X}) - \sigma_{\varepsilon} \eta(U | \tilde{U}_{\varepsilon}^{-X}) \right) d\xi \\
&= \int_{\mathbb{R}} a'^{-X} (p(v) - p(\tilde{v}_{\varepsilon}^{-X})) (h - \tilde{h}_{\varepsilon}^{-X}) d\xi - \frac{\sigma_{\varepsilon}}{2} \int_{\mathbb{R}} a'^{-X} |h - \tilde{h}_{\varepsilon}^{-X}|^2 d\xi - \sigma_{\varepsilon} \int_{\mathbb{R}} a'^{-X} Q(v | \tilde{v}_{\varepsilon}^{-X}) d\xi, \\
I_2 &= - \int_{\mathbb{R}} a^{-X} \partial_{\xi} \tilde{h}_{\varepsilon}^{-X} p(v | \tilde{v}_{\varepsilon}^{-X}) d\xi.
\end{aligned}$$

By integration by parts, we have

$$\begin{aligned}
I_3 &= \int_{\mathbb{R}} a^{-X} (p(v) - p(\tilde{v}_{\varepsilon}^{-X})) \partial_{\xi} \left( v^{\beta} \partial_{\xi} (p(v) - p(\tilde{v}_{\varepsilon}^{-X})) \right) d\xi \\
&= - \int_{\mathbb{R}} a^{-X} v^{\beta} |\partial_{\xi} (p(v) - p(\tilde{v}_{\varepsilon}^{-X}))|^2 d\xi - \int_{\mathbb{R}} a'^{-X} v^{\beta} (p(v) - p(\tilde{v}_{\varepsilon}^{-X})) \partial_{\xi} (p(v) - p(\tilde{v}_{\varepsilon}^{-X})) d\xi, \\
I_4 &= \int_{\mathbb{R}} a^{-X} (p(v) - p(\tilde{v}_{\varepsilon}^{-X})) \partial_{\xi} \left( (v^{\beta} - \tilde{v}_{\varepsilon}^{\beta}) \partial_{\xi} p(\tilde{v}_{\varepsilon}^{-X}) \right) d\xi \\
&= - \int_{\mathbb{R}} a'^{-X} (p(v) - p(\tilde{v}_{\varepsilon}^{-X})) (v^{\beta} - \tilde{v}_{\varepsilon}^{\beta}) \partial_{\xi} p(\tilde{v}_{\varepsilon}^{-X}) d\xi \\
&\quad - \int_{\mathbb{R}} a^{-X} \partial_{\xi} (p(v) - p(\tilde{v}_{\varepsilon}^{-X})) (v^{\beta} - \tilde{v}_{\varepsilon}^{\beta}) \partial_{\xi} p(\tilde{v}_{\varepsilon}^{-X}) d\xi.
\end{aligned}$$

Since it follows from (4.6) and (4.4) that

$$I_5 = \int_{\mathbb{R}} a^{-X} (\nabla \eta)(U | \tilde{U}_{\varepsilon}^{-X}) \partial_{\xi} A(\tilde{U}_{\varepsilon}^{-X}) d\xi = \int_{\mathbb{R}} a^{-X} p(v | \tilde{v}_{\varepsilon}^{-X}) \left( \partial_{\xi} \tilde{h}_{\varepsilon}^{-X} + \sigma_{\varepsilon} \partial_{\xi} \tilde{v}_{\varepsilon}^{-X} \right) d\xi,$$

we have some cancellation

$$I_2 + I_5 = \sigma_{\varepsilon} \int_{\mathbb{R}} a^{-X} \partial_{\xi} \tilde{v}_{\varepsilon}^{-X} p(v | \tilde{v}_{\varepsilon}^{-X}) d\xi.$$

Therefore, we have

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}} a^{-X} \eta(U|\tilde{U}_{\varepsilon}^{-X}) d\xi \\
&= \dot{X} \left( - \int_{\mathbb{R}} a'^{-X} \eta(U|\tilde{U}_{\varepsilon}^{-X}) d\xi + \int_{\mathbb{R}} a^{-X} \partial_{\xi} \nabla \eta(\tilde{U}_{\varepsilon}^{-X})(U - \tilde{U}_{\varepsilon}^{-X}) d\xi \right) \\
&+ \int_{\mathbb{R}} a'^{-X} (p(v) - p(\tilde{v}_{\varepsilon}^{-X})) (h - \tilde{h}_{\varepsilon}^{-X}) d\xi - \frac{\sigma_{\varepsilon}}{2} \int_{\mathbb{R}} a'^{-X} |h - \tilde{h}_{\varepsilon}|^2 d\xi - \sigma_{\varepsilon} \int_{\mathbb{R}} a'^{-X} Q(v|\tilde{v}_{\varepsilon}) d\xi \\
&+ \sigma_{\varepsilon} \int_{\mathbb{R}} a^{-X} \partial_{\xi} \tilde{v}_{\varepsilon}^{-X} p(v|\tilde{v}_{\varepsilon}^{-X}) d\xi - \int_{\mathbb{R}} a'^{-X} v^{\beta} (p(v) - p(\tilde{v}_{\varepsilon}^{-X})) \partial_{\xi} (p(v) - p(\tilde{v}_{\varepsilon}^{-X})) d\xi \\
&- \int_{\mathbb{R}} a'^{-X} (p(v) - p(\tilde{v}_{\varepsilon}^{-X})) \left( v^{\beta} - (\tilde{v}_{\varepsilon}^{\beta})^{-X} \right) \partial_{\xi} p(\tilde{v}_{\varepsilon}^{-X}) d\xi \\
&- \int_{\mathbb{R}} a^{-X} \partial_{\xi} (p(v) - p(\tilde{v}_{\varepsilon}^{-X})) \left( v^{\beta} - (\tilde{v}_{\varepsilon}^{\beta})^{-X} \right) \partial_{\xi} p(\tilde{v}_{\varepsilon}^{-X}) d\xi - \int_{\mathbb{R}} a^{-X} v^{\beta} |\partial_{\xi} (p(v) - p(\tilde{v}_{\varepsilon}^{-X}))|^2 d\xi.
\end{aligned}$$

Again, we use a change of variable  $\xi \mapsto \xi + X(t)$  to have

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathbb{R}} a \eta(U^X|\tilde{U}_{\varepsilon}) d\xi \\
&= \dot{X} \left( - \int_{\mathbb{R}} a' \eta(U^X|\tilde{U}_{\varepsilon}) d\xi + \int_{\mathbb{R}} a \partial_{\xi} \nabla \eta(\tilde{U}_{\varepsilon})(U^X - \tilde{U}_{\varepsilon}) d\xi \right) \\
&+ \int_{\mathbb{R}} a' (p(v^X) - p(\tilde{v}_{\varepsilon})) (h^X - \tilde{h}_{\varepsilon}) d\xi - \frac{\sigma_{\varepsilon}}{2} \int_{\mathbb{R}} a' |h^X - \tilde{h}_{\varepsilon}|^2 d\xi - \sigma_{\varepsilon} \int_{\mathbb{R}} a' Q(v^X|\tilde{v}_{\varepsilon}) d\xi \\
&+ \sigma_{\varepsilon} \int_{\mathbb{R}} a \partial_{\xi} \tilde{v}_{\varepsilon} p(v^X|\tilde{v}_{\varepsilon}) d\xi - \int_{\mathbb{R}} a' (v^{\beta})^X (p(v^X) - p(\tilde{v}_{\varepsilon})) \partial_{\xi} (p(v^X) - p(\tilde{v}_{\varepsilon})) d\xi \\
&- \int_{\mathbb{R}} a' (p(v^X) - p(\tilde{v}_{\varepsilon})) \left( (v^{\beta})^X - \tilde{v}_{\varepsilon}^{\beta} \right) \partial_{\xi} p(\tilde{v}_{\varepsilon}) d\xi \\
&- \int_{\mathbb{R}} a \partial_{\xi} (p(v^X) - p(\tilde{v}_{\varepsilon})) \left( (v^{\beta})^X - \tilde{v}_{\varepsilon}^{\beta} \right) \partial_{\xi} p(\tilde{v}_{\varepsilon}) d\xi - \int_{\mathbb{R}} a (v^{\beta})^X |\partial_{\xi} (p(v^X) - p(\tilde{v}_{\varepsilon}))|^2 d\xi,
\end{aligned}$$

which provides the desired representation.  $\square$

**4.3. Construction of the weight function.** We define the weight function  $a$  by

$$(4.12) \quad a(\xi) = 1 - \lambda \frac{p(\tilde{v}_{\varepsilon}(\xi)) - p(v_-)}{[p]},$$

where  $[p] := p(v_+) - p(v_-)$ . We briefly present some useful properties on the weight  $a$ . First of all, the weight function  $a$  is positive and decreasing, and satisfies  $1 - \lambda \leq a \leq 1$ . Since  $[p] = \varepsilon$ ,  $p'(v_-/2) \leq p'(\tilde{v}_{\varepsilon}) \leq p'(v_-)$  and

$$(4.13) \quad a' = -\lambda \frac{\partial_{\xi} p(\tilde{v}_{\varepsilon})}{[p]},$$

we have

$$(4.14) \quad |a'| \sim \frac{\lambda}{\varepsilon} |\tilde{v}'_{\varepsilon}|.$$

**4.4. Maximization in terms of  $h - \tilde{h}_\varepsilon$ .** In order to estimate the right-hand side of (4.15), we will use Proposition 4.2, i.e., a sharp estimate with respect to  $v - \tilde{v}_\varepsilon$  when  $v - \tilde{v}_\varepsilon \ll 1$ , for which we need to rewrite  $\mathcal{J}^{bad}$  on the right-hand side of (4.9) only in terms of  $v$  near  $\tilde{v}_\varepsilon$ , by separating  $h - \tilde{h}_\varepsilon$  from the first term of  $\mathcal{J}^{bad}$ . Therefore, we will rewrite  $\mathcal{J}^{bad}$  into the maximized representation in terms of  $h - \tilde{h}_\varepsilon$  in the following lemma. However, we will keep all terms of  $\mathcal{J}^{bad}$  in a region  $\{p(v) - p(\tilde{v}_\varepsilon) > \delta\}$  for small values of  $v$ , since we use the estimate (4.68) to control the first term of  $\mathcal{J}^{bad}$  in that region.

**Lemma 4.3.** *Let  $a : \mathbb{R} \rightarrow \mathbb{R}^+$  be as in (4.12), and  $\tilde{U}_\varepsilon = (\frac{\tilde{v}_\varepsilon}{\tilde{h}_\varepsilon})$  be the viscous shock in (3.2). Let  $\delta$  be any positive constant. For any solution  $U \in C(0, T; \mathcal{H})$  to (4.5), we have*

$$(4.15) \quad \frac{d}{dt} \int_{\mathbb{R}} a(\xi) \eta(U^X(t, \xi)) \tilde{U}_\varepsilon(\xi) d\xi = \dot{X}(t) Y(U^X) + \mathcal{B}_\delta(U^X) - \mathcal{G}_\delta(U^X),$$

where

$$(4.16) \quad \begin{aligned} Y(U) &:= - \int_{\mathbb{R}} a' \eta(U | \tilde{U}_\varepsilon) d\xi + \int_{\mathbb{R}} a \partial_\xi \nabla \eta(\tilde{U}_\varepsilon) (U - \tilde{U}_\varepsilon) d\xi, \\ \mathcal{B}_\delta(U) &:= \sigma_\varepsilon \int_{\mathbb{R}} a \partial_\xi \tilde{v}_\varepsilon p(v | \tilde{v}_\varepsilon) d\xi + \frac{1}{2\sigma_\varepsilon} \int_{\mathbb{R}} a' |p(v) - p(\tilde{v}_\varepsilon)|^2 \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) \leq \delta\}} d\xi \\ &\quad + \int_{\mathbb{R}} a' (p(v) - p(\tilde{v}_\varepsilon)) (h - \tilde{h}_\varepsilon) \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) > \delta\}} d\xi \\ &\quad - \int_{\mathbb{R}} a' v^\beta (p(v) - p(\tilde{v}_\varepsilon)) \partial_\xi (p(v) - p(\tilde{v}_\varepsilon)) d\xi - \int_{\mathbb{R}} a' (p(v) - p(\tilde{v}_\varepsilon)) (v^\beta - \tilde{v}_\varepsilon^\beta) \partial_\xi p(\tilde{v}_\varepsilon) d\xi \\ &\quad - \int_{\mathbb{R}} a \partial_\xi (p(v) - p(\tilde{v}_\varepsilon)) (v^\beta - \tilde{v}_\varepsilon^\beta) \partial_\xi p(\tilde{v}_\varepsilon) d\xi, \\ \mathcal{G}_\delta(U) &:= \frac{\sigma_\varepsilon}{2} \int_{\mathbb{R}} a' \left| h - \tilde{h}_\varepsilon - \frac{p(v) - p(\tilde{v}_\varepsilon)}{\sigma_\varepsilon} \right|^2 \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) \leq \delta\}} d\xi + \frac{\sigma_\varepsilon}{2} \int_{\mathbb{R}} a' \left| h - \tilde{h}_\varepsilon \right|^2 \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) > \delta\}} d\xi \\ &\quad + \sigma_\varepsilon \int_{\mathbb{R}} a' Q(v | \tilde{v}_\varepsilon) d\xi + \int_{\mathbb{R}} a v^\beta |\partial_\xi (p(v) - p(\tilde{v}_\varepsilon))|^2 d\xi. \end{aligned}$$

**Remark 4.3.** *Since  $\sigma_\varepsilon a' > 0$  and  $a > 0$ ,  $-\mathcal{G}_\delta$  consists of four good terms.*

*Proof.* For any fixed  $\delta > 0$ , we first rewrite  $\mathcal{J}^{bad}$  and  $-\mathcal{J}^{good}$  into

$$\begin{aligned} \mathcal{J}^{bad}(U) &:= \underbrace{\int_{\mathbb{R}} a' (p(v) - p(\tilde{v}_\varepsilon)) (h - \tilde{h}_\varepsilon) \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) \leq \delta\}} d\xi}_{=: J_1} \\ &\quad + \int_{\mathbb{R}} a' (p(v) - p(\tilde{v}_\varepsilon)) (h - \tilde{h}_\varepsilon) \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) > \delta\}} d\xi + \sigma_\varepsilon \int_{\mathbb{R}} a \partial_\xi \tilde{v}_\varepsilon p(v | \tilde{v}_\varepsilon) d\xi \\ &\quad - \int_{\mathbb{R}} a' v^\beta (p(v) - p(\tilde{v}_\varepsilon)) \partial_\xi (p(v) - p(\tilde{v}_\varepsilon)) d\xi - \int_{\mathbb{R}} a' (p(v) - p(\tilde{v}_\varepsilon)) (v^\beta - \tilde{v}_\varepsilon^\beta) \partial_\xi p(\tilde{v}_\varepsilon) d\xi \\ &\quad - \int_{\mathbb{R}} a \partial_\xi (p(v) - p(\tilde{v}_\varepsilon)) (v^\beta - \tilde{v}_\varepsilon^\beta) \partial_\xi p(\tilde{v}_\varepsilon) d\xi, \end{aligned}$$

and

$$\begin{aligned} -\mathcal{J}^{good}(U) &:= \underbrace{-\frac{\sigma_\varepsilon}{2} \int_{\mathbb{R}} a' |h - \tilde{h}_\varepsilon|^2 \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon) \leq \delta\}} d\xi}_{=: J_2} - \frac{\sigma_\varepsilon}{2} \int_{\mathbb{R}} a' |h - \tilde{h}_\varepsilon|^2 d\xi \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon) > \delta\}} d\xi \\ &\quad - \sigma_\varepsilon \int_{\mathbb{R}} a' Q(v|\tilde{v}_\varepsilon) d\xi - \int_{\mathbb{R}} av^\beta |\partial_\xi(p(v) - p(\tilde{v}_\varepsilon))|^2 d\xi. \end{aligned}$$

Applying the quadratic identity  $\alpha x^2 + \beta x = \alpha(x + \frac{\beta}{2\alpha})^2 - \frac{\beta^2}{4\alpha}$  with  $x := h - \tilde{h}_\varepsilon$  to the integrands of  $J_1 + J_2$ , we find

$$-\frac{\sigma_\varepsilon}{2} |h - \tilde{h}_\varepsilon|^2 + (p(v) - p(\tilde{v}_\varepsilon))(h - \tilde{h}_\varepsilon) = -\frac{\sigma_\varepsilon}{2} \left| h - \tilde{h}_\varepsilon - \frac{p(v) - p(\tilde{v}_\varepsilon)}{\sigma_\varepsilon} \right|^2 + \frac{1}{2\sigma_\varepsilon} |p(v) - p(\tilde{v}_\varepsilon)|^2.$$

Therefore, we have the desired representation (4.15)-(4.16).  $\square$

**4.5. Main proposition.** The main proposition for the proof of Theorem 3.1 is the following.

**Proposition 4.1.** *There exist  $\varepsilon_0, \delta_0, \delta_3 > 0$  such that for any  $\varepsilon < \varepsilon_0$  and  $\delta_0^{-1}\varepsilon < \lambda < \delta_0 < 1/2$ , the following is true.*

*For any  $U \in \mathcal{H} \cap \{U \mid |Y(U)| \leq \varepsilon^2\}$ ,*

$$\begin{aligned} (4.17) \quad \mathcal{R}(U) &:= -\frac{1}{\varepsilon^4} Y^2(U) + \left(1 + \delta_0 \frac{\varepsilon}{\lambda}\right) |\mathcal{B}_{\delta_3}(U)| \\ &\quad - \mathcal{G}_1^-(U) - \mathcal{G}_1^+(U) - \left(1 - \delta_0 \frac{\varepsilon}{\lambda}\right) \mathcal{G}_2(U) - (1 - \delta_0) \mathcal{D}(U) \leq 0, \end{aligned}$$

where  $Y$  and  $\mathcal{B}_{\delta_3}$  are as in (4.16), and  $\mathcal{G}_1^-, \mathcal{G}_1^+, \mathcal{G}_2, \mathcal{D}$  denote the four terms of  $\mathcal{G}_{\delta_3}$  as follows:

$$\begin{aligned} (4.18) \quad \mathcal{G}_1^-(U) &:= \frac{\sigma_\varepsilon}{2} \int_{\Omega^c} a' |h - \tilde{h}_\varepsilon|^2 \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon) > \delta_3\}} d\xi, \\ \mathcal{G}_1^+(U) &:= \frac{\sigma_\varepsilon}{2} \int_{\Omega} a' \left( h - \tilde{h}_\varepsilon - \frac{p(v) - p(\tilde{v}_\varepsilon)}{\sigma_\varepsilon} \right)^2 \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon) \leq \delta_3\}} d\xi, \\ \mathcal{G}_2(U) &:= \sigma_\varepsilon \int_{\mathbb{R}} a' Q(v|\tilde{v}_\varepsilon) d\xi, \\ \mathcal{D}(U) &:= \int_{\mathbb{R}} av^\beta |\partial_\xi(p(v) - p(\tilde{v}_\varepsilon))|^2 d\xi. \end{aligned}$$

**4.6. Proof of Theorem 3.1 from Proposition 4.1.** We will first show how Proposition 4.1 implies Theorem 3.1.

For any fixed  $\varepsilon > 0$ , we consider a continuous function  $\Phi_\varepsilon$  defined by

$$(4.19) \quad \Phi_\varepsilon(y) = \begin{cases} \frac{1}{\varepsilon^2}, & \text{if } y \leq -\varepsilon^2, \\ -\frac{1}{\varepsilon^4} y, & \text{if } |y| \leq \varepsilon^2, \\ -\frac{1}{\varepsilon^2}, & \text{if } y \geq \varepsilon^2. \end{cases}$$

Let  $\varepsilon_0, \delta_0, \delta_3$  be the constants in Proposition 4.1. Then  $\varepsilon, \lambda$  be any constants such that  $0 < \varepsilon < \varepsilon_0$  and  $\delta_0^{-1}\varepsilon < \lambda < \delta_0 < 1/2$ .

We define a shift function  $X(t)$  as a solution of the nonlinear ODE:

$$(4.20) \quad \begin{cases} \dot{X}(t) = \Phi_\varepsilon(Y(U^X)) \left( 2|\mathcal{B}_{\delta_3}(U^X)| + 1 \right), \\ X(0) = 0, \end{cases}$$

where  $Y$  and  $\mathcal{B}_\delta$  are as in (4.16).

Therefore, for the solution  $U \in C(0, T; \mathcal{H})$ , the shift  $X$  exists and is unique at least locally by the Cauchy-Lipschitz theorem. Indeed, since  $\tilde{v}'_\varepsilon, \tilde{h}'_\varepsilon, a'$  are bounded, smooth and integrable, using  $U \in C(0, T; \mathcal{H})$  together with the change of variables  $\xi \mapsto \xi - X(t)$  as in (4.11), we find that the right-hand side of the ODE (4.20) is uniformly Lipschitz continuous in  $X$ , and is continuous in  $t$ .

Moreover, the global-in-time existence and uniqueness of the shift are verified by the a priori estimate (4.24).

Based on (4.15) and (4.20), to get the contraction estimate, it is enough to prove that for almost every time  $t > 0$

$$(4.21) \quad \Phi_\varepsilon(Y(U^X)) \left( 2|\mathcal{B}_{\delta_3}(U^X)| + 1 \right) Y(U^X) + \mathcal{B}_{\delta_3}(U^X) - \mathcal{G}_{\delta_3}(U^X) \leq 0.$$

We define

$$\mathcal{F}(U) := \Phi_\varepsilon(Y(U)) \left( 2|\mathcal{B}_{\delta_3}(U)| + 1 \right) Y(U) + |\mathcal{B}_{\delta_3}(U)| - \mathcal{G}_{\delta_3}(U), \quad \forall U \in \mathcal{H}.$$

From (4.19), we have

$$(4.22) \quad \Phi_\varepsilon(Y) \left( 2|\mathcal{B}_{\delta_3}| + 1 \right) Y \leq \begin{cases} -2|\mathcal{B}_{\delta_3}|, & \text{if } |Y| \geq \varepsilon^2, \\ -\frac{1}{\varepsilon^4} Y^2, & \text{if } |Y| \leq \varepsilon^2. \end{cases}$$

Hence, for all  $U \in \mathcal{H}$  satisfying  $|Y(U)| \geq \varepsilon^2$ , we have

$$\mathcal{F}(U) \leq -|\mathcal{B}_{\delta_3}(U)| - \mathcal{G}_{\delta_3}(U) \leq 0.$$

Using both (4.22) and Proposition 4.1, we find that for all  $U \in \mathcal{H}$  satisfying  $|Y(U)| \leq \varepsilon^2$ ,

$$\mathcal{F}(U) \leq -\delta_0 \frac{\varepsilon}{\lambda} |\mathcal{B}_{\delta_3}(U)| - \delta_0 \frac{\varepsilon}{\lambda} \mathcal{G}_2(U) - \delta_0 \mathcal{D}(U) \leq 0.$$

Since  $\delta_0 < 1/2$ , these two estimates show that for every  $U \in \mathcal{H}$  we have

$$\mathcal{F}(U) \leq -\delta_0 \frac{\varepsilon}{\lambda} |\mathcal{B}_{\delta_3}(U)| - \delta_0 \frac{\varepsilon}{\lambda} \mathcal{G}_2(U) - \delta_0 \mathcal{D}(U).$$

For every fixed  $t > 0$ , using this estimate with  $U = U^X(t, \cdot)$ , together with (4.15) and (4.21), we have

$$(4.23) \quad \frac{d}{dt} \int_{\mathbb{R}} a\eta(U^X | \tilde{U}_\varepsilon) d\xi \leq \mathcal{F}(U^X) \leq -\delta_0 \frac{\varepsilon}{\lambda} |\mathcal{B}_{\delta_3}(U^X)| - \delta_0 \frac{\varepsilon}{\lambda} \mathcal{G}_2(U^X) - \delta_0 \mathcal{D}(U^X).$$

Therefore we have

$$\frac{d}{dt} \int_{\mathbb{R}} a\eta(U^X | \tilde{U}_\varepsilon) d\xi + \delta_0 \frac{\varepsilon}{\lambda} \mathcal{G}_2(U^X) + \delta_0 \mathcal{D}(U) \leq 0,$$

which completes (3.5).

In particular, since it follows from (4.23) that

$$\delta_0 \frac{\varepsilon}{\lambda} \int_0^\infty |\mathcal{B}_{\delta_3}(U^X)| dt \leq \int_{\mathbb{R}} \eta(U_0 | \tilde{U}_\varepsilon) d\xi < \infty \quad \text{by the initial condition,}$$

using (4.20) and  $\|\Phi_\varepsilon\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\varepsilon^2}$  by (4.19), we have

$$(4.24) \quad |\dot{X}| \leq \frac{1}{\varepsilon^2} + \frac{2}{\varepsilon^2} |\mathcal{B}_{\delta_3}|, \quad \|\mathcal{B}_{\delta_3}\|_{L^1(0,\infty)} \leq \frac{1}{\delta_0} \frac{\lambda}{\varepsilon} \int_{\mathbb{R}} \eta(U_0|\tilde{U}_\varepsilon) d\xi.$$

This provides the global-in-time estimate (3.6), thus  $X \in W^{1,1}(\mathbb{R}^+)$ . This completes the proof of Theorem 3.1.

The rest of this section is dedicated to the proof of Proposition 4.1.

**4.7. Expansion in the size of the shock.** We define the following functionals:

$$(4.25) \quad \begin{aligned} Y_g(v) &:= -\frac{1}{2\sigma_\varepsilon^2} \int_{\mathbb{R}} a' |p(v) - p(\tilde{v}_\varepsilon)|^2 d\xi - \int_{\mathbb{R}} a' Q(v|\tilde{v}_\varepsilon) d\xi - \int_{\mathbb{R}} a \partial_\xi p(\tilde{v}_\varepsilon)(v - \tilde{v}_\varepsilon) d\xi \\ &\quad + \frac{1}{\sigma_\varepsilon} \int_{\mathbb{R}} a \partial_\xi \tilde{h}_\varepsilon(p(v) - p(\tilde{v}_\varepsilon)) d\xi, \\ \mathcal{I}_1(v) &:= \sigma_\varepsilon \int_{\mathbb{R}} a \partial_\xi \tilde{v}_\varepsilon p(v|\tilde{v}_\varepsilon) d\xi, \\ \mathcal{I}_2(v) &:= \frac{1}{2\sigma_\varepsilon} \int_{\mathbb{R}} a' |p(v) - p(\tilde{v}_\varepsilon)|^2 d\xi, \\ \mathcal{G}_2(v) &:= \sigma_\varepsilon \int_{\mathbb{R}} a' Q(v|\tilde{v}_\varepsilon) d\xi, \\ \mathcal{D}(v) &:= \int_{\mathbb{R}} a v^\beta |\partial_\xi(p(v) - p(\tilde{v}_\varepsilon))|^2 d\xi. \end{aligned}$$

Note that all these quantities depend only on  $v$  (not on  $h$ ).

**Proposition 4.2.** *For any constant  $C_2 > 0$ , there exist  $\varepsilon_0, \delta_3 > 0$ , such that for any  $\varepsilon \in (0, \varepsilon_0)$ , and any  $\lambda, \delta \in (0, \delta_3)$  such that  $\varepsilon \leq \lambda$ , the following is true.*

*For any function  $v : \mathbb{R} \rightarrow \mathbb{R}^+$  such that  $\mathcal{D}(v) + \mathcal{G}_2(v)$  is finite, if*

$$(4.26) \quad |Y_g(v)| \leq C_2 \frac{\varepsilon^2}{\lambda}, \quad \|p(v) - p(\tilde{v}_\varepsilon)\|_{L^\infty(\mathbb{R})} \leq \delta_3,$$

*then*

$$(4.27) \quad \begin{aligned} \mathcal{R}_{\varepsilon,\delta}(v) &:= -\frac{1}{\varepsilon\delta} |Y_g(v)|^2 + (1 + \delta) |\mathcal{I}_1(v)| \\ &\quad + \left(1 + \delta \left(\frac{\varepsilon}{\lambda}\right)\right) |\mathcal{I}_2(v)| - \left(1 - \delta \left(\frac{\varepsilon}{\lambda}\right)\right) \mathcal{G}_2(v) - (1 - \delta) \mathcal{D}(v) \leq 0. \end{aligned}$$

*Proof.* The proof is almost the same as that of [32, Proposition 3.4], because  $Y_g, \mathcal{I}_1, \mathcal{I}_2, \mathcal{G}_2$  defined in (4.25) are the exactly same functionals as in [32, Proposition 3.4], and the diffusion  $\mathcal{D}$  is slightly different but has the same expansion. Note that this proposition corresponds to an expansion in  $p(v)$  near  $p(\tilde{v}_\varepsilon)$  (up to  $\delta_3$ ) for  $p(\tilde{v}_\varepsilon)$  close to  $p(v_-)$  (up to  $\varepsilon$ ). It is therefore natural that the expansion is similar to the case of  $\alpha = \gamma$  as in [32, Proposition 3.4], since the viscosity is almost constant near  $p(v_-)$ . For completeness, the main part of the proof is given in Appendix A.  $\square$

**4.8. Truncation of the big values of  $|p(v) - p(\tilde{v}_\varepsilon)|$ .** In order to use Proposition 4.2 in the proof of Proposition 4.1, we need to show that the values for  $p(v)$  such that  $|p(v) - p(\tilde{v}_\varepsilon)| \geq \delta_3$  have a small effect. However, the value of  $\delta_3$  is itself conditioned to the constant  $C_2$  in the proposition. Therefore, we need first to find a uniform bound on  $Y_g$  which is not yet conditioned on the level of truncation  $k$ .

We consider a truncation on  $|p(v) - p(\tilde{v}_\varepsilon)|$  with a constant  $k > 0$ . Later we will consider the case  $k = \delta_3$  as in Proposition 4.2. But for now, we consider the general case  $k$  to estimate the constant  $C_2$ . For that, let  $\psi_k$  be a continuous function defined by

$$(4.28) \quad \psi_k(y) = \inf(k, \sup(-k, y)).$$

We then define the function  $\bar{v}_k$  uniquely (since the function  $p$  is one to one) as

$$(4.29) \quad p(\bar{v}_k) - p(\tilde{v}_\varepsilon) = \psi_k(p(v) - p(\tilde{v}_\varepsilon)).$$

We have the following lemma (see [32, Lemma 3.2]).

**Lemma 4.4.** *For a fixed  $v_- \geq 0$ ,  $u_- \in \mathbb{R}$ , there exists  $C_2, k_0, \varepsilon_0, \delta_0 > 0$  such that for any  $\varepsilon \leq \varepsilon_0$ ,  $\varepsilon/\lambda \leq \delta_0$  with  $\lambda < 1/2$ , the following is true whenever  $|Y(U)| \leq \varepsilon^2$ :*

$$(4.30) \quad \int_{\mathbb{R}} |a'| |h - \tilde{h}_\varepsilon|^2 d\xi + \int_{\mathbb{R}} |a'| Q(v|\tilde{v}_\varepsilon) d\xi \leq C \frac{\varepsilon^2}{\lambda},$$

$$(4.31) \quad |Y_g(\bar{v}_k)| \leq C_2 \frac{\varepsilon^2}{\lambda}, \quad \text{for every } k \leq k_0.$$

We now fix the constant  $\delta_3$  of Proposition 4.2 associated to the constant  $C_2$  of Lemma 4.4. Without loss of generality, we can assume that  $\delta_3 < k_0$  (since Proposition 4.2 is valid for any smaller  $\delta_3$ ). From now on, we set (without confusion)

$$\bar{v} := \bar{v}_{\delta_3}, \quad \bar{U} := (\bar{v}, h), \quad \mathcal{B} := \mathcal{B}_{\delta_3}, \quad \mathcal{G} := \mathcal{G}_{\delta_3}.$$

Note from Lemma 4.4 that

$$(4.32) \quad |Y_g(\bar{v})| \leq C_2 \frac{\varepsilon^2}{\lambda}.$$

We first recall the terms  $Y$  in (4.16) as

$$Y = - \int_{\mathbb{R}} a' \frac{|h - \tilde{h}_\varepsilon|^2}{2} d\xi - \int_{\mathbb{R}} a' Q(v|\tilde{v}_\varepsilon) d\xi + \int_{\mathbb{R}} a \left( -\partial_\xi p(\tilde{v}_\varepsilon)(v - \tilde{v}_\varepsilon) + \partial_\xi \tilde{h}_\varepsilon(h - \tilde{h}_\varepsilon) \right) d\xi.$$

In what follows, for simplification, we use the notation:

$$\Omega := \{\xi \mid (p(v) - p(\tilde{v}_\varepsilon))(\xi) \leq \delta_3\}.$$

We split  $Y$  into four parts  $Y_g, Y_b, Y_l$  and  $Y_s$  as follows:

$$Y = Y_g + Y_b + Y_l + Y_s,$$

where

$$\begin{aligned}
Y_g &:= -\frac{1}{2\sigma_\varepsilon^2} \int_{\Omega} a' |p(v) - p(\tilde{v}_\varepsilon)|^2 d\xi - \int_{\Omega} a' Q(v|\tilde{v}_\varepsilon) d\xi - \int_{\Omega} a \partial_\xi p(\tilde{v}_\varepsilon)(v - \tilde{v}_\varepsilon) d\xi \\
&\quad + \frac{1}{\sigma_\varepsilon} \int_{\Omega} a \partial_\xi \tilde{h}_\varepsilon (p(v) - p(\tilde{v}_\varepsilon)) d\xi, \\
Y_b &:= -\frac{1}{2} \int_{\Omega} a' \left( h - \tilde{h}_\varepsilon - \frac{p(v) - p(\tilde{v}_\varepsilon)}{\sigma_\varepsilon} \right)^2 d\xi \\
&\quad - \frac{1}{\sigma_\varepsilon} \int_{\Omega} a' (p(v) - p(\tilde{v}_\varepsilon)) \left( h - \tilde{h}_\varepsilon - \frac{p(v) - p(\tilde{v}_\varepsilon)}{\sigma_\varepsilon} \right) d\xi, \\
Y_l &= \int_{\Omega} a \partial_\xi \tilde{h}_\varepsilon \left( h - \tilde{h}_\varepsilon - \frac{p(v) - p(\tilde{v}_\varepsilon)}{\sigma_\varepsilon} \right) d\xi, \\
Y_s &= - \int_{\Omega^c} a' Q(v|\tilde{v}_\varepsilon) d\xi - \int_{\Omega^c} a \partial_\xi p(\tilde{v}_\varepsilon)(v - \tilde{v}_\varepsilon) d\xi \\
&\quad - \int_{\Omega^c} a' \frac{|h - \tilde{h}_\varepsilon|^2}{2} d\xi + \int_{\Omega^c} a \partial_\xi \tilde{h}_\varepsilon (h - \tilde{h}_\varepsilon) d\xi.
\end{aligned}$$

Notice that  $Y_g$  consists of the terms related to  $v - \tilde{v}_\varepsilon$ , while  $Y_b$  and  $Y_l$  consist of terms related to  $h - \tilde{h}_\varepsilon$ . While  $Y_b$  is quadratic, and  $Y_l$  is linear in  $h - \tilde{h}_\varepsilon$ . Since  $\{\xi \mid |p(v) - p(\tilde{v}_\varepsilon)| \leq \delta_3\} \subset \Omega$ ,  $Y_g(\bar{U})$  is the same as  $Y_g(v)$  in Proposition 4.2. Therefore we need show that  $Y_g(U) - Y_g(\bar{U})$ ,  $Y_b(U)$ ,  $Y_l(U)$  and  $Y_s(U)$  are negligible by the good term  $\mathcal{G}$ .

For the bad terms  $\mathcal{B}$  in (4.16), we will use the following notations :

$$(4.33) \quad \mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2^- + \mathcal{B}_2^+ + \mathcal{B}_3 + \mathcal{B}_4 + \mathcal{B}_5,$$

where

$$\begin{aligned}
\mathcal{B}_1 &= \sigma_\varepsilon \int_{\mathbb{R}} a \partial_\xi \tilde{v}_\varepsilon p(v|\tilde{v}_\varepsilon) d\xi, \\
\mathcal{B}_2^- &= \int_{\Omega^c} a' (p(v) - p(\tilde{v}_\varepsilon)) (h - \tilde{h}_\varepsilon) d\xi, \quad \mathcal{B}_2^+ = \frac{1}{2\sigma_\varepsilon} \int_{\Omega} a' |p(v) - p(\tilde{v}_\varepsilon)|^2 d\xi, \\
\mathcal{B}_3 &= - \int_{\mathbb{R}} a' v^\beta (p(v) - p(\tilde{v}_\varepsilon)) \partial_\xi (p(v) - p(\tilde{v}_\varepsilon)) d\xi, \\
\mathcal{B}_4 &= - \int_{\mathbb{R}} a' (p(v) - p(\tilde{v}_\varepsilon)) (v^\beta - \tilde{v}_\varepsilon^\beta) \partial_\xi p(\tilde{v}_\varepsilon) d\xi, \\
\mathcal{B}_5 &= - \int_{\mathbb{R}} a \partial_\xi (p(v) - p(\tilde{v}_\varepsilon)) (v^\beta - \tilde{v}_\varepsilon^\beta) \partial_\xi p(\tilde{v}_\varepsilon) d\xi.
\end{aligned}$$

We also recall the notations  $\mathcal{G}_1^-, \mathcal{G}_1^+, \mathcal{G}_2, \mathcal{D}$  in (4.18) for the good terms.

We now state the following proposition.

**Proposition 4.3.** *There exist constants  $\varepsilon_0, \delta_0, C, C^* > 0$  (in particular,  $C$  depends on the constant  $\delta_3$  in Proposition 4.1) such that for any  $\varepsilon < \varepsilon_0$  and  $\delta_0^{-1}\varepsilon < \lambda < \delta_0 < 1/2$ , the following statements hold true.*



1. For any  $U$  such that  $|Y(U)| \leq \varepsilon^2$ ,

$$(4.34) \quad |\mathcal{B}_1(U) - \mathcal{B}_1(\bar{U})| \leq C \frac{\varepsilon}{\lambda} \left( \mathcal{D}(U) + (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(\bar{U}) \right),$$

$$(4.35) \quad |\mathcal{B}_2^-(U)| \leq \delta_0 \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right) + \frac{1}{2} \mathcal{G}_1^-(U),$$

$$(4.36) \quad |\mathcal{B}_2^+(U) - \mathcal{B}_2^+(\bar{U})| \leq \sqrt{\frac{\varepsilon}{\lambda}} \mathcal{D}(U),$$

$$(4.37) \quad |\mathcal{B}_3(U)| + |\mathcal{B}_4(U)| + |\mathcal{B}_5(U)| \leq C \delta_0 \left( \mathcal{D}(U) + (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})) + \frac{\varepsilon}{\lambda} \mathcal{G}_2(\bar{U}) \right),$$

$$(4.38) \quad |\mathcal{B}(U)| \leq C^* \frac{\varepsilon^2}{\lambda} + C \sqrt{\delta_0} \mathcal{D}(U).$$

2. For any  $U$  such that  $|Y(U)| \leq \varepsilon^2$  and  $\mathcal{D}(U) \leq \frac{C^*}{4} \frac{\varepsilon^2}{\lambda}$ ,

$$(4.39) \quad \begin{aligned} & |Y_g(U) - Y_g(\bar{U})|^2 + |Y_b(U)|^2 + |Y_l(U)|^2 + |Y_s(U)|^2 \\ & \leq C \frac{\varepsilon^2}{\lambda} \left( \sqrt{\frac{\varepsilon}{\lambda}} \mathcal{D}(U) + (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})) + \mathcal{G}_1^-(U) + \left( \frac{\lambda}{\varepsilon} \right)^{1/4} \mathcal{G}_1^+(U) + \left( \frac{\varepsilon}{\lambda} \right)^{1/4} \mathcal{G}_2(\bar{U}) \right). \end{aligned}$$

To prove this proposition, we will control the bad terms in different ways for each case of small or big values of  $v$ , which all correspond to the big values of  $|p(v) - p(\tilde{v}_\varepsilon)|$  (as  $|p(v) - p(\tilde{v}_\varepsilon)| \geq \delta_3$ ). For that, we set

$$(4.40) \quad p(\bar{v}_s) - p(\tilde{v}_\varepsilon) := \psi_{\delta_3}^s(p(v) - p(\tilde{v}_\varepsilon)), \quad p(\bar{v}_b) - p(\tilde{v}_\varepsilon) := \psi_{\delta_3}^b(p(v) - p(\tilde{v}_\varepsilon)),$$

where  $\psi_{\delta_3}^s$  and  $\psi_{\delta_3}^b$  are one-sided truncations of  $\psi_{\delta_3}$  defined in (4.28), i.e.,

$$\psi_{\delta_3}^s(y) = \inf(\delta_3, y), \quad \psi_{\delta_3}^b(y) = \sup(-\delta_3, y).$$

Notice that the function  $\bar{v}_s$  (resp.  $\bar{v}_b$ ) represents the truncation of small (resp. big) values of  $v$  corresponding to  $|p(v) - p(\tilde{v}_\varepsilon)| \geq \delta_3$ .

By comparing the definitions of (4.29) and (4.40), we see

$$(4.41) \quad \begin{aligned} (p(\bar{v}_s) - p(\tilde{v}_\varepsilon)) \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) \geq -\delta_3\}} &= (p(\bar{v}) - p(\tilde{v}_\varepsilon)) \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) \geq -\delta_3\}}, \\ (p(\bar{v}_b) - p(\tilde{v}_\varepsilon)) \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) \leq \delta_3\}} &= (p(\bar{v}) - p(\tilde{v}_\varepsilon)) \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) \leq \delta_3\}}. \end{aligned}$$

We also note that

$$(4.42) \quad \begin{aligned} p(v) - p(\bar{v}_s) &= (p(v) - p(\tilde{v}_\varepsilon)) + (p(\tilde{v}_\varepsilon) - p(\bar{v}_s)) \\ &= (I - \psi_{\delta_3}^s)(p(v) - p(\tilde{v}_\varepsilon)) \\ &= ((p(v) - p(\tilde{v}_\varepsilon)) - \delta_3)_+, \\ p(\bar{v}_b) - p(v) &= (p(\bar{v}_b) - p(\tilde{v}_\varepsilon)) + (p(\tilde{v}_\varepsilon) - p(v)) \\ &= (\psi_{\delta_3}^b - I)(p(v) - p(\tilde{v}_\varepsilon)) \\ &= (-(p(v) - p(\tilde{v}_\varepsilon)) - \delta_3)_+, \\ |p(v) - p(\bar{v})| &= |(p(v) - p(\tilde{v}_\varepsilon)) + (p(\tilde{v}_\varepsilon) - p(\bar{v}))| \\ &= |(I - \psi_{\delta_3})(p(v) - p(\tilde{v}_\varepsilon))| \\ &= (|p(v) - p(\tilde{v}_\varepsilon)| - \delta_3)_+. \end{aligned}$$

Therefore, using (4.29), (4.40) and (4.42), we have

(4.43)

$$\begin{aligned} \mathcal{D}(U) &= \int_{\mathbb{R}} av^\beta |\partial_\xi(p(v) - p(\tilde{v}_\varepsilon))|^2 d\xi \\ &= \int_{\mathbb{R}} av^\beta |\partial_\xi(p(v) - p(\tilde{v}_\varepsilon))|^2 (\mathbf{1}_{\{|p(v)-p(\tilde{v}_\varepsilon)| \leq \delta_3\}} + \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon) > \delta_3\}} + \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon) < -\delta_3\}}) d\xi \\ &= \mathcal{D}(\bar{U}) + \int_{\mathbb{R}} av^\beta |\partial_\xi(p(v) - p(\bar{v}_s))|^2 d\xi + \int_{\mathbb{R}} av^\beta |\partial_\xi(p(v) - p(\bar{v}_b))|^2 d\xi \\ &\geq \int_{\mathbb{R}} av^\beta |\partial_\xi(p(v) - p(\bar{v}_s))|^2 d\xi + \int_{\mathbb{R}} av^\beta |\partial_\xi(p(v) - p(\bar{v}_b))|^2 d\xi. \end{aligned}$$

On the other hand, since  $Q(v|\tilde{v}_\varepsilon) \geq Q(\bar{v}|\tilde{v}_\varepsilon)$ , we have

$$(4.44) \quad |\sigma_\varepsilon| \int_{\mathbb{R}} |a'| Q(v|\tilde{v}_\varepsilon) d\xi \geq \mathcal{G}_2(U) - \mathcal{G}_2(\bar{U}) = |\sigma_\varepsilon| \int_{\mathbb{R}} |a'| (Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon)) d\xi \geq 0,$$

which together with (4.30) yields

$$(4.45) \quad 0 \leq \mathcal{G}_2(U) - \mathcal{G}_2(\bar{U}) \leq \mathcal{G}_2(U) \leq C \frac{\varepsilon^2}{\lambda}.$$

We first present a series of following lemmas.

**Lemma 4.5.** *Under the same assumption as Proposition 4.3, we have*

$$(4.46) \quad \int_{\mathbb{R}} |a'| |p(v) - p(\bar{v}_b)|^2 d\xi + \int_{\mathbb{R}} |a'| |p(v) - p(\bar{v}_b)| d\xi \leq \sqrt{\frac{\varepsilon}{\lambda}} \mathcal{D}(U),$$

$$(4.47) \quad \int_{\mathbb{R}} |a'| \left| |p(v) - p(\tilde{v}_\varepsilon)|^2 - |p(\bar{v}_b) - p(\tilde{v}_\varepsilon)|^2 \right| \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon) \leq \delta_3\}} d\xi \leq \sqrt{\frac{\varepsilon}{\lambda}} \mathcal{D}(U).$$

$$(4.48) \quad \int_{\mathbb{R}} |a'|^2 v^\beta |p(v) - p(\bar{v})|^2 d\xi + \int_{\mathbb{R}} |a'|^2 v^\beta |p(v) - p(\bar{v})| d\xi \leq C \lambda^2 \left( \mathcal{D}(U) + \frac{\varepsilon}{\lambda} \mathcal{G}_2(U) \right),$$

$$(4.49) \quad \int_{\mathbb{R}} |a'|^2 \left| v^\beta |p(v) - p(\tilde{v}_\varepsilon)|^2 - \bar{v}^\beta |p(\bar{v}) - p(\tilde{v}_\varepsilon)|^2 \right| d\xi \leq C \lambda^2 \left( \mathcal{D}(U) + \frac{\varepsilon}{\lambda} \mathcal{G}_2(U) \right).$$

*Proof. Proof of (4.46):* We split the proof into two steps.

*Step 1:* Note first that since  $(y - \delta_3/2)_+ \geq \delta_3/2$  whenever  $(y - \delta_3)_+ > 0$ , we have

$$(4.50) \quad (y - \delta_3)_+ \leq (y - \delta_3/2)_+ \mathbf{1}_{\{y - \delta_3 > 0\}} \leq (y - \delta_3/2)_+ \left( \frac{(y - \delta_3/2)_+}{\delta_3/2} \right) \leq \frac{2}{\delta_3} (y - \delta_3/2)_+^2.$$

Hence, to show (4.46), it is enough to show it only for the quadratic part, with  $\bar{v}_b$  defined with  $\delta_3/2$  instead of  $\delta_3$ . We will keep the notation  $\bar{v}_b$  in Step 2 below.

*Step 2:* First, using (4.43), we find that for any  $\xi \in \mathbb{R}$ ,

$$\begin{aligned} |p(v) - p(\bar{v}_b)(\xi)| &\leq \int_{\xi_0}^{\xi} |\partial_\xi(p(v) - p(\bar{v}_b))| \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon) < -\delta_3\}} d\xi \\ (4.51) \quad &\leq C \int_{\xi_0}^{\xi} v^{\beta/2} |\partial_\xi(p(v) - p(\bar{v}_b))| \mathbf{1}_{\{p(v)-p(\tilde{v}_\varepsilon) < -\delta_3\}} d\xi \\ &\leq C \sqrt{|\xi| + \frac{1}{\varepsilon}} \sqrt{\mathcal{D}(U)}. \end{aligned}$$

For any  $\xi$  such that  $|(p(v) - p(\bar{v}))(\xi)| > 0$ , we have from (4.42) that  $|(p(v) - p(\tilde{v}_\varepsilon))(\xi)| > \delta_3$ . Thus using (3.11) and (3.8), we have  $Q(v(\xi)|\tilde{v}_\varepsilon(\xi)) \geq \alpha$ , for some constant  $\alpha > 0$  depending only on  $\delta_3$ . Thus,

$$(4.52) \quad \mathbf{1}_{\{|p(v) - p(\bar{v})| > 0\}} \leq \frac{Q(v|\tilde{v}_\varepsilon)}{\alpha}.$$

Since  $\mathbf{1}_{\{|p(v) - p(\bar{v}_b)| > 0\}} \leq \mathbf{1}_{\{|p(v) - p(\bar{v})| > 0\}}$ , using (4.51), (4.52) and (4.30), we estimate

$$(4.53) \quad \begin{aligned} & \int_{\mathbb{R}} |a'| |p(v) - p(\bar{v}_b)|^2 d\xi \\ & \leq \int_{|\xi| \leq \frac{1}{\varepsilon} \sqrt{\frac{\lambda}{\varepsilon}}} |a'| |p(v) - p(\bar{v}_b)|^2 d\xi + \int_{|\xi| \geq \frac{1}{\varepsilon} \sqrt{\frac{\lambda}{\varepsilon}}} |a'| |p(v) - p(\bar{v}_b)|^2 d\xi \\ & \leq \left( \sup_{[-\sqrt{\frac{\lambda}{\varepsilon^3}}, \sqrt{\frac{\lambda}{\varepsilon^3}}]} |p(v) - p(\bar{v}_s)| \right) \int_{|\xi| \leq \frac{1}{\varepsilon} \sqrt{\frac{\lambda}{\varepsilon}}} |a'| \mathbf{1}_{\{|p(v) - p(\bar{v})| > 0\}} d\xi \\ & \quad + C\mathcal{D}(U) \int_{|\xi| \geq \frac{1}{\varepsilon} \sqrt{\frac{\lambda}{\varepsilon}}} |a'| \left( |\xi| + \frac{1}{\varepsilon} \right) d\xi \\ & \leq C\mathcal{D}(U) \left( \sqrt{\frac{\lambda}{\varepsilon^3}} \int_{\mathbb{R}} |a'| \frac{Q(v|\tilde{v}_\varepsilon)}{\alpha} d\xi + 2 \int_{|\xi| \geq \frac{1}{\varepsilon} \sqrt{\frac{\lambda}{\varepsilon}}} |a'| |\xi| d\xi \right). \end{aligned}$$

Therefore we have

$$\int_{\mathbb{R}} |a'| |p(v) - p(\bar{v}_b)|^2 d\xi \leq C \sqrt{\frac{\varepsilon}{\lambda}} \mathcal{D}(U).$$

Indeed, using (4.30) and (4.1) (recalling  $|a'| = (\lambda/\varepsilon)|\tilde{v}'_\varepsilon|$ ), we have

$$\int_{\mathbb{R}} |a'| |p(v) - p(\bar{v})|^2 d\xi \leq C\mathcal{D}(U) \left( \sqrt{\frac{\varepsilon}{\lambda}} + \lambda\varepsilon \int_{|\xi| \geq \frac{1}{\varepsilon} \sqrt{\frac{\lambda}{\varepsilon}}} e^{-c\varepsilon|\xi|} |\xi| d\xi \right),$$

and for the last term, we take  $\delta_0$  small enough such that for any  $\varepsilon/\lambda \leq \delta_0$ ,

$$\lambda\varepsilon \int_{|\xi| \geq \frac{1}{\varepsilon} \sqrt{\frac{\lambda}{\varepsilon}}} e^{-c\varepsilon|\xi|} |\xi| d\xi = \frac{\lambda}{\varepsilon} \int_{|\xi| \geq \sqrt{\frac{\lambda}{\varepsilon}}} e^{-c|\xi|} |\xi| d\xi \leq \frac{\lambda}{\varepsilon} \int_{|\xi| \geq \sqrt{\frac{\lambda}{\varepsilon}}} e^{-\frac{c}{2}|\xi|} d\xi = \frac{2\lambda}{c\varepsilon} e^{-\frac{c}{2}\sqrt{\frac{\lambda}{\varepsilon}}} \leq \sqrt{\frac{\varepsilon}{\lambda}}.$$

As mentioned in Step 1, recall that  $\bar{v}_b = \bar{v}_{b_{\delta_3/2}}$  in the above estimate. Then using (4.42), we have

$$(4.54) \quad \begin{aligned} \int_{\mathbb{R}} |a'|^2 v^\beta |p(v) - p(\bar{v}_{b_{\delta_3}})|^2 d\xi &= \int_{\mathbb{R}} |a'|^2 v^\beta (-(p(v) - p(\tilde{v}_\varepsilon)) - \delta_3)_+^2 d\xi \\ &\leq \int_{\mathbb{R}} |a'|^2 v^\beta (-(p(v) - p(\tilde{v}_\varepsilon)) - \delta_3/2)_+^2 d\xi \\ &= \int_{\mathbb{R}} |a'|^2 v^\beta |p(v) - p(\bar{v}_{b_{\delta_3/2}})|^2 d\xi \\ &\leq C \sqrt{\frac{\varepsilon}{\lambda}} \mathcal{D}(U). \end{aligned}$$

For the linear part, using (4.42) and (4.50) with  $y := -(p(v) - p(\tilde{v}_\varepsilon))$ , we have

$$(4.55) \quad \begin{aligned} \int_{\mathbb{R}} |a'|^2 v^\beta |p(v) - p(\bar{v}_{b_{\delta_3}})| d\xi &\leq \frac{2}{\delta_3} \int_{\mathbb{R}} |a'|^2 v^\beta |p(v) - p(\bar{v}_{b_{\delta_3/2}})|^2 d\xi \\ &\leq C \sqrt{\frac{\varepsilon}{\lambda}} \mathcal{D}(U). \end{aligned}$$

Hence, we obtain (4.46).

**Proof of (4.47):** Since it follows from (4.41) that

$$|p(\bar{v}_b) - p(\tilde{v}_\varepsilon)| \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) \leq \delta_3\}} = |p(\bar{v}) - p(\tilde{v}_\varepsilon)| \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) \leq \delta_3\}} \leq \delta_3,$$

using (4.46), we have

$$(4.56) \quad \begin{aligned} &\int_{\mathbb{R}} |a'| \left| |p(v) - p(\tilde{v}_\varepsilon)|^2 - |p(\bar{v}_b) - p(\tilde{v}_\varepsilon)|^2 \right| \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) \leq \delta_3\}} d\xi \\ &= \int_{\mathbb{R}} |a'| |p(v) - p(\bar{v}_b)| |p(v) + p(\bar{v}_b) - 2p(\tilde{v}_\varepsilon)| \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) \leq \delta_3\}} d\xi \\ &\leq \int_{\mathbb{R}} |a'| |p(v) - p(\bar{v}_b)| (|p(v) - p(\bar{v}_b)| + 2|p(\bar{v}_b) - p(\tilde{v}_\varepsilon)|) \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) \leq \delta_3\}} d\xi \\ &\leq \int_{\mathbb{R}} |a'| (|p(v) - p(\bar{v}_b)|^2 + 2\delta_3 |p(v) - p(\bar{v}_b)|) d\xi \leq C \sqrt{\frac{\varepsilon}{\lambda}} \mathcal{D}(U). \end{aligned}$$

**Proof of (4.48):** Thanks to (4.50), it is enough to show the quadratic part, with  $\bar{v}$  defined with  $\delta_3/2$  instead of  $\delta_3$ . For this case, we will keep the notations  $\bar{v}_s$  and  $\bar{v}_b$  below without confusion.

We first decompose the quadratic part into two parts:

$$\int_{\mathbb{R}} |a'|^2 v^\beta |p(v) - p(\bar{v})|^2 d\xi = \underbrace{\int_{\mathbb{R}} |a'|^2 v^\beta |p(v) - p(\bar{v}_b)|^2 d\xi}_{=: Q_b} + \underbrace{\int_{\mathbb{R}} |a'|^2 v^\beta |p(v) - p(\bar{v}_s)|^2 d\xi}_{=: Q_s}.$$

First, using the condition  $\beta = \gamma - \alpha \leq 1$ , we have

$$\begin{aligned} |Q_b| &= \int_{\mathbb{R}} |a'|^2 \frac{v^\beta |p(v) - p(\bar{v}_b)|^2}{|v - \bar{v}|} |v - \bar{v}| \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) < -\delta_3/2\}} d\xi \\ &\leq C \int_{\mathbb{R}} |a'|^2 |v - \bar{v}| \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) < -\delta_3/2\}} d\xi. \end{aligned}$$

To control the right hand side, we use (3.10) as follows: If  $|v - \bar{v}| > 0$ , using (3.11), we find

$$|\bar{v} - \tilde{v}_\varepsilon| \geq \min(c_5^{-1} \delta_3/2, v_-/2 - \varepsilon_0).$$

Taking  $\delta_*$  in 2) of Lemma 3.1 such that  $\varepsilon_0 \leq \delta_*/2$  and  $\min(c_5^{-1} \delta_3, v_-/2 - \varepsilon_0) \geq \delta_*$ , we use (3.10) with  $w = \tilde{v}_\varepsilon$ ,  $u = \bar{v}$  and  $v = v$  to find that there exists a constant  $C > 0$  such that

$$(4.57) \quad C|v - \bar{v}| \leq Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon).$$

Therefore, using  $|a'| \leq \varepsilon \lambda$ , we find

$$(4.58) \quad |Q_b| \leq C \int_{\mathbb{R}} |a'|^2 (Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon)) d\xi \leq C \varepsilon \lambda \mathcal{G}_2(U).$$

On the other hand, to control  $Q_s$ , we will first derive a point-wise estimate (4.61) as below:

Using  $|a'| = (\lambda/\varepsilon)|\tilde{v}'_\varepsilon|$ , together with (4.2) and (4.30), we get

$$\begin{aligned} 2\varepsilon \int_{-1/\varepsilon}^{1/\varepsilon} Q(v|\tilde{v}_\varepsilon) d\xi &\leq \frac{2\varepsilon}{\inf_{[-1/\varepsilon, 1/\varepsilon]} |a'|} \int_{\mathbb{R}} |a'| Q(v|\tilde{v}_\varepsilon) d\xi \\ &\leq C \frac{\varepsilon}{\lambda \varepsilon} \frac{\varepsilon^2}{\lambda} = C \left( \frac{\varepsilon}{\lambda} \right)^2. \end{aligned}$$

Therefore, there exists  $\xi_0 \in [-1/\varepsilon, 1/\varepsilon]$  such that  $Q(v(\xi_0), \tilde{v}_\varepsilon(\xi_0)) \leq C(\varepsilon/\lambda)^2$ . For  $\delta_0$  small enough, and using (3.16), we have

$$|(p(v) - p(\tilde{v}_\varepsilon))(\xi_0)| \leq C \frac{\varepsilon}{\lambda}.$$

Thus, if  $\delta_0$  is small enough such that  $C\varepsilon/\lambda \leq \delta_3/2$ , then we have from the definition of  $\bar{v}_s$  that

$$(p(v) - p(\bar{v}_s))(\xi_0) = 0.$$

Therefore, for any  $\xi \in \mathbb{R}$ ,

$$|v^{\beta/2}(p(v) - p(\bar{v}_s))(\xi)| = \left| \int_{\xi_0}^{\xi} \partial_\xi (v^{\beta/2}(p(v) - p(\bar{v}_s))) d\zeta \right|.$$

To control the right-hand side by the good terms, we observe that since  $v^{\beta/2} = p(v)^{-(\gamma-\alpha)/2\gamma}$ , we have

$$\begin{aligned} \partial_\xi (v^{\beta/2}(p(v) - p(\bar{v}_s))) &= \partial_\xi (p(v)^{-(\gamma-\alpha)/2\gamma} ((p(v) - p(\bar{v}_s)))) \\ &= p(v)^{-(\gamma-\alpha)/2\gamma} \partial_\xi ((p(v) - p(\bar{v}_s))) \\ &\quad - \frac{\gamma-\alpha}{2\gamma} p(v)^{-(\gamma-\alpha)/2\gamma} \frac{p(v) - p(\bar{v}_s)}{p(v)} \partial_\xi [(p(v) - p(\tilde{v}_\varepsilon)) + p(\tilde{v}_\varepsilon)] \\ &= v^{\beta/2} \partial_\xi ((p(v) - p(\bar{v}_s))) - \underbrace{\frac{\gamma-\alpha}{2\gamma} v^{\beta/2} \frac{p(v) - p(\bar{v}_s)}{p(v)} \partial_\xi ((p(v) - p(\tilde{v}_\varepsilon)))}_{=:K} \\ &\quad - \frac{\gamma-\alpha}{2\gamma} v^{\beta/2} \frac{p(v) - p(\bar{v}_s)}{p(v)} \partial_\xi p(\tilde{v}_\varepsilon). \end{aligned}$$

In particular, note that (by the definition of  $\bar{v}_s$ ) the part  $K$  above can be rewritten by

$$K = \frac{p(v) - p(\bar{v}_s)}{p(v)} \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) > \delta_3/2\}} \partial_\xi ((p(v) - p(\tilde{v}_\varepsilon))) = \frac{p(v) - p(\bar{v}_s)}{p(v)} \partial_\xi ((p(v) - p(\bar{v}_s))).$$

Then, using  $|\partial_\xi p(\tilde{v}_\varepsilon)| \leq C|\tilde{v}'_\varepsilon|$  and

$$\frac{p(v) - p(\bar{v}_s)}{p(v)} \leq C,$$

we have

$$|\partial_\xi (v^{\beta/2}(p(v) - p(\bar{v}_s)))| \leq C v^{\beta/2} (|\partial_\xi ((p(v) - p(\bar{v}_s)))| + |\tilde{v}'_\varepsilon|).$$

Therefore, using (4.43), we have that for any  $\xi \in \mathbb{R}$ ,

$$\begin{aligned}
|v^{\beta/2}(p(v) - p(\bar{v}_s))(\xi)| &= \left| \int_{\xi_0}^{\xi} \partial_{\xi} (v^{\beta/2}(p(v) - p(\bar{v}_s))) d\xi \right| \\
&\leq \int_{\xi_0}^{\xi} |\partial_{\xi} (v^{\beta/2}(p(v) - p(\bar{v}_s)))| \mathbf{1}_{\{p(v) - p(\bar{v}_\varepsilon) > \delta_3/2\}} d\xi \\
&\leq \int_{\xi_0}^{\xi} v^{\beta/2} (|\partial_{\xi} (p(v) - p(\bar{v}_s))| + |\tilde{v}'_{\varepsilon}|) \mathbf{1}_{\{p(v) - p(\bar{v}_\varepsilon) > \delta_3/2\}} d\xi \\
&\leq C \sqrt{|\xi| + \frac{1}{\varepsilon}} \left( \sqrt{\mathcal{D}(U)} + \sqrt{\int_{\mathbb{R}} a |\tilde{v}'_{\varepsilon}|^2 v^{\beta} \mathbf{1}_{\{p(v) - p(\bar{v}_\varepsilon) > \delta_3/2\}} d\xi} \right).
\end{aligned}$$

Using the condition  $\beta = \gamma - \alpha > 0$ , we have

$$\begin{aligned}
(4.59) \quad \int_{\mathbb{R}} a |\tilde{v}'_{\varepsilon}|^2 v^{\beta} \mathbf{1}_{\{p(v) - p(\bar{v}_\varepsilon) > \delta_3/2\}} d\xi &= \int_{\mathbb{R}} a |\tilde{v}'_{\varepsilon}|^2 \frac{v^{\beta}}{|v - \tilde{v}_{\varepsilon}|^2} |v - \tilde{v}_{\varepsilon}|^2 \mathbf{1}_{\{p(v) - p(\bar{v}_\varepsilon) > \delta_3/2\}} d\xi \\
&\leq C \int_{\mathbb{R}} a |\tilde{v}'_{\varepsilon}|^2 |v - \tilde{v}_{\varepsilon}|^2 \mathbf{1}_{\{p(v) - p(\bar{v}_\varepsilon) > \delta_3/2\}} d\xi.
\end{aligned}$$

In addition, using (3.8) and  $|\tilde{v}'_{\varepsilon}| \leq C \frac{\varepsilon}{\lambda} |a'|$ , we have

$$(4.60) \quad \int_{\mathbb{R}} a |\tilde{v}'_{\varepsilon}|^2 |v - \tilde{v}_{\varepsilon}|^2 \mathbf{1}_{\{p(v) - p(\bar{v}_\varepsilon) > \delta_3/2\}} d\xi \leq C \frac{\varepsilon^2}{\lambda^2} \int_{\mathbb{R}} |a'| Q(v|\tilde{v}_{\varepsilon}) d\xi.$$

Therefore we obtain that

$$(4.61) \quad \forall \xi \in \mathbb{R}, \quad |v^{\beta/2}(p(v) - p(\bar{v}_s))(\xi)| \leq C \sqrt{|\xi| + \frac{1}{\varepsilon}} \left( \sqrt{\mathcal{D}(U)} + \frac{\varepsilon}{\lambda} \sqrt{\mathcal{G}_2(U)} \right).$$

Now, using (4.14) with (4.1), we have

$$\begin{aligned}
Q_s &\leq C \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right) \lambda^2 \varepsilon \int_{\mathbb{R}} e^{-C|\varepsilon\xi|} (|\varepsilon\xi| + 1) d\xi \\
&\leq C \lambda^2 \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right).
\end{aligned}$$

Therefore, this and (4.58) complete the estimate:

$$\int_{\mathbb{R}} |a'|^2 v^{\beta} |p(v) - p(\bar{v})|^2 d\xi \leq C \lambda^2 \left( \mathcal{D}(U) + \frac{\varepsilon}{\lambda} \mathcal{G}_2(U) \right).$$

Hence using the similar estimates as in (4.54) and (4.55) (i.e., using (4.42) and (4.50) with  $y := |p(v) - p(\bar{v}_\varepsilon)|$ ), we obtain (4.48).

**Proof of (4.49):** We first separate it into two parts:

$$\begin{aligned}
&\int_{\mathbb{R}} |a'|^2 \left| v^{\beta} |p(v) - p(\bar{v}_\varepsilon)|^2 - \bar{v}^{\beta} |p(\bar{v}) - p(\bar{v}_\varepsilon)|^2 \right| d\xi \\
&\leq \underbrace{\int_{\mathbb{R}} |a'|^2 v^{\beta} \left| |p(v) - p(\bar{v}_\varepsilon)|^2 - |p(\bar{v}) - p(\bar{v}_\varepsilon)|^2 \right| d\xi}_{=: I_1} + \underbrace{\int_{\mathbb{R}} |a'|^2 v^{\beta} - \bar{v}^{\beta} |p(\bar{v}) - p(\bar{v}_\varepsilon)|^2 d\xi}_{=: I_2}.
\end{aligned}$$

Using the same arguments as in (4.56), it follows from (4.48) that

$$\begin{aligned} I_1 &\leq \int_{\mathbb{R}} |a'|^2 v^\beta (|p(v) - p(\bar{v})|^2 + 2\delta_3 |p(v) - p(\bar{v})|) d\xi \\ &\leq C\lambda^2 \left( \mathcal{D}(U) + \frac{\varepsilon}{\lambda} \mathcal{G}_2(U) \right). \end{aligned}$$

For  $I_2$ , we first separate  $I_2$  into two parts:

$$I_2 = \underbrace{\int_{\mathbb{R}} |a'|^2 |v^\beta - \bar{v}_b^\beta| |p(\bar{v}_b) - p(\tilde{v}_\varepsilon)|^2 d\xi}_{=: I_2^b} + \underbrace{\int_{\mathbb{R}} |a'|^2 |v^\beta - \bar{v}_s^\beta| |p(\bar{v}_s) - p(\tilde{v}_\varepsilon)|^2 d\xi}_{=: I_2^s}.$$

Using the assumption  $\beta \leq 1$ , we have

$$\begin{aligned} I_2^b &\leq \delta_3^2 \int_{\mathbb{R}} |a'|^2 \frac{|v^\beta - \bar{v}_b^\beta|}{|v - \bar{v}|} |v - \bar{v}| \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) < -\delta_3\}} d\xi \\ &\leq C \int_{\mathbb{R}} |a'|^2 |v - \bar{v}| \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) < -\delta_3\}} d\xi \\ &\leq C \int_{\mathbb{R}} |a'|^2 (Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon)) d\xi \leq C\varepsilon\lambda\mathcal{G}_2(U). \end{aligned}$$

Likewise, we use  $\beta > 0$  to have

$$\begin{aligned} I_2^s &\leq \delta_3^2 \int_{\mathbb{R}} |a'|^2 \frac{|v^\beta - \bar{v}_s^\beta|}{|v - \bar{v}|} |v - \bar{v}| \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) > \delta_3\}} d\xi \\ &\leq C \int_{\mathbb{R}} |a'|^2 |v - \bar{v}| \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) > \delta_3\}} d\xi \leq C\varepsilon\lambda\mathcal{G}_2(U). \end{aligned}$$

□

**Lemma 4.6.** *Under the same assumption as Proposition 4.3, we have*

$$\begin{aligned} (4.62) \quad &\int_{\mathbb{R}} |a'| |p(v|\tilde{v}_\varepsilon) - p(\bar{v}|\tilde{v}_\varepsilon)| d\xi \\ &\leq C\sqrt{\frac{\varepsilon}{\lambda}} \left( \mathcal{D}(U) + \left(\frac{\varepsilon}{\lambda}\right)^2 \mathcal{G}_2(\bar{U}) \right) + C (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})), \end{aligned}$$

$$(4.63) \quad \int_{\mathbb{R}} |a'| |Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon)| d\xi + \int_{\mathbb{R}} |a'| |v - \bar{v}| d\xi \leq C (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})).$$

*Proof.* Following the proof of [32, Lemma 3.3] together with (4.45), we have

$$\begin{aligned} &\int_{\mathbb{R}} |a'| |Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon)| d\xi + \int_{\mathbb{R}} |a'| |v - \bar{v}| d\xi \\ &\leq C \int_{\mathbb{R}} |a'| (Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon)) d\xi \leq C (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})). \end{aligned}$$

Following the proof of [32, Lemma 3.3], we have

$$\begin{aligned} &\int_{\mathbb{R}} |a'| |p(v|\tilde{v}_\varepsilon) - p(\bar{v}|\tilde{v}_\varepsilon)| d\xi \\ &\leq \underbrace{\int_{\mathbb{R}} |a'| |p(v) - p(\bar{v})| d\xi}_{=: I_1} + C \underbrace{\int_{\mathbb{R}} |a'| (Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon)) d\xi}_{=: I_2}. \end{aligned}$$

First, using (4.45), we have

$$I_2 \leq C \left( \mathcal{G}_2(U) - \mathcal{G}_2(\bar{U}) \right).$$

We separate  $I_1$  into three parts:

$$I_1 = \underbrace{\int_{\mathbb{R}} |a'| |p(v) - p(\bar{v}_b)| d\xi}_{=: I_{11}} + \underbrace{\int_{\mathbb{R}} |a'| |p(v) - p(\bar{v}_s)| \mathbf{1}_{\{v < v_-/2\}} d\xi}_{=: I_{12}} + \underbrace{\int_{\mathbb{R}} |a'| |p(v) - p(\bar{v}_s)| \mathbf{1}_{\{v \geq v_-/2\}} d\xi}_{=: I_{13}}.$$

By (4.46), we have

$$I_{11} \leq C \sqrt{\frac{\varepsilon}{\lambda}} \mathcal{D}(U).$$

For  $I_{12}$ , we first observe that

$$\begin{aligned} v^\beta |p(v) - p(\bar{v}_s)|^2 &= p(v)^{-\frac{\gamma-\alpha}{\gamma}} |p(v) - p(\bar{v}_s)|^2 \mathbf{1}_{\{p(v)-p(\bar{v}_\varepsilon) > \delta_3\}} \\ (4.64) \quad &= \left( \frac{|p(v) - p(\bar{v}_s)|}{p(v)} \right)^{\frac{\gamma-\alpha}{\gamma}} \mathbf{1}_{\{p(v)-p(\bar{v}_\varepsilon) > \delta_3\}} |p(v) - p(\bar{v}_s)|^{\frac{\gamma+\alpha}{\gamma}} \\ &\geq C \delta_3^{\frac{\gamma-\alpha}{\gamma}} |p(v) - p(\bar{v}_s)|^{\frac{\gamma+\alpha}{\gamma}}. \end{aligned}$$

Since (by the smallness of  $\varepsilon_0$  and  $\delta_3$ )

$$\begin{aligned} |p(v) - p(\bar{v}_s)| \mathbf{1}_{\{v < v_-/2\}} &\geq |p(v) - p(\bar{v}_\varepsilon)| \mathbf{1}_{\{v < v_-/2\}} - |p(\bar{v}_s) - p(\bar{v}_\varepsilon)| \mathbf{1}_{\{v < v_-/2\}} \\ (4.65) \quad &\geq |p(v) - p(\bar{v}_\varepsilon)| \mathbf{1}_{\{v < v_-/2\}} - \delta_3 \\ &\geq |p(v_-/2) - p(3v_-/4)| - \delta_3 \\ &\geq \frac{1}{2} |p(v_-/2) - p(3v_-/4)|, \end{aligned}$$

using (4.64), we have

$$|p(v) - p(\bar{v}_s)| \mathbf{1}_{\{v < v_-/2\}} \leq C |p(v) - p(\bar{v}_s)|^{\frac{\gamma+\alpha}{\gamma}} \leq C v^\beta |p(v) - p(\bar{v}_s)|^2.$$

Then it follows from (4.61) that for all  $\xi \in \mathbb{R}$ ,

$$(4.66) \quad |p(v) - p(\bar{v}_s)| \mathbf{1}_{\{v < v_-/2\}}(\xi) \leq C \left( |\xi| + \frac{1}{\varepsilon} \right) \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right).$$

Therefore, using (4.66) together with the same estimate as in (4.53), we have

$$\begin{aligned} (4.67) \quad I_{12} &\leq \int_{|\xi| \leq \frac{1}{\varepsilon} \sqrt{\frac{\lambda}{\varepsilon}}} |a'| |p(v) - p(\bar{v}_s)| \mathbf{1}_{\{v < v_-/2\}} d\xi + \int_{|\xi| \geq \frac{1}{\varepsilon} \sqrt{\frac{\lambda}{\varepsilon}}} |a'| |p(v) - p(\bar{v}_s)| \mathbf{1}_{\{v < v_-/2\}} d\xi \\ &\leq C \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right) \left( \sqrt{\frac{\lambda}{\varepsilon^3}} \int_{\mathbb{R}} |a'| Q(v|\bar{v}_\varepsilon) d\xi + 2 \int_{|\xi| \geq \frac{1}{\varepsilon} \sqrt{\frac{\lambda}{\varepsilon}}} |a'| |\xi| d\xi \right) \\ &\leq C \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right) \left( \sqrt{\frac{\varepsilon}{\lambda}} + \lambda \varepsilon \int_{|\xi| \geq \frac{1}{\varepsilon} \sqrt{\frac{\lambda}{\varepsilon}}} e^{-c\varepsilon|\xi|} |\xi| d\xi \right) \\ &\leq C \sqrt{\frac{\varepsilon}{\lambda}} \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right). \end{aligned}$$



For  $I_{13}$ , since  $|p(v) - p(\bar{v}_s)| \mathbf{1}_{\{v \geq v_-/2\}} \leq p(v_-/2)$ , we have

$$\begin{aligned} I_{13} &= \int_{\mathbb{R}} |a'| \frac{|p(v) - p(\bar{v}_s)|}{|v - \bar{v}|} |v - \bar{v}| \mathbf{1}_{\{v \geq v_-/2\} \cap \{p(v) - p(\bar{v}_\varepsilon) > \delta_3\}} d\xi \\ &\leq C \int_{\mathbb{R}} |a'| |v - \bar{v}| \mathbf{1}_{\{v \geq v_-/2\} \cap \{p(v) - p(\bar{v}_\varepsilon) > \delta_3\}} d\xi \leq C (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})). \end{aligned}$$

Hence, we have

$$I_1 \leq C \sqrt{\frac{\varepsilon}{\lambda}} \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right) + C (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})),$$

which gives (4.62). □

**Lemma 4.7.** *Under the same assumption as Proposition 4.3, we have*

$$(4.68) \quad \int_{\Omega^c} |a'| |p(v) - p(\bar{v}_\varepsilon)| |h - \tilde{h}_\varepsilon| d\xi \leq \delta_0 \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right) + \frac{1}{2} \mathcal{G}_1^-(U),$$

$$(4.69) \quad \int_{\Omega^c} |a'| (Q(\bar{v}|\tilde{v}_\varepsilon) + |\bar{v} - \tilde{v}_\varepsilon|) d\xi \leq C \sqrt{\frac{\varepsilon}{\lambda}} \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right).$$

*Proof.* **Proof of (4.68):** We first separate it into two parts:

$$\begin{aligned} &\int_{\Omega^c} |a'| |p(v) - p(\bar{v}_\varepsilon)| |h - \tilde{h}_\varepsilon| d\xi \\ &\leq \underbrace{\int_{\Omega^c} |a'| |p(v) - p(\bar{v})| |h - \tilde{h}_\varepsilon| d\xi}_{=: J_1} + \underbrace{\int_{\Omega^c} |a'| |p(\bar{v}) - p(\bar{v}_\varepsilon)| |h - \tilde{h}_\varepsilon| d\xi}_{=: J_2} \end{aligned}$$

We use the definition of  $\bar{v}_s$  and Hölder's inequality to have

$$\begin{aligned} J_1 &= \int_{\Omega^c} |a'| |p(v) - p(\bar{v}_s)| |h - \tilde{h}_\varepsilon| d\xi \\ &\leq \left( \int_{\Omega^c} |a'| |p(v) - p(\bar{v}_s)|^2 d\xi \right)^{1/2} \left( \int_{\Omega^c} |a'| |h - \tilde{h}_\varepsilon|^2 d\xi \right)^{1/2}. \end{aligned}$$

To estimate  $\int_{\mathbb{R}} |a'| |p(v) - p(\bar{v}_s)|^2 d\xi$ , using (4.64) and (4.61), we find that for any  $\xi \in \mathbb{R}$ ,

$$(4.70) \quad |(p(v) - p(\bar{v}_s))(\xi)|^2 \leq C \left( |\xi| + \frac{1}{\varepsilon} \right)^{\frac{2\gamma}{\gamma+\alpha}} \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right)^{\frac{2\gamma}{\gamma+\alpha}}.$$

Following the similar arguments as in (4.53), and using (4.70) with  $q := \frac{2\gamma}{\gamma+\alpha}$  (note  $1 < q < 2$  by  $0 < \alpha < \gamma$ ), we obtain

$$\begin{aligned}
& \int_{\mathbb{R}} |a'| |p(v) - p(\bar{v}_s)|^2 d\xi \\
& \leq \int_{|\xi| \leq \frac{1}{\varepsilon} (\frac{\lambda}{\varepsilon})^{1/q}} |a'| |p(v) - p(\bar{v}_s)|^2 d\xi + \int_{|\xi| \geq \frac{1}{\varepsilon} (\frac{\lambda}{\varepsilon})^{1/q}} |a'| |p(v) - p(\bar{v}_s)|^2 d\xi \\
& \leq C \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right)^q \left( \frac{1}{\varepsilon^q} \int_{\mathbb{R}} |a'| Q(v|\bar{v}_\varepsilon) d\xi + 2 \int_{|\xi| \geq (\frac{\lambda}{\varepsilon})^{1/q}} |a'| |\xi|^q d\xi \right) \\
& \leq C \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right)^q \left( \frac{1}{\varepsilon^q} \frac{\varepsilon^2}{\lambda} + \frac{\lambda}{\varepsilon^q} \int_{|\xi| \geq (\frac{\lambda}{\varepsilon})^{1/q}} |\xi|^q e^{-c|\xi|} d\xi \right) \\
& \leq C \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right)^q \frac{\varepsilon^{2-q}}{\lambda}.
\end{aligned}$$

Therefore,

$$J_1 \leq C \sqrt{\frac{\varepsilon^{2-q}}{\lambda}} \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right)^{q/2} \left( \int_{\Omega^c} |a'| |h - \tilde{h}_\varepsilon|^2 d\xi \right)^{1/2}.$$

Using the Young's inequality (recall  $1 < q < 2$ ), we have

$$J_1 \leq \delta_0 \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right) + \frac{C}{\delta_0} \left( \frac{\varepsilon^{2-q}}{\lambda} \right)^{\frac{1}{2-q}} \left( \int_{\Omega^c} |a'| |h - \tilde{h}_\varepsilon|^2 d\xi \right)^{\frac{1}{2-q}}.$$

Since (4.30) yields

$$\begin{aligned}
\left( \frac{\varepsilon^{2-q}}{\lambda} \right)^{\frac{1}{2-q}} \left( \int_{\Omega^c} |a'| |h - \tilde{h}_\varepsilon|^2 d\xi \right)^{\frac{1}{2-q}} & \leq C \left( \frac{\varepsilon^{2-q}}{\lambda} \right)^{\frac{1}{2-q}} \left( \frac{\varepsilon^2}{\lambda} \right)^{\frac{q-1}{2-q}} \int_{\Omega^c} |a'| |h - \tilde{h}_\varepsilon|^2 d\xi \\
& = C \left( \frac{\varepsilon}{\lambda} \right)^{\frac{2}{2-q}} \int_{\Omega^c} |a'| |h - \tilde{h}_\varepsilon|^2 d\xi \\
& \leq C \left( \frac{\varepsilon}{\lambda} \right)^2 \int_{\Omega^c} |a'| |h - \tilde{h}_\varepsilon|^2 d\xi,
\end{aligned}$$

we have

$$J_1 \leq \delta_0 \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right) + \frac{C}{\delta_0} \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_1^-(U).$$

For  $J_2$ , we use  $|p(\bar{v}) - p(\bar{v}_\varepsilon)| \leq \delta_3$  and Young's inequality to have

$$J_2 \leq \delta_3 \int_{\Omega^c} |a'| |h - \tilde{h}_\varepsilon| d\xi \leq \frac{1}{2} \mathcal{G}_1^-(U) + C \underbrace{\int_{\mathbb{R}} |a'| \mathbf{1}_{\{p(v) - p(\bar{v}_\varepsilon) > \delta_3\}} d\xi}_{=: J_{21}}.$$

To control  $J_{21}$ , we observe that since  $(y - \delta_3/2)_+ \geq \delta_3/2$  whenever  $(y - \delta_3)_+ > 0$ , we have

$$(4.71) \quad |p(v) - p(\bar{v}_{\delta_3/2})| = (|p(v) - p(\bar{v}_\varepsilon)| - \delta_3/2)_+ \geq \frac{\delta_3}{2} \mathbf{1}_{\{p(v) - p(\bar{v}_\varepsilon) > \delta_3\}}.$$

Then, using (4.71) and (4.70) (with  $q := \frac{2\gamma}{\gamma+\alpha}$ ) and following the same estimates as in (4.53), we have

$$\begin{aligned} J_{21} &\leq C \int_{\mathbb{R}} |a'| |p(v) - p(\bar{v}_{\delta_3/2})|^{2/q} \mathbf{1}_{\{p(v) - p(\bar{v}_\varepsilon) > \delta_3\}} d\xi \\ &\leq C \sqrt{\frac{\varepsilon}{\lambda}} \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right). \end{aligned}$$

Therefore,

$$J_2 \leq \frac{1}{2} \mathcal{G}_1^-(U) + C \sqrt{\frac{\varepsilon}{\lambda}} \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right).$$

Hence we obtain (4.68).

**Proof of (4.69):** The proof follows from the above estimate for  $J_{12}$  as follows:

$$\int_{\Omega^c} |a'| (Q(\bar{v}|\tilde{v}_\varepsilon) + |\bar{v} - \tilde{v}_\varepsilon|) d\xi \leq C \int_{\mathbb{R}} |a'| \mathbf{1}_{\{p(v) - p(\bar{v}_\varepsilon) > \delta_3\}} d\xi \leq C \sqrt{\frac{\varepsilon}{\lambda}} \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right).$$

□

**Lemma 4.8.** *Under the same assumption as Proposition 4.3, we have*

$$(4.72) \quad \int_{\mathbb{R}} |a'|^2 \frac{|v^\beta - \bar{v}^\beta|^2}{v^\beta} d\xi \leq C\lambda \left( \mathcal{D}(U) + (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(\bar{U}) \right),$$

$$\begin{aligned} (4.73) \quad &\int_{\mathbb{R}} |a'|^2 \left| \frac{|v^\beta - \tilde{v}_\varepsilon^\beta|^2}{v^\beta} - \frac{|\bar{v}^\beta - \tilde{v}_\varepsilon^\beta|^2}{\bar{v}^\beta} \right| d\xi \\ &\leq C\lambda \left( \mathcal{D}(U) + (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(\bar{U}) \right). \end{aligned}$$

*Proof.* **Proof of (4.72):** We first have

$$\int_{\mathbb{R}} |a'|^2 \frac{|v^\beta - \bar{v}^\beta|^2}{v^\beta} d\xi = \underbrace{\int_{\mathbb{R}} |a'|^2 \frac{|v^\beta - \bar{v}_b^\beta|^2}{v^\beta} d\xi}_{=: I_b} + \underbrace{\int_{\mathbb{R}} |a'|^2 \frac{|v^\beta - \bar{v}_s^\beta|^2}{v^\beta} d\xi}_{=: I_s}.$$

Since  $0 < \beta \leq 1$ , we have

$$\begin{aligned} I_b &\leq C \int_{\mathbb{R}} |a'|^2 |v^\beta - \bar{v}_b^\beta| \mathbf{1}_{\{p(v) - p(\bar{v}_\varepsilon) < -\delta_3\}} d\xi = C \int_{\mathbb{R}} |a'|^2 \frac{|v^\beta - \bar{v}_b^\beta|}{|v - \bar{v}|} |v - \bar{v}| \mathbf{1}_{\{p(v) - p(\bar{v}_\varepsilon) < -\delta_3\}} d\xi \\ &\leq C \int_{\mathbb{R}} |a'|^2 |v - \bar{v}| \mathbf{1}_{\{p(v) - p(\bar{v}_\varepsilon) < -\delta_3\}} d\xi \leq C \int_{\mathbb{R}} |a'|^2 (Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon)) d\xi \\ &\leq C\varepsilon\lambda (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})). \end{aligned}$$

For  $I_s$ , we separate it into two cases of  $\alpha \geq 1$  and  $\alpha < 1$ .

*Case of  $\alpha \geq 1$  :* Since  $\beta \leq \gamma - 1$  by  $\alpha \geq 1$ , we observe

$$v^{-\beta} \leq v^{-\gamma+1} = Q(v) \quad \text{as } v \rightarrow 0,$$

we have

$$\begin{aligned} I_s &= \int_{\mathbb{R}} |a'|^2 \frac{v^{-\beta} |v^\beta - \bar{v}_s^\beta|^2}{Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon)} \mathbf{1}_{\{p(v) - p(\bar{v}_\varepsilon) > \delta_3\}} (Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon)) d\xi \\ &\leq C \int_{\mathbb{R}} |a'|^2 (Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon)) d\xi \leq C\varepsilon\lambda (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})). \end{aligned}$$

Case of  $0 < \alpha < 1$  : Since  $\frac{1-\alpha}{\gamma} < 1$  and

$$v^\beta p(v)^{\frac{1-\alpha}{\gamma}} Q(v) = v^\beta (v^{-\gamma})^{\frac{1-\alpha}{\gamma}} v^{-\gamma+1} = v^0 = 1,$$

using (4.57), we have

$$\begin{aligned} I_s &= \int_{\mathbb{R}} |a'|^2 \frac{|v^\beta - \bar{v}_s^\beta|^2 \mathbf{1}_{\{p(v) - p(\bar{v}_\varepsilon) > \delta_3\}}}{v^\beta |p(v) - p(\bar{v}_s)|^{\frac{1-\alpha}{\gamma}}} |p(v) - p(\bar{v}_s)|^{\frac{1-\alpha}{\gamma}} (Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon)) d\xi \\ &\leq C \int_{\mathbb{R}} |a'|^2 |p(v) - p(\bar{v}_s)|^{\frac{1-\alpha}{\gamma}} (Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon)) d\xi. \end{aligned}$$

Then, using the fact from (4.64) and (4.61) that

$$|p(v) - p(\bar{v}_s)|^{\frac{\gamma+\alpha}{\gamma}} \leq C \left( |\xi| + \frac{1}{\varepsilon} \right) \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right),$$

we have

$$I_s \leq C \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right)^{\frac{1-\alpha}{\gamma+\alpha}} \int_{\mathbb{R}} |a'|^2 \left( |\xi| + \frac{1}{\varepsilon} \right)^{\frac{1-\alpha}{\gamma+\alpha}} (Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon)) d\xi.$$

Notice that since  $0 < \frac{1-\alpha}{\gamma+\alpha} < 1$  by  $0 < \alpha < 1$ , we have

$$\begin{aligned} |a'| \left( |\xi| + \frac{1}{\varepsilon} \right)^{\frac{1-\alpha}{\gamma+\alpha}} &\leq C\varepsilon\lambda e^{-c\varepsilon|\xi|} \left( |\xi| + \frac{1}{\varepsilon} \right)^{\frac{1-\alpha}{\gamma+\alpha}} \\ &\leq C\varepsilon^{1-\frac{1-\alpha}{\gamma+\alpha}} \lambda e^{-c\varepsilon|\xi|} (|\varepsilon\xi| + 1)^{\frac{1-\alpha}{\gamma+\alpha}} \leq C\lambda. \end{aligned}$$

Thus, we have

$$I_s \leq C\lambda \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right)^{\frac{1-\alpha}{\gamma+\alpha}} (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})).$$

Now, using the Young's inequality with  $\frac{1}{p} := \frac{1-\alpha}{\gamma+\alpha}$  and  $\frac{1}{p} + \frac{1}{p'} = 1$ , and then (4.30), we have

$$\begin{aligned} I_s &\leq C\lambda \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right) + C\lambda (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U}))^{p'} \\ &\leq C\lambda \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right) + C\lambda (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})). \end{aligned}$$

Hence we complete the proof.

**Proof of (4.73):** Since  $C^{-1} \leq \bar{v}^\beta \leq C$  and  $|\bar{v}^\beta - \tilde{v}_\varepsilon^\beta| \leq C$ , we have

$$\begin{aligned}
& \int_{\mathbb{R}} |a'|^2 \left| \frac{|v^\beta - \tilde{v}_\varepsilon^\beta|^2}{v^\beta} - \frac{|\bar{v}^\beta - \tilde{v}_\varepsilon^\beta|^2}{\bar{v}^\beta} \right| d\xi \\
& \leq \int_{\mathbb{R}} |a'|^2 \left( \frac{1}{v^\beta} \left| |v^\beta - \tilde{v}_\varepsilon^\beta|^2 - |\bar{v}^\beta - \tilde{v}_\varepsilon^\beta|^2 \right| + \frac{|v^\beta - \bar{v}^\beta|}{v^\beta \bar{v}^\beta} |\bar{v}^\beta - \tilde{v}_\varepsilon^\beta|^2 \right) d\xi \\
& \leq C \int_{\mathbb{R}} |a'|^2 \left( \frac{1}{v^\beta} |v^\beta - \bar{v}^\beta| \left( |v^\beta - \bar{v}^\beta| + 2|\bar{v}^\beta - \tilde{v}_\varepsilon^\beta| \right) + \frac{|v^\beta - \bar{v}^\beta|}{v^\beta} \right) d\xi \\
& \leq C \int_{\mathbb{R}} |a'|^2 \frac{1}{v^\beta} |v^\beta - \bar{v}^\beta|^2 d\xi + C \underbrace{\int_{\mathbb{R}} |a'|^2 \frac{1}{v^\beta} |v^\beta - \bar{v}^\beta| d\xi}_{=:J}.
\end{aligned}$$

By (4.72), it remains to estimate the term  $J$ . For that, we separate it into two parts:

$$J = \underbrace{\int_{\mathbb{R}} |a'|^2 \frac{1}{v^\beta} |v^\beta - \bar{v}^\beta| \mathbf{1}_{\{v < v_-/2\} \cup \{v > 2v_-\}} d\xi}_{=:J_1} + \underbrace{\int_{\mathbb{R}} |a'|^2 \frac{1}{v^\beta} |v^\beta - \bar{v}^\beta| \mathbf{1}_{\{v_-/2 \leq v \leq 2v_-\}} d\xi}_{=:J_2}.$$

Using the same argument as (4.65) together with the definition of  $\bar{v}$ , we have

$$|v^\beta - \bar{v}^\beta| \mathbf{1}_{\{v < v_-/2\} \cup \{v > 2v_-\}} > C > 0,$$

which yields

$$J_1 \leq C \int_{\mathbb{R}} |a'|^2 \frac{1}{v^\beta} |v^\beta - \bar{v}^\beta|^2 d\xi.$$

Since

$$|v - \bar{v}| \mathbf{1}_{\{v_-/2 \leq v \leq 2v_-\}} \leq |p'(v_-/2)| |p(v) - p(\bar{v})|,$$

we have

$$\begin{aligned}
J_2 &= \int_{\mathbb{R}} |a'|^2 \frac{1}{v^\beta} \frac{|v^\beta - \bar{v}^\beta|}{|v - \bar{v}|} |v - \bar{v}| \mathbf{1}_{\{v_-/2 \leq v \leq 2v_-\}} d\xi \\
&\leq C \int_{\mathbb{R}} |a'|^2 |v - \bar{v}| \mathbf{1}_{\{v_-/2 \leq v \leq 2v_-\}} d\xi \\
&\leq C \int_{\mathbb{R}} |a'|^2 v^\beta |p(v) - p(\bar{v})| \mathbf{1}_{\{v_-/2 \leq v \leq 2v_-\}} d\xi.
\end{aligned}$$

Therefore, (4.48) and (4.72) give the desired result.  $\square$

**4.8.1. Proof of Proposition 4.3. Proof of (4.34):** It follows from (4.62) together with  $|\tilde{v}'_\varepsilon| \leq C \frac{\varepsilon}{\lambda} |a'|$  that

$$|\mathcal{B}_1(U) - \mathcal{B}_1(\bar{U})| \leq C \frac{\varepsilon}{\lambda} \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(\bar{U}) + (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})) \right).$$

**Proof of (4.35):** By (4.68), we have

$$|\mathcal{B}_2^-(U)| \leq \delta_0 \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right) + \frac{1}{2} \mathcal{G}_1^-(U).$$

**Proof of (4.36):** We use (4.41) and (4.47) to have

$$|\mathcal{B}_2^+(U) - \mathcal{B}_2^+(\bar{U})| = \left| \int_{\mathbb{R}} a' (|p(v) - p(\tilde{v}_\varepsilon)|^2 - |p(\bar{v}_b) - p(\tilde{v}_\varepsilon)|) \mathbf{1}_{\{p(v) - p(\tilde{v}_\varepsilon) \leq \delta_3\}} d\xi \right| \leq \sqrt{\frac{\varepsilon}{\lambda}} \mathcal{D}(U).$$

**Proof of (4.37):** Using Young's inequality together with  $\frac{1}{2} \leq a \leq 1$ , we first find

$$\begin{aligned} |\mathcal{B}_3(U)| &\leq \delta_0 \mathcal{D}(U) + \underbrace{\frac{C}{\delta_0} \int_{\mathbb{R}} |a'|^2 v^\beta |p(v) - p(\tilde{v}_\varepsilon)|^2 d\xi}_{=:\mathcal{B}_6}, \\ |\mathcal{B}_4(U)| &\leq \underbrace{\int_{\mathbb{R}} |a'| |\partial_\xi p(\tilde{v}_\varepsilon)| v^\beta |p(v) - p(\tilde{v}_\varepsilon)|^2 d\xi}_{=:\mathcal{B}_7} + \underbrace{\int_{\mathbb{R}} |a'| |\partial_\xi p(\tilde{v}_\varepsilon)| \frac{|v^\beta - \tilde{v}_\varepsilon^\beta|^2}{v^\beta} d\xi}_{=:\mathcal{B}_8}, \\ |\mathcal{B}_5(U)| &\leq \delta_0 \mathcal{D}(U) + \underbrace{\frac{C}{\delta_0} \int_{\mathbb{R}} |\partial_\xi p(\tilde{v}_\varepsilon)|^2 \frac{|v^\beta - \tilde{v}_\varepsilon^\beta|^2}{v^\beta} d\xi}_{=:\mathcal{B}_9}. \end{aligned}$$

Using (4.49) and (4.73) together with  $|\partial_\xi p(\tilde{v}_\varepsilon)| \leq C \frac{\varepsilon}{\lambda} |a'|$  and  $\delta_0^{-1} \varepsilon < \lambda < \delta_0$ , we have

$$\begin{aligned} |\mathcal{B}_6(U) - \mathcal{B}_6(\bar{U})| + |\mathcal{B}_7(U) - \mathcal{B}_7(\bar{U})| &\leq C \delta_0 \left( \mathcal{D}(U) + \frac{\varepsilon}{\lambda} \mathcal{G}_2(U) \right), \\ |\mathcal{B}_8(U) - \mathcal{B}_8(\bar{U})| + |\mathcal{B}_9(U) - \mathcal{B}_9(\bar{U})| &\leq C \delta_0 \left( \mathcal{D}(U) + (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(\bar{U}) \right). \end{aligned}$$

Therefore, we have

$$\sum_{i=3}^5 |\mathcal{B}_i(U)| \leq \sum_{i=6}^9 |\mathcal{B}_i(\bar{U})| + C \delta_0 \mathcal{D}(U) + C \delta_0 \left( (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})) + \frac{\varepsilon}{\lambda} \mathcal{G}_2(\bar{U}) \right).$$

Since  $|a'| \leq C \varepsilon \lambda$ , we have

$$\sum_{i=6}^9 |\mathcal{B}_i(\bar{U})| \leq C \frac{\varepsilon \lambda}{\delta_0} \int_{\mathbb{R}} |a'| \bar{v}^\beta |p(\bar{v}) - p(\tilde{v}_\varepsilon)|^2 d\xi + C \varepsilon^2 \int_{\mathbb{R}} |a'| \frac{|\bar{v}^\beta - \tilde{v}_\varepsilon^\beta|^2}{\bar{v}^\beta} d\xi.$$

Using  $C^{-1} \leq \bar{v} \leq C$  and (3.16) together with  $|p(\bar{v}) - p(\tilde{v}_\varepsilon)| \leq \delta_3$ , we have

$$\int_{\mathbb{R}} |a'| \bar{v}^\beta |p(\bar{v}) - p(\tilde{v}_\varepsilon)|^2 d\xi \leq C \int_{\mathbb{R}} |a'| Q(\bar{v}|\tilde{v}_\varepsilon) d\xi \leq C \mathcal{G}_2(\bar{U}).$$

Moreover, since

$$\begin{aligned} \int_{\mathbb{R}} |a'| \frac{|\bar{v}^\beta - \tilde{v}_\varepsilon^\beta|^2}{\bar{v}^\beta} d\xi &\leq C \int_{\mathbb{R}} |a'| \frac{|\bar{v}^\beta - \tilde{v}_\varepsilon^\beta|^2}{Q(\bar{v}|\tilde{v}_\varepsilon)} Q(\bar{v}|\tilde{v}_\varepsilon) d\xi \\ &\leq C \int_{\mathbb{R}} |a'| Q(\bar{v}|\tilde{v}_\varepsilon) d\xi \leq C \mathcal{G}_2(\bar{U}), \end{aligned}$$

we have

$$\sum_{i=6}^9 |\mathcal{B}_i(\bar{U})| \leq C \delta_0 \frac{\varepsilon}{\lambda} \mathcal{G}_2(\bar{U}).$$

Hence we have the desired estimate (4.37).

**Proof of (4.38):** First, using (3.15), (3.16) and (4.30), we have

$$|\mathcal{B}_1(\bar{U})| + |\mathcal{B}_2^+(\bar{U})| \leq C \int_{\mathbb{R}} |a'| Q(\bar{v}|\tilde{v}_\varepsilon) d\xi \leq C \int_{\mathbb{R}} |a'| Q(v|\tilde{v}_\varepsilon) d\xi \leq C \frac{\varepsilon^2}{\lambda}.$$

Then, it follows from (4.33), (4.34)-(4.37) and (4.45) that

$$|\mathcal{B}(U)| \leq C \frac{\varepsilon^2}{\lambda} + C \sqrt{\delta_0} (\mathcal{D}(U) + \mathcal{G}_2(U)) \leq C^* \frac{\varepsilon^2}{\lambda} + C \sqrt{\delta_0} \mathcal{D}(U).$$

**Proof of (4.39):** We split the proof in three steps.

*Step 1:* First of all, we use the notations  $Y_1^s, Y_2^s, Y_3^s$  and  $Y_4^s$  for the terms of  $Y_s$  as follows:

$$Y_s = \underbrace{- \int_{\Omega^c} a' Q(v|\tilde{v}_\varepsilon) d\xi}_{=:Y_1^s} - \underbrace{\int_{\Omega^c} a \partial_\xi p(\tilde{v}_\varepsilon)(v - \tilde{v}_\varepsilon) d\xi}_{=:Y_2^s} - \underbrace{\int_{\Omega^c} a' \frac{|h - \tilde{h}_\varepsilon|^2}{2} d\xi}_{=:Y_3^s} + \underbrace{\int_{\Omega^c} a \partial_\xi \tilde{h}_\varepsilon (h - \tilde{h}_\varepsilon) d\xi}_{=:Y_4^s}.$$

We use (4.46), (4.47), (4.63) together with (4.41) to have

$$\begin{aligned} & |Y_g(U) - Y_g(\bar{U})| + |Y_1^s(U) - Y_1^s(\bar{U})| + |Y_2^s(U) - Y_2^s(\bar{U})| \\ (4.74) \quad & \leq C \int_{\mathbb{R}} |a'| \left( |p(v) - p(\tilde{v}_\varepsilon)|^2 - |p(\bar{v}_b) - p(\tilde{v}_\varepsilon)|^2 + |Q(v|\tilde{v}_\varepsilon) - Q(\bar{v}|\tilde{v}_\varepsilon)| \right. \\ & \quad \left. + |v - \bar{v}| + |p(v) - p(\bar{v}_b)| \right) d\xi \leq C \sqrt{\frac{\varepsilon}{\lambda}} \mathcal{D}(U) + C (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})). \end{aligned}$$

On the other hand, (4.69) yields

$$(4.75) \quad |Y_1^s(\bar{U})| + |Y_2^s(\bar{U})| \leq \int_{\Omega^c} |a'| (Q(\bar{v}|\tilde{v}_\varepsilon) + |\bar{v} - \tilde{v}_\varepsilon|) d\xi \leq C \sqrt{\frac{\varepsilon}{\lambda}} \left( \mathcal{D}(U) + \left( \frac{\varepsilon}{\lambda} \right)^2 \mathcal{G}_2(U) \right).$$

Next, by the definitions of  $\mathcal{G}_1^\pm$  in (4.18), we have

$$\begin{aligned} |Y_3^s(U)| + |Y_b(U)| & \leq C \mathcal{G}_1^-(U) + C \mathcal{G}_1^+(U) + C \int_{\Omega} |a'| |p(v) - p(\tilde{v}_\varepsilon)|^2 d\xi \\ & \leq C (\mathcal{G}_1^-(U) + \mathcal{G}_1^+(U) + |\mathcal{B}(U)|), \end{aligned}$$

Moreover, since

$$\mathcal{G}_1^+(U) \leq C \int_{\Omega} |a'| \left( |h - \tilde{h}_\varepsilon|^2 + |p(v) - p(\tilde{v}_\varepsilon)|^2 \right) d\xi \leq C \int_{\Omega} |a'| |h - \tilde{h}_\varepsilon|^2 d\xi + C |\mathcal{B}(U)|,$$

using (4.38), we have

$$|Y_3^s(U)| + |Y_b(U)| \leq C \int_{\mathbb{R}} |a'| |h - \tilde{h}_\varepsilon|^2 d\xi + C^* \frac{\varepsilon^2}{\lambda} + C \sqrt{\delta_0} \mathcal{D}(U).$$

Therefore, using (4.30), (4.45), and the assumption  $\mathcal{D}(U) \leq C^* \varepsilon^2 / \lambda$ , it follows from (4.74), (4.75) and the above estimate that

$$|Y_g(U) - Y_g(\bar{U})| + |Y_1^s(U)| + |Y_2^s(U)| + |Y_3^s(U)| + |Y_b(U)| \leq C \frac{\varepsilon^2}{\lambda}.$$

*Step 2:* First of all, using Young's inequality and (3.16), (4.36), we estimate

$$\begin{aligned}
& \left| \int_{\Omega} a' (p(v) - p(\tilde{v}_{\varepsilon})) \left( h - \tilde{h}_{\varepsilon} - \frac{p(v) - p(\tilde{v}_{\varepsilon})}{\sigma_{\varepsilon}} \right) d\xi \right| \\
& \leq \left( \frac{\lambda}{\varepsilon} \right)^{1/4} \mathcal{G}_1^+(U) + C \left( \frac{\varepsilon}{\lambda} \right)^{1/4} \int_{\Omega} |a'| |p(v) - p(\tilde{v}_{\varepsilon})|^2 d\xi \\
& \leq \left( \frac{\lambda}{\varepsilon} \right)^{1/4} \mathcal{G}_1^+(U) + C \left( \frac{\varepsilon}{\lambda} \right)^{1/4} (\mathcal{B}_2^+(\bar{U}) + (\mathcal{B}_2^+(U) - \mathcal{B}_2^+(\bar{U}))) \\
& \leq \left( \frac{\lambda}{\varepsilon} \right)^{1/4} \mathcal{G}_1^+(U) + C \left( \frac{\varepsilon}{\lambda} \right)^{1/4} \left( \mathcal{G}_2(\bar{U}) + \sqrt{\frac{\varepsilon}{\lambda}} \mathcal{D}(U) \right).
\end{aligned}$$

Therefore, this estimate, (4.74) and (4.75) yield

$$\begin{aligned}
& |Y_g(U) - Y_g(\bar{U})| + |Y_1^s(U)| + |Y_2^s(U)| + |Y_3^s(U)| + |Y_b(U)| \\
& \leq C \sqrt{\frac{\varepsilon}{\lambda}} \mathcal{D}(U) + C (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})) + C \mathcal{G}_1^-(U) + 2 \left( \frac{\lambda}{\varepsilon} \right)^{1/4} \mathcal{G}_1^+(U) + C \left( \frac{\varepsilon}{\lambda} \right)^{1/4} \mathcal{G}_2(\bar{U}).
\end{aligned}$$

*Step 3:* For the remaining terms, using Hölder's inequality together with  $|\tilde{h}'_{\varepsilon}| \leq C \frac{\varepsilon}{\lambda} |a'|$ , we estimate

$$\begin{aligned}
|Y_4^s(U)|^2 & \leq C \left( \frac{\varepsilon}{\lambda} \right)^2 \left( \int_{\mathbb{R}} |a'| d\xi \right) \int_{\Omega^c} |a'| |h - \tilde{h}_{\varepsilon}|^2 d\xi \leq C \frac{\varepsilon^2}{\lambda} \mathcal{G}_1^-(U), \\
|Y_l(U)|^2 & \leq C \left( \frac{\varepsilon}{\lambda} \right)^2 \left( \int_{\mathbb{R}} |a'| d\xi \right) \int_{\Omega} |a'| \left( h - \tilde{h}_{\varepsilon} - \frac{p(\bar{v}) - p(\tilde{v}_{\varepsilon})}{\sigma_{\varepsilon}} \right)^2 d\xi \leq C \frac{\varepsilon^2}{\lambda} \mathcal{G}_1^+(U).
\end{aligned}$$

Therefore, this together with Step1 and Step2 yield

$$\begin{aligned}
& |Y_g(U) - Y_g(\bar{U})|^2 + |Y_b(U)|^2 + |Y_l(U)|^2 + |Y_s(U)|^2 \\
& \leq (|Y_g(U) - Y_g(\bar{U})| + |Y_1^s(U)| + |Y_2^s(U)| + |Y_3^s(U)| + |Y_b(U)|)^2 + |Y_4^s(U)|^2 + |Y_l(U)|^2 \\
& \leq C \frac{\varepsilon^2}{\lambda} \left( \sqrt{\frac{\varepsilon}{\lambda}} \mathcal{D}(U) + (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})) + \mathcal{G}_1^-(U) + \left( \frac{\lambda}{\varepsilon} \right)^{1/4} \mathcal{G}_1^+(U) + \left( \frac{\varepsilon}{\lambda} \right)^{1/4} \mathcal{G}_2(\bar{U}) \right).
\end{aligned}$$

**4.9. Proof of Proposition 4.1.** We now prove the main Proposition of the paper. We split the proof into two steps, depending on the strength of the dissipation term  $\mathcal{D}(U)$ .

*Step 1:* We first consider the case of  $\mathcal{D}(U) \geq 4C^* \frac{\varepsilon^2}{\lambda}$ , where the constant  $C^*$  is defined as in Proposition 4.3. Then using (4.38) and taking  $\delta_0$  small enough, we have

$$\begin{aligned}
\mathcal{R}(U) & := -\frac{|Y(U)|^2}{\varepsilon^4} + \left( 1 + \delta_0 \frac{\varepsilon}{\lambda} \right) |\mathcal{B}(U)| - \mathcal{G}_1^-(U) - \mathcal{G}_1^+(U) - \left( 1 - \delta_0 \frac{\varepsilon}{\lambda} \right) \mathcal{G}_2(U) - (1 - \delta_0) \mathcal{D}(U) \\
& \leq 2|\mathcal{B}(U)| - (1 - \delta_0) \mathcal{D}(U) \\
& \leq 2C^* \frac{\varepsilon^2}{\lambda} - \left( 1 - \delta_0 - 2C\sqrt{\delta_0} \right) \mathcal{D}(U) \\
& \leq 2C^* \frac{\varepsilon^2}{\lambda} - \frac{1}{2} \mathcal{D}(U) \leq 0,
\end{aligned}$$

which gives the desired result.



*Step 2:* We now assume the other alternative, i.e.,  $\mathcal{D}(U) \leq 4C^* \frac{\varepsilon^2}{\lambda}$ .

We will use Proposition 4.2 to get the desired result. First of all, we have (4.32), and for the small constant  $\delta_3$  of Proposition 4.2 associated to the constant  $C_2$  of (4.32), we have  $|p(\bar{v}) - p(\bar{v}_\varepsilon)| \leq \delta_3$ .

Using

$$Y_g(\bar{U}) = Y(U) - (Y_g(U) - Y_g(\bar{U})) - Y_b(U) - Y_l(U) - Y_s(U),$$

we have

$$|Y_g(\bar{U})|^2 \leq 4(|Y(U)|^2 + |Y_g(U) - Y_g(\bar{U})|^2 + |Y_b(U)|^2 + |Y_l(U)|^2 + |Y_s(U)|^2),$$

which can be written as

$$-4|Y(U)|^2 \leq -|Y_g(\bar{U})|^2 + 4|Y_g(U) - Y_g(\bar{U})|^2 + 4|Y_b(U)|^2 + 4|Y_l(U)|^2 + 4|Y_s(U)|^2.$$

Now, let us take  $\delta_0$  small enough such that  $\delta_0 \leq \delta_3^9$ . (In fact, since we see from the proofs of Lemma 4.5-4.8 that the constants  $C$  in Proposition 4.3 depend on  $\delta_3$  as algebraically negative power of it, we take  $\delta_0$  smaller enough if needed.)

Then we find that for any  $\varepsilon < \varepsilon_0(\leq \delta_3)$  and  $\varepsilon/\lambda < \delta_0(\leq \delta_3^9)$ ,

$$\begin{aligned} \mathcal{R}(U) &\leq -\frac{4|Y(U)|^2}{\varepsilon\delta_3} + \left(1 + \delta_0 \frac{\varepsilon}{\lambda}\right) |\mathcal{B}(U)| \\ &\quad - \mathcal{G}_1^-(U) - \mathcal{G}_1^+(U) - \left(1 - \delta_0 \frac{\varepsilon}{\lambda}\right) \mathcal{G}_2(U) - (1 - \delta_0) \mathcal{D}(U) \\ &\leq -\frac{|Y_g(\bar{U})|^2}{\varepsilon\delta_3} + \left(1 + \delta_0 \frac{\varepsilon}{\lambda}\right) (|\mathcal{B}_1(\bar{U})| + |\mathcal{B}_2^+(\bar{U})|) - \left(1 - \delta_3 \frac{\varepsilon}{\lambda}\right) \mathcal{G}_2(\bar{U}) - (1 - \delta_3) \mathcal{D}(\bar{U}) \\ &\quad + \underbrace{\frac{4}{\varepsilon\delta_3} (|Y_g(U) - Y_g(\bar{U})|^2 + |Y_b(U)|^2 + |Y_l(U)|^2 + |Y_s(U)|^2)}_{=: J_1} \\ &\quad + \underbrace{\left(1 + \delta_0 \frac{\varepsilon}{\lambda}\right) \left(|\mathcal{B}_1(U) - \mathcal{B}_1(\bar{U})| + |\mathcal{B}_2^+(U) - \mathcal{B}_2^+(\bar{U})| + |\mathcal{B}_2^-(U)| + \sum_{i=3}^5 |\mathcal{B}_i(U)|\right)}_{=: J_2} \\ &\quad - \mathcal{G}_1^-(U) - \mathcal{G}_1^+(U) - \frac{1}{2} (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})) - \frac{\delta_3 \varepsilon}{2 \lambda} \mathcal{G}_2(\bar{U}) - (\delta_3 - \delta_0) \mathcal{D}(U), \end{aligned}$$

where we used  $\mathcal{D}(\bar{U}) \leq \mathcal{D}(U)$  by (4.43). We claim that  $J_1, J_2$  are controlled by the last line above. Indeed, it follows from (4.34)-(4.37) and (4.39) that for any  $\varepsilon/\lambda < \delta_0(\leq \delta_3^9)$ ,

$$\begin{aligned} J_1 &\leq \frac{C}{\delta_3} \frac{\varepsilon}{\lambda} \left( \sqrt{\frac{\varepsilon}{\lambda}} \mathcal{D}(U) + (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})) + \mathcal{G}_1^-(U) + \left(\frac{\lambda}{\varepsilon}\right)^{1/4} \mathcal{G}_1^+(U) + \left(\frac{\varepsilon}{\lambda}\right)^{1/4} \mathcal{G}_2(\bar{U}) \right) \\ &\leq \frac{C}{\delta_3} \left(\frac{\varepsilon}{\lambda}\right)^{1/4} \left( \mathcal{D}(U) + (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})) + \mathcal{G}_1^-(U) + \mathcal{G}_1^+(U) + \frac{\varepsilon}{\lambda} \mathcal{G}_2(\bar{U}) \right) \\ &\leq \frac{1}{4} \delta_3 \left( \mathcal{D}(U) + (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})) + \mathcal{G}_1^-(U) + \mathcal{G}_1^+(U) + \frac{\varepsilon}{\lambda} \mathcal{G}_2(\bar{U}) \right), \\ J_2 &\leq C \sqrt{\delta_0} \left( \mathcal{D}(U) + (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})) + \frac{\varepsilon}{\lambda} \mathcal{G}_2(\bar{U}) \right) \\ &\leq \frac{1}{4} \delta_3 \left( \mathcal{D}(U) + (\mathcal{G}_2(U) - \mathcal{G}_2(\bar{U})) + \frac{\varepsilon}{\lambda} \mathcal{G}_2(\bar{U}) \right). \end{aligned}$$

Therefore, we have

$$\mathcal{R}(U) \leq -\frac{|Y_g(\bar{U})|^2}{\varepsilon\delta_3} + \left(1 + \delta_0 \frac{\varepsilon}{\lambda}\right) (|\mathcal{B}_1(\bar{U})| + |\mathcal{B}_2^+(\bar{U})|) - \left(1 - \delta_3 \frac{\varepsilon}{\lambda}\right) \mathcal{G}_2(\bar{U}) - (1 - \delta_3)\mathcal{D}(\bar{U}).$$

Since the above quantities  $Y_g(\bar{U}), \mathcal{B}_1, \mathcal{B}_2^+(\bar{U}), \mathcal{G}_2(\bar{U})$  and  $\mathcal{D}(\bar{U})$  depends only on  $\bar{v}$  through  $\bar{U}$ , it follows from Proposition 4.2 that  $\mathcal{R}(U) \leq 0$ .

Hence we complete the proof of Proposition 4.1.

## 5. PROOF OF THEOREM 1.1

**5.1. Proof of (1.11) : Well-prepared initial data.** For a given datum  $(v^0, u^0)$  satisfying (1.10), let  $\{(v_0^r, u_0^r)\}_{r>0}$  be a sequence of truncations defined by

$$v_0^r = \begin{cases} v^0 \mathbf{1}_{\{r \leq v^0 \leq r^{-1}\}} & \text{if } -r^{-1} \leq x \leq r^{-1}, \\ v_- & \text{if } x \leq -r^{-1}, \\ v_+ & \text{if } x \geq r^{-1}, \end{cases}$$

and

$$u_0^r = \begin{cases} u^0 \mathbf{1}_{\{-r^{-1} \leq u^0 \leq r^{-1}\}} & \text{if } -r^{-1} \leq x \leq r^{-1}, \\ u_- & \text{if } x \leq -r^{-1}, \\ u_+ & \text{if } x \geq r^{-1}. \end{cases}$$

Then, we consider a mollification of the above sequence: using  $\phi_\nu(x) := \frac{1}{\sqrt{\nu}}\phi_1\left(\frac{x}{\sqrt{\nu}}\right)$  where  $\phi_1$  is a smooth mollifier with  $\text{supp}\phi_1 = [-1, 1]$ , consider a double sequence  $\{(v_0^{r,\nu}, u_0^{r,\nu})\}_{r,\nu>0}$  defined by

$$v_0^{r,\nu} = v_0^r * \phi_\nu, \quad u_0^{r,\nu} = u_0^r * \phi_\nu.$$

First, we will show

$$(5.1) \quad \lim_{r \rightarrow 0} \lim_{\nu \rightarrow 0} \int_{\mathbb{R}} Q(v_0^{r,\nu} | \tilde{v}(\frac{x}{\nu})) dx = \int_{-\infty}^0 Q(v^0 | v_-) dx + \int_0^\infty Q(v^0 | v_+) dx.$$

For a fixed  $r$ , since  $v_0^{r,\nu} \rightarrow v_0^r$  a.e., and  $\tilde{v}(\frac{x}{\nu}) \rightarrow v_-$  a.e.  $x < 0$  as  $\nu \rightarrow 0$ , using

$$\begin{aligned} Q(v_0^{r,\nu} | \tilde{v}(\frac{x}{\nu})) - Q(v_0^r | v_-) &= (Q(v_0^{r,\nu}) - Q(v_0^r)) + (Q(v_-) - Q(\tilde{v}(\frac{x}{\nu}))) \\ &\quad - \left(Q'(\tilde{v}(\frac{x}{\nu})) - Q'(v_-)\right)(v_0^{r,\nu} - \tilde{v}(\frac{x}{\nu})) - Q'(v_-)((v_0^{r,\nu} - v_0^r) - (\tilde{v}(\frac{x}{\nu}) - v_-)), \end{aligned}$$

we have

$$Q(v_0^{r,\nu} | \tilde{v}(\frac{x}{\nu})) \rightarrow Q(v_0^r | v_-) \quad \text{a.e. } x < 0, \quad \text{as } \nu \rightarrow 0.$$

Likewise,

$$Q(v_0^{r,\nu} | \tilde{v}(\frac{x}{\nu})) \rightarrow Q(v_0^r | v_+) \quad \text{a.e. } x > 0, \quad \text{as } \nu \rightarrow 0.$$

Moreover, since

$$|v_0^r - \bar{v}| \leq \max(r^{-1}, v_\pm) \mathbf{1}_{\{|x| \leq r^{-1}\}},$$

we have

$$\begin{aligned} Q(v_0^{r,\nu} | \tilde{v}(\frac{x}{\nu})) &\leq C_r |v_0^{r,\nu} - \tilde{v}(\frac{x}{\nu})|^2 \leq C_r (|v_0^{r,\nu} - \bar{v}|^2 + |\tilde{v}(\frac{x}{\nu}) - \bar{v}|^2) \\ &\leq \max(r^{-2}, v_\pm^2) \mathbf{1}_{\{|x| \leq r^{-1}+1\}} + |\tilde{v} - \bar{v}|^2, \quad \nu < 1. \end{aligned}$$

Since  $\mathbf{1}_{\{|x| \leq r^{-1}+1\}} + |\tilde{v} - \bar{v}|^2 \in L^1(\mathbb{R})$ , the dominated convergence theorem implies

$$\lim_{\nu \rightarrow 0} \int_{\mathbb{R}} Q(v_0^{r,\nu} | \tilde{v}(\frac{x}{\nu})) dx = \int_{-\infty}^0 Q(v_0^r | v_-) dx + \int_0^{\infty} Q(v_0^r | v_+) dx.$$

Furthermore, since  $Q(v_0^r | \bar{v}) \leq Q(v^0 | \bar{v}) \in L^1(\mathbb{R})$  and  $v_0^r \rightarrow v^0$  a.e. as  $r \rightarrow 0$ , we have

$$\lim_{r \rightarrow 0} \left( \int_{-\infty}^0 Q(v_0^r | v_-) dx + \int_0^{\infty} Q(v_0^r | v_+) dx \right) = \int_{-\infty}^0 Q(v^0 | v_-) dx + \int_0^{\infty} Q(v^0 | v_+) dx,$$

which completes (5.1).

Hence by the diagonal extraction of (5.1), there exists a sequence (still denoted by  $v_0^\nu$ ) such that

$$\lim_{\nu \rightarrow 0} \int_{\mathbb{R}} Q(v_0^\nu | \tilde{v}(\frac{x}{\nu})) dx = \int_{-\infty}^0 Q(v^0 | v_-) dx + \int_0^{\infty} Q(v^0 | v_+) dx.$$

In particular, we have from the above construction that  $v_0^\nu$  converges to  $v^0$  in  $L_{loc}^1(\mathbb{R})$ , and especially :

$$(5.2) \quad v_0^\nu \rightarrow v^0 \quad \text{in } W_{loc}^{-s,1}(\mathbb{R}), \quad s > 0.$$

where this convergence will be used in the proof of (5.23).

Using the same argument as above, we show

$$(5.3) \quad \lim_{r \rightarrow 0} \lim_{\nu \rightarrow 0} \int_{\mathbb{R}} \frac{1}{2} \left( u_0^{r,\nu} + \nu \left( p(v_0^{r,\nu})^{\frac{\alpha}{\gamma}} \right)_x - \tilde{u}^\nu(x) - \nu \left( p(\tilde{v}^\nu(x))^{\frac{\alpha}{\gamma}} \right)_x \right)^2 dx = \int_{-\infty}^{\infty} \frac{|u^0 - \bar{u}|^2}{2} dx,$$

where  $\tilde{u}^\nu(x) = \tilde{u}(x/\nu)$  and  $\tilde{v}^\nu(x) = \tilde{v}(x/\nu)$ .

Indeed, since  $|v_0^r| \leq r^{-1}$  for any small  $r > 0$ , we have

$$\begin{aligned} \left| \nu \left( p(v_0^{r,\nu})^{\frac{\alpha}{\gamma}} \right)_x \right| &= \nu \frac{\alpha}{\gamma} p(v_0^{r,\nu})^{\frac{\alpha}{\gamma}-1} |p'(v_0^{r,\nu})| |v_0^r * (\phi_\nu(x))_x| \\ &\leq \frac{\alpha}{\gamma} p(r^{-1})^{\frac{\alpha}{\gamma}-1} |p'(r)| r^{-1} \int_{\mathbb{R}} |\phi_1'(\frac{y}{\sqrt{\nu}})| dy \\ &\leq C(r) \sqrt{\nu}, \end{aligned}$$

which means

$$\left\| \nu \left( p(v_0^{r,\nu})^{\frac{\alpha}{\gamma}} \right)_x \right\|_{L^\infty(\mathbb{R})} \rightarrow 0 \quad \text{as } \nu \rightarrow 0.$$

Moreover, since  $\tilde{v}'(\frac{x}{\nu}) \rightarrow 0$  a.e. as  $\nu \rightarrow 0$ , we have

$$\left( u_0^{r,\nu} + \nu \left( p(v_0^{r,\nu})^{\frac{\alpha}{\gamma}} \right)_x - \tilde{u} \left( \frac{x}{\nu} \right) - \nu \left( p \left( \tilde{v} \left( \frac{x}{\nu} \right) \right)^{\frac{\alpha}{\gamma}} \right)_x \right)^2 \rightarrow \begin{cases} |u_0^r - u_-|^2 & \text{for a.e. } x < 0, \\ |u_0^r - u_+|^2 & \text{for a.e. } x > 0. \end{cases}$$

Furthermore, since

$$\begin{aligned}
& \left( u_0^{r,\nu} + \nu \left( p(v_0^{r,\nu})^{\frac{\alpha}{\gamma}} \right)_x - \tilde{u} \left( \frac{x}{\nu} \right) - \nu \left( p \left( \tilde{v} \left( \frac{x}{\nu} \right) \right)^{\frac{\alpha}{\gamma}} \right)_x \right)^2 \\
& \leq 2 \left( |u_0^{r,\nu} - \bar{u}|^2 + \left| \tilde{u} \left( \frac{x}{\nu} \right) - \bar{u} \right|^2 + \left| \nu \left( p(v_0^{r,\nu})^{\frac{\alpha}{\gamma}} \right)_x \right|^2 + \left| \nu \left( p \left( \tilde{v} \left( \frac{x}{\nu} \right) \right)^{\frac{\alpha}{\gamma}} \right)_x \right|^2 \right) \\
& \leq C(r) \left( \mathbf{1}_{\{|x| \leq r^{-1}+1\}} + |\tilde{u} - \bar{u}|^2 + \left| \phi_1' \left( \frac{x}{\sqrt{\nu}} \right) \right|^2 + \left| \tilde{v}' \left( \frac{x}{\nu} \right) \right|^2 \right) \\
& \leq C(r) \left( \mathbf{1}_{\{|x| \leq r^{-1}+1\}} + |\tilde{u} - \bar{u}|^2 + |\phi_1'(x)|^2 + |\tilde{v}'(x)|^2 \right) =: g(x),
\end{aligned}$$

and  $g \in L^1(\mathbb{R})$ , the dominated convergence theorem implies

$$\begin{aligned}
& \lim_{\nu \rightarrow 0} \int_{\mathbb{R}} \frac{1}{2} \left( u_0^{r,\nu} + \nu \left( p(v_0^{r,\nu})^{\frac{\alpha}{\gamma}} \right)_x - \tilde{u} \left( \frac{x}{\nu} \right) - \nu \left( p \left( \tilde{v} \left( \frac{x}{\nu} \right) \right)^{\frac{\alpha}{\gamma}} \right)_x \right)^2 dx \\
& = \int_{-\infty}^0 \frac{|u_0^r - u_-|^2}{2} dx + \int_0^{\infty} \frac{|u_0^r - u_+|^2}{2} dx.
\end{aligned}$$

Furthermore, since  $|u_0^r - \bar{u}|^2 \leq |u^0 - \bar{u}|^2 \in L^1(\mathbb{R})$  and  $u_0^r \rightarrow u^0$  a.e. as  $r \rightarrow 0$ , we have

$$\lim_{r \rightarrow 0} \left( \int_{-\infty}^0 \frac{|u_0^r - u_-|^2}{2} dx + \int_0^{\infty} \frac{|u_0^r - u_+|^2}{2} dx \right) = \int_{-\infty}^{\infty} \frac{|u^0 - \bar{u}|^2}{2} dx,$$

which completes (5.3). Hence, using the diagonal extraction as before, there exists a sequence (still denoted by  $u_\nu^0$ ) such that

$$\lim_{\nu \rightarrow 0} \int_{\mathbb{R}} \frac{1}{2} \left( u_\nu^0 + \nu \left( p(v_\nu^0)^{\frac{\alpha}{\gamma}} \right)_x - \tilde{u}^\nu(x) - \nu \left( p(\tilde{v}^\nu(x))^{\frac{\alpha}{\gamma}} \right)_x \right)^2 dx = \int_{-\infty}^{\infty} \frac{|u^0 - \bar{u}|^2}{2} dx.$$

**5.2. Proof for the main part of Theorem 1.1.** We here present a proof for the second part (ii) of Theorem 1.1.

**5.2.1. Uniform estimates in  $\nu$ .** Let  $\{(v^\nu, u^\nu)\}_{\nu>0}$  be a sequence of solutions on  $(0, T)$  to (1.1) with the initial datum  $(v_0^\nu, u_0^\nu)$ . Our starting point is to apply Theorem 1.2 to the below functions:

$$v(t, x) = v^\nu(\nu t, \nu x), \quad \tilde{v}(x) := \tilde{v}^\nu(\nu x), \quad u(t, x) = u^\nu(\nu t, \nu x), \quad \tilde{u}(x) := \tilde{u}^\nu(\nu x).$$

That is, using (1.17) in Theorem 1.2 together with (4.14), we have

$$\begin{aligned}
& \int_{-\infty}^{\infty} E((v, u)(t, x) | (\tilde{v}, \tilde{u})(x - X(t))) dx \\
& \quad + \int_0^{T/\nu} \int_{-\infty}^{\infty} |\tilde{v}'(x)| Q(v(t, x) | \tilde{v}(x - X(t))) dx dt \\
& \quad + \int_0^{T/\nu} \int_{-\infty}^{\infty} v^{\gamma-\alpha}(t, x) |\partial_x(p(v(t, x)) - p(\tilde{v}(x - X(t))))|^2 dx dt \\
& \leq C \int_{-\infty}^{\infty} E((v_0^\nu, u_0^\nu)(\nu x) | (\tilde{v}, \tilde{u})(x)) dx.
\end{aligned}$$

Then by the change of variables  $t \mapsto t/\nu, x \mapsto x/\nu$ , we have

$$\begin{aligned}
& \int_{-\infty}^{\infty} E_{\nu}((v^{\nu}, u^{\nu})(t, x)|(\tilde{v}^{\nu}, \tilde{u}^{\nu})(x - X_{\nu}(t))) dx \\
& + \int_0^T \int_{-\infty}^{\infty} |(\tilde{v}^{\nu})'(x)| Q(v^{\nu}(t, x)|\tilde{v}^{\nu}(x - X_{\nu}(t))) dx dt \\
& + \nu \int_0^T \int_{-\infty}^{\infty} (v^{\nu})^{\gamma-\alpha}(t, x) |\partial_x(p(v^{\nu}(t, x)) - p(\tilde{v}^{\nu}(x - X_{\nu}(t))))|^2 dx dt \\
& \leq C \int_{-\infty}^{\infty} E_{\nu}((v_0^{\nu}, u_0^{\nu})(x)|(\tilde{v}^{\nu}, \tilde{u}^{\nu})(x)) dx,
\end{aligned}$$

where  $X_{\nu}(t) := \nu X(t/\nu)$ , and

$$(5.4) \quad E_{\nu}((v_1, u_1)|(v_2, u_2)) := \frac{1}{2} \left( u_1 + \nu \left( p(v_1)^{\frac{\alpha}{\gamma}} \right)_x - u_2 - \nu \left( p(v_2)^{\frac{\alpha}{\gamma}} \right)_x \right)^2 + Q(v_1|v_2).$$

For simplification, we introduce the variables:

$$(5.5) \quad h^{\nu} := u^{\nu} + \nu \left( p(v^{\nu})^{\frac{\alpha}{\gamma}} \right)_x, \quad \tilde{h}^{\nu} := \tilde{u}^{\nu} + \nu \left( p(\tilde{v}^{\nu})^{\frac{\alpha}{\gamma}} \right)_x,$$

Then, recalling (1.8), the above estimate implies

$$\begin{aligned}
& \int_{-\infty}^{\infty} \eta((v^{\nu}, h^{\nu})(t, x)|(\tilde{v}^{\nu}, \tilde{h}^{\nu})(x - X_{\nu}(t))) dx \\
& + \int_0^T \int_{-\infty}^{\infty} |(\tilde{v}^{\nu})'| Q(v^{\nu}(t, x)|\tilde{v}^{\nu}(x - X_{\nu}(t))) dx dt \\
& + \nu \int_0^T \int_{-\infty}^{\infty} (v^{\nu})^{\gamma-\alpha} |\partial_x(p(v^{\nu}(t, x)) - p(\tilde{v}^{\nu}(x - X_{\nu}(t))))|^2 dx dt \\
& \leq C \int_{-\infty}^{\infty} E_{\nu}((v_0^{\nu}, u_0^{\nu})(x)|(\tilde{v}^{\nu}, \tilde{u}^{\nu})(x)) dx.
\end{aligned}$$

Therefore, using (1.11), we find that

for any  $\delta \in (0, 1)$ , there exists  $\nu_*$  such that for all  $\nu < \nu_*$ ,

$$\begin{aligned}
& \int_{-\infty}^{\infty} \eta((v^{\nu}, h^{\nu})(t, x)|(\tilde{v}^{\nu}, \tilde{h}^{\nu})(x - X_{\nu}(t))) dx \\
(5.6) \quad & + \int_0^T \int_{-\infty}^{\infty} |(\tilde{v}^{\nu})'| Q(v^{\nu}(t, x)|\tilde{v}^{\nu}(x - X_{\nu}(t))) dx dt \\
& + \nu \int_0^T \int_{-\infty}^{\infty} (v^{\nu})^{\gamma-\alpha} |\partial_x(p(v^{\nu}(t, x)) - p(\tilde{v}^{\nu}(x - X_{\nu}(t))))|^2 dx dt \\
& \leq C \mathcal{E}_0 + \delta,
\end{aligned}$$

where

$$\mathcal{E}_0 := \int_{-\infty}^{\infty} \eta((v^0, u^0)|(\bar{v}, \bar{u})) dx.$$

5.2.2. **Proof of (1.12).** We first prove the weak convergence (1.12).

• **Convergence of  $\{v^\nu\}_{\nu>0}$  :** For the given two end states  $v_\pm$ , we first fix a constant  $M > 1$  such that

$$(\min\{v_-, v_+\}, \max\{v_-, v_+\}) \subset (M^{-1}, M).$$

Then we fix the constant  $k_0 > 1$  in Lemma 3.2, and set

$$(5.7) \quad K := \max\{3M, k_0\}.$$

For the constant  $K > 1$ , let  $\bar{\psi}$  be a continuous function defined by

$$(5.8) \quad \bar{\psi}(x) = \begin{cases} x, & \text{if } K^{-1} \leq x \leq K, \\ K^{-1}, & \text{if } x < K^{-1}, \\ K, & \text{if } x > K. \end{cases}$$

Then we set

$$(5.9) \quad \underline{v}^\nu := \bar{\psi}(v^\nu), \quad v_e^\nu := v^\nu - \underline{v}^\nu.$$

Note that the truncation  $\underline{v}^\nu$  will be used in the proof of (1.13) below.

Since

$$|v_e^\nu| \leq \max\{(K^{-1} - v^\nu)_+, (v^\nu - K)_+\},$$

and  $M^{-1} < \tilde{v}^\nu(x - X_\nu) < M$ , we use (3.12) in Lemma 3.2 to have

$$|v_e^\nu| \leq CQ(v^\nu|\tilde{v}^\nu(x - X_\nu)).$$

Then it follows from (5.6) that for all  $\nu < \nu_*$ ,

$$\int_{-\infty}^{\infty} |v_e^\nu| dx \leq C(\mathcal{E}_0 + 1).$$

Therefore,  $\{v_e^\nu\}_{\nu>0}$  is bounded in  $L^\infty(0, T; L^1(\mathbb{R}))$ . Moreover, since the definition (5.9) implies that  $\{\underline{v}^\nu\}_{\nu>0}$  is bounded in  $L^\infty((0, T) \times \mathbb{R}) \subset L^\infty(0, T; L_{loc}^1(\mathbb{R}))$ , we obtain that

$$(5.10) \quad \{v^\nu\}_{\nu>0} \text{ is bounded in } L^\infty(0, T; L_{loc}^1(\mathbb{R})).$$

Therefore, there exists  $v_\infty$  such that

$$(5.11) \quad v^\nu \rightharpoonup v_\infty \quad \text{in } \mathcal{M}((0, T) \times \mathbb{R}).$$

• **Convergence of  $\{u^\nu\}_{\nu>0}$  :**

We split the proof into two steps.

*Step 1:* We will first show convergence of  $\{h^\nu\}_{\nu>0}$ .

For the given two end states  $u_\pm$ , we first fix a constant  $L > 1$  such that

$$(\min\{u_-, u_+\}, \max\{u_-, u_+\}) \subset (-\frac{L}{2}, \frac{L}{2}).$$

Then let  $\bar{\varphi}$  be a continuous function defined by

$$(5.12) \quad \bar{\varphi}(x) = \begin{cases} x, & \text{if } |x| \leq L, \\ -L, & \text{if } x < -L, \\ L, & \text{if } x > L. \end{cases}$$

Then we set

$$(5.13) \quad \underline{h}^\nu := \bar{\varphi}(h^\nu), \quad h_e^\nu := h^\nu - \underline{h}^\nu.$$

Note that the truncation  $\underline{h}^\nu$  will be used in the proof of (1.13) below. Likewise, since

$$|h_e^\nu| \leq \max \left\{ (-h^\nu - L)_+, (h^\nu - L)_+ \right\},$$

and  $-L < \tilde{h}^\nu(x - X_\nu) < L$ , we have

$$|h_e^\nu| \leq \max \left\{ (-h^\nu - L)_+, (h^\nu - L)_+ \right\} \leq |h^\nu - \tilde{h}^\nu(x - X_\nu)|.$$

Then it follows from (5.6) that for all  $\nu < \nu_*$ ,

$$\int_{-\infty}^{\infty} |h_e^\nu|^2 dx \leq C(\mathcal{E}_0 + 1).$$

Therefore,  $\{h_e^\nu\}_{\nu>0}$  is bounded in  $L^\infty(0, T; L^2(\mathbb{R}))$ . Moreover, since  $\{\underline{h}^\nu\}_{\nu>0}$  is bounded in  $L^\infty((0, T) \times \mathbb{R}) \subset L^\infty(0, T; L_{loc}^2(\mathbb{R}))$ , we have

$$(5.14) \quad \{h^\nu\}_{\nu>0} \text{ is bounded in } L^\infty(0, T; L_{loc}^2(\mathbb{R})).$$

Therefore, there exists  $u_\infty$  such that

$$(5.15) \quad h^\nu \rightharpoonup u_\infty \quad \text{in } \mathcal{M}((0, T) \times \mathbb{R}).$$

*Step 2:* We now prove that  $u^\nu \rightharpoonup u_\infty$  in  $\mathcal{M}((0, T) \times \mathbb{R})$ .

Since  $u^\nu = h^\nu - \nu \left( p(v^\nu)^{\frac{\alpha}{\gamma}} \right)_x$ , it is enough to show that

$$(5.16) \quad \nu \left( p(v^\nu)^{\frac{\alpha}{\gamma}} \right)_x \rightharpoonup 0 \quad \text{in } \mathcal{M}((0, T) \times \mathbb{R}).$$

In fact, since

$$\left( p(v^\nu)^{\frac{\alpha}{\gamma}} \right)_x = \frac{\alpha}{\gamma} p(v^\nu)^{\frac{\alpha-\gamma}{\gamma}} p(v^\nu)_x = \frac{\alpha}{\gamma} (v^\nu)^\beta p(v^\nu)_x, \quad (\text{recall } \beta = \gamma - \alpha),$$

it is enough to show that

$$(5.17) \quad \nu (v^\nu)^\beta p(v^\nu)_x \rightharpoonup 0 \quad \text{in } \mathcal{M}((0, T) \times \mathbb{R}).$$

For that, we separate  $\nu (v^\nu)^\beta p(v^\nu)_x$  into two parts:

$$\nu (v^\nu)^\beta p(v^\nu)_x = \underbrace{\nu (v^\nu)^\beta \left( p(v^\nu) - p(\tilde{v}^\nu(x - X_\nu(t))) \right)}_{=: J_1} + \underbrace{\nu (v^\nu)^\beta p(\tilde{v}^\nu(x - X_\nu(t)))}_x_{=: J_2}$$

For any  $\Psi \in C_c((0, T) \times \mathbb{R})$ , using (5.6) and (3.8) together with the condition  $0 < \beta \leq 1$ , we find that for all  $\nu < \nu_*$ ,

$$\begin{aligned}
\int_{\mathbb{R}^+ \times \mathbb{R}} J_1 \Psi dx dt &\leq \nu \sqrt{\int_{\mathbb{R}^+ \times \mathbb{R}} (v^\nu)^\beta \left| (p(v^\nu) - p(\tilde{v}^\nu(x - X_\nu(t))))_x \right|^2 \Psi dx dt} \sqrt{\int_{\mathbb{R}^+ \times \mathbb{R}} (v^\nu)^\beta \Psi dx dt} \\
&\leq C \sqrt{\nu} \sqrt{\mathcal{E}_0 + 1} \sqrt{\int_{\mathbb{R}^+ \times \mathbb{R}} (v^\nu)^\beta (\mathbf{1}_{\{v^\nu \leq 3v_-\}} + \mathbf{1}_{\{v^\nu \geq 3v_-\}}) \Psi dx dt} \\
&\leq C \sqrt{\nu} \sqrt{\mathcal{E}_0 + 1} \sqrt{1 + \int_{\mathbb{R}^+ \times \mathbb{R}} |v^\nu - \tilde{v}^\nu(x - X_\nu(t))| \mathbf{1}_{\{v^\nu \geq 3v_-\}} \Psi dx dt} \\
&\leq C \sqrt{\nu} \sqrt{\mathcal{E}_0 + 1} \sqrt{1 + \int_{\mathbb{R}^+ \times \mathbb{R}} Q(v^\nu | \tilde{v}^\nu(x - X_\nu(t))) \Psi dx dt} \\
&\leq C \sqrt{\nu} \sqrt{\mathcal{E}_0 + 1} \sqrt{1 + \int_{\text{supp}(\Psi)} \eta((v^\nu, h^\nu)(t, x)) |(\tilde{v}^\nu, \tilde{h}^\nu)(x - X_\nu(t))| dx dt} \\
&\leq C \sqrt{\nu} (\mathcal{E}_0 + 1).
\end{aligned}$$

Likewise, we find that for all  $\nu < \nu_*$ ,

$$\begin{aligned}
\int_{\mathbb{R}^+ \times \mathbb{R}} J_2 \Psi dx dt &\leq \nu C \int_{\mathbb{R}^+ \times \mathbb{R}} p(\tilde{v}^\nu(x - X_\nu(t)))_x \mathbf{1}_{\{v^\nu \leq 3v_-\}} \Psi dx dt \\
&\quad + \nu \int_{\mathbb{R}^+ \times \mathbb{R}} (v^\nu)^\beta p(\tilde{v}^\nu(x - X_\nu(t)))_x \mathbf{1}_{\{v^\nu \geq 3v_-\}} \Psi dx dt \\
&\leq \nu C \int_{\mathbb{R}^+ \times \mathbb{R}} |(\tilde{v}^\nu)'(x)| \Psi dx dt \\
&\quad + C \nu \int_{\mathbb{R}^+ \times \mathbb{R}} |(\tilde{v}^\nu)'(x)| Q(v^\nu | \tilde{v}^\nu(x - X_\nu(t))) \mathbf{1}_{\{v^\nu \geq 3v_-\}} \Psi dx dt \\
&\leq \nu C \int_{\mathbb{R}} |\tilde{v}'(x)| dx + C \nu (\mathcal{E}_0 + 1).
\end{aligned}$$

Therefore we have

$$\int_{\mathbb{R}^+ \times \mathbb{R}} \nu (v^\nu)^\beta p(v^\nu)_x \Psi dx dt \rightarrow 0 \quad \text{as } \nu \rightarrow 0,$$

which implies (5.17), and thus,

$$u^\nu \rightharpoonup u_\infty \quad \text{in } \mathcal{M}((0, T) \times \mathbb{R}).$$

Hence we complete the proof of (1.12).

### 5.2.3. Convergence of $\{X_\nu\}_{\nu>0}$ .

**Lemma 5.1.** *There exists  $X_\infty \in BV(0, T)$  such that*

$$(5.18) \quad X_\nu \rightarrow X_\infty \quad \text{in } L^1(0, T), \quad \text{up to subsequence as } \nu \rightarrow 0.$$

*Proof.* First, since  $X'_\nu(t) = X'(t/\nu)$ , it follows from (1.18) that

$$|X'_\nu(t)| \leq C(1 + f_\nu(t)),$$



where  $f_\nu(t) := f(\frac{t}{\nu})$ . Notice that (1.18) and (1.11) imply that for any  $\nu < \nu_*$ ,

$$\begin{aligned} \|f_\nu\|_{L^1(0,T)} &= \nu \|f\|_{L^1(0,T/\nu)} \leq \nu C \int_{-\infty}^{\infty} \eta((v_0^1, h_0^1)|(\tilde{v}, \tilde{h})) dx \\ &\leq C \left( \int_{-\infty}^{\infty} \eta((v^0, u^0)|(\bar{v}, \bar{u})) dx + 1 \right). \end{aligned}$$

Thus,  $f_\nu$  is uniformly bounded in  $L^1(0, T)$ . Therefore,  $X'_\nu$  is uniformly bounded in  $L^1(0, T)$ . Moreover, since  $X_\nu(0) = 0$  and thus,

$$|X_\nu(t)| \leq Ct + C \int_0^t f_\nu(s) ds,$$

$X_\nu$  is also uniformly bounded in  $L^1(0, T)$ .

Therefore, by the compactness of BV (see for example [1, Theorem 3.23]), we have the desired convergence.  $\square$

**5.2.4. Proof of (1.13).** For any fixed constants  $t_1, t_2 \in (0, T)$  such that  $0 < t_2 - t_1 < 1$ , we define

$$(5.19) \quad \phi(s) := \mathbf{1}_{[t_1, t_2]}(s).$$

For the truncations  $\underline{v}^\nu, \underline{h}^\nu$  defined by (5.9), (5.13) with  $L, K$  fixed, we let

$$L_\nu := \int_0^T \phi(s) \int_{\mathbb{R}} \eta((\underline{v}^\nu, \underline{h}^\nu)(s, x)|(\tilde{v}^\nu, \tilde{h}^\nu)(x - X_\nu(s))) dx ds,$$

Using the definition of the truncations together with (5.6) and  $\int_0^T \phi \leq 1$ , we find that for all  $\nu < \nu_*$ ,

$$(5.20) \quad L_\nu \leq \int_0^T \phi(s) \int_{\mathbb{R}} \eta((v^\nu, h^\nu)(s, x)|(\tilde{v}^\nu, \tilde{h}^\nu)(x - X_\nu(s))) dx ds \leq C\mathcal{E}_0 + \delta.$$

Then we have the following.

**Lemma 5.2.** *For the fixed constants  $L, K$ , let*

$$R_\nu := \int_0^T \phi(s) \int_{\mathbb{R}} \eta((\underline{v}^\nu, \underline{h}^\nu)(s, x)|(\bar{v}, \bar{u})(x - X_\infty(s))) dx ds.$$

*Then*

$$|L_\nu - R_\nu| \rightarrow 0 \quad \text{up to a subsequence as } \nu \rightarrow 0.$$

*Proof.* Since  $\underline{h}^\nu, \tilde{h}^\nu, \bar{u}$  are bounded, we have

$$||\underline{h}^\nu - \tilde{h}^\nu(x - X_\nu)|^2 - |\underline{h}^\nu - \bar{u}(x - X_\infty)|^2| \leq C |\tilde{h}^\nu(x - X_\nu) - \bar{u}(x - X_\infty)|.$$

We separate the right-hand side into two parts:

$$|\tilde{h}^\nu(x - X_\nu) - \bar{u}(x - X_\infty)| \leq \underbrace{|\tilde{h}^\nu(x - X_\nu) - \bar{u}(x - X_\nu)|}_{=: I_1} + \underbrace{|\bar{u}(x - X_\nu) - \bar{u}(x - X_\infty)|}_{=: I_2}.$$

Since  $\tilde{h}^\nu = \tilde{u}^\nu + \nu p(\tilde{v}^\nu)_x$ , using  $\|\tilde{u}^\nu - \bar{u}\|_{L^1(\mathbb{R})} = \nu \|\tilde{u} - \bar{u}\|_{L^1(\mathbb{R})}$ , we have

$$\|I_1\|_{L^1(\mathbb{R})} = \|\tilde{h}^\nu - \bar{u}\|_{L^1(\mathbb{R})} \leq \|\tilde{u}^\nu - \bar{u}\|_{L^1(\mathbb{R})} + \nu \|p(\tilde{v}^\nu)_x\|_{L^1(\mathbb{R})} \leq C\nu.$$

Moreover, since  $\|I_2\|_{L^1(\mathbb{R})} = |u_- - u_+| |X_\nu - X_\infty|$ , it follows from Lemma 5.1 that

$$\begin{aligned} & \int_0^T \phi(s) \int_{\mathbb{R}} \frac{1}{2} \left| |\underline{h}^\nu - \tilde{h}^\nu(x - X_\nu(s))|^2 - |\underline{h}^\nu - \bar{u}(x - X_\infty(s))|^2 \right| dx ds \\ & \leq C\nu + C \int_0^\infty \phi(s) |X_\nu(s) - X_\infty(s)| ds \rightarrow 0. \end{aligned}$$

Likewise, since  $\underline{v}^\nu$  is bounded, using the definition of  $Q(\cdot|\cdot)$ , we have

$$\begin{aligned} & |Q(\underline{v}^\nu|\tilde{v}^\nu(x - X_\nu)) - Q(\underline{v}^\nu|\bar{v}(x - X_\infty))| \\ & \leq |Q(\tilde{v}^\nu(x - X_\nu)) - Q(\bar{v}(x - X_\infty))| + |Q'(\tilde{v}^\nu(x - X_\nu))| |\tilde{v}^\nu(x - X_\nu) - \bar{v}(x - X_\infty)| \\ & \quad + (|\underline{v}^\nu| + |\bar{v}(x - X_\infty)|) |Q'(\tilde{v}^\nu(x - X_\nu)) - Q'(\bar{v}(x - X_\infty))| \\ & \leq C |\tilde{v}^\nu(x - X_\nu) - \bar{v}(x - X_\infty)|. \end{aligned}$$

Therefore, following the same computations as above, we have the desired result.  $\square$

Recalling (5.8) and (5.12), we now consider

$$\begin{aligned} & \iint_{\mathbb{R}^+ \times \mathbb{R}} \phi(s) \eta((v^\nu, h^\nu)(s, x)) (\bar{v}, \bar{u})(x - X_\infty(s)) dx ds \\ & = \underbrace{\iint_{h^\nu \in [-L, L]} \phi \frac{|h^\nu - \bar{u}(x - X_\infty)|^2}{2} dx ds}_{=: J_1} + \underbrace{\iint_{h^\nu \notin [-L, L]} \phi \frac{|h^\nu - \bar{u}(x - X_\infty)|^2}{2} dx ds}_{=: J_2} \\ & \quad + \underbrace{\iint_{v^\nu \in [K^{-1}, K]} \phi Q(\underline{v}^\nu|\bar{v}(x - X_\infty)) dx ds}_{=: J_3} + \underbrace{\iint_{v^\nu \notin [K^{-1}, K]} \phi Q(v^\nu|\bar{v}(x - X_\infty)) dx ds}_{=: J_4}. \end{aligned}$$

Note that (using (5.20))

$$J_1 + J_3 \leq R_\nu = (R_\nu - L_\nu) + L_\nu \leq (R_\nu - L_\nu) + C\mathcal{E}_0 + \delta.$$

For  $J_2$ , we use the fact that since  $\bar{u}, \tilde{h}^\nu \in (\min\{u_-, u_+\}, \max\{u_-, u_+\}) \subset (-L/2, L/2)$ , we find

$$|h^\nu - \bar{u}(x - X_\infty)| \leq 3|h^\nu - \tilde{h}^\nu(x - X_\nu)| \quad \text{for all } h^\nu \notin [-L, L].$$

Then using (5.6),

$$J_2 \leq \frac{9}{2} \iint_{h^\nu \notin [-L, L]} \phi |h^\nu - \tilde{h}^\nu(x - X_\nu)|^2 dx ds \leq C(\mathcal{E}_0 + \delta).$$

Likewise for  $J_4$ , since  $\bar{v}, \tilde{v}^\nu \in (\min\{v_-, v_+\}, \max\{v_-, v_+\}) \subset (M^{-1}, M)$ , using (3.13) in Lemma 3.2 with the choice (5.7), we have

$$J_4 \leq C \iint_{v^\nu \notin [K^{-1}, K]} \phi Q(v^\nu|\tilde{v}^\nu(x - X_\nu)) dx ds \leq C(\mathcal{E}_0 + \delta).$$

Therefore, we have

$$(5.21) \quad \iint_{\mathbb{R}^+ \times \mathbb{R}} \phi(s) \eta((v^\nu, h^\nu)(s, x)) (\bar{v}, \bar{u})(x - X_\infty(s)) dx ds \leq |R_\nu - L_\nu| + C(\mathcal{E}_0 + \delta).$$

To complete the proof, we use the following lemma.

**Lemma 5.3.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}^+$  be a nonnegative convex function, and  $\Omega$  be a measurable subset of  $\mathbb{R}^d$  where  $d \geq 1$ . Then, the functional  $u \mapsto \int_{\Omega} f(u(y))dy$  is weakly lower semi-continuous with respect to weak convergence in  $\mathcal{M}(\Omega)$ . That is, if  $u_n \rightharpoonup u$  in  $\mathcal{M}(\Omega)$ , then*

$$\int_{\Omega} f(u)dy \leq \liminf_n \int_{\Omega} f(u_n)dy.$$

This lemma looks classical result. But we can not find any references for its proof. So we present the proof in Appendix B for completeness.

To apply Lemma 5.3 to (5.21), we let

$$\begin{aligned}\Omega_- &:= \{(s, x) \mid t_1 \leq s \leq t_2, \quad x < X_{\infty}(s)\}, \\ \Omega_+ &:= \{(s, x) \mid t_1 \leq s \leq t_2, \quad x > X_{\infty}(s)\},\end{aligned}$$

where  $t_1, t_2$  are as in (5.19).

For the two end states  $(v_{\pm}, u_{\pm})$ , we consider the following convex functions  $f_{\pm}, g_{\pm}$ :

$$g_{\pm}(v) := Q(v|v_{\pm}), \quad h_{\pm}(u) := \frac{|u - u_{\pm}|^2}{2}.$$

Then, Lemma 5.3 implies that the following functionals are weakly lower semi-continuous:

$$G_{\pm}(v) := \iint_{\Omega_{\pm}} Q(v|v_{\pm})dxds, \quad H_{\pm}(u) := \iint_{\Omega_{\pm}} \frac{|u - u_{\pm}|^2}{2}dxds,$$

We set

$$\mathcal{F}_{\pm}(v, u) := \iint_{\Omega_{\pm}} \eta((v, u)|(v_{\pm}, u_{\pm}))dxds.$$

Then,  $\mathcal{F}_{\pm}$  are weakly lower semi-continuous.

Therefore, (5.11), (5.15) and (5.21) together with Lemma 5.2 imply that

$$\mathcal{F}_{\pm}(v_{\infty}, u_{\infty}) \leq \liminf_{\nu \rightarrow 0} \mathcal{F}_{\pm}(v^{\nu}, h^{\nu}) \leq C(\mathcal{E}_0 + \delta).$$

Since  $\delta > 0$  is arbitrary, we obtain

$$\int_{t_1}^{t_2} \int_{\mathbb{R}} \eta((v_{\infty}, h_{\infty})(s, x)|(\bar{v}, \bar{u})(x - X_{\infty}(s)))dxds \leq C\mathcal{E}_0.$$

Since  $t_1, t_2$  are also arbitrary, we find

$$(5.22) \quad \int_{\mathbb{R}} \eta((v_{\infty}, h_{\infty})(t, x)|(\bar{v}, \bar{u})(x - X_{\infty}(t)))dx \leq C\mathcal{E}_0, \quad \text{for a.e. } t \in (0, T),$$

which gives (1.13).

**5.2.5. Proof of  $v_{\infty}(t = 0) = v^0$ .** In order to prove (1.14), we will first prove the continuity of  $v_{\infty}$  in time, and

$$(5.23) \quad v_{\infty}(t = 0) = v^0.$$

We first claim that

$$(5.24) \quad \{u^{\nu}\}_{\nu > 0} \text{ is bounded in } L^2(0, T; L^1_{loc}(\mathbb{R})).$$

For that, recall from (5.5) that  $u^\nu = h^\nu - \nu(p(v^\nu)^{\frac{\alpha}{\gamma}})_x$ . First, we have (5.14). To get a uniform boundedness of  $\nu(p(v^\nu)^{\frac{\alpha}{\gamma}})_x$ , we use the same estimates as in Step 2 for the proof of (5.16). Indeed, since

$$\begin{aligned} \nu \left| (p(v^\nu)^{\frac{\alpha}{\gamma}})_x \right| &= \nu \frac{\alpha}{\gamma} (v^\nu)^\beta |p(v^\nu)_x| \\ &\leq \nu (v^\nu)^\beta |(p(v^\nu) - p(\tilde{v}^\nu(x - X_\nu(t))))_x| + \nu (v^\nu)^\beta |p(\tilde{v}^\nu(x - X_\nu(t)))_x| \\ &\leq C \left( \sqrt{\nu} (v^\nu)^\beta |(p(v^\nu) - p(\tilde{v}^\nu(x - X_\nu(t))))_x|^2 + (1 + Q(v^\nu |\tilde{v}^\nu(x - X_\nu(t)))) \right. \\ &\quad \left. + |(\tilde{v}^\nu)'| (1 + Q(v^\nu |\tilde{v}^\nu(x - X_\nu(t)))) \right), \end{aligned}$$

using (5.6), we have

$$\nu(p(v^\nu)^{\frac{\alpha}{\gamma}})_x \text{ is uniformly bounded in } L^2(0, T; L^1_{loc}(\mathbb{R})).$$

Therefore, we have (5.24).

Then, (5.24) together with the equation  $v_t^\nu - u_x^\nu = 0$  in (1.1) implies

$$v_t^\nu \text{ is uniformly bounded in } L^2(0, T; W_{loc}^{-1,1}(\mathbb{R})).$$

Hence, by Aubin-Lions lemma, this and (5.10) together with (5.11) imply that (up to a subsequence)

$$v^\nu \rightarrow v_\infty \text{ in } C([0, T]; W_{loc}^{-s,1}(\mathbb{R})), \quad s > 0,$$

which together with (5.2) completes the proof of (5.23).

**5.2.6. Proof of (1.14).** First of all, since  $X_\infty \in BV((0, T))$ , there exists a positive constant  $r = r(T)$  such that  $\|X_\infty\|_{L^\infty((0, T))} = r$ . Then we consider a nonnegative smooth function  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\psi(x) = \psi(-x)$ , and  $\psi'(x) \leq 0$  for all  $x \geq 0$ , and  $|\psi'(x)| \leq 2/r$  for all  $x \in \mathbb{R}$ , and

$$\psi(x) = \begin{cases} 1, & \text{if } |x| \leq r, \\ 0, & \text{if } |x| \geq 2r. \end{cases}$$

On the other hand, let  $\theta : \mathbb{R} \rightarrow \mathbb{R}$  be a nonnegative smooth function such that  $\theta(s) = \theta(-s)$ ,  $\int_{\mathbb{R}} \theta = 1$  and  $\text{supp } \theta = [-1, 1]$ , and let

$$\theta_\delta(s) := \frac{1}{\delta} \theta\left(\frac{s - \delta}{\delta}\right) \quad \text{for any } \delta > 0.$$

Then for a given  $t \in (0, T)$ , and any  $\delta < t/2$ , we define a nonnegative smooth function

$$\varphi_{t,\delta}(s) := \int_0^s (\theta_\delta(\tau) - \theta_\delta(\tau - t)) d\tau.$$

Since  $v_t^\nu - u_x^\nu = 0$  by (1.1)<sub>1</sub>, it follows from (1.12) that the limit functions  $v_\infty, u_\infty$  satisfy

$$(5.25) \quad \int_0^T \int_{\mathbb{R}} (\varphi'_{t,\delta}(s) \psi(x) v_\infty(s, x) - \varphi_{t,\delta}(s) \psi'(x) u_\infty(s, x)) dx ds = 0.$$

Since  $\varphi'_{t,\delta}(s) = \theta_\delta(s) - \theta_\delta(s - t)$ , we decompose the left-hand side above into three parts as

$$I_1^\delta + I_2^\delta + I_3^\delta = 0,$$

where

$$\begin{aligned} I_1^\delta &:= \int_0^T \theta_\delta(s) \left( \int_{\mathbb{R}} \psi(x) v_\infty(s, x) dx \right) ds \\ I_2^\delta &:= - \int_0^T \theta_\delta(s-t) \left( \int_{\mathbb{R}} \psi(x) v_\infty(s, x) dx \right) ds \\ I_3^\delta &:= - \int_0^T \int_{\mathbb{R}} \varphi_{t,\delta}(s) \psi'(x) u_\infty(s, x) dx ds. \end{aligned}$$

Using (5.23) and the fact that  $\int_{\mathbb{R}} \psi(x) v_\infty(s, x) dx$  is continuous in  $s$ , we find that as  $\delta \rightarrow 0$  :

$$I_1^\delta \rightarrow \int_{\mathbb{R}} \psi(x) v^0(x) dx, \quad I_2^\delta \rightarrow - \int_{\mathbb{R}} \psi(x) v_\infty(t, x) dx,$$

and

$$I_3^\delta \rightarrow - \int_0^T \int_{\mathbb{R}} \psi'(x) u_\infty(s, x) dx ds.$$

Therefore, it follows from (5.25) that

$$\underbrace{\int_{\mathbb{R}} \psi(x) (v_\infty(t, x) - v^0(x)) dx}_{=: J_1} + \underbrace{\int_0^T \int_{\mathbb{R}} \psi'(x) u_\infty(s, x) dx ds}_{=: J_2} = 0.$$

To show (1.14) from the above equation, we will use the stability estimate (5.22) and the Rankine-Hugoniot condition.

For that, we decompose  $J_1$  into three parts:

$$J_1 = J_{11} + J_{12} + J_{13},$$

where

$$\begin{aligned} J_{11} &= \int_{\mathbb{R}} \psi(x) (v_\infty(t, x) - \bar{v}(x - X_\infty(t))) dx, \\ J_{12} &= \int_{\mathbb{R}} \psi(x) (\bar{v}(x - X_\infty(t)) - \bar{v}(x)) dx, \\ J_{13} &= \int_{\mathbb{R}} \psi(x) (\bar{v}(x) - v^0(x)) dx. \end{aligned}$$

Likewise, we decompose  $J_2$  into two parts:

$$J_2 = J_{21} + J_{22},$$

where

$$\begin{aligned} J_{21} &= \int_0^t \int_{\mathbb{R}} \psi'(x) (u_\infty(s, x) - \bar{u}(x - X_\infty(s))) dx ds, \\ J_{22} &= \int_0^t \int_{\mathbb{R}} \psi'(x) \bar{u}(x - X_\infty(s)) dx ds. \end{aligned}$$

Since  $|X_\infty(t)| \leq r$  for all  $t \in (0, T)$ , we have

$$J_{12} + J_{22} = (v_- - v_+) X_\infty(t) + t(u_- - u_+).$$

Then using the Rankine-Hugoniot condition (1.5)<sub>1</sub>, i.e.,  $\sigma = -\frac{u_- - u_+}{v_- - v_+}$ , we have

$$J_{12} + J_{22} = (X_\infty(t) - \sigma t)(v_- - v_+).$$

To control  $J_{11}$ ,  $J_{13}$  and  $J_{21}$  by the initial perturbation  $\mathcal{E}_0 = \int_{-\infty}^{\infty} \eta((v^0, u^0)|(\bar{v}, \bar{u}))dx$ , we use the stability estimate (1.13) as follows: First, we use (3.8) to have

$$\begin{aligned} |J_{11}| &\leq \int_{-2r}^{2r} |v_{\infty}(t, x) - \bar{v}(x - X_{\infty}(t))| \mathbf{1}_{\{v \leq 3v_{-}\}} dx + \int_{-2r}^{2r} |v_{\infty}(t, x) - \bar{v}(x - X_{\infty}(t))| \mathbf{1}_{\{v \geq 3v_{-}\}} dx \\ &\leq \frac{1}{\sqrt{c_1}} \int_{-2r}^{2r} \sqrt{Q(v_{\infty}(t, x)|\bar{v}(x - X_{\infty}(t)))} dx + \frac{1}{c_2} \int_{\mathbb{R}} Q(v_{\infty}(t, x)|\bar{v}(x - X_{\infty}(t))) dx \\ &\leq C\sqrt{r}\sqrt{\mathcal{E}_0} + C\mathcal{E}_0. \end{aligned}$$

Using the same estimates as above, and (1.11), we have

$$|J_{13}| \leq C\sqrt{r}\sqrt{\mathcal{E}_0} + C\mathcal{E}_0.$$

Likewise,

$$|J_{21}| \leq \frac{2}{r} \int_0^t \int_{[-2r, -r] \cup [r, 2r]} |u_{\infty}(s, x) - \bar{u}(x - X_{\infty}(s))| dx ds \leq \frac{C}{\sqrt{r}} t \sqrt{\mathcal{E}_0}.$$

Hence we have

$$(v_{-} - v_{+})|X_{\infty}(t) - \sigma t| \leq C(\mathcal{E}_0 + (1+t)\sqrt{\mathcal{E}_0}),$$

which completes the proof.

## APPENDIX A. PROOF OF PROPOSITION 4.2

We rewrite the functionals  $Y_g, \mathcal{I}_1, \mathcal{I}_2, \mathcal{G}_2, \mathcal{D}$  with respect to the following variables

$$w := p(v) - p(\tilde{v}_{\varepsilon}), \quad y := \frac{p(\tilde{v}_{\varepsilon}(\xi)) - p(v_{-})}{p(v_{+}) - p(v_{-})}.$$

Since  $p(\tilde{v}_{\varepsilon}(\xi))$  is increasing in  $\xi$ , we use the change of variable  $\xi \in \mathbb{R} \mapsto y \in [0, 1]$ . Notice that  $a = 1 - \lambda y$  and  $|a - 1| \leq \delta_3$  by (4.12), and

$$(A.1) \quad \frac{dy}{d\xi} = \frac{p(\tilde{v}_{\varepsilon})'}{p(v_{+}) - p(v_{-})}, \quad \text{where } |p(v_{+}) - p(v_{-})| = \varepsilon.$$

As in [32, Proposition 3.4], we use the same notations:

$$W := \frac{\lambda}{\varepsilon} w, \quad \alpha_{\gamma} := \frac{\gamma \sqrt{-p'(v_{-})p(v_{-})}}{\gamma + 1} > 0.$$

First of all, note that  $Y_g, \mathcal{I}_1, \mathcal{I}_2, \mathcal{G}_2$  are respectively the same functionals as  $Y_g, \mathcal{B}_1, \mathcal{B}_2, \mathcal{G}_2$  in [32, Proposition 3.4] except for the term  $\frac{1}{2} \int_{\mathbb{R}} a'' |p(v) - p(\tilde{v}_{\varepsilon})|^2 d\xi$  in  $\mathcal{B}_2$ , which is negligible by  $\mathcal{I}_2$  because of  $|a''| \leq C\varepsilon|a'|$  (see [32, (2.29) and (3.31)]).

Thus, it follows from [32, (3.39), (3.34), (3.33), (3.35)] that

$$\begin{aligned}
 (A.2) \quad & -2\alpha_\gamma \frac{\lambda^2}{\varepsilon^3} \frac{|Y_g|^2}{\varepsilon \delta_3} \leq -\frac{\alpha_\gamma}{\delta_3 \sigma^4} \left| \int_0^1 W^2 dy + 2 \int_0^1 W dy \right|^2 + C\delta_3 \int_0^1 W^2 dy, \\
 & 2\alpha_\gamma \frac{\lambda^2}{\varepsilon^3} |\mathcal{I}_1| \leq (1 + C(\varepsilon_0 + \delta_3)) \int_0^1 W^2 dy, \\
 & 2\alpha_\gamma \frac{\lambda^2}{\varepsilon^3} |\mathcal{I}_2| \leq \left( \frac{\alpha_\gamma}{\sigma} \left( \frac{\lambda}{\varepsilon} \right) + C(\varepsilon_0 + \delta_3) \right) \int_0^1 W^2 dy, \\
 & -2\alpha_\gamma \frac{\lambda^2}{\varepsilon^3} \mathcal{G}_2 \leq \left( -\frac{\alpha_\gamma}{\sigma} \left( \frac{\lambda}{\varepsilon} \right) + C\delta_3 \right) \int_0^1 W^2 dy + \frac{2}{3} \int_0^1 W^3 dy + C\varepsilon_0 \int_0^1 |W|^3 dy.
 \end{aligned}$$

Therefore, it remains to estimate the diffusion  $\mathcal{D}$  as follows:

First, by the change of variable, we have

$$\mathcal{D} = \int_0^1 (1 - \lambda y) |\partial_y w|^2 v^\beta \left( \frac{dy}{d\xi} \right) dy.$$

Since it follows from (3.2) that

$$\tilde{v}_\varepsilon^\beta p(\tilde{v}_\varepsilon)' = \sigma_\varepsilon (\tilde{v}_\varepsilon - v_-) + \frac{p(\tilde{v}_\varepsilon) - p(v_-)}{\sigma_\varepsilon},$$

using (A.1), we find

$$(A.3) \quad \varepsilon \tilde{v}_\varepsilon^\beta \frac{dy}{d\xi} = \frac{1}{\sigma_\varepsilon} \left( \sigma_\varepsilon^2 (\tilde{v}_\varepsilon - v_-) + p(\tilde{v}_\varepsilon) - p(v_-) \right).$$

Since the right-hand side of (A.3) is the same as the one in the proof of [32, Lemma 3.1], we have

$$\frac{\tilde{v}_\varepsilon^\beta}{y(1-y)} \frac{dy}{d\xi} = \frac{\varepsilon}{\sigma_\varepsilon (v_- - v_+)} \left( \frac{\tilde{v}_\varepsilon - v_-}{p(\tilde{v}_\varepsilon) - p(v_-)} + \frac{\tilde{v}_\varepsilon - v_+}{p(v_+) - p(\tilde{v}_\varepsilon)} \right).$$

Thus, it follows from the proof of [32, Lemma 3.1] that

$$\left| \frac{\tilde{v}_\varepsilon^\beta}{y(1-y)} \frac{dy}{d\xi} - \frac{\varepsilon}{2\alpha_\gamma} \right| \leq C\varepsilon^2.$$

Then, using  $|(v^\beta/\tilde{v}_\varepsilon^\beta) - 1| \leq C\delta_3$ , we have

$$\begin{aligned}
 \mathcal{D} & \geq (1 - \lambda) \int_0^1 |\partial_y w|^2 v^\beta \left( \frac{dy}{d\xi} \right) dy = (1 - \lambda) \int_0^1 |\partial_y w|^2 \frac{v^\beta}{\tilde{v}_\varepsilon^\beta} \tilde{v}_\varepsilon^\beta \left( \frac{dy}{d\xi} \right) dy \\
 & \geq (1 - \lambda) \left( \frac{\varepsilon}{2\alpha_\gamma} - C\varepsilon^2 - C\delta_3 \right) \int_0^1 y(1-y) |\partial_y w|^2 dy \\
 & \geq \frac{\varepsilon}{2\alpha_\gamma} (1 - C(\delta_3 + \varepsilon_0)) \int_0^1 y(1-y) |\partial_y w|^2 dy.
 \end{aligned}$$

After the normalization, we obtain

$$(A.4) \quad -2\alpha_\gamma \frac{\lambda^2}{\varepsilon^3} \mathcal{D} \leq -(1 - C(\varepsilon_0 + \delta_3)) \int_0^1 y(1-y) |\partial_y W|^2 dy.$$

To finish the proof, we first observe that for any  $\delta < \delta_3$ ,

$$\begin{aligned} \mathcal{R}_{\varepsilon, \delta}(v) &\leq -\frac{1}{\varepsilon \delta_3} |Y_g(v)|^2 + (1 + \delta_3) |\mathcal{I}_1(v)| \\ &\quad + \left(1 + \delta_3 \left(\frac{\varepsilon}{\lambda}\right)\right) |\mathcal{I}_2(v)| - \left(1 - \delta_3 \left(\frac{\varepsilon}{\lambda}\right)\right) \mathcal{G}_2(v) - (1 - \delta_3) \mathcal{D}(v). \end{aligned}$$

Then, (A.2) and (A.4) together with  $\varepsilon_0 \leq \delta_3$  imply

$$\begin{aligned} 2\alpha_\gamma \left(\frac{\lambda^2}{\varepsilon^3}\right) \mathcal{R}_{\varepsilon, \delta}(v) &\leq -\frac{1}{C_\gamma \delta_3} \left(\int_0^1 W^2 dy + 2 \int_0^1 W dy\right)^2 + (1 + C_* \delta_3) \int_0^1 W^2 dy \\ &\quad + \frac{2}{3} \int_0^1 W^3 dy + C_* \delta_3 \int_0^1 |W|^3 dy - (1 - C_* \delta_3) \int_0^1 y(1-y) |\partial_y W|^2 dy. \end{aligned}$$

To finish the proof, we use the nonlinear Poincaré type inequality [32, Proposition 3.3] as follow:

**Proposition A.1.** [32, Proposition 3.3] *For a given  $C_1 > 0$ , there exists  $\delta_2 > 0$ , such that for any  $\delta < \delta_2$  the following is true.*

*For any  $W \in L^2(0, 1)$  such that  $\sqrt{y(1-y)} \partial_y W \in L^2(0, 1)$ , if  $\int_0^1 |W(y)|^2 dy \leq C_1$ , then*

$$\begin{aligned} (A.5) \quad &-\frac{1}{\delta} \left(\int_0^1 W^2 dy + 2 \int_0^1 W dy\right)^2 + (1 + \delta) \int_0^1 W^2 dy \\ &+ \frac{2}{3} \int_0^1 W^3 dy + \delta \int_0^1 |W|^3 dy - (1 - \delta) \int_0^1 y(1-y) |\partial_y W|^2 dy \leq 0. \end{aligned}$$

First, using the same estimate as in [32, (3.38)], we find the constant  $C_1 > 0$  such that

$$\int_0^1 W^2 dy \leq C_1.$$

Then, let us fix the value of the  $\delta_2$  of Proposition A.1 corresponding to the constant  $C_1$ . We consider  $\bar{\delta} = \max(C_\gamma, C_*) \delta_3$ , and choose  $\delta_3$  small enough, such that  $\bar{\delta}$  is smaller than  $\delta_2$ . Then we have

$$\begin{aligned} 2\alpha_\gamma \left(\frac{\lambda^2}{\varepsilon^3}\right) \mathcal{R}_{\varepsilon, \delta}(v) &\leq -\frac{1}{\delta_2} \left(\int_0^1 W^2 dy + 2 \int_0^1 W dy\right)^2 + (1 + \delta_2) \int_0^1 W^2 dy \\ &\quad + \frac{2}{3} \int_0^1 W^3 dy + \delta_2 \int_0^1 |W|^3 dy - (1 - \delta_2) \int_0^1 y(1-y) |\partial_y W|^2 dy. \end{aligned}$$

Therefore, using Proposition A.1, we have

$$2\alpha_\gamma \left(\frac{\lambda^2}{\varepsilon^3}\right) \mathcal{R}_{\varepsilon, \delta}(v) \leq 0,$$

which completes the proof.

## APPENDIX B. PROOF OF LEMMA 5.3

For the given measurable set  $\Omega \subset \mathbb{R}^d$ , we first choose a sequence of bounded measurable sets  $\{\Omega_k\}_{k=1}^\infty$  such that

$$\Omega_k \subset \Omega, \quad d(\Omega_k, \Omega^c) \geq \frac{1}{k} \quad \text{for all } k \geq 1.$$



Indeed, we can construct  $\Omega_k$  by  $\Omega_k = \{y \in \Omega \mid |y| \leq k, d(y, \Omega^c) \geq \frac{1}{k}\}$ .

Then,  $\Omega_k \nearrow \Omega$ , and thus,  $\mathbf{1}_{\Omega_k}(y) \rightarrow \mathbf{1}_{\Omega}(y)$  for a.e.  $y \in \mathbb{R}^d$ .

Therefore, for the given sequence  $\{u_n\}$  such that  $u_n \rightarrow u$  in  $\mathcal{M}(\Omega)$ , once we prove that

$$(B.6) \quad \forall k \geq 1, \quad \int_{\Omega_k} f(u) dy \leq \liminf_n \int_{\Omega} f(u_n) dy,$$

using Fatou's lemma together with  $f \geq 0$ , we have

$$\int_{\Omega} f(u) dy \leq \liminf_k \int_{\Omega_k} f(u) dy \leq \liminf_n \int_{\Omega} f(u_n) dy,$$

which completes the proof.

To show (B.6), we first fix  $k \geq 1$ , and set

$$\theta_{\delta}(y) := \frac{1}{\delta^d} \theta_1\left(\frac{y}{\delta}\right) \quad \text{for any } \delta \in (0, 1/k),$$

where  $\theta_1 : \mathbb{R}^d \rightarrow \mathbb{R}$  is a nonnegative smooth function such that  $\theta_1(y) = \theta_1(-y)$ ,  $\int_{\mathbb{R}^d} \theta_1 = 1$  and  $\text{supp } \theta_1 = \{y \mid |y| \leq 1\}$ . Then we consider the mollifications of  $u_n, u$  as

$$u_n^{\delta} := u_n * \theta_{\delta}, \quad u^{\delta} := u * \theta_{\delta}.$$

First, using the fact that  $u_n \rightarrow u$  in  $\mathcal{M}(\Omega_k)$ , and  $\Omega_k$  is bounded, we find that

$$\forall \delta \in (0, 1/k), \quad u_n^{\delta} \rightarrow u^{\delta} \quad \text{in } C^0(\Omega_k).$$

Therefore, using the fact that  $f$  is convex (thus continuous), we have

$$(B.7) \quad \forall \delta \in (0, 1/k), \quad \int_{\Omega_k} |f(u_n^{\delta}) - f(u^{\delta})| dy \rightarrow 0.$$

On the other hand, we claim that

$$(B.8) \quad \forall \delta \in (0, 1/k), \quad \forall n, \quad \int_{\Omega_k} f(u_n^{\delta}) dy \leq \int_{\Omega} f(u_n) dy.$$

Indeed, using the fact that  $\theta_{\delta}(y) = \theta_{\delta}(-y)$ ,  $\int_{\mathbb{R}^d} \theta_{\delta} = 1$  and  $\text{supp } \theta_{\delta} = \{y \mid |y| \leq \delta\}$  together with  $d(\Omega_k, \Omega^c) \geq \frac{1}{k} > \delta$ , Jensen's inequality implies that for any  $y \in \Omega_k$ ,

$$f(u_n^{\delta}(y)) = f\left(\int_{\mathbb{R}^d} u_n(z) \theta_{\delta}(y - z) dz\right) \leq \int_{\mathbb{R}^d} f(u_n(z)) \theta_{\delta}(z - y) dz = \int_{\Omega} f(u_n(z)) \theta_{\delta}(z - y) dz.$$

Therefore, using Fubini's theorem, we have

$$\begin{aligned} \int_{\Omega_k} f(u_n^{\delta}(y)) dy &\leq \int_{\Omega_k} \int_{\Omega} f(u_n(z)) \theta_{\delta}(z - y) dz dy \\ &= \int_{\Omega} f(u_n(z)) \left( \int_{\Omega_k} \theta_{\delta}(y - z) dy \right) dz \leq \int_{\Omega} f(u_n(z)) dz, \end{aligned}$$

which provides (B.8).

Since  $u^{\delta}(y) \rightarrow u(y)$  a.e.  $y$ , and  $f \geq 0$ , we use Fatou's lemma to have

$$(B.9) \quad \int_{\Omega_k} f(u) dy \leq \liminf_{\delta} \int_{\Omega_k} f(u^{\delta}) dy.$$

Hence, using (B.9), (B.7) and (B.8), we have

$$\begin{aligned}
\int_{\Omega_k} f(u) dx &\leq \liminf_{\delta} \int_{\Omega_k} f(u^{\delta}) dx \\
&\leq \liminf_{\delta} \left( \liminf_n \int_{\Omega_k} |f(u^{\delta}) - f(u_n^{\delta})| dx + \liminf_n \int_{\Omega_k} f(u_n^{\delta}) dx \right) \\
&\leq \liminf_{\delta} \liminf_n \int_{\Omega_k} f(u_n^{\delta}) dx \leq \liminf_n \int_{\Omega} f(u_n) dx,
\end{aligned}$$

which gives (B.6).

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