A new proof of partial regularity of solutions to Navier Stokes equations

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Abstract: In this paper we give a new proof of the partial regularity of solutions to the incompressible Navier Stokes equation in dimension 3 first proved by Caffarelli, Kohn and Nirenberg. The proof relies on a method introduced by De Giorgi for elliptic equations.

1 Introduction

This paper deals with the partial regularity of solutions to the incompressible Navier Stokes equation in dimension 3, namely:

$$\begin{aligned} &\partial_t u + \mathrm{div}(u \otimes u) + \nabla P - \Delta u = 0 \qquad t \in]0, \infty[, \ x \in \Omega, \\ &\mathrm{div} u = 0, \end{aligned} \tag{1}$$

where Ω is a regular subset of $\mathbb{R}^3$.

The initial boundary value problem is endowed with the conditions:

$$\begin{aligned} &u(0, \cdot) = u^0 \in L^2(\Omega), \\ &u(t, x) = 0, \qquad x \in \partial\Omega \ \ t \in]0, \infty[. \end{aligned}$$

The existence of weak solutions for this problem was proved long ago by Leray [12] and Hopf [9]. For this Leray introduce a notion of weak solution. He shows that for any initial value with finite energy $u^0 \in L^2(\mathbb{R}^3)$ there exists a function $u \in L^\infty(0, \infty; L^2(\Omega)) \times L^2(0, \infty; H^1_0(\Omega))$ verifying (1) in the sense of distribution. From that time on, much effort has been made to establish results on the uniqueness and regularity of weak solutions. However those two questions remains yet mostly open. Especially it is not known until now if such a weak solution can develop singularities in finite time, even considering smooth initial data. The question of uniqueness is related to the one of regularity. Indeed it is well known that if the solution is smooth enough, then it is unique. Several steps has already been performed concerning the regularity of weak solutions. In [21],

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Serrin showed that a solution of (1) lying in $L^p(0,\infty;L^q(\Omega))$ with $p,q\geq 1$ such that $2/p+3/q<1$ is smooth in the spatial directions. This result was later extend in [23] and [3] to the case of equality for $p<\infty$. Notice that the case of $L^\infty(0,\infty;L^3(\Omega))$ was proven only very recently by Escauriaza, Seregin and Sverak [10]. In a series of papers [16, 17, 18, 19], Scheffer began to develop the analysis about the possible singular points set, and established various partial regularity results for a class weak solutions named "suitable weak solutions". Those solutions verifies in addition of (1) the generalized energy inequality in the sense of distribution:

$$\partial_t \frac{|u|^2}{2} + \mathrm{div}(u\frac{|u|^2}{2}) + \mathrm{div}(uP) + |\nabla u|^2 - \Delta\frac{|u|^2}{2} \leq 0 \qquad t\in]0,\infty[,\ x\in\Omega. \quad (2)$$

Let us mention also related interesting works done by Foias and Temam [4], Giga [6] and Sohr and von Wahl [22]. The result of Scheffer was later improved in the stunning result of Caffarelli, Kohn and Nirenberg [1]. They showed in this paper that the set of possible singular points of a suitable weak solution is of measure 0 for the 1 dimensional Hausdorff measure in position-time space. This work gave rise to a lot of activities in the area. A simplified proof was proposed by Lin in [13]. Let us mention also the related works of Maremonti [15], Grunau [7] and Struwe [23] in the case of five dimensional stationary Navier-Stokes equations. Recently Tian and Xin established the local theory regularity for the suitable weak solutions with slightly different hypothesis in [24]. Seregin and Sverak showed the full regularity of suitable solutions under a natural (but still unproved) condition on the pressure in [20]. Finally, let us cite the result of He [8] where the partial regularity result is obtained for any weak solutions (not only suitable).

Our result still used the notion of suitable solution, but it is more constructivist (in the same spirit than the one of [1]). Following [1] (see also [13]) we split the proof into two parts. We denote $B(r)$ the ball of radius r and center 0 in $\mathbb{R}^3$. First we show the following theorem:

Theorem 1 *For every $p>1$, there exists a universal constant C^*, such that any solution u of (1) (2) in $[-1,1]\times B(1)$ verifying:*

$$\sup_{t\in[-1,1]}\left(\int_{B(1)}|u|^2\,dx\right)+\int_{-1}^{1}\int_{B(1)}|\nabla u|^2\,dx\,dt+\left[\int_{-1}^{1}\left(\int_{B(1)}|P|\,dx\right)^p dt\right]^{\frac{2}{p}}\leq C^*, \quad (3)$$

is bounded by 1 on $[-1/2,1]\times B(1/2)$.

In a second part this result is used in a local way to show the second theorem:

Theorem 2 *There exists a universal constant δ^* such that the following property holds for any u solution to (1) (2) in $]0,\infty[\times\Omega$. Let (t_0,x_0) lying in the interior of $]0,\infty[\times\Omega$ and be such that:*

$$\limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t_0-\varepsilon^2}^{t_0+\varepsilon^2}\int_{x_0+B(\varepsilon)}|\nabla u|^2\,dx\,dt\leq\delta^*. \quad (4)$$

Then u is bounded on a neighborhood of (t_0, x_0).

It is well known since [1] that, using classical covering lemmas, this result gives the partial regularity result. Namely that the one dimensional (in space time) Hausdorff measure of the set of singular points is 0 for any suitable solution of (1) (2) lying in $L^\infty(0,\infty; L^2(\Omega)) \times L^2(0,\infty; H^1_0(\Omega))$.

We do not claim any originality in the proof of this last Theorem 2. Its proof relies on Theorem 1. The statement of this first result is slightly different that the usual ones in [1] or [13]. For the sake of completeness we then give a proof of the second Theorem as well.

All the novelty of this paper lies in the proof of Theorem 1. It uses a method first introduced by De Giorgi to show regularity of solutions to elliptic equations with rough diffusion coefficients [2]. As in [1], we consider the change of a quantity depending on u from set Q_k to set Q_{k+1} with $Q_{k+1} \subset Q_k$ (except that in our context they do not shrink to 0). It is striking that this feature which was already in the Schaeffer paper follows the physical principle of transfer of turbulent energy from scales to scales (known as "Kolmogorov cascade"). But instead of tracking the total energy $\int |u|^2\,dx$ from a set to an other one, we are considering the transfer of energy from a level set $\int (|u| - C_k)_+^2\,dx$ to an other one $\int (|u| - C_{k+1})_+^2\,dx$ where C_k is an increasing sequence. The estimate of this transfer relies on the equation verified by $v_k^2 = (|u| - C_k)_+^2$. The main difference with inequality (2) is that the force term involving the pressure cannot be expressed as a divergence term anymore. In the proof we will decompose the pressure force acting on v_k^2 in Q_k into two parts: The "non local" part which depends (from the Riesz transform) of values of u outside Q_{k-1} and the "local part" which depends only on the values of u inside Q_{k-1}. This "local part" is itself split into two parts: one part which can be expressed as a divergence term and the rest which cannot. Each term of the equation on v_k^2 will be characterize by a power exponent (see Proposition 3). We show in an appendix the importance of the value of this exponent. Actually the full regularity result for any suitable weak solution of Navier Stokes equation would be fulfilled provided that those exponents are bigger than a critical value. All the terms can be controlled in that way (including the transport term, the non local pressure term and the local pressure term which can be written in a divergence form) but one: the local pressure term which cannot be written in a divergence form. At this stage the estimate of this term is too loose. This result is at most a curiosity. But it characterizes in a cute way the obstruction to full regularity. At least it singles out the bad part of the pressure term.

Let us finish this introduction by a very general remark. The scaling of the Navier Stokes equation gives a striking invariance, namely the fact that u is solution to Navier Stokes equations if and only if u_λ given by:

$$u_\lambda(t,x) = u(\lambda^2 t, \lambda x)$$

is also solution for every λ. This property is used in Theorem 2 but not in Theorem 1. Kato introduced the notion of mild solutions (see [11], [5]) and shows the importance of functional space invariant by the scaling. A lot of works followed which can be summarized very roughly in the following way: if the initial datum is small in an invariant norm (with respect to the scaling) then the solution is smooth for all time. After rescaling properly the equation those studies can be seen as results in the low Reynolds number regime. In physics, the turbulence (and so the possible singularities of the solutions) is expected for high Reynolds number, that is when the advection term and the pressure term are big compared to the diffusion term. From this very general remark, it could be wise to try to emancipate ourselves from this strong invariance structure. This result can be seen as a first attempt in this direction. The claim is that the introduction of the level set of energy gives more richness of scales. This allows to single out some pressure terms from others even if they share the same fundamental scaling property. No doubt that this remark is highly controversial. After all, this paper gives no final result which was not proven before.

2 Main propositions

As said in the introduction we introduce a sequence of decreasing sets:

$$\begin{aligned}
&B_k = B(1/2(1+2^{-3k})) \qquad T_k = 1/2(-1-2^{-k}),\\
&Q_k = [T_k, 1] \times B_k,\\
&B_{k-1/3} = B_{1/2(1+2*2^{-3k})}.
\end{aligned}$$

To deal with the non locality of the pressure we will also introduce:

$$B_{k-2/3} = B_{1/2(1+4*2^{-3k})}.$$

Then we introduce a new function:

$$v_k = [|u| - (1-2^{-k})]_+.$$

Notice that v_k^2 can be seen as a level set of energy since $v_k^2 = 0$ for $|u| < 1-2^{-k}$ and is of the order of $|u|^2$ for $|u| \gg 1-2^{-k}$.

Let us define:

$$U_k = \sup_{t\in[T_k,1]} \left(\int_{B_k} |v_k(t,x)|^2\,dx \right) + \int_{Q_k} |d_k(t,x)|^2\,dx\,dt,$$

where:

$$d_k^2 = \frac{(1-2^{-k})\mathbf{1}_{\{|u|\geq(1-2^{-k})\}}}{|u|}|\nabla|u||^2 + \frac{v_k}{|u|}|\nabla u|^2.$$

Notice that:

$$U_0 = \sup_{t\in[-1,1]} \left(\int_{B(1)} |u(t,x)|^2\,dx \right) + \int_{-1}^{1}\int_{B(1)} |\nabla u(t,x)|^2\,dx\,dt.$$

We want to study the limit when k goes to infinity of U_k. Notice that there is no pressure term in U_k. This feature differs from the proof of [1] and [13]. We can focus only on v_k and the gradient term d_k thanks to the fact that the sets Q_k do not shrink to 0. Indeed we have for every k:

$$\begin{aligned}&[-1/2,1]\times B(1/2)\subset Q_k,\\&B(1/2)\subset B_k.\end{aligned}$$

Thus the global control of the pressure on Q_0 will be sufficient. This also justify the norm for the pressure chosen in Theorem 1. In the paper of Lin for instance, the norm chosen on P was the $L^{3/2}$ norm in space time. This was chosen that way to have the same homogeneity than the L^3 norm of u. For our purpose there is no reason to do that since the pressure is not handled in a similar way than u. In our case the norm $L^p(L^1)$ comes more naturally. Anyway this does not change the final result since Theorem 1 is equivalent to the corresponding result in [1] or [13]. Notice in particular that if (u,P) is solution to (1) with $u\in L^\infty(0,\infty;L^2(\Omega))\times L^2(0,\infty,H^1_0(\Omega))$ then P lies in $L^p_{\text{loc}}(0,\infty;L^1_{\text{loc}}(\Omega))$ (see for instance [14]).

The main result of this paper is the following:

Proposition 3 *let $p>1$. There exists universal constants C_p, $\beta_p>1$ depending only on p such that for any solution to* (1), (2) *in* $[-1,1]\times B(1)$, *if* $U_0\leq 1$ *then we have for every* $k>0$*:*

$$U_k\leq C_p^k(1+\|P\|_{L^p(0,1;L^1(B_0))})U_{k-1}^{\beta_p}. \tag{5}$$

As mentioned in the introduction the value of the exponent β_p is of great importance if we are interesting to the full regularity of the solutions. We show in the appendix that if the Proposition 3 holds true for a p with $\beta_p>3/2$ then this implies the full regularity of any suitable weak solutions of Navier-Stokes equations in $]0,\infty[\times\mathbb{R}^3$. Notice that 3/2 corresponds to the scale of the equation. The idea of De Giorgi (applied on elliptic equations) was to used the Sobolev imbedding Theorem together with the Tchebichev inequality to increase the power beyond the natural scale of the equation. We will explicit all the exponents we have in the proof. For p big enough, the only term for which the exponent is below the rod is the part of the local pressure term which cannot be written in a divergence form. By local, we mean the term acting on the set Q_k which depends only on the values of u on Q_{k-1}. This term has an exponent strictly smaller than 4/3.

For any $p>1$ this proposition leads to the Theorem 1 thanks to the following lemma:

Lemma 4 *For $C>1$ and $\beta>1$ there exists a constant C_0^* such that for every sequence verifying $0<W_0<C_0^*$ and for every k:*

$$0\leq W_{k+1}\leq C^kW_k^\beta,$$

we have

$$\lim_{k\to+\infty}W_k=0.$$

Let us first check that Lemma 4 and Proposition 3 imply Theorem 1. If we consider a $C^* \leq 1$ we have $U_0 \leq 1$. Notice that from the definition of U_k we have $U_k \leq U_0$ so $U_k \leq 1$ for every k. This also gives $\|P\|_{L^p(0,1;L^1(B_0))}$ smaller than 1. So Proposition 3 gives that for every $k > 1$:

$$U_k \leq (2C_p)^k U_{k-1}^{\beta_p}. \tag{6}$$

Notice that $\beta_p > 1$ for $p > 1$. So, if we set $C^* = \inf(1, C_0^*)$, Lemma 4 implies that U_k converges to 0. But for every k and every $-1/2 \leq t \leq 1$:

$$\int_{B(1/2)} [|u(t,x)| - 1]_+^2 \, dx \leq U_k.$$

So the left hand side of the inequality is equal to 0, which applies that $|u(t,x)| \leq 1$ almost everywhere on $[-1/2, 1] \times B(1/2)$.

Proof of Lemma 4. Let us denote:

$$\overline{W}_k = C^{\frac{k}{\beta-1}} C^{\frac{1}{(\beta-1)^2}} W_k.$$

The hypothesis of the lemma gives:

$$0 \leq \overline{W}_{k+1} \leq \overline{W}_k^{\beta}.$$

So if $W_0 \leq C_0^* = C^{-1/(\beta-1)^2}$, we have $\overline{W}_0 \leq 1$ and by induction $\overline{W}_k \leq 1$ for every k. This gives:

$$W_k \leq C^{\frac{-k}{\beta-1}} C^{\frac{-1}{(\beta-1)^2}}.$$

Since $C > 1$, this shows that W_k converges to 0 when k goes to infinity. □

To show Theorem 2 we use the usual scaling of the Navier-Stokes equation. For that we introduce the rescaled solutions:

$$\begin{aligned} u_k(t,x) &= \lambda^k u(\lambda^{2k} t + t_0, \lambda^k x + x_0), \\ P_k(t,x) &= \lambda^{2k} P(\lambda^{2k} t + t_0, \lambda^k x + x_0), \end{aligned}$$

for a fixed $\lambda < 1$. Notice that u_k is still a vector whose components will be denoted u_{ki}. For every fixed (t_0, x_0) in the interior of $]0, \infty[\times\Omega$, for k big enough (u_k, P_k) is solution to (1) (2) in Q_0. We define the time dependent mean value pressure function:

$$\overline{P}_k(t) = \frac{1}{|B_0|} \int_{B_0} P_k(t,x) \, dx,$$

and a sequence:

$$V_k = \|u_k\|^2_{L^\infty(-1,1;L^2(B_0))} + \frac{1}{\lambda^8} \|P_k - \overline{P}_k\|^2_{L^p(-1,1;L^2(B_0))}, \tag{7}$$

for $1 < p < 4/3$. Notice that for any $u \in L^\infty(0,\infty; L^2(\Omega)) \times L^2(0,\infty; H_0^1(\Omega))$ solution to (1) the corresponding pressure P lies in $L^p_{\rm loc}(L^2_{\rm loc})$ for this range of p (see for instance [14]). We have the following proposition:

Proposition 5 *For $1 < p < 4/3$, there exists $\lambda < 1$ and $\delta_p^* \leq C^*/2$ small enough such that the following property holds true for any solution u to (1) (2) lying in $L^\infty(0,\infty; L^2(\Omega)) \times L^2(0,\infty; H_0^1(\Omega))$. For any (t_0, x_0) in the interior of $]0,\infty[\times\Omega$ verifying (4), there is a $k_0 > 0$ such that the sequence V_k defined by (7) verifies:*

$$V_{k+1} \leq \frac{V_k}{4} + \frac{C^*}{4},$$

for any $k \geq k_0$.

This proposition gives Theorem 2. Indeed since for k bigger than a k_0 we have:

$$V_{k+1} \leq \frac{V_k}{4} + \frac{C^*}{4},$$

this implies that $\limsup V_k \leq C^*/3$. Moreover:

$$\|\nabla u_k\|^2_{L^2(L^2)} = \frac{1}{\lambda^k} \int_{t_0-\lambda^{2k}}^{t_0+\lambda^{2k}} \int_{x_0+B(\lambda^k)} |\nabla u|^2 \, dx \, dt.$$

So, from (4) and the bound on δ_p^* of Proposition 5, there is a k_1 big enough such that:

$$V_{k_1} + \|\nabla u_{k_1}\|^2_{L^2(L^2)} < \frac{C^*}{3} + \frac{C^*}{2} + \varepsilon \leq C^*.$$

Notice that $(u_{k_1}, P_{k_1} - \overline{P}_{k_1})$ is solution to (1) (2) in $Q_0 = [-1,1] \times B(1)$. So, from Theorem 1, $|u_{k_1}| \leq 1$ on $[-1/2, 1] \times B(1/2)$.

We can notice that Proposition 3 deals with a nonlinear sequence whose exponent is expected to be as high as possible. In contrast Proposition 5 consider a linear relation. The reason is that U_0 is supposed to be small. So we can use in a full extent the smallness of power function near 0. In contrast the purpose of Proposition 5 is to bring V_k small enough coming from a V_0 which can be very large.

3 Preliminaries and pressure decomposition

This section is devoted to preliminaries and to the decomposition of the pressure into the local and non local parts.

Lemma 6 *There exists a constant C such that for every k, and every $F \in L^\infty(T_k, 1; L^2(B_k))$ and $\nabla F \in L^2(Q_k)$:*

$$\|F\|_{L^{10/3}(Q_k)} \leq C \left(\|F\|_{L^\infty(T_k,1;L^2(B_k))} + \|F\|^{2/5}_{L^\infty(T_k,1;L^2(B_k))} \|\nabla F\|^{3/5}_{L^2(Q_k)} \right).$$

Proof. From Sobolev imbedding we have:

$$\|F\|_{L^2(T_k,1;L^6(B_k))} \leq C\left(\|F\|_{L^\infty(T_k,1;L^2(B_k))} + \|\nabla F\|_{L^2(Q_k)}\right).$$

Notice that we can choose the same constant for every k since $B(1/2) \subset B_k \subset B(1)$. Holder inequality gives:

$$\begin{aligned}\|F\|_{L^{10/3}(Q_k)} &\leq \|F\|^{2/5}_{L^\infty(T_k,1;L^2(B_k))}\|F\|^{3/5}_{L^2(T_k,1;L^6(B_k))}\\ &\leq C\left(\|F\|_{L^\infty(T_k,1;L^2(B_k))} + \|F\|^{2/5}_{L^\infty(T_k,1;L^2(B_k))}\|\nabla F\|^{3/5}_{L^2(Q_k)}\right).\end{aligned}$$

□

We introduce functions $\phi_k \in C^\infty(\mathbb{R}^3)$ verifying:

$$\begin{aligned}&\phi_k(x) = 1 \qquad \text{in } B_{k-2/3}\\ &\phi_k(x) = 0 \qquad \text{in } B^C_{k-1}\\ &0 \leq \phi_k(x) \leq 1\\ &|\nabla\phi_k| \leq C2^{3k}\\ &|\nabla^2\phi_k| \leq C2^{6k}.\end{aligned}$$

We have the following lemma:

Lemma 7 *For $p > 1$, let $G_{ij} \in L^\infty(T_{k-1}, 1; L^1(B_{k-1}))$, $1 \leq i,j \leq 3$ and $P \in L^p(T_{k-1}, 1; L^1(B_{k-1}))$ verifying in Q_{k-1}:*

$$-\Delta P = \sum_{ij} \partial_i\partial_j G_{ij}.$$

Then we can decompose $P|_{B_{k-2/3}}$ into two parts:

$$P|_{B_{k-2/3}} = P_{k1}|_{B_{k-2/3}} + P_{k2}|_{B_{k-2/3}},$$

where P_{k1} verifies:

$$-\Delta P_{k1} = 0 \qquad \text{in} \qquad [T_{k-1}, 1] \times B_{k-2/3},$$

and the following estimates on the closer set $B_{k-1/3}$:

$$\begin{aligned}&\|\nabla P_{k1}\|_{L^p(T_{k-1},1;L^\infty(B_{k-1/3}))} + \|P_{k1}\|_{L^p(T_{k-1},1;L^\infty(B_{k-1/3}))}\\ &\qquad \leq C2^{12k}\left(\|P\|_{L^p(T_{k-1},1;L^1(B_{k-1}))} + \sum_{ij}\|G_{ij}\|_{L^\infty(T_{k-1},1;L^1(B_{k-1}))}\right).\end{aligned}$$

The second part P_{k2} is solution on $[T_{k-1}, 1] \times \mathbb{R}^3$ to:

$$-\Delta P_{k2} = \sum_{i,j} \partial_i\partial_j(\phi_k G_{ij}).$$

Notice that the support of ϕ_k is contained in B_{k-1} so we can define $\phi_k G_{ij}$ in $\mathbb{R}^3$ by extending it to 0 on B_{k-1}^c.

Proof. Since $\phi_k = 1$ on $B_{k-2/3}$, we have $P = \phi_k P$ in $[T_{k-1}, 1] \times B_{k-2/3}$. Moreover:

$$\begin{aligned}
-\Delta(\phi_k P) &= -\phi_k \Delta P - 2\mathrm{div}((\nabla\phi_k)P) + P\Delta\phi_k \\
-\phi_k \Delta P &= \phi_k \sum_{i,j} \partial_i\partial_j G_{ij} \\
&= +\sum_{i,j} \partial_i\partial_j(\phi_k G_{ij}) - \sum_{i,j} \partial_j[(\partial_i\phi_k)(G_{ij})] \\
&\qquad - \sum_{i,j} \partial_i[(\partial_j\phi_k)(G_{ij})] + \sum_{i,j}(\partial_i\partial_j\phi_k)(G_{ij}).
\end{aligned}$$

Let us define P_{k2} by:

$$-\Delta P_{k2} = \sum_{i,j} \partial_i\partial_j \left[\phi_k G_{ij}\right].$$

and P_{k1} by:

$$\begin{aligned}
-\Delta P_{k1} &= -2\mathrm{div}((\nabla\phi_k)P) + P\Delta\phi_k + \sum_{i,j} D_{ij}(G_{ij}) \\
&= \sum_i D_{ii}P + \sum_{i,j} D_{ij}(G_{ij}), \\
D_{ij}f &= -\partial_j((\partial_i\phi_k)f) - \partial_i((\partial_j\phi_k)f) + (\partial_i\partial_j\phi_k)f.
\end{aligned}$$

Notice that for every f, $D_{ij}f$ vanishes on $B_{k-2/3}$ since $\nabla\phi_k = 0$ on this set. This implies first that:

$$-\Delta P_{k1} = 0 \qquad\qquad \text{in} \qquad B_{k-2/3}.$$

Moreover, for every $x \in B_{k-1/3}$, using the representation:

$$P_{k1} = \frac{1}{4\pi}\frac{1}{|x|} * [\sum_i D_{ii}P + \sum_{i,j} D_{ij}(G_{ij})],$$

we find:

$$\begin{aligned} P_{k1}(t,x) &= \frac{1}{4\pi}\int_{B^C_{k-2/3}} \frac{1}{|x-y|}(-2\mathrm{div}((\nabla\phi_k)P) + P\Delta\phi_k)(t,y)\,dy \\ &\quad +\frac{1}{4\pi}\int_{B^C_{k-2/3}} \frac{1}{|x-y|}(\sum_{i,j} D_{ij}(G_{ij}))(t,y)\,dy \\ &= \frac{1}{4\pi}\int_{B^C_{k-2/3}} 2\frac{x-y}{|x-y|^3}\cdot(\nabla\phi_k(y))P(t,y)\,dy \\ &\quad +\frac{1}{4\pi}\int_{B^C_{k-2/3}} \frac{1}{|x-y|}P(t,y)\Delta\phi_k(t,y)\,dy \\ &\quad +\frac{1}{4\pi}\sum_{i,j}\int_{B^C_{k-2/3}} \frac{(x-y)_i}{|x-y|^3}(\partial_j\phi_k(y))G_{ij}(t,y)\,dy \\ &\quad +\frac{1}{4\pi}\sum_{i,j}\int_{B^C_{k-2/3}} \frac{(x-y)_j}{|x-y|^3}(\partial_i\phi_k(y))G_{ij}(t,y)\,dy \\ &\quad +\frac{1}{4\pi}\sum_{i,j}\int_{B^C_{k-2/3}} \frac{1}{|x-y|}(\partial_i\partial_j\phi_k(y))G_{ij}(t,y)\,dy. \end{aligned}$$

Since the distance between $B_{k-1/3}$ and $B^C_{k-2/3}$ is bigger than 2^{-3k} and:

$$\begin{aligned} &|\nabla\phi_k| \leq C2^{3k} \\ &|\nabla^2\phi_k| \leq C2^{6k}, \end{aligned}$$

We have for every $x\in B_{k-1/3}$:

$$|P_{k1}(t,x)| \leq C2^{9k}\left(\int_{B_{k-1}}|P|\,dy + \int_{B_{k-1}}|G|\,dy\right),$$

where $|G| = \sum_{i,j}|G_{ij}|$. In the same way we can write:

$$\nabla P_{k1} = \frac{-1}{4\pi}\frac{x}{|x|^3} * [\sum_i D_{ii}P + \sum_{i,j} D_{ij}(G_{ij})],$$

to find that for every $x\in B_{k-1/3}$:

$$|\nabla P_{k1}(t,x)| \leq C2^{12k}\left(\int_{B_{k-1}}|P|\,dy + \int_{B_{k-1}}|G|\,dy\right).$$

Taking the L^p norm in time leads to the desired bound on P_{k1}. Indeed, since $1-T_k\leq 2$, we have:

$$\|G_{ij}\|_{L^p(T_k,1;L^1(B_{k-1}))} \leq 2^{1/p}\|G_{ij}\|_{L^\infty(T_k,1;L^1(B_{k-1}))}.$$

□

Let us state two straightforward corollaries which will be useful in the next sections:

Corollary 8 *Let (u, P) be a solution to (1) (2) in Q_{k-1}. Then we can decompose $P|_{B_{k-2/3}}$ into two parts:*

$$P|_{B_{k-2/3}} = P_{k1}|_{B_{k-2/3}} + P_{k2}|_{B_{k-2/3}},$$

where:

$$\begin{aligned}&\|\nabla P_{k1}\|_{L^p(T_{k-1},1;L^\infty(B_{k-1/3}))} + \|P_{k1}\|_{L^p(T_{k-1},1;L^\infty(B_{k-1/3}))}\\ &\quad \leq C2^{12k}\left(\|P\|_{L^p(T_{k-1},1;L^1(B_{k-1}))} + \|u\|^2_{L^\infty(T_{k-1},1;L^2(B_{k-1}))}\right).\end{aligned}$$

and P_{k2} is solution on $[T_{k-1}, 1] \times \mathbb{R}^3$ to:

$$-\Delta P_{k2} = \sum_{i,j} \partial_i\partial_j \left[\phi_k u_j u_i\right].$$

Proof. Taking the divergence of equation (1) we find:

$$-\Delta P = \sum_{i,j} \partial_i\partial_j (u_j u_i).$$

We set $G_{ij} = u_i u_j$ to find the result. □

Corollary 9 *Let (u, P) be a solution to (1) (2) in $[-1, 1] \times B(1)$. We define:*

$$\overline{u}(t) = \int_{B(1)} u(t, x)\, dx.$$
$$\overline{P}(t) = \int_{B(1)} P(t, x)\, dx.$$

Then we can decompose $(P - \overline{P})|_{[-1/2,1/2]\times B(1/2)}$ into two parts:

$$(P - \overline{P})|_{[-1/2,1/2]\times B(1/2)} = P_1|_{[-1/2,1/2]\times B(1/2)} + P_2|_{[-1/2,1/2]\times B(1/2)},$$

where:

$$\begin{aligned}&\|P_1\|_{L^p(-1/2,1/2;L^\infty(B(1/2)))}\\ &\leq C\left(\|P - \overline{P}\|_{L^p(-1,1;L^1(B(1)))} + \|u - \overline{u}\|^2_{L^\infty(-1,1;L^2(B(1)))}\right),\\ &-\Delta P_1 = 0 \qquad \text{in} \qquad [-1/2, 1/2] \times B(1/2),\end{aligned}$$

and P_2 is solution on $\mathbb{R}^3$ to:

$$-\Delta P_2 = \sum_{i,j} \partial_i\partial_j \left[\phi_1(u_j - \overline{u}_j)(u_i - \overline{u}_i)\right].$$

Proof. Taking the divergence of equation (1) we find:

$$\begin{aligned}-\Delta P &= \sum_{i,j} \partial_i\partial_j (u_j u_i)\\ &= \sum_{i,j} \partial_i\partial_j \left[(u_j - \overline{u}_j)(u_i - \overline{u}_i)\right].\end{aligned}$$

We use Lemma 10 with $k = 1$ replacing P by $P - \overline{P}$ and setting $G_{ij} = (u_j - \overline{u}_j)(u_i - \overline{u}_i)$. Notice that we have $B_0 = B(1)$ and:

$$B(1/2) \subset B_{2/3} \subset B_{1/3},$$

so this gives the result. $\square$

We finish this section by a lemma which gives the links between d_k and the the gradient of v_k.

Lemma 10 *The function u can be split in the following way:*

$$u = u\frac{v_k}{|u|} + u\left(1 - \frac{v_k}{|u|}\right),$$

where:

$$\left|u\left(1 - \frac{v_k}{|u|}\right)\right| \leq 1 - 2^{-k}.$$

Moreover we can bound the following gradients with respect to d_k:

$$\begin{aligned}
&\frac{v_k}{|u|}|\nabla u| \leq d_k,\\
&\mathbf{1}_{\{|u|\geq 1-2^{-k}\}}|\nabla|u|| \leq d_k,\\
&|\nabla v_k| \leq d_k,\\
&\left|\nabla\frac{uv_k}{|u|}\right| \leq 3d_k.
\end{aligned}$$

Proof. The function $(1 - v_k/|u|)$ is Lipshitz and equal to:

$$\begin{aligned}
1 - \frac{v_k}{|u|} &= 1 \qquad \text{if} \qquad |u| \leq 1 - 2^{-k}\\
&= \frac{1 - 2^{-k}}{|u|} \qquad \text{if} \qquad |u| \geq 1 - 2^{-k}.
\end{aligned}$$

Therefore:

$$\left|u\left(1 - \frac{v_k}{|u|}\right)\right| \leq 1 - 2^{-k}.$$

Let us first show that:

$$\frac{v_k}{|u|}|\nabla u| \leq d_k \tag{8}$$

$$\mathbf{1}_{\{|u|\geq 1-2^{-k}\}}|\nabla|u|| \leq d_k. \tag{9}$$

Statement (8) comes from the definition of d_k and the fact that $v_k \leq |u|$:

$$d_k^2 \geq \frac{v_k}{|u|}|\nabla u|^2 \geq \left(\frac{v_k}{|u|}|\nabla u|\right)^2.$$

To show (9), notice that:

$$|\nabla|u||^2 = \left|\frac{u}{|u|}\nabla u\right|^2 \leq |\nabla u|^2.$$

So:

$$d_k^2 \geq \frac{(1-2^{-k})\mathbf{1}_{\{|u|\geq(1-2^{-k})\}} + v_k}{|u|}|\nabla|u||^2,$$

with:

$$((1-2^{-k}) + v_k)\mathbf{1}_{\{|u|\geq(1-2^{-k})\}} = |u|\mathbf{1}_{\{|u|\geq(1-2^{-k})\}}.$$

So:

$$d_k^2 \geq \mathbf{1}_{\{|u|\geq(1-2^{-k})\}}|\nabla|u||^2.$$

Then the bound on ∇v_k follows (9) since:

$$|\nabla v_k| = |\nabla|u||\mathbf{1}_{\{|u|\geq(1-2^{-k})\}}.$$

To find the last inequality we fist write:

$$\nabla\left(\frac{uv_k}{|u|}\right) = \frac{u}{|u|}\nabla v_k + v_k\nabla\left(\frac{u}{|u|}\right).$$

The first term can be bounded by:

$$\left|\frac{u}{|u|}\nabla v_k\right| \leq |\nabla v_k| \leq d_k.$$

The second one can be rewritten in the following way:

$$v_k\nabla\left(\frac{u}{|u|}\right) = \frac{v_k}{|u|}\nabla u - \frac{v_k u}{|u|^2}\nabla|u|.$$

So, thanks to (8) and (9):

$$\begin{aligned}\left|v_k\nabla\left(\frac{u}{|u|}\right)\right| &\leq \frac{v_k}{|u|}|\nabla u| + \mathbf{1}_{\{|u|\geq(1-2^{-k})\}}|\nabla|u|| \\ &\leq 2d_k.\end{aligned}$$

This gives:

$$\left|\nabla\left(\frac{uv_k}{|u|}\right)\right| \leq 3d_k.$$

This ends the proof of the lemma. □

Remark. From Lemma 6, Lemma 10, and the definition of U_k we see:

$$\begin{aligned}&\|v_{k-1}\|_{L^{10/3}(Q_{k-1})} \\ &\leq C\left(\|v_{k-1}\|_{L^\infty(T_{k-1},1;L^2(B_{k-1}))} + \|v_{k-1}\|^{2/5}_{L^\infty(T_{k-1},1;L^2(B_{k-1}))}\|\nabla v_{k-1}\|^{3/5}_{L^2}\right) \\ &\leq C\left(\|v_{k-1}\|_{L^\infty(T_{k-1},1;L^2(B_{k-1}))} + \|v_{k-1}\|^{2/5}_{L^\infty(T_{k-1},1;L^2(B_{k-1}))}\|d_k\|^{3/5}_{L^2(Q_{k-1})}\right) \\ &\leq CU_{k-1}^{1/2}.\end{aligned}$$

Hence:

$$\|v_{k-1}\|_{L^{10/3}(Q_{k-1})} \leq CU_{k-1}^{1/2}. \tag{10}$$

4 Proof of Proposition 3

This section is devoted to the proof of Proposition 3. We remind the reader that it is the key point to Theorem 1 (see section 2). We split the proof into several steps.

Step 1: Evolution of v_k^2. The first step is to derive the equation verified by the level set energy function v_k. We summarize the result in the following lemma.

Lemma 11 *Let u be a solution of (1) (2) in $Q =]0,\infty[\times\Omega$. We define the level set energy function v_k as in Section 2. The function v_k verifies in the sense of distribution:*

$$\begin{aligned}\partial_t \frac{v_k^2}{2} + \mathrm{div}\left(u\frac{v_k^2}{2}\right) + d_k^2 - \Delta\frac{v_k^2}{2}&\\ +\mathrm{div}(uP) + (v_k/|u|-1)u\cdot\nabla_x P \leq 0.&\end{aligned} \quad (11)$$

Remark: The lemma is formally obtained multiplying (1) by $uv_k/|u|$. We have to show that all the derivation is valid if u verifies only the natural bound of Navier Stokes solution. Especially, we have to carefully check what happened for big values of u. Indeed, we cannot derive the equation on v_k^2 only from (1) for the same reason that we cannot derive (2) from (1) if u verifies only the usual bounds. Since v_k^2 behaves like $|u|^2$ for big values of $|u|$, we can use (2) to solve this problem.

Proof. First we can rewrite v_k^2 in the following way:

$$\frac{v_k^2}{2} = \frac{|u|^2}{2} + \frac{v_k^2 - |u|^2}{2}.$$

Equation (2) gives the evolution of $|u|^2/2$. For the second term we notice that for any (time or space) derivative ∂_α we have:

$$\begin{aligned}\partial_\alpha\left(\frac{v_k^2-|u|^2}{2}\right) &= v_k\partial_\alpha v_k - u\partial_\alpha u\\ &= v_k\partial_\alpha|u| - u\partial_\alpha u\\ &= v_k\frac{u}{|u|}\partial_\alpha u - u\partial_\alpha u\\ &= u\left(\frac{v_k}{|u|}-1\right)\partial_\alpha u.\end{aligned}$$

Lemma 10 ensures that $\left|u\left(\frac{v_k}{|u|}-1\right)\right|$ is bounded by 1. Using that $\mathrm{div}(u\otimes u) =$

$u \cdot \nabla u$, multiplying (1) by $u(v_k/|u|-1)$ we find:

$$\partial_t \frac{v_k^2-|u|^2}{2} + \mathrm{div}(u\frac{v_k^2-|u|^2}{2}) + u\left(\frac{v_k}{|u|}-1\right)\nabla P - u\left(\frac{v_k}{|u|}-1\right)\Delta u = 0. \quad (12)$$

Notice that the bound on $u\left(\frac{v_k}{|u|}-1\right)$ and the natural bounds on u ensures the validity of those calculations. Moreover we have:

$$\begin{aligned}
&-u\left(\frac{v_k}{|u|}-1\right)\Delta u \\
&= -\mathrm{div}\left(u\left(\frac{v_k}{|u|}-1\right)\nabla u\right) + \nabla\left(\frac{uv_k}{|u|}\right)\nabla u - |\nabla u|^2 \\
&= -\Delta\frac{v_k^2-|u|^2}{2} + \left(\frac{v_k}{|u|}-1\right)|\nabla u|^2 + (u\nabla u)\,\nabla\left(\frac{v_k}{|u|}\right),
\end{aligned}$$

with:

$$\begin{aligned}
&\left(\frac{v_k}{|u|}-1\right)|\nabla u|^2 + \left(|u|\frac{u}{|u|}\nabla u\right)\nabla\left(\frac{v_k}{|u|}\right) \\
&= \left(\frac{v_k}{|u|}-1\right)|\nabla u|^2 + (\nabla|u|)|u|\nabla\left(1-\frac{1-2^{-k}}{|u|}\right)_+ \\
&= \left(\frac{v_k}{|u|}-1\right)|\nabla u|^2 + \frac{(1-2^{-k})\mathbf{1}_{|\{|u|\geq 1-2^{-k}\}}}{|u|}|\nabla|u||^2 \\
&= d_k^2 - |\nabla u|^2.
\end{aligned}$$

So summing (2) and (12) leads to:

$$\begin{aligned}
\partial_t\frac{v_k^2}{2} + \mathrm{div}\left(u\frac{v_k^2}{2}\right) + d_k^2 - \Delta\frac{v_k^2}{2}& \\
+\mathrm{div}(uP) + u\left(\frac{v_k}{|u|}-1\right)\nabla P &\leq 0.
\end{aligned}$$

□

Step 2: Bound on U_k.

Let us introduce functions $\eta_k \in C^\infty(\mathbb{R}^3)$ verifying:

$$\begin{aligned}
&\eta_k(x) = 1 \qquad \text{in } B_k \\
&\eta_k(x) = 0 \qquad \text{in } B_{k-1/3}^C \\
&0 \leq \eta_k(x) \leq 1 \\
&|\nabla\eta_k| \leq C2^{3k} \\
&|\nabla^2\eta_k| \leq C2^{6k}.
\end{aligned}$$

We multiply (11) by $\eta_k(x)$ and integrate on $[\sigma,t]\times\mathbb{R}^3$ for $T_{k-1}\le\sigma\le T_k\le t\le 1$ to find:

$$\begin{aligned}
&\int \eta_k(x)\frac{|v_k(t,x)|^2}{2}\,dx+\int_\sigma^t\int \eta_k(x)d_k^2(s,x)\,dx\,ds\\
&\quad\le \int \eta_k(x)\frac{|v_k(\sigma,x)|^2}{2}\,dx\\
&\qquad+\int_\sigma^t\int \nabla\eta_k(x)u\frac{|v_k(s,x)|^2}{2}\,dx\,ds\\
&\qquad+\int_\sigma^t\int \Delta\eta_k(x)\frac{|v_k(s,x)|^2}{2}\,dx\,ds\\
&\qquad-\int_\sigma^t\int \eta_k(x)\left\{\mathrm{div}(uP)+\left(\frac{v_k}{|u|}-1\right)u\nabla P\right\}\,dx\,dt.
\end{aligned}$$

Integrating in σ between T_{k-1} and T_k and divided by $T_{k-1}-T_k=2^{-(k+1)}$, we find:

$$\begin{aligned}
&\sup_{t\in[T_k,1]}\left(\int \eta_k(x)\frac{|v_k(t,x)|^2}{2}\,dx+\int_{T_k}^t\int \eta_k(x)d_k^2(s,x)\,dx\,ds\right)\\
&\quad\le 2^{k+1}\int_{T_{k-1}}^{T_k}\int \eta_k(x)\frac{|v_k(\sigma,x)|^2}{2}\,dx\,d\sigma\\
&\qquad+\int_{T_{k-1}}^1\left|\int \nabla\eta_k(x)u\frac{|v_k(s,x)|^2}{2}\,dx\right|\,ds\\
&\qquad+\int_{T_{k-1}}^1\left|\int \Delta\eta_k(x)\frac{|v_k(s,x)|^2}{2}\,dx\right|\,ds\\
&\qquad+\int_{T_{k-1}}^1\left|\int \eta_k(x)\left\{\mathrm{div}(uP)+\left(\frac{v_k}{|u|}-1\right)u\nabla P\right\}\,dx\right|\,dt.
\end{aligned}$$

Since $\eta_k\equiv 1$ on B_k,

$$\begin{aligned}
U_k&\le \sup_{t\in[T_k,1]}\left(\int \eta_k(x)\frac{|v_k(t,x)|^2}{2}\,dx\right)+\int_{T_k}^1\int \eta_k(x)d_k^2(s,x)\,dx\,ds\\
&\le 2\sup_{t\in[T_k,1]}\left(\int \eta_k(x)\frac{|v_k(t,x)|^2}{2}\,dx+\int_{T_k}^t\int \eta_k(x)d_k^2(s,x)\,dx\,ds\right).
\end{aligned}$$

We claim that:

$$\begin{aligned}
U_k&\le C2^{6k}\int_{Q_{k-1}}|v_k(s,x)|^2\,dx\,ds\\
&\quad+C2^{3k}\int_{Q_{k-1}}|v_k(s,x)|^3\,dx\,ds\\
&\quad+2\int_{T_{k-1}}^1\left|\int \eta_k(x)\left\{\mathrm{div}(uP)+\left(\frac{v_k}{|u|}-1\right)u\nabla P\right\}\,dx\right|\,dt.
\end{aligned}\tag{13}$$

We use the bound on $\nabla\eta_k$ and $\Delta\eta_k$, the fact that η_k is supported in Q_{k-1}, and the decomposition:

$$u\frac{v_k^2}{2} = \left\{ u\left(1-\frac{v_k}{|u|}\right) + \frac{uv_k}{|u|} \right\} \frac{v_k^2}{2}.$$

Thanks to Lemma 10:

$$\begin{aligned} \left| u\left(1-\frac{v_k}{|u|}\right)\frac{v_k^2}{2} \right| &\leq \frac{v_k^2}{2} \\ \left| \frac{u}{|u|} v_k \frac{v_k^2}{2} \right| &\leq \frac{v_k^3}{2}. \end{aligned}$$

Step 3: Raise of the power exponents.

We want to bound the right-hand side term of (13) with nonlinear power of U_{k-1} bigger than 1. To do so we use the following method due to De Giorgi.

Lemma 12 *There exists a constant C such that for all $k > 1$ and $q > 1$ we have:*

$$\|1_{\{v_k>0\}}\|_{L^q(Q_{k-1})} \leq C2^{\frac{10k}{3q}} U_{k-1}^{\frac{5}{3q}},$$
$$\|1_{\{v_k>0\}}\|_{L^\infty(T_{k-1},1;L^q(B_{k-1}))} \leq C2^{\frac{2k}{q}} U_{k-1}^{\frac{1}{q}}.$$

Proof. If $v_k > 0$ then $|u| - 1 + 2^{-k} > 0$ and:

$$\begin{aligned} v_{k-1} &= [|u| - 1 + 2^{-k+1}]_+ \\ &= [|u| - 1 + 2^{-k} + (2^{-k+1} - 2^{-k})]_+ \\ &> 2^{-k+1} - 2^{-k} = 2^{-k}. \end{aligned}$$

Using Tchebichev inequality and (10), we find:

$$\begin{aligned} \|1_{\{v_k>0\}}\|_{L^q(Q_{k-1})}^q &= \int_{Q_{k-1}} 1_{\{v_k>0\}}\,dx\,dt \\ &\leq \int_{Q_{k-1}} 1_{\{v_{k-1}>2^{-k}\}}\,dx\,dt \\ &\leq |\{v_{k-1} > 2^{-k}\}| \cap Q_{k-1} \\ &\leq 2^{10k/3} \int_{Q_{k-1}} |v_{k-1}|^{10/3}\,dx\,dt \\ &\leq 2^{10k/3} \|v_{k-1}\|_{L^{10/3}(Q_{k-1})}^{10/3} \\ &\leq 2^{10k/3} U_{k-1}^{5/3}. \end{aligned}$$

The proof of the second statement is similar. Indeed for every $t \in [T_{k-1}, 1]$:

$$\begin{aligned}
\|1_{\{v_k(t,\cdot)>0\}}\|^q_{L^q(B_{k-1})} &\le \int_{B_{k-1}} 1_{\{v_k(t,\cdot)>0\}}\,dx \\
&\le \int_{B_{k-1}} 1_{\{v_{k-1}(t,\cdot)>2^{-k}\}}\,dx \\
&\le |\{v_{k-1}(t,\cdot)>2^{-k}\}| \cap B_{k-1} \\
&\le 2^{2k}\int_{B_{k-1}} |v_{k-1}(t,x)|^2\,dx \\
&\le 2^{2k} \sup_{s\in[T_{k-1},1]} \int_{B_{k-1}} v_{k-1}^2(s,x)\,dx \\
&\le 2^{2k} U_{k-1}.
\end{aligned}$$

Therefore:

$$\|1_{\{v_k>0\}}\|_{L^\infty(T_k-1,1;L^q(B_{k-1}))} \le 2^{\frac{2k}{q}} U_{k-1}^{\frac{1}{q}}.$$

□

This lemma allows us to control the two first terms of the righthand side of (13):

$$\begin{aligned}
&C2^{6k}\int_{Q_{k-1}} |v_k(s,x)|^2\,dx\,ds + C2^{3k}\int_{Q_{k-1}} |v_k(s,x)|^3\,dx\,ds \\
&\quad \le C2^{6k}\|v_k^2\|_{L^{5/3}(Q_{k-1})}\|1_{\{v_k>0\}}\|_{L^{5/2}(Q_{k-1})} \\
&\qquad + C2^{3k}\|v_k^3\|_{L^{10/9}(Q_{k-1})}\|1_{\{v_k>0\}}\|_{L^{10}(Q_{k-1})}
\end{aligned}$$

From the definition of v_k we have that $v_k \le v_{k-1}$, and so:

$$\|v_k^2\|_{L^{5/3}(Q_{k-1})} = \|v_k\|^2_{L^{10/3}(Q_{k-1})} \le \|v_{k-1}\|^2_{L^{10/3}(Q_{k-1})}.$$

This quantity is bounded by CU_{k-1} thanks to (10). In the same way we have:

$$\||v_k|^3\|_{L^{10/9}(Q_{k-1})} = \|v_k\|^3_{L^{10/3}(Q_{k-1})} \le U_{k-1}^{3/2}.$$

Therefore, thanks to Lemma 12:

$$\begin{aligned}
&C2^{6k}\int_{Q_{k-1}} |v_k(s,x)|^2\,dx\,ds + C2^{3k}\int_{Q_{k-1}} |v_k(s,x)|^3\,dx\,ds \\
&\quad \le C2^{6k+4k/3}U_k^{5/3}.
\end{aligned} \tag{14}$$

Notice that the exponent $5/3$ is bigger than $3/2$. Therefore we have succeeded to overtake the scale of the equation.

We want now consider the pressure terms in (13). Since Supp $\eta_k \subset B_{k-1/3}$:

$$\begin{aligned}
&\int_{T_{k-1}}^1 \left|\int \eta_k \{\mathrm{div}(uP) + (v_k/|u| - 1)u\nabla P\}\, dx\right|\, dt \\
&= \int_{T_{k-1}}^1 \left|\int_{B_{k-1/3}} \eta_k \{\mathrm{div}(uP) + (v_k/|u| - 1)u\nabla P\}\, dx\right|\, dt \\
&\leq \int_{T_{k-1}}^1 \left|\int_{B_{k-1/3}} \eta_k \frac{v_k u}{|u|}\nabla P_{k1}\, dx\right|\, dt \qquad (15)\\
&\quad + \int_{T_{k-1}}^1 \left|\int_{B_{k-1/3}} \eta_k \{\mathrm{div}(uP_{k2}) + (v_k/|u| - 1)u\nabla P_{k2}\}\, dx\right|\, dt. \qquad (16)
\end{aligned}$$

We have used the decomposition of Corollary 8 and the fact that ∇P_{k1} is bounded in x. Therefore:

$$\mathrm{div}(uP_{k1}) = u\nabla P_{k1} \in L^p(L^2).$$

Step 4: Bound of the pressure term involving P_{k1} (non local term).

We want to bound the term (15). We discuss about the value of the index p. If $p > 10$, then we bound it by:

$$\begin{aligned}
&C\|v_k\|_{L^{\frac{10}{3}}(Q_{k-1})}\|\nabla P_{k1}\|_{L^p(T_{k-1},1;L^\infty(B_{k-1/3}))}\|\mathbf{1}_{\{v_k>0\}}\|_{L^q(T_{k-1},1;L^{\frac{10}{7}}(B_{k-1}))}\\
&\quad \leq C\|v_k\|_{L^{10/3}(Q_{k-1})}\|\nabla P_{k1}\|_{L^p(T_{k-1},1;L^\infty(B_{k-1/3}))}\|\mathbf{1}_{\{v_k>0\}}\|_{L^q(Q_{k-1})},
\end{aligned}$$

where $\frac{1}{q} = \frac{7}{10} - \frac{1}{p}$. From (10) and Lemma 12 we find that it is bounded by:

$$C2^{7k/3-10k/(3p)}\|\nabla P_{k1}\|_{L^p(T_{k-1},1;L^\infty(B_{k-1}))}U_{k-1}^{5/3(1-1/p)}.$$

Thanks to the bound of ∇P_{k1} given by Corollary 8 we find that it is smaller than:

$$C2^{12k+\frac{7k}{3}-\frac{10k}{3p}}U_{k-1}^{5/3(1-1/p)}\left(\|P\|_{L^p(T_{k-1},1;L^1(B_{k-1}))} + \|u\|^2_{L^\infty(T_{k-1},1;L^2(B_{k-1}))}\right).$$

The power of U_{k-1} is bigger than $3/2$ for those values of p. Therefore, for $p > 10$, we still can overtake the typical scale of the equation for the non local pressure term. For the proof of Theorem 2 we need to consider also the small p.

For $p \leq 10$ we bound the term (15) by:

$$C\|v_k\|_{L^\infty(L^2)}\|\nabla P_{k1}\|_{L^p(T_{k-1},1;L^\infty(B_{k-1/3}))}\|\mathbf{1}_{\{v_k>0\}}\|_{L^{p'}(T_{k-1},1;L^2(B_{k-1}))}. \qquad (17)$$

For $2 \leq p \leq 10$ we control this term by:

$$\begin{aligned}
&C\|v_k\|_{L^\infty(T_{k-1},1;L^2(B_{k-1}))}\|\nabla P_{k1}\|_{L^p(T_{k-1},1;L^\infty(B_{k-1/3}))}\|\mathbf{1}_{\{v_k>0\}}\|_{L^2(Q_{k-1})}\\
&\quad \leq C2^{5k/3}\|\nabla P_{k1}\|_{L^p(T_{k-1},1;L^\infty(B_{k-1/3}))}U_{k-1}^{4/3},
\end{aligned}$$

thanks to Lemma 12. Notice that the power of U_{k-1} is still bigger than 1 for those values of p.

For $p < 2$, we control (17) by:

$$C\|v_k\|_{L^\infty(T_{k-1},1;L^2(B_{k-1}))}\|\nabla P_{k1}\|_{L^p(T_{k-1},1;L^\infty(B_{k-1/3}))}$$
$$\times\|\mathbf{1}_{\{v_k>0\}}\|_{L^{p'}(Q_{k-1})}\|\mathbf{1}_{\{v_k>0\}}\|_{L^\infty(T_{k-1},1;L^{\frac{2p}{2-p}}(B_{k-1}))}.$$

Lemma 12 shows that (17) is bounded by

$$C2^{7k/3-4k/(3p)}(\|P\|_{L^p(L^1)}+\|u\|^2_{L^\infty(L^2)})U_{k-1}^{5/3-2/(3p)}.$$

Notice that the power of U_{k-1} is still bigger than 1 for any $p > 1$. Hence for any $p > 1$ there exists $\alpha_p > 0$ and $\beta_p > 1$ such that the term (15) is bounded by:

$$C2^{k\alpha_p}U_{k-1}^{\beta_p}(\|P\|_{L^p(L^1)}+\|u\|^2_{L^\infty(L^2)}),$$

which is smaller than:

$$C2^{k\alpha_p}U_{k-1}^{\beta_p}(\|P\|_{L^p(L^1)}+1) \tag{18}$$

if $U_0 \le 1$. Moreover $\beta_p > 3/2$ if $p > 10$.

Step 5: Bound of the pressure term involving P_{k2} (local term).
To control the term involving P_{k2} we split it into three terms:

$$P_{k2} = P_{k21} + P_{k22} + P_{k23},$$

where P_{k21}, P_{k22}, P_{k23} are defined by:

$$\begin{aligned}
-\Delta P_{k21} &= \sum_{i,j}\partial_i\partial_j\left\{\phi_k u_j\left(1-\frac{v_k}{|u|}\right)u_i\left(1-\frac{v_k}{|u|}\right)\right\}\\
-\Delta P_{k22} &= \sum_{i,j}\partial_i\partial_j\left\{2\phi_k u_j\left(1-\frac{v_k}{|u|}\right)u_i\frac{v_k}{|u|}\right\}\\
-\Delta P_{k23} &= \sum_{i,j}\partial_i\partial_j\left\{\phi_k u_j\frac{v_k}{|u|}u_i\frac{v_k}{|u|}\right\},
\end{aligned}$$

We just used $1 = (1 - v_k/|u|) + v_k/|u|$. Thanks to Lemma 10, $u\,(1 - v_k/|u|)$ is bounded by 1. So, from Riesz Theorem:

$$\|P_{k21}\|_{L^q(Q_{k-1})} \le C_q \qquad \forall 1 < q < \infty.$$

We have:

$$\begin{aligned}
&\mathrm{div}(uP_{k21}) + u\,(v_k/|u| - 1)\,\nabla P_{k21}\\
&\quad= \mathrm{div}\left(v_k\frac{u}{|u|}P_{k21}\right) - P_{k21}\mathrm{div}\left(\frac{uv_k}{|u|}\right).
\end{aligned} \tag{19}$$

From Lemma 10, we have:

$$\left|\nabla \frac{uv_k}{|u|}\right| \leq 3d_k.$$

Therefore for $q > 2$:

$$\begin{aligned}
&\int_{T_{k-1}}^{1} \left| \int_{B_{k-1/3}} \eta_k \{\mathrm{div}(uP_{k21}) + (v_k/|u| - 1)u\nabla P_{k21}\}\, dx \right| dt \\
&\leq 2^{3k} C_q \|v_k\|_{L^{10/3}} \|P_{k21}\|_{L^q} \|\mathbf{1}_{\{|u|\geq 1-2^{-k}\}}\|_{L^{10q/(7q-10)}} \\
&\quad + C_q \|P_{k21}\|_{L^q} \|d_k\|_{L^2} \|\mathbf{1}_{\{|u|\geq 1-2^{-k}\}}\|_{L^{2q/(q-2)}} \\
&\leq C_q 2^{k\alpha_q} (U_{k-1}^{5/3(1-1/q)} + U_{k-1}^{4/3-5/(3q)}).
\end{aligned} \tag{20}$$

Notice that the second term (with the exponent smaller than 3/2) comes from the second term of the right hand side of (19), namely the pressure term which is not in a divergence form.

We now turn to the terms involving P_{k22} and P_{k23}. By Riesz Theorem and (10):

$$\begin{aligned}
\|P_{k22}\|_{L^{10/3}} &\leq C \sum_{i,j} \|u_j(1 - v_k/|u|)\|_{L^\infty} \|v_k u_i/|u|\|_{L^{10/3}} \\
&\leq C\|v_k\|_{L^{10/3}} \leq C U_{k-1}^{1/2}, \\
\|P_{k23}\|_{L^{5/3}} &\leq C \sum_{i,j} \|u_j v_k/|u|\|_{L^{10/3}} \|v_k u_i/|u|\|_{L^{10/3}} \\
&\leq C\|v_k\|_{L^{10/3}}^2 \leq U_{k-1}.
\end{aligned}$$

We need to control their gradients too:

Lemma 13 *We can decompose ∇P_{k22} and ∇P_{k23} in the following way:*

$$\begin{aligned}
\nabla P_{k22} &= G_{221} + G_{222} + G_{223}, \\
\nabla P_{k23} &= G_{231} + G_{232},
\end{aligned}$$

where:

$$\begin{aligned}
&\|G_{221}\|_{L^{10/3}(Q_{k-1/3})} \leq C 2^{3k} \|v_k\|_{L^{10/3}(Q_{k-1})} \\
&\|G_{222}\|_{L^2(Q_{k-1/3})} \leq C\|d_k\|_{L^2(Q_{k-1})} \\
&\|G_{223}\|_{L^{5/4}(Q_{k-1/3})} \leq C\|v_k\|_{L^{10/3}(Q_{k-1})} \|d_k\|_{L^2(Q_{k-1})} \\
&\|G_{231}\|_{L^{5/3}(Q_{k-1/3})} \leq C 2^{3k} \|v_k\|_{L^{10/3}(Q_{k-1})}^2 \\
&\|G_{232}\|_{L^{5/4}(Q_{k-1/3})} \leq C\|v_k\|_{L^{10/3}(Q_{k-1})} \|d_k\|_{L^2(Q_{k-1})}
\end{aligned}$$

Proof. We have:

$$\nabla\left(\phi_k\frac{u_jv_k}{|u|}\frac{u_iv_k}{|u|}\right)$$
$$=\nabla\phi_k\frac{u_jv_k}{|u|}\frac{u_iv_k}{|u|}$$
$$+\phi_k\nabla\left(\frac{u_jv_k}{|u|}\right)\frac{u_iv_k}{|u|}$$
$$+\phi_k\nabla\left(\frac{u_iv_k}{|u|}\right)\frac{u_jv_k}{|u|}.$$

Thanks to the bound on $\nabla\phi_k$ and Lemma 10 we have:

$$\left|\phi_k\nabla\left(\frac{u_jv_k}{|u|}\right)\frac{u_iv_k}{|u|}+\phi_k\nabla\left(\frac{u_iv_k}{|u|}\right)\frac{u_jv_k}{|u|}\right|$$
$$\leq Cd_kv_k,$$
$$\left|\nabla\phi_k\frac{u_jv_k}{|u|}\frac{u_iv_k}{|u|}\right|\leq C2^{3k}|v_k|^2.$$

So if we denote G_{231} and G_{232} solutions to:

$$-\Delta G_{231}=\sum_{i,j}\partial_i\partial_j\left(\nabla\phi_k\frac{u_jv_k}{|u|}\frac{u_iv_k}{|u|}\right)$$
$$-\Delta G_{232}=\sum_{i,j}\partial_i\partial_j\left(\phi_k\nabla\left(\frac{u_jv_k}{|u|}\right)\frac{u_iv_k}{|u|}+\phi_k\nabla\left(\frac{u_iv_k}{|u|}\right)\frac{u_jv_k}{|u|}\right),$$

We have $\nabla P_{k23}=G_{231}+G_{232}$, and from Riesz Theorem:

$$\|G_{231}\|_{L^{5/3}(Q_{k-1/3})}\leq C2^{3k}\|v_k\|^2_{L^{10/3}(Q_{k-1})}$$
$$\|G_{232}\|_{L^{5/4}(Q_{k-1/3})}\leq C\|v_k\|_{L^{10/3}(Q_{k-1})}\|d_k\|_{L^2(Q_{k-1})}.$$

For ∇P_{k22} we first compute:

$$\nabla\left(\phi_ku_j\left(1-\frac{v_k}{|u|}\right)\frac{u_iv_k}{|u|}\right)$$
$$=\nabla\phi_ku_j\left(1-\frac{v_k}{|u|}\right)\frac{u_iv_k}{|u|}$$
$$+\phi_ku_j\left(1-\frac{v_k}{|u|}\right)\nabla\frac{u_iv_k}{|u|}$$
$$+\phi_ku_i\left(1-\frac{v_k}{|u|}\right)(\nabla u_j)\frac{v_k}{|u|}$$
$$-\phi_ku_j\nabla\left(\frac{v_k}{|u|}\right)\frac{u_iv_k}{|u|}.$$

Notice that:

$$u_j\nabla\left(\frac{v_k}{|u|}\right)=\frac{u_j}{|u|}\nabla v_k-\frac{v_ku_j}{|u|^2}\nabla|u|.$$

So, thanks to Lemma 10:

$$\left|u_j\nabla\left(\frac{v_k}{|u|}\right)\right| \le |\nabla v_k| + \mathbf{1}_{\{|u|\ge 1-2^{-k}\}}|\nabla|u|| \le 2d_k.$$

Hence, if we denote G_{221}, G_{222}, and G_{223} solutions to:

$$\begin{aligned}
-\Delta G_{221} &= \sum_{i,j}\partial_i\partial_j\left(\nabla\phi_k u_j\left(1-\frac{v_k}{|u|}\right)\frac{u_i v_k}{|u|}\right)\\
-\Delta G_{222} &= \sum_{i,j}\partial_i\partial_j\left(\phi_k u_j\left(1-\frac{v_k}{|u|}\right)\nabla\frac{u_i v_k}{|u|}+\phi_k u_i\left(1-\frac{v_k}{|u|}\right)(\nabla u_j)\frac{v_k}{|u|}\right)\\
-\Delta G_{223} &= -\sum_{i,j}\partial_i\partial_j\left(\phi_k u_j\nabla\left(\frac{v_k}{|u|}\right)\frac{u_i v_k}{|u|}\right),
\end{aligned}$$

then we have $\nabla P_{k22} = G_{221}+G_{222}+G_{223}$, and from Riesz Theorem and Lemma 10:

$$\begin{aligned}
\|G_{221}\|_{L^{10/3}(Q_{k-1/3})} &\le C2^{3k}\|v_k\|_{L^{10/3}(Q_{k-1})}\\
\|G_{222}\|_{L^2(Q_{k-1/3})} &\le C\|d_k\|_{L^2(Q_{k-1})}\\
\|G_{223}\|_{L^{5/4}(Q_{k-1/3})} &\le C\|v_k\|_{L^{10/3}(Q_{k-1})}\|d_k\|_{L^2(Q_{k-1})}.
\end{aligned}$$

□

Using this lemma we can bound the term (16) in the following way:

$$\int_{T_{k-1}}^{1}\left|\int_{B_{k-\frac{1}{3}}}\eta_k\{\mathrm{div}(u(P_{k22}+P_{k23}))+(v_k/|u|-1)u\nabla(P_{k22}+P_{k23})\}\,dx\right|\,dt$$
$$\leq \int_{T_{k-1}}^{1}\int_{B_{k-1/3}}|\nabla\eta_k||u||P_{k22}+P_{k23}|\,dx\,dt$$
$$+\int_{T_{k-1}}^{1}\int_{B_{k-1/3}}\eta_k|(\nabla P_{k22}|+|\nabla P_{k23}|)\,dx\,dt$$
$$\leq C2^{3k}\int_{T_{k-1}}^{1}\int_{B_{k-1/3}}(1+v_k)(|P_{k22}|+|P_{k23}|)\,dx\,dt$$
$$+\int_{T_{k-1}}^{1}\int_{B_{k-1/3}}(|\nabla P_{k22}|+|\nabla P_{k23}|)\,dx\,dt$$
$$\leq C2^{3k}(\|\mathbf{1}_{\{|u|\geq 1-2^{-k}\}}\|_{L^{10/7}(B_{k-1})}\|P_{k22}\|_{L^{10/3}(B_{k-1/3})}$$
$$+\|\mathbf{1}_{\{|u|\geq 1-2^{-k}\}}\|_{L^{5/2}(B_{k-1})}\|P_{k23}\|_{L^{5/3}(B_{k-1/3})})$$
$$+C2^{3k}\|v_k\|_{L^{10/3}(B_{k-1})}(\|\mathbf{1}_{\{|u|\geq 1-2^{-k}\}}\|_{L^{5/2}(B_{k-1})}\|P_{k22}\|_{L^{10/3}(B_{k-1/3})}$$
$$+\|\mathbf{1}_{\{|u|\geq 1-2^{-k}\}}\|_{L^{10}(B_{k-1})}\|P_{k23}\|_{L^{5/3}(B_{k-1/3})})$$
$$+C(\|\mathbf{1}_{\{|u|\geq 1-2^{-k}\}}\|_{L^{10/7}(B_{k-1})}\|G_{221}\|_{L^{10/3}(B_{k-1/3})}$$
$$+\|\mathbf{1}_{\{|u|\geq 1-2^{-k}\}}\|_{L^{5/2}(B_{k-1})}\|G_{231}\|_{L^{5/3}(B_{k-1/3})})$$
$$+C\|\mathbf{1}_{\{|u|\geq 1-2^{-k}\}}\|_{L^{5}(B_{k-1})}(\|G_{223}\|_{L^{5/4}(B_{k-1/3})}+\|G_{232}\|_{L^{5/4}(B_{k-1/3})})$$
$$+C\|\mathbf{1}_{\{|u|\geq 1-2^{-k}\}}\|_{L^{2}(B_{k-1})}\|G_{222}\|_{L^{2}(B_{k-1/3})}$$
$$\leq C2^{\alpha k}U_{k-1}^{5/3}+CU_{k-1}^{4/3}.$$

Again the exponent $4/3 < 3/2$ comes from the pressure term which is not in a divergence form in (16).

Step 6: Conclusion.

From (13), (14), (18) and the last inequality of Step 5 we see that for every $p > 1$ there exists $\alpha_p > 0$, $\beta_p > 1$ such that for any solution to (1), (2) in $[-1,1]\times B(1)$, if $U_0 \leq 1$ then we have for every $k > 0$:

$$U_k \leq C_p^k(1+\|P\|_{L^p(0,1;L^1(B_0))})U_{k-1}^{\beta_p}.$$

This concludes the proof of Proposition 3. Moreover, for $p > 10$, the only bad terms (with exponent smaller than 3/2) comes from the local pressure term which cannot be written in a divergence form.

5 Proof of Proposition 5.

We consider (u_k, P_k) defined in the introduction, where $\lambda < 2^{-3}$ will be chosen later. Notice that for any $t, x \in Q_0 = [-1,1] \times B(1)$:

$$u_{k+1}(t,x) = \lambda u_k(\lambda^2 t, \lambda x).$$

We introduce for $-1 \le t \le \lambda^2, x \in B(1)$:

$$\psi_\lambda(t,x) = \frac{1}{(2\lambda^2 - t)^{3/2}} e^{-\frac{|x|^2}{4(2\lambda^2 - t)}}.$$

The function ψ_λ is solution to:

$$\partial_t \psi_\lambda + \Delta \psi_\lambda = 0 \qquad \text{in }]-\infty, \lambda^2] \times \mathbb{R}^3,$$

and verify:

$$|\psi_\lambda(-1,x)| \le 1 \qquad \text{for } x \in \mathbb{R}^3 \tag{21}$$

$$|\psi_\lambda(t,x)| \ge \frac{C}{\lambda^3} \qquad \text{for } |x| \le \lambda,\ -\lambda^2 \le t \le \lambda^2 \tag{22}$$

$$|\Delta\psi_\lambda| + |\nabla\psi_\lambda| \le C \qquad \text{for } x \in B_1^c,\ \ -1 \le t \le \lambda^2 \tag{23}$$

$$|\nabla\psi_\lambda| \le \frac{C}{\lambda^4} \qquad \text{for } x \in \mathbb{R}^3,\ \ -1 \le t \le \lambda^2. \tag{24}$$

We define:

$$\begin{aligned}
\overline{u}_k(t) &= \frac{1}{|B(1)|} \int_{B(1)} u_k(t,x)\,dx \\
\overline{P}_k(t) &= \frac{1}{|B(1)|} \int_{B(1)} P_k(t,x)\,dx \\
\overline{u^2}_k(t) &= \frac{1}{|B(1)|} \int_{B(1)} |u_k|^2(t,x)\,dx.
\end{aligned}$$

Multiplying (2) by $\eta_1(x)\psi_\lambda(t,x)$ and integrating on $[-1,s] \times \mathbb{R}^3$ for $-1 \le s \le \lambda^2$ we find:

$$\begin{aligned}
&\int \psi_\lambda(s,x)\eta_1(x)\frac{|u_k(s,x)|^2}{2}\,dx \\
\le &\int \psi_\lambda(-1,x)\eta_1(x)\frac{|u_k(-1,x)|^2}{2}\,dx \\
&+ \int_{-1}^{\lambda^2} \left| \int \nabla(\eta_1\psi_\lambda) \cdot u_k \left(\frac{|u_k|^2}{2} - \frac{\overline{u_k^2}}{2} \right) dx \right| dt \\
&+ \int_{-1}^{\lambda^2} \left| \int (\psi_\lambda \Delta\eta_1 + 2\nabla\eta_1 \cdot \nabla\psi_\lambda) \frac{|u_k|^2(t,x)}{2}\,dx \right| dt \\
&+ \int_{-1}^{\lambda^2} \left| \int \nabla(\eta_1\psi_\lambda) \cdot u_k \left(P_k - \overline{P}_k \right) dx \right| dt.
\end{aligned} \tag{25}$$

We have used the facts that:

$$\operatorname{div}\left(u_k\frac{|u_k|^2}{2}\right)=\operatorname{div}\left(u_k\left(\frac{|u_k|^2}{2}-\frac{\overline{u_k^2}}{2}\right)\right)$$
$$\operatorname{div}(u_kP_k)=\operatorname{div}(u_k(P_k-\overline{P}_k)),$$

since $\operatorname{div} u_k=0$. Thanks to (22) we have for $s\in[-\lambda^2,\lambda^2]$:

$$\begin{aligned}\int\psi_\lambda(s,x)\eta_1(x)\frac{|u_k(s,x)|^2}{2}\,dx&\geq\frac{C}{\lambda^3}\int_{B(\lambda)}\frac{|u_k(s,x)|^2}{2}\,dx\\&\geq\frac{C}{\lambda^2}\int|u_{k+1}(s,x)|^2\,dx.\end{aligned}\tag{26}$$

We have used that $\eta_1=1$ on $B(\lambda)$ (since $\lambda<2^{-3}$). So, we can get by this way some information on $\|u_{k+1}\|^2_{L^\infty(L^2)}$. We want to control the right hand side of the inequality (25). First we have from (21):

$$\int\psi_\lambda(-1,x)\eta_1(x)\frac{|u_k(-1,x)|^2}{2}\,dx\leq V_k.$$

Since:

$$\begin{aligned}&\left\|u_k\left(\frac{|u_k|^2}{2}-\frac{\overline{u_k^2}}{2}\right)\right\|_{L^1(-1,1;L^1(B(1)))}\\&\leq\|u_k\|_{L^\infty(-1,1;L^2(B(1)))}\left\|\frac{|u_k|^2}{2}-\frac{\overline{u_k^2}}{2}\right\|_{L^1(-1,1;L^2(B(1)))},\end{aligned}$$

and:

$$\|\nabla\frac{|u_k|^2}{2}\|_{L^1(-1,1;L^{3/2}(B(1)))}\leq\|u_k\|_{L^2(-1,1;L^6(B(1)))}\|\nabla u_k\|_{L^2([-1,1]\times B(1))},$$

by Sobolev Imbedding and Holder inequality in $[-1,1]\times B(1)$:

$$\begin{aligned}\left\|\frac{|u_k|^2}{2}-\frac{\overline{u_k^2}}{2}\right\|_{L^1(L^2)}&\leq\left\|\frac{|u_k|^2}{2}-\frac{\overline{u_k^2}}{2}\right\|_{L^1(L^3)}\\&\leq C\|\nabla\frac{|u_k|^2}{2}\|_{L^1(-1,1;L^{3/2}(B(1)))}\\&\leq C\|u_k\|_{L^2(L^6)}\|\nabla u_k\|_{L^2(L^2)}\\&\leq C\|\nabla u_k\|_{L^2(L^2)}\left(\|u_k\|_{L^\infty(L^2)}+\|\nabla u_k\|_{L^2(L^2)}\right).\end{aligned}$$

Therefore, using (24):

$$\begin{aligned}&\int\left|\nabla(\eta_1\psi_\lambda)u_k\left(\frac{|u_k|^2}{2}-\frac{\overline{u_k^2}}{2}\right)\right|\,dx\,dt\\&\leq\|\nabla\psi\|_{L^\infty}\|u_k\|_{L^\infty(L^2)}\left\|\frac{|u_k|^2}{2}-\frac{\overline{u_k^2}}{2}\right\|_{L^1(L^2)}\\&\leq C\frac{\|\nabla u_k\|_{L^2(L^2)}}{\lambda^4}V_k+\frac{C}{\lambda^4}\|\nabla u_k\|^2_{L^2(L^2)}\sqrt{V_k}.\end{aligned}\tag{27}$$

Thanks to (23), and noticing that $\nabla\eta_1 = 0$ on B_1^c:

$$\int_{-1}^{\lambda^2}\int \left(|\nabla\eta_1\nabla\psi_\lambda| + \psi_\lambda|\Delta\eta_1|\right)\frac{|u_k|^2}{2}\,dx\,dt \\ \leq C\|u_k\|^2_{L^\infty(L^2)} \leq CV_k. \tag{28}$$

The last term of (25) is bounded by

$$\frac{C}{\lambda^4}\|u_k\|_{L^\infty(L^2)}\|P_k - \overline{P}_k\|_{L^p(L^2)} \\ \leq C\left(\|u_k\|^2_{L^\infty(L^2)} + \frac{\|P_k - \overline{P}_k\|^2_{L^p(L^2)}}{\lambda^8}\right) \leq CV_k.$$

This together with equations (25), (26), (27), and (28) gives:

$$\|u_{k+1}\|^2_{L^\infty(-1,1;B(1))} \leq C\lambda^2 V_k + \frac{C}{\lambda^2}\|\nabla u_k\|_{L^2}V_k + \frac{C}{\lambda^2}\|\nabla u_k\|^2_{L^2}\sqrt{V_k}. \tag{29}$$

We need now to bound $\|P_{k+1} - \overline{P}_{k+1}\|^2_{L^p(L^2)}$. We use the decomposition of the pressure term of Corollary 9:

$$P_k - \overline{P}_k = P_{1k} + P_{2k}.$$

Since $\lambda < 1/2$: $[-\lambda^2, \lambda^2] \times B(\lambda)$ is contained in $[-1/2, 1/2] \times B(1/2)$. Since P_{k1} is harmonic in this latter set, we have for every $t \in [-\lambda^2, \lambda^2]$:

$$\frac{1}{|B(\lambda)|}\int_{B(\lambda)}\left|P_{1k}(t,x) - \frac{1}{|B(\lambda)|}\int_{B(\lambda)}P_{1k}(t,y)\,dy\right|^2 dx \\ \leq (2\lambda)^2\frac{1}{|B(1/2)|}\int_{B(1/2)}\left|P_{1k}(t,x) - \frac{1}{|B(1/2)|}\int_{B(1/2)}P_{1k}(t,y)\,dy\right|^2 dx \\ \leq C\lambda^2\int_{B(1/2)}|P_{1k}(t,x)|^2\,dx.$$

Consider now the term P_{k2}. It is solution to:

$$-\Delta P_{2k} = \sum_{i,j}\partial_i\partial_j[\phi_1(u_{kj} - \overline{u}_{kj})(u_{ki} - \overline{u}_{ki})].$$

If $p \leq 4/3$ then $4/(2-p) \leq 6$ and

$$\|u_k - \overline{u}_k\|^{2/p}_{L^2(L^{\frac{4}{2-p}})} \leq \|\nabla u_k\|^{2/p}_{L^2(L^2)}.$$

Then Riesz Theorem together with Holder inequality gives (if $p < 4/3$):

$$\begin{aligned} \|P_{k2}\|_{L^p(L^2)} &\leq C\||u_k - \overline{u}_k|^{2(1-1/p)}\|_{L^\infty(L^{\frac{p}{p-1}})}\||u_k - \overline{u}_k|^{2/p}\|_{L^p(L^{\frac{2p}{2-p}})} \\ &\leq C\|u_k - \overline{u}_k\|^{2(1-1/p)}_{L^\infty(L^2)}\|u_k - \overline{u}_k\|^{2/p}_{L^2(L^{\frac{4}{2-p}})} \\ &\leq C\|u_k - \overline{u}_k\|^{2(1-1/p)}_{L^\infty(L^2)}\|\nabla u_k\|^{2/p}_{L^2(L^2)} \\ &\leq CV_k^{1-1/p}\|\nabla u_k\|^{2/p}_{L^2(L^2)}. \end{aligned}$$

For every $t \in [-1,1]$:

$$\begin{aligned}
&\|P_{k+1}-\overline{P}_{k+1}\|^2_{(L^2(B(1)))}\\
&\le 2\lambda^4\frac{1}{|B(\lambda)|}\int_{B(\lambda)}\left|P_{1k}(\lambda^2 t,x)-\frac{1}{|B(\lambda)|}\int_{B(\lambda)}P_{1k}(\lambda^2 t,y)\,dy\right|^2\,dx\\
&\qquad +2\lambda^4\frac{1}{|B(\lambda)|}\int_{B(\lambda)}\left|P_{2k}(\lambda^2 t,x)-\frac{1}{|B(\lambda)|}\int_{B(\lambda)}P_{2k}(\lambda^2 t,y)\,dy\right|^2\,dx\\
&\le C\lambda^6\int_{B(1/2)}\left|P_{1k}(\lambda^2 t,x)\right|^2\,dx\\
&\qquad +4\lambda^4\frac{1}{|B(\lambda)|}\int_{B(\lambda)}\left|P_{2k}(\lambda^2 t,x)\right|^2\,dx\\
&\le C\lambda^6\int_{B(1)}\left|P_{1k}(\lambda^2 t,x)\right|^2\,dx+C\lambda\int_{B(\lambda)}\left|P_{2k}(\lambda^2 t,x)\right|^2\,dx.
\end{aligned}$$

Therefore:

$$\|P_{k+1}-\overline{P}_{k+1}\|^2_{L^p(L^2)}\le C\lambda^{6-\frac{4}{p}}\|P_{1k}\|^2_{L^p(L^2)}+C\lambda^{1-\frac{4}{p}}\|P_{2k}\|^2_{L^p(L^2)}.$$

The corollary gives the bound:

$$\begin{aligned}
&\|P_{1k}\|_{L^p(-1/2,1/2;L^\infty(B(1/2)))}\\
&\le C\left(\|P_k-\overline{P}_k\|_{L^p(-1,1;L^1(B(1)))}+\|u_k-\overline{u}_k\|^2_{L^\infty(-1,1;L^2(B(1)))}\right)\\
&\le C\left(\|P_k-\overline{P}_k\|_{L^p(-1,1;L^1(B(1)))}+\|\nabla u_k\|^2_{L^2}\right).
\end{aligned}$$

Using the bound computed for P_{k2} we find:

$$\begin{aligned}
&\|P_{k+1}-\overline{P}_{k+1}\|^2_{L^p(L^2)}\\
&\le C\lambda^{6-4/p}(\|P_k-\overline{P}_k\|^2_{L^p(-1,1;L^2(B(1)))}+\|\nabla u_k\|^4_{L^2(L^2)})\\
&\qquad +C\lambda^{1-4/p}V_k^{2-2/p}\|\nabla u_k\|^{4/p}_{L^2(L^2)}.
\end{aligned}$$

Hence:

$$\begin{aligned}
V_{k+1}\quad\le\quad & C(\lambda^2+\lambda^{6-4/p})V_k+\frac{C\|\nabla u_k\|_{L^2(L^2)}}{\lambda^2}V_k\\
&+\frac{C\|\nabla u_k\|^2_{L^2(L^2)}}{\lambda^2}\sqrt{V_k}+C\frac{\|\nabla u_k\|^4_{L^2(L^2)}}{\lambda^{2+4/p}}+C\frac{\|\nabla u_k\|^{4/p}_{L^2(L^2)}}{\lambda^{7+4/p}}V_k^{2-2/p}.
\end{aligned}$$

First notice that for $p<4/3$ we have $2-2/p\le 1/2$, and for any $0\le q\le 1$:

$$V_k^q\le 1+V_k.$$

Moreover we have $6-4/p>0$ so we can fix λ such that $C(\lambda^2+\lambda^{6-4/p})<1/8$. Then for every $1<p<4/3$ there exists δ_p^* small enough (depending on λ) such that if

$$\|\nabla u_k\|^2_{L^2(L^2)}\le 2\delta_p^*$$

then:

$$V_{k+1} \leq \frac{V_k}{4} + \frac{C^*}{4}.$$

This gives the result noticing that:

$$\|\nabla u_k\|^2_{L^2(L^2)} = \frac{1}{\lambda^k} \int_{-\lambda^{2k}+t_0}^{\lambda^{2k}+t_0} \int_{x+B(\lambda^k)} |\nabla u|^2 \, dx.$$

A Appendix

We introduce a rescaled Navier Stokes equation for $\varepsilon < 1$:

$$\begin{aligned} &\partial_t u + \frac{1}{\varepsilon}\mathrm{div}(u \otimes u) + \frac{1}{\varepsilon}\nabla P - \Delta u = 0 \qquad t \in [-1,1],\ x \in B(1), \\ &\mathrm{div} u = 0, \end{aligned} \tag{30}$$

with the local energy inequality:

$$\partial_t \frac{|u|^2}{2} + \frac{1}{\varepsilon}\mathrm{div}(u\frac{|u|^2}{2}) + \frac{1}{\varepsilon}\mathrm{div}(uP) + |\nabla u|^2 - \Delta\frac{|u|^2}{2} \leq 0 \qquad t \in [-1,1],\ x \in B(1). \tag{31}$$

Let us assume the following conjecture.

Conjecture 14 *There exists universal constants $p > 1$, $C, \beta > 3/2$ such that for any solution to (30) (31) in $[-1,1] \times B(1)$ we have for every $k > 0$*

$$U_k \leq \frac{C^k}{\varepsilon}\left(1 + \|P\|_{L^p(0,1;L^1(B(1)))}\right) U_{k-1}^\beta.$$

Notice that after proper scaling, Proposition 3 is the equivalent to this conjecture but with a $\beta < 3/2$. Let us prove that this conjecture implies that all the solutions to (1)(2) lying in $L^\infty(L^2) \times L^2(H^1_0)$ are locally bounded (and so regular). Consider such a solution, and any point $(t_0, x_0) \in]0,\infty[\times\mathbb{R}^3$. For $\lambda < \sqrt{t_0}$, we have $(\lambda^2[-1,1]+t_0) \times (x_0+\lambda B(1))$ which is included in the domain $]0,\infty[\times\mathbb{R}^3$. We define a family of solution to (30) (31) in $]-1,1[\times B(1)$ in the following way:

$$\begin{aligned} u_\varepsilon(t,x) &= \varepsilon\lambda u(t_0 + \lambda^2 t, x_0 + \lambda x), \\ P_\varepsilon(t,x) &= \varepsilon^2\lambda^2 P(t_0 + \lambda^2 t, x_0 + \lambda x). \end{aligned}$$

Notice that:

$$\|P_\varepsilon\|_{L^p(L^1)} \leq \frac{\varepsilon^2}{\lambda^{1+2/p}} \|P\|_{L^p(L^1)},$$

is bounded for every $1 < p < \infty$ and $\varepsilon > 0$ since $P \in L^p_{\mathrm{loc}}(L^1_{\mathrm{loc}})$ (see for instance [14]). So from the conjecture the $U_{\varepsilon,k}$ associated to u_ε verifies for ε small enough:

$$U_{\varepsilon,k} \leq 2\frac{C^k}{\varepsilon} U^\beta_{\varepsilon,k-1}.$$

Let us denote $W_{\varepsilon,k} = U_{\varepsilon,k}\varepsilon^{-\frac{1}{\beta-1}}$. Whenever $W_{\varepsilon,k-1} \le 1$ we have $W_{\varepsilon,k} \le 2C^k W_{\varepsilon,k-1}^{\beta}$. So, from Lemma 4, if $W_{\varepsilon,0} \le C_0^*$ then $\lim W_{\varepsilon,k} = 0$. So if

$$U_{\varepsilon,0} \le \varepsilon^{1/(\beta-1)} = \varepsilon^2 \varepsilon^{-\frac{2\beta-3}{\beta-1}}, \tag{32}$$

then $U_{\varepsilon,k}$ converges to 0 when k goes to infinity. But:

$$\begin{aligned} U_{\varepsilon,0} &= \|u_\varepsilon\|^2_{L^\infty(L^2)} + \|\nabla u_\varepsilon\|^2_{L^2} \\ &\le \frac{\varepsilon^2}{\lambda}\left(\|u\|^2_{L^\infty(L^2)} + \|\nabla u\|^2_{L^2}\right). \end{aligned}$$

Therefore, since $(2\beta-3)/(\beta-1) > 0$, for ε small enough, (32) is verified and $|u_\varepsilon| \le 1$ on $]-1/2, 1/2[\times B(1/2)$. This means that $|u|$ is bounded by $1/(\lambda\varepsilon)$ on a neighborhood of (t_0, x_0). □

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