Almost sure convergence for transport in random media

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Abstract

We go back to the analysis of the transport of (classical or quantum) particles in time-dependent random media as proposed in [F. Poupaud, A. Vasseur, J. Math. Pures Appl., 82, 711–748, 2003], where the convergence of the expectation of solutions of the Liouville equation (resp. Wigner equation) to solution of the Fokker-Planck equation (resp. Boltzmann equation) is established. Here we show that the convergence holds pointwise with respect to the random variable. This result relies on certain scale separation properties of the scaling, as indicated by several counter-examples.

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1 Introduction

We are interested in asymptotic problems for models describing the transport of particles subjected to a strong and oscillating velocity or force field. This question dates back to Taylor Tay [17] who made a connection to the Brownian motion characterized by the correlations of the randomly oscillating field. The problem has been analyzed in details in KP1, KP2 [9, 10]. Instead of dealing with particles' trajectories, it is also possible to consider the Liouville equation satisfied by the particle distribution functions. This

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viewpoint is developed in [5], and, dealing with quantum particles, in [3]. One is then led asymptotically to Fokker-Planck or Boltzmann equation, with a kernel that depends on the statistics of the applied potential. This question has been revisited in [15], exploiting the time-decorrellation of the oscillating potential. The approach is quite systematic and it has been applied to various contexts: plasma physics [13], semi-classical regimes and hyperbolic systems [1], modeling of laden-flows [7], turbulent transport [8], etc. In these papers, the convergence (in a suitable weak sense) of the expectation of the particle distribution is established. One may wonder whether or not the convergence holds pointwise with respect to the random variable.

This is the question addressed in this paper, which is organized as follows. In Section 2 we remind the basis of Poupaud-Vasseur's approach [15]. The simple example presented in Section 2 also shows that, in general, the almost sure convergence does not hold. In Section 3, we go back to the Liouville equation as dealt with in [15]. We show that, for this problem, a *scale separation* property allows us to establish the almost sure convergence, which strengthens the original result. The idea of the proof consists in considering non linear quantities, constructed by doubling the phase space variable. Identifying the limiting equation shows that non linear quantities pass to the limit, thus proving the almost sure convergence. We describe further examples in Section 4; in particular we establish the almost sure convergence for the case of quantum transport.

2 Overview of the Poupaud-Vasseur (PV) approach, and a counter-example to the almost sure convergence

We shall deal with random fields. Namely, given a probability space $(\Omega, \mathscr{T}, \mu)$, with μ a σ-finite probability measure, we are going to consider random variables $X : \omega \to \mathbb{R}$. The expectation of such a random variable is nothing but

$$\mathbb{E}(X) = \int_\Omega X(\omega)\, \mathrm{d}\mu(\omega).$$

We will consider evolution equations where the coefficients are such random variables, considered as given. Accordingly, the solutions of the evolution equations are random variables too. As it is usual we will omit to denote the dependence with respect to the randomness variable $\omega \in \Omega$.

Roughly speaking, the problem we are interested in has the form of an evolution equation, whose unknown is u_ε, and which involves an oscillating quantity a_ε. The parameter $0 < \varepsilon \ll 1$ characterizes the scales of variation of the field. The equation is linear, but the analysis of the regime $\varepsilon \to 0$, leads to consider the limit of the product $a_\varepsilon u_\varepsilon$. We set up the basis of the PV approach [15] which relies on

- the Duhamel formula in order to analyze the behavior of $\mathbb{E}[a_\varepsilon u_\varepsilon]$; it leads to identify a leading term which is quadratic with respect to a_ε;
- an assumption on the time decorrellation of a_ε.

We start with a toy model, inspired from F. Poupaud's lecture notes `poupaud_Gr` [14]. Consider the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}u_\varepsilon(t) = \frac{\mathrm{i}}{\varepsilon}a(t/\varepsilon^2)u_\varepsilon(t) \tag{1} \quad \boxed{\texttt{ode}}$$

where $a(t) : \Omega \to \mathbb{R}$ is a random variable with zero mean

$$\mathbb{E}a(t) = 0 \quad \text{for any } t \geq 0. \tag{2} \quad \boxed{\texttt{centre}}$$

We suppose it has finite variance, and fulfils the following stationarity property

$$\mathbb{E}\big[a(t)a(s)\big] = R(s-t) \quad \text{for any } t, s \geq 0. \tag{3} \quad \boxed{\texttt{statio}}$$

Obviously, the equation preserves the modulus

$$|u_\varepsilon(t)| = \Big|\exp\Big(\mathrm{i}\int_0^t a(s/\varepsilon^2)\,\mathrm{d}s\Big)\Big|\ |u(0)| = |u(0)|. \tag{4} \quad \boxed{\texttt{mod}}$$

We assume that the initial data $u(0)$ is deterministic (it does not depend on the "hidden" variable $\omega \in \Omega$) and we wish to determine the asymptotic behavior of the expection value $\mathbb{E}u_\varepsilon(t)$ as ε goes to 0. The crucial assumption consists in the following finite time decorrelation hypothesis:

$$\begin{cases} a(t) \text{ and } a(s) \text{ decorrellate when } |t-s| \geq 1: \\ \mathbb{E}\big[a(t)\ a(s)\big] = R(s-t) = 0 \quad \text{for } |t-s| \geq 1. \end{cases} \tag{5} \quad \boxed{\texttt{dec}}$$

The analysis is based on the Duhamel formula:

$$u_\varepsilon(t) = u_\varepsilon(s) + \frac{\mathrm{i}}{\varepsilon}\int_s^t a(\sigma/\varepsilon^2)u_\varepsilon(\sigma)\,\mathrm{d}\sigma. \tag{6} \quad \boxed{\texttt{Duhamel0}}$$

It has two immediate consequences:

a) first of all, with $s = 0$ and bearing in mind that $u_\varepsilon(0)$ is deterministic, we realize that $u_\varepsilon(t)$ depends only on realization of $a(s/\varepsilon^2)$ for $0 \leq s \leq t$; hence due to (5) `dec`, $u_\varepsilon(t)$ and $a(t'/\varepsilon^2)$ are independent when $t' - t \geq \varepsilon^2$;

b) second of all, it already provides the continuity estimate

$$|u_\varepsilon(t) - u_\varepsilon(s)| \leq \|a\|_{L^\infty}|u(0)|\ \frac{|t-s|}{\varepsilon}, \tag{7} \quad \boxed{\texttt{cont}}$$

where we have used (4) `mod`.
Then, let us specify (6) `Duhamel0` to $s = t - \varepsilon^2$, which involves the decorrelation time scale:

$$u_\varepsilon(t) = u_\varepsilon(t-\varepsilon^2) + \frac{1}{\varepsilon}\int_{t-\varepsilon^2}^t \mathrm{i}a(\sigma/\varepsilon^2)u_\varepsilon(\sigma)\,\mathrm{d}\sigma. \tag{8} \quad \boxed{\texttt{Duhamel1}}$$

It allows us to rewrite

$$\frac{a(t/\varepsilon^2)}{\varepsilon}u_\varepsilon(t) = \frac{a(t/\varepsilon^2)}{\varepsilon}u_\varepsilon(t-\varepsilon^2) + \frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^t \mathrm{i}a(t/\varepsilon^2)a(\sigma/\varepsilon^2)u_\varepsilon(\sigma)\,\mathrm{d}\sigma.$$

In the right hand side, due to (2) and property a) stated above, the expectation of the first term vanishes

$$\begin{aligned}\mathbb{E}\Big(\frac{a(t/\varepsilon^2)}{\varepsilon}u_\varepsilon(t-\varepsilon^2)\Big) &= \mathbb{E}\frac{a(t/\varepsilon^2)}{\varepsilon}\ \mathbb{E}u_\varepsilon(t-\varepsilon^2) \quad \text{by decorrellation}\\ &= 0 \qquad\qquad\qquad\qquad\qquad\quad \text{by (2),}\end{aligned}$$

while the second term is of order

$$\Big(\frac{1}{\varepsilon^2}\times\text{length of the integration interval}\Big) = \Big(\frac{1}{\varepsilon^2}\times\varepsilon^2\Big) = \mathcal{O}(1).$$

In particular, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}u_\varepsilon(t) = \mathrm{i}\mathbb{E}\frac{a(t/\varepsilon^2)}{\varepsilon}u_\varepsilon(t) = -\mathbb{E}\frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^{t} a(t/\varepsilon^2)a(\sigma/\varepsilon^2)u_\varepsilon(\sigma)\,\mathrm{d}\sigma = \mathcal{O}(1).$$

To be more specific, by (4), we have

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}u_\varepsilon(t)\Big| \le \|a\|^2_{L^\infty}|u(0)|.$$

Therefore, the family $\{t\mapsto\mathbb{E}u_\varepsilon(t),\ \varepsilon>0\}$ is equibounded and equicontinuous and, by applying the Arzela-Ascoli theorem, we deduce that it belongs to a compact set of $C^0([0,T])$ for any $0<T<\infty$. The next step consists in replacing in the last integral $u_\varepsilon(\sigma)$, with $t-\varepsilon^2\le\sigma\le t$, by $\mathbb{E}u_\varepsilon(t)$. The error can indeed be controlled and vanishes as ε goes to 0, as a consequence of (5) and (7). To this end, we write

$$\begin{aligned}&\mathbb{E}\frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^{t} a(t/\varepsilon^2)a(\sigma/\varepsilon^2)u_\varepsilon(\sigma)\,\mathrm{d}\sigma\\ &\quad = \mathbb{E}\frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^{t} a(t/\varepsilon^2)a(\sigma/\varepsilon^2)\,\mathrm{d}\sigma\ \mathbb{E}u_\varepsilon(t)\\ &\qquad +\mathbb{E}\frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^{t} a(t/\varepsilon^2)a(\sigma/\varepsilon^2)\big(u_\varepsilon(\sigma)-u_\varepsilon(t-2\varepsilon^2)\big)\,\mathrm{d}\sigma\\ &\qquad +\mathbb{E}\frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^{t} a(t/\varepsilon^2)a(\sigma/\varepsilon^2)\big(u_\varepsilon(t-2\varepsilon^2)-\mathbb{E}u_\varepsilon(t-2\varepsilon^2)\big)\,\mathrm{d}\sigma\\ &\qquad +\mathbb{E}\frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^{t} a(t/\varepsilon^2)a(\sigma/\varepsilon^2)\big(\mathbb{E}u_\varepsilon(t-2\varepsilon^2)-\mathbb{E}u_\varepsilon(t)\big)\,\mathrm{d}\sigma.\end{aligned}\tag{9}$$

Since, by (5), $u_\varepsilon(t-2\varepsilon^2)$ is independent of $\{a(\sigma/\varepsilon^2),\ t-\varepsilon^2\le\sigma\le t\}$, the third term can be recast as

$$\frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^{t}\mathbb{E}\Big(a(t/\varepsilon^2)a(\sigma/\varepsilon^2)\Big)\mathbb{E}\big(u_\varepsilon(t-2\varepsilon^2)-\mathbb{E}u_\varepsilon(t-2\varepsilon^2)\big)\,\mathrm{d}\sigma = 0$$

and it vanishes. The second term is estimated by using (7): for any $t-\varepsilon^2\le\sigma\le t$, we have $|u_\varepsilon(\sigma)-u_\varepsilon(t-2\varepsilon^2)|\le\|a\|_{L^\infty}|u(0)|\frac{|\sigma-(t-2\varepsilon^2)|}{\varepsilon}\le 2\|a\|_{L^\infty}|u(0)|\varepsilon$. Hence the second term in (21) is dominated by

$$2\|a\|^3_{L^\infty}|u(0)|\times\frac{1}{\varepsilon^2}\times\varepsilon\times\varepsilon^2 = 2\|a\|^3_{L^\infty}|u(0)|\varepsilon.$$

A similar estimate holds for the last term in ($\overset{\texttt{mn}}{21}$). We thus have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}u_\varepsilon(t) = \mathrm{i}\mathbb{E}\Big(\frac{a(t/\varepsilon^2)}{\varepsilon}u_\varepsilon(t)\Big) = -\mathbb{E}\left(\frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^{t} a(t/\varepsilon^2)a(\sigma/\varepsilon^2)\,\mathrm{d}\sigma\right)\ \mathbb{E}u_\varepsilon(t) + r_\varepsilon,$$

with $r_\varepsilon \xrightarrow[\varepsilon\to 0]{} 0.$

$$(10) \quad \boxed{\texttt{Duhamel2}}$$

We end up with the following statement.

Theorem 2.1 *The expectation $\mathbb{E}u_\varepsilon$ converges uniformly on $[0,T]$ to u, solution of the ODE*

$$\frac{\mathrm{d}}{\mathrm{d}t}u = -\lambda u$$

where the effective coefficient is

$$\lambda = \frac{1}{2}\int_{-\infty}^{+\infty} R(\tau)\,\mathrm{d}\tau \geq 0.$$

Proof. It only remains to identify the coefficient λ. First, let us check the positivity of λ which is not completely direct. The proof relies on the following observation: for any $F \in L^1(\mathbb{R})$, we have

$$\int_{\mathbb{R}} F(\tau)\,\mathrm{d}\tau = \lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{+L}\int_{-L}^{+L} F(\sigma-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau. \qquad (11) \quad \boxed{\texttt{lemalex}}$$

Therefore, λ becomes

$$\begin{aligned}\lambda &= \lim_{L\to\infty}\frac{1}{4L}\int_{-L}^{+L}\int_{-L}^{+L} R(\sigma-\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau = \lim_{L\to\infty}\frac{1}{4L}\int_{-L}^{+L}\int_{-L}^{+L}\mathbb{E}\big[a(\tau)a(\sigma)\big]\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ &= \lim_{L\to\infty}\frac{1}{4L}\mathbb{E}\left(\int_{-L}^{+L} a(\sigma)\,\mathrm{d}\sigma\right)^2 \geq 0.\end{aligned}$$

Consequently, the modulus of the limit u is not conserved anymore, but it decays as time grows. It indicates that the passage to the limit and the stochasticity effects have induced a loss of irreversibility. We prove ($\overset{\texttt{lemalex}}{11}$) by writing

$$\int_{\mathbb{R}} F(s)\,\mathrm{d}s = \frac{1}{2L}\int_{-L}^{+L}\int_{\mathbb{R}} F(s)\,\mathrm{d}s\,\mathrm{d}t = \frac{1}{2L}\int_{-L}^{+L}\left(\int_{\mathbb{R}} F(\sigma-t)\,\mathrm{d}\sigma\right)\,\mathrm{d}t.$$

Therefore, it suffices to show

$$\lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{+L}\left(\int_{|\sigma|\geq L}|F(\sigma-t)|\,\mathrm{d}\sigma\right)\,\mathrm{d}t = 0.$$

Changing variables again, we reduce the problem to investigating the behavior of

$$\frac{1}{2L}\int_{-L}^{+L}\left(\int_{s+t\geq L}|F(s)|\,\mathrm{d}s\right)\,\mathrm{d}t,$$

for large L's, and similarly for the quantity obtained by replacing $s+t \geq L$ by $s+t \leq -L$. The Fubini theorem yields

$$\begin{aligned}
&\frac{1}{2L}\int_0^\infty |F(s)| \left(\int_{\mathbb{R}} \mathbf{1}_{-L\leq t\leq L}\mathbf{1}_{L-s\leq t}\,\mathrm{d}t\right)\mathrm{d}s \\
&\qquad = \frac{1}{2L}\int_0^{2L} |F(s)| \left(\int_{L-s}^{L} \mathrm{d}t\right)\mathrm{d}s + \frac{1}{2L}\int_0^{2L} |F(s)| \left(\int_{-L}^{L} \mathrm{d}t\right)\mathrm{d}s \\
&\qquad = \int_0^{2L} |F(s)| \frac{s}{2L}\,\mathrm{d}s + \int_{2L}^{\infty} |F(s)|\,\mathrm{d}s,
\end{aligned}$$

and we conclude by applying the Lebesgue theorem.

We go back to the definition of the effective coefficient. In (10), we make the following quantity appear

$$\frac{1}{\varepsilon^2}\mathbb{E}\int_{t-\varepsilon^2}^{t} a(t/\varepsilon^2)a(s/\varepsilon^2)\,\mathrm{d}s = \int_{t-\varepsilon^2}^{t} R\Big(\frac{s-t}{\varepsilon^2}\Big)\frac{\mathrm{d}s}{\varepsilon^2} = \int_{-1}^{0} R(\tau)\,\mathrm{d}\tau.$$

as a consequence of (3). Hence this quantity does not depend on ε anymore and it can be rewritten as

$$\begin{aligned}
\int_0^1 R(-\tau)\,\mathrm{d}\tau &= \mathbb{E}\int_0^1 a(\tau)a(0)\,\mathrm{d}\tau = \mathbb{E}\int_0^1 a(0)a(\tau)\,\mathrm{d}\tau = \mathbb{E}\int_0^1 R(\tau)\,\mathrm{d}\tau \\
&= \frac{1}{2}\int_{-1}^{+1} R(\tau)\,\mathrm{d}\tau = \frac{1}{2}\int_{-\infty}^{+\infty} R(\tau)\,\mathrm{d}\tau
\end{aligned}$$

due to the support property of the function R in (5). ■

However with this simple example, we immediately see an obstruction to the almost sure convergence: we have

$$\mathbb{E}|u_\varepsilon(t)|^2 = 1$$

whose limit as $\varepsilon \to 0$ does not coincide with

$$|u(t)|^2 = \lim_{\varepsilon\to 0} |\mathbb{E}u_\varepsilon(t)|^2 = e^{-2\lambda t}|u(0)|^2.$$

The general strategy described on this basic example — use of time-compactness, the Duhamel formula and the decorrelation property of the oscillating field — applies to further, more intricate, contexts. In particular, we will be interested in partial differential equations describing the transport of classical or quantum particles. We shall see that the almost sure convergence can be obtained, owing to certain scale separation induced by the scaling.

3 The Liouville equation with a randomly oscillating potential

liouv

We are interested in the Liouville equation

$$\partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon + \mathscr{E}_\varepsilon\cdot\nabla_v f_\varepsilon = 0, \tag{12}$$

Liou

endowed with the initial data

$$f_\varepsilon\big|_{t=0} = f_0.$$

The initial data f_0 is deterministic, while the force field $(t,x) \mapsto \mathscr{E}_\varepsilon(t,x)$ is a random variable described through three scales:

- η which measures the strength of the force field,
- τ which is the scale of the time variation of the force field,
- λ which is the scale of the space variation of the force field.

At the end of the day, certain relations will be imposed between these parameters, so that we can assume they are all functions of $\varepsilon > 0$, which is intended to become small. We thus denote

$$\mathscr{E}_\varepsilon(t,x) = \eta E(t/\tau, x/\lambda).$$

We shall assume throughout the discussion

(H0) $E \in W^{1,\infty}(\mathbb{R} \times \mathbb{R}^N)$,

(H1) $\mathbb{E}E(t,x) = 0$ for any $(t,x) \in \mathbb{R} \times \mathbb{R}^N$,

(H2) $\mathbb{E}\big[E(t,x) \otimes E(s,y)\big] = 0$ when $|t-s| \geq 1$, for any $x, y \in \mathbb{R}^N$.

We will make the assumptions on the statistical properties of the force field more precise later on. We start by writing the Duhamel formula

$$\begin{aligned} f_\varepsilon(t,x,v) &= f_\varepsilon(t-s, x-sv, v) + \int_{t-s}^{t} \mathscr{E}_\varepsilon(\sigma, x-(t-\sigma)v) \cdot \nabla_v f_\varepsilon(\sigma, x-(t-\sigma)v, v)\,\mathrm{d}\sigma \\ &= S_s f_\varepsilon(t-s,x,v) + \int_{t-s}^{t} S_{t-\sigma}\big[\mathscr{E}_\varepsilon \cdot \nabla_v f_\varepsilon\big](\sigma,x,v)\,\mathrm{d}\sigma, \end{aligned} \tag{13}$$

`Duhamel`

where S_t stands for the semi-group associated to the transport operator

$$S_t g(x,v) = g(x-tv, v).$$

For further purposes, note that its adjoint is defined by

$$S_t^\star \phi(x,v) = \phi(x+tv, v),$$

and we have

$$\nabla_v\big[S_t^\star \phi(x,v)\big] = (\nabla_v \phi)(x+tv,v) + t(\nabla_x \varphi)(x+tv,v) = S_t^\star(\nabla_v + t\nabla_x)\varphi(x,v).$$

By using (`Duhamel`13) with $s = t$, we see that $f_\varepsilon(t)$ only depends on the realizations of $\mathscr{E}_\varepsilon(\sigma)$ for $0 \leq \sigma \leq t$. Since the initial data is deterministic, we deduce the following independence property.

`l:indep` **Lemma 3.1** *Assume (H2). For any $h \geq \tau$, $f_\varepsilon(t)$ and $\mathscr{E}_\varepsilon(t+h)$ are independent.*

Owing to (13), we can compute the force term in the Liouville equation with the formula

$$\mathscr{E}_\varepsilon(t,x)f_\varepsilon(t,x,v)=\mathscr{E}_\varepsilon(t,x)S_s f_\varepsilon(t-s,x,v)+\int_{t-s}^{t}\mathscr{E}_\varepsilon(t,x)S_{t-\sigma}\big[\mathscr{E}\cdot\nabla_v f_\varepsilon\big](\sigma,x,v)\,\mathrm{d}\sigma.$$

In fact, we shall work on a weak form of this relation. Let $\varphi\in C_c^\infty(\mathbb{R}^N\times\mathbb{R}^N)$. We have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\iint f_\varepsilon(t,x,v)\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x\\&\qquad=\iint f_\varepsilon(t,x,v)v\cdot\nabla_x\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x+\iint f_\varepsilon(t,x,v)\mathscr{E}_\varepsilon(t,x)\cdot\nabla_v\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x,\end{aligned}$$

where only the last term can present difficulties for the asymptotic issues $\varepsilon\to 0$. It can be cast as follows

$$\begin{aligned}&\iint \mathscr{E}_\varepsilon(t,x)f_\varepsilon(t,x,v)\cdot\nabla_v\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x\\&=\iint f_\varepsilon(t-s,x,v)S_s^\star\big[\mathscr{E}_\varepsilon(t)\cdot\nabla_v\varphi\big](x,v)\,\mathrm{d}v\,\mathrm{d}x\\&\qquad+\int_{t-s}^{t}\iint \nabla_v\cdot\big(\mathscr{E}_\varepsilon(\sigma)S_{t-\sigma}^\star\big[\mathscr{E}_\varepsilon(t)\cdot\nabla_v\varphi\big]\big)(x,v)\ f_\varepsilon(\sigma,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\sigma\\&=\iint f_\varepsilon(t-s,x,v)S_s^\star\big[\mathscr{E}_\varepsilon(t)\cdot\nabla_v\varphi\big](x,v)\,\mathrm{d}v\,\mathrm{d}x\\&\qquad+\int_{t-s}^{t}\iint \mathscr{E}_\varepsilon(\sigma,x)\cdot S_{t-\sigma}^\star\big[(\nabla_v+(t-\sigma)\nabla_x)(\mathscr{E}_\varepsilon(t)\cdot\nabla_v\varphi)\big](x,v)\ f_\varepsilon(\sigma,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\sigma.\end{aligned}\tag{14}$$

`formula`

We observe that

$$\begin{aligned}&(\nabla_v+(t-\sigma)\nabla_x)(\mathscr{E}_\varepsilon(t)\cdot\nabla_v\varphi)(x,v)\\&\quad=\sum_{j=1}^{N}(\mathscr{E}_\varepsilon)_j(t,x)(\nabla_v+(t-\sigma)\nabla_x)\partial_{v_j}\varphi(x,v)+(t-\sigma)\sum_{j=1}^{N}\partial_{v_j}\varphi\nabla_x(\mathscr{E}_\varepsilon)_j(t,x).\end{aligned}\tag{15}$$

`formula2`

We look at the order of the different terms with respect to the scaling parameters. We shall use (14) with $s=\tau$ and take the expectation. By virtue of Lemma 3.1 and (H1), the expectation of $\iint f_\varepsilon(t-\tau,x,v)S_\tau^\star\big[\mathscr{E}_\varepsilon(t)\varphi\big](x,v)\,\mathrm{d}v\,\mathrm{d}x$ vanishes. The second term in the right hand side of (14) is quadratic with respect to the force field and integrated over a time interval with amplitude τ, which yields a factor $\tau\eta^2$. This is the order of the term with a velocity derivative coming from (15). Concerning the term involving $\nabla_x\varphi$, it is multiplied by $t-\sigma$, which, again, is of order τ; it yields a contribution of order $\tau^2\eta^2$. Finally, we have

$$(t-\sigma)\nabla_x\mathscr{E}_\varepsilon(t,x)=\frac{t-\sigma}{\lambda}\ (\nabla_x E)\Big(\frac{t}{\tau},\frac{x}{\lambda}\Big).$$

It yields a contribution of order $\frac{\tau^2\eta^2}{\lambda}$.

We shall deal with the situation where

$$\eta\to+\infty,\qquad \tau\to 0,\qquad \lambda\to 0\tag{16}$$

`scal1`

assuming that

$$\tau\eta^2 = 1, \qquad \tau = \lambda. \tag{17} \qquad \boxed{\texttt{scal2}}$$

In other words, we can set

$$\eta = \frac{1}{\varepsilon}, \quad \tau = \varepsilon^2, \quad \lambda = \varepsilon^2, \quad 0 < \varepsilon \ll 1.$$

We recover the scaling dealt with in `PV`[15]. With these relations, the leading term when taking the expectation in (`formula`14) thus reads

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\mathbb{E}\int_{t-\varepsilon^2}^{t}\iint E\Big(\frac{\sigma}{\varepsilon^2},\frac{x}{\varepsilon^2}\Big)\otimes E\Big(\frac{t}{\varepsilon^2},\frac{x+(t-\sigma)v}{\varepsilon^2}\Big)\\
&\qquad\qquad\qquad : (D_v^2\varphi)(x+(t-\sigma)v,v)f_\varepsilon(\sigma,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\sigma\\
&+\frac{1}{\varepsilon^2}\mathbb{E}\int_{t-\varepsilon^2}^{t}\iint \frac{t-\sigma}{\varepsilon^2}(\nabla_x E)\Big(\frac{t}{\varepsilon^2},\frac{x+(t-\sigma)v}{\varepsilon^2}\Big)\,E\Big(\frac{\sigma}{\varepsilon^2},\frac{x}{\varepsilon^2}\Big)\\
&\qquad\qquad\qquad \cdot(\nabla_v\varphi)(x+(t-\sigma)v,v)f_\varepsilon(\sigma,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\sigma.
\end{aligned} \tag{18} \qquad \boxed{\texttt{leading}}$$

In order to identify the limit equation, we combine two ingredients. First of all, we replace $f_\varepsilon(\sigma,x,v)$ in (`leading`18), where $t-\tau \le \sigma \le t$, by $\mathbb{E}f_\varepsilon(t,x,v)$; this is possible at the price of error terms that tend to 0 as $\tau \to 0$, in the same spirit as in (`nn`21) and (`Duhamel2`10). Second of all, we use a more precise information on the statistics of the force field. Namely, we assume

(H3) $\mathbb{E}\big[E(s,y)\otimes E(t,x)\big] = R(t-s,x-y)$ with $\mathrm{supp}(R)\subset[-1,+1]\times B(0,1)$ and $\partial_x^\alpha R$ lies in $C^0(\mathbb{R};L^1\cap L^\infty(\mathbb{R}^N))$ for $|\alpha|\le 3$.

We have already used the fact that the correllation function R is supported in $[-1,+1]\times\mathbb{R}^N$ see (H2); here we additionnally assume a stationarity assumption with respect to the space variable, with a correlation function that decays at infinity. We refer the reader to `PV`[15, Example 2] for the construction of potentials that satisfy such an assumption.

We are thus led to compute the coefficients

$$\mathrm{A}_\varepsilon[\varphi](x,v) = \frac{1}{\varepsilon^2}\mathbb{E}\int_{t-\varepsilon^2}^{t} E\Big(\frac{\sigma}{\varepsilon^2},\frac{x}{\varepsilon^2}\Big)\otimes E\Big(\frac{t}{\varepsilon^2},\frac{x+(t-\sigma)v}{\varepsilon^2}\Big) : (D_v^2\varphi)(x+(t-\sigma)v,v)\,\mathrm{d}\sigma,$$

and

$$\mathrm{B}_\varepsilon[\varphi](x.v) = \frac{1}{\varepsilon^2}\mathbb{E}\int_{t-\varepsilon^2}^{t} \frac{t-\sigma}{\varepsilon}(\nabla_x E)\Big(\frac{t}{\varepsilon^2},\frac{x+(t-\sigma)v}{\varepsilon^2}\Big)\,E\Big(\frac{\sigma}{\varepsilon^2},\frac{x}{\varepsilon^2}\Big)\cdot(\nabla_v\varphi)(x+(t-\sigma)v,v)\,\mathrm{d}\sigma.$$

With (H3), we obtain

$$\begin{aligned}
\mathrm{A}_\varepsilon[\varphi](x,v) &= \frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^{t} R\Big(\frac{t-\sigma}{\varepsilon^2},\frac{(t-\sigma)v}{\varepsilon^2}\Big) : (D_v^2\varphi)(x+(t-\sigma)v,v)\,\mathrm{d}\sigma\\
&= \int_0^1 R(\theta,\theta v) : (D_v^2\varphi)(x+\varepsilon^2\theta v,v)\,\mathrm{d}\theta.
\end{aligned}$$

We conclude that

$$\mathrm{A}_\varepsilon[\varphi](x,v) \xrightarrow[\varepsilon\to 0]{} \mathrm{A}[\varphi](x,v) = \int_0^1 R(\theta,\theta v)\,\mathrm{d}\theta : D_v^2\varphi(x,v).$$

Next, we remark that

$$\begin{aligned}
\partial_{v_i}\Big[R\Big(\frac{t-\sigma}{\varepsilon^2},\frac{(t-\sigma)v}{\varepsilon^2}\Big)\Big] &= \frac{(t-\sigma)}{\varepsilon}(\partial_{x_i}R)\Big(\frac{t-\sigma}{\varepsilon^2},\frac{(t-\sigma)v}{\varepsilon^2}\Big)\\
&= \partial_{v_i}\mathbb{E}\left[E\Big(\frac{\sigma}{\varepsilon^2},\frac{x}{\varepsilon^2}\Big)\otimes E\Big(\frac{t}{\varepsilon^2},\frac{x+(t-\sigma)v}{\varepsilon^2}\Big)\right]\\
&= \mathbb{E}\left[E\Big(\frac{\sigma}{\varepsilon^2},\frac{x}{\varepsilon^2}\Big)\otimes \partial_{v_i}\left(E\Big(\frac{t}{\varepsilon^2},\frac{x+(t-\sigma)v}{\varepsilon^2}\Big)\right)\right]\\
&= \mathbb{E}E\Big(\frac{\sigma}{\varepsilon^2},\frac{x}{\varepsilon^2}\Big)\otimes\frac{(t-\sigma)}{\lambda}(\partial_{x_i}E)\Big(\frac{t}{\varepsilon^2},\frac{x+(t-\sigma)v}{\varepsilon^2}\Big).
\end{aligned}$$

We deduce that

$$\begin{aligned}
\mathrm{B}_\varepsilon[\varphi](x,v) &= \sum_{i,j=1}^N\int_0^1 \partial_{v_i}\left[R_{ij}\Big(\theta,\frac{\tau\theta v}{\lambda}\Big)\right](\partial_{v_j}\varphi)(x+\varepsilon^2\theta v,v)\,\mathrm{d}\theta\\
&\xrightarrow[\tau\to 0]{} \mathrm{B}[\varphi](x,v) = \sum_{i,j=1}^N \partial_{v_i}\left[\int_0^1 R_{ij}(\theta,\theta v)\,\mathrm{d}\theta\right]\partial_{v_j}\varphi(x,v)
\end{aligned}.$$

It only remains to explain how the compactness of $\mathbb{E}f_\varepsilon$ can be obtained. We suppose

(H4) f_0 lies in $L^1\cap L^\infty(\mathbb{R}^N\times\mathbb{R}^N)$.

Since the field $(t,x,v)\mapsto(v,\mathscr{E}_\varepsilon(t,x))$ is smooth, we can define the associated characteristics $(t,s,x,v)\mapsto(\xi(t,s,x,v),\Theta(t,s,x,v))$, solution of the ODE system

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\xi(t)=\Theta(t), && \frac{\mathrm{d}}{\mathrm{d}t}\Theta(t)=\mathscr{E}_\varepsilon(t,\xi(t))\\
&\xi(s)=x, && \Theta(s)=v.
\end{aligned}$$

Then, the solution of ($\overset{\texttt{Liou}}{12}$) is simply given by

$$f_\varepsilon(t,x,v)=f_0(\xi(0,t,x,v),\Theta(0,t,x,v)),$$

which clearly implies

$$\|f_\varepsilon(t,\cdot)\|_{L^p(\mathbb{R}^N\times\mathbb{R}^N)}\le\|f_0\|_{L^p(\mathbb{R}^N\times\mathbb{R}^N)}$$

(using the fact that $\mathrm{div}_{x,v}(v,\mathscr{E}_\varepsilon(t,x))=0$). Next, let $\varphi\in C_c^\infty(\mathbb{R}^N\times\mathbb{R}^N)$, with $\mathrm{supp}(\varphi)\subset B(0,M)\times B(0,M)$. We have

$$\left|\iint f_\varepsilon(t,x,v)\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x\right|\le\|f_0\|_{L^p}\|\varphi\|_{L^{p'}} \tag{19}$$

$\boxed{\texttt{contlp}}$

for any $1 \le p \le \infty$, $1/p + 1/p' = 1$, and, by using (14) and Lemma 3.1,

$$\begin{aligned}
&\Big| \frac{\mathrm{d}}{\mathrm{d}t} \mathbb{E} \iint f_\varepsilon(t,x,v)\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x \Big| \\
&\quad \le \Big| \mathbb{E} \iint f_\varepsilon(t,x,v) v \cdot \nabla_x \varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x \Big| + \Big| \mathbb{E} \iint f_\varepsilon(t,x,v) \mathscr{E}_\varepsilon(t,x) \cdot \nabla_v \varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x \Big| \\
&\quad \le \|f_\varepsilon(t,\cdot)\|_{L^1} M \|\nabla_x \varphi\|_{L^\infty} \\
&\qquad + \Big| \mathbb{E} \int_{t-\varepsilon^2}^{t} \iint f_\varepsilon(\sigma,x,v) \mathscr{E}_\varepsilon(\sigma,x) \cdot S^\star_{t-\sigma} \big[(\nabla_v + (t-\sigma)\nabla_x) \mathscr{E}_\varepsilon(t) \cdot \nabla_v \varphi \big](x,v)\,\mathrm{d}v\,\mathrm{d}x \Big| \\
&\quad \le \|f_\varepsilon(t,\cdot)\|_{L^1} M \|\nabla_x \varphi\|_{L^\infty} + 3\varepsilon^2 \|\varphi\|_{W^{2,\infty}} \frac{\|E\|^2_{W^{1,\infty}}}{\varepsilon^2} \|f_\varepsilon(t,\cdot)\|_{L^1} \\
&\quad \le \|f_0\|_{L^1} \|\varphi\|_{W^{2,\infty}} \|E\|^2_{W^{1,\infty}} (3 + M).
\end{aligned}$$

By virtue of the Arzela-Ascoli theorem, the family $\{t \mapsto \mathbb{E} \iint f_\varepsilon(t,x,v)\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x, \varepsilon > 0\}$ lies in a compact set of $C^0([0,T])$ for any $0 < T < \infty$. Coming back to (19), this conclusion applies for a trial function $\varphi \in L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$. We finally appeal to the diagonal argument and we conclude that we can find $f \in L^\infty(0,T; L^p(\mathbb{R}^N \times \mathbb{R}^N))$ and we can extract a subsequence, still labelled by ε, such that for any $\varphi \in L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$, we have

$$\lim_{\varepsilon \to 0} \mathbb{E} \iint f_\varepsilon(t,x,v)\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x = \iint f(t,x,v)\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x \quad \text{uniformly on } [0,T].$$

We conclude that the limit of the leading term (18) reads

$$\iint (A[\varphi] + B[\varphi])(x,v) f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x = \iint \nabla_v \cdot \Big(\int_0^1 R(\theta, \theta v)\,\mathrm{d}\theta \nabla_v \varphi(x,v) \Big) f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x.$$

We recap the analysis in the following statement [15, Theorem 3.1 and Proposition 3.1 for the properties of the diffusion matrix].

`th:PV`

Theorem 3.2 *We suppose (H0–H4). Then, up to a subsequence, $\mathbb{E} f_\varepsilon$ converges in $C^0([0,T]; L^p(\mathbb{R}^N \times \mathbb{R}^N)-\text{weak})$ to $f \in L^\infty(0,T; L^p(\mathbb{R}^N \times \mathbb{R}^N))$ solution of*

$$\partial_t f + v \cdot \nabla_x f - \nabla_v \cdot (D \nabla_v f) = 0, \qquad f\big|_{t=0} = f_0,$$

where

$$D(v) = \int_0^1 R(\theta, \theta v)\,\mathrm{d}\theta.$$

The matrix D is symmetric, and non negative; its coefficients are even and belong to $L^\infty(\mathbb{R}^N)$, with, furthermore, $\partial_v^\alpha D_{ij} \in L^\infty(\mathbb{R}^N)$ for $|\alpha| \le 2$ and $\lim_{|v| \to \infty} D(v) = 0$.

We turn to discuss the almost sure convergence: the main result of the paper states as follows.

`th:nous`

Theorem 3.3 *In Theorem 3.2, extracting further subsequences if necessary, f_ε^ω converges in $C^0([0,T]; L^p(\mathbb{R}^N \times \mathbb{R}^N)-\text{weak})$ to f for $\mathrm{d}\mu$-almost every $\omega \in \Omega$.*

Proof. Let us introduce the extended phase-space variables

$$X = (x, y) \in \mathbb{R}^N \times \mathbb{R}^N, \qquad V = (v, w) \in \mathbb{R}^N \times \mathbb{R}^N$$

and set

$$F_\varepsilon(t, X, V) = f_\varepsilon(t, x, v) f_\varepsilon(t, y, w).$$

It satisfies

$$\partial_t F_\varepsilon + V \cdot \nabla_x F_\varepsilon - \mathfrak{E}_\varepsilon \cdot \nabla_v F = 0,$$

where

$$\mathfrak{E}_\varepsilon(t, X) = \begin{pmatrix} \mathscr{E}_\varepsilon(t, x) \\ \mathscr{E}_\varepsilon(t, y) \end{pmatrix} \in \mathbb{R}^{2N}.$$

The $(2N \times 2N)$ correllation matrix is thus defined blockwise by

$$\mathbb{E}\big[\mathfrak{E}_\varepsilon(t, X) \otimes \mathfrak{E}_\varepsilon(s, X')\big] = \eta^2 \begin{pmatrix} R(\frac{t-s}{\varepsilon^2}, \frac{x-x'}{\varepsilon^2}) & R(\frac{t-s}{\varepsilon^2}, \frac{x-y'}{\varepsilon^2}) \\ R(\frac{t-s}{\varepsilon^2}, \frac{y-x'}{\varepsilon^2}) & R(\frac{t-s}{\varepsilon^2}, \frac{y-y'}{\varepsilon^2}) \end{pmatrix} = \mathfrak{R}\Big(\frac{t-s}{\varepsilon^2}, \frac{X}{\varepsilon^2}, \frac{X'}{\varepsilon^2}\Big).$$

In particular, the decorrellation property (H2) still holds for the field $\mathfrak{E}_\varepsilon$. However, (H3) is not satisfied by the extended matrix $\mathfrak{R}$: the extra-diagonal terms do not depend on the difference $(X - X') = (x - x', y - y')$.

Nevertheless, we can repeat the analysis made above, that includes the identification of the leading term, with the same form as (18). We can still replace $F_\varepsilon(\sigma, X, V)$ by $\mathbb{E}F_\varepsilon(t, X, V)$, up to small error terms and we are thus led to study

$$\begin{aligned}
&\mathfrak{A}_\varepsilon[\varphi](X, V) = \\
&\quad \frac{1}{\varepsilon^2}\mathbb{E}\int_{t-\varepsilon^2}^{t} \mathfrak{E}\Big(\frac{\sigma}{\varepsilon^2}, \frac{X}{\varepsilon^2}\Big) \otimes \mathfrak{E}\Big(\frac{t}{\varepsilon^2}, \frac{X + (t-\sigma)V}{\varepsilon^2}\Big) : (D_V^2\varphi)(X + (t-\sigma)V, V)\, \mathrm{d}\sigma \\
&\quad = \frac{1}{\varepsilon^2}\int_{t-\varepsilon^2}^{t} \mathfrak{R}\Big(\frac{t-\sigma}{\varepsilon^2}, \frac{X}{\varepsilon^2}, \frac{X + (t-\sigma)V}{\varepsilon^2}\Big) : (D_V^2\varphi)(X + (t-\sigma)V, V)\, \mathrm{d}\sigma \\
&\quad = \int_0^1 \mathfrak{R}\Big(\theta, \frac{X}{\varepsilon^2}, \frac{X + \varepsilon^2\theta V}{\varepsilon^2}\Big) : (D_V^2\varphi)(X + \varepsilon^2\theta V, V)\, \mathrm{d}\theta.
\end{aligned}$$

The matrix reads

$$\mathfrak{R}\Big(\theta, \frac{X}{\varepsilon^2}, \frac{X + \varepsilon^2\theta V}{\varepsilon^2}\Big) = \begin{pmatrix} R(\theta, \theta v) & R(\theta, \frac{x-y}{\varepsilon^2} - \theta w) \\ R(\theta, \frac{y-x}{\varepsilon^2} - \theta v) & R(\theta, \theta w) \end{pmatrix}.$$

For almost every $(X, V) = (x, y, v, w) \in \mathbb{R}^{2N} \times \mathbb{R}^{2N}$, it tends to the diagonal matrix

$$\begin{pmatrix} R(\theta, \theta v) & 0 \\ 0 & R(\theta, \theta w) \end{pmatrix}$$

as $\varepsilon \to 0$, owing to the fact that $\lim_{|z|\to\infty} R(t, z) = 0$ from (H3). Thus, we obtain

$$\lim_{\varepsilon\to 0} \mathfrak{A}_\varepsilon[\varphi](X, V) = \mathfrak{D}(V) : D_V^2\varphi(X, V),$$

with

$$\mathfrak{D}(V) = \int_0^1 \begin{pmatrix} R(\theta, \theta v) & 0 \\ 0 & R(\theta, \theta w) \end{pmatrix} \,\mathrm{d}\theta.$$

Similarly, we check that

$$\begin{aligned} &\mathfrak{B}_\varepsilon[\varphi](X, V) \\ &\quad = \frac{1}{\varepsilon^2}\mathbb{E}\int_{t-\varepsilon^2}^{t} \frac{t-\sigma}{\varepsilon^2} (\nabla_X \mathfrak{E})\Big(\frac{t}{\varepsilon^2}, \frac{X + (t-\sigma)V}{\varepsilon^2}\Big)\ \mathfrak{E}\Big(\frac{\sigma}{\varepsilon^2}, \frac{x}{\varepsilon^2}\Big) \cdot (\nabla_V \varphi)(X + (t-\sigma)v, V)\,\mathrm{d}\sigma, \end{aligned}$$

tends to

$$\sum_{i,j=1}^{N} \partial_{V_i}[\mathfrak{D}_{ij}(V)]\partial_{V_j}\varphi(X, V).$$

We conclude that, possibly at the price of extracting a suitable subsequence, $\mathbb{E}F_\varepsilon(t, X, V)$ converges to $F(t, X, V)$ which satisfies

$$\partial_t F + V \cdot \nabla_X F - \nabla_V \cdot (\mathfrak{D}\nabla_V F) = 0, \quad F(0, X, V) = f_0(x, v)f_0(y, w).$$

Then we observe that $f(t, x, v)f(t, y, w)$ satisfies the same equation, with the same initial data. By uniqueness (see the discussion below), we thus get $F(t, X, V) = f(t, x, v)f(t, y, w)$.

We explain how this result implies the almost sure convergence, in a sense that we are going to make precise. Pick $\varphi \in L^{p'}(\mathbb{R}^N \times \mathbb{R}^N)$ and set

$$\mathscr{X}_\varepsilon(t) = \iint f_\varepsilon(t, x, v)\varphi(x, v)\,\mathrm{d}v\,\mathrm{d}x.$$

This is a random variable and we have shown that, on the one hand,

$$\mathbb{E}\mathscr{X}_\varepsilon(t) \xrightarrow[\varepsilon\to 0]{} \mathscr{X}(t) = \iint f(t, x, v)\varphi(x, v)\,\mathrm{d}v\,\mathrm{d}x \quad \text{uniformly on } [0, T],$$

and, on the other hand

$$\begin{aligned} \mathbb{E}[\mathscr{X}_\varepsilon(t)^2] = \mathbb{E}\Big[\iiiint F_\varepsilon(t, X, V)\varphi(x, v)\varphi(y, w)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}v\,\mathrm{d}w\Big] \\ \xrightarrow[\varepsilon\to 0]{} \mathscr{X}(t)^2 = \Big(\iint f(t, x, v)\varphi(x, v)\,\mathrm{d}v\,\mathrm{d}x\Big)^2. \end{aligned}$$

It follows that

$$\mathbb{E}\big[|\mathscr{X}_\varepsilon(t) - \mathbb{E}\mathscr{X}_\varepsilon(t)|^2\big] = \mathbb{E}\big[\mathscr{X}_\varepsilon(t)^2 - (\mathbb{E}\mathscr{X}_\varepsilon(t))^2\big] \xrightarrow[\varepsilon\to 0]{} 0.$$

We infer that

$$\mathbb{E}\big[|\mathscr{X}_\varepsilon(t) - \mathscr{X}(t)|^2\big] \le 2\Big(\mathbb{E}\big[|\mathscr{X}_\varepsilon(t) - \mathbb{E}\mathscr{X}_\varepsilon(t)|^2\big] + \big[|\mathbb{E}\mathscr{X}_\varepsilon(t) - \mathscr{X}(t)|^2\big]\Big) \xrightarrow[\varepsilon\to 0]{} 0.$$

In other words $\mathscr{X}_\varepsilon(t)$ converges to $\mathscr{X}(t)$ as $\varepsilon \to 0$ in $L^2(\Omega, \mathrm{d}\mu(\omega))$. The partial reciprocal to the Lebesgue theorem, see e. g. [6, Lemma 3.31] tells us, extracting further subsequences if necessary, that $\mathscr{X}_\varepsilon(t)$ converges to $\mathscr{X}(t)$ as $\varepsilon \to 0$ for $\mathrm{d}\mu$-a. e. $\omega \in \Omega$.

The validity of the analysis thus relies on a uniqueness statement for the limiting equation, which can be cast as a Fokker-Planck equation on $(0,\infty) \times \mathbb{R}^{2d}$

$$\partial_t f + \nabla_y \cdot (\mathcal{C} f) - \nabla_y \cdot (\mathcal{A} \nabla_y f) = 0$$

where $y \in \mathbb{R}^{2d}$ stands for the extended variable (x, v), and

$$\mathcal{C}(y) = \begin{pmatrix} v \\ 0 \end{pmatrix}, \qquad \mathcal{A}(y) = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & D(v) \end{array}\right).$$

Note that $\nabla_y \cdot \mathcal{C} = 0$. The difficulty is two-fold; it relies

- on the degeneracy of the diffusion coefficient $\mathcal{A}$: it is a non-negative matrix, but it does not fulfil an ellipticity criterion, even locally (the best that can be expected is to find, for any $0 < R < \infty$, a constant α_R such that $D(v)\xi \cdot \xi \geq \alpha_R |\xi|^2$ holds for any $|v| \leq R$ and $\xi \in \mathbb{R}^N$);
- on the fact that the limiting process provides a solution in, say, $L^\infty((0,\infty); L^1 \cap L^\infty(\mathbb{R}^{2d}))$, satisfying some weak continuity with respect to time, but it tells nothing about $\mathcal{A}^{1/2}\nabla_y f$, a quantity which would naturally appear in energy estimates for the Fokker-Planck equation.

The issue of the uniqueness in such a framework has been addressed quite recently: we refer the reader for instance to [11] where conditions on $\mathcal{A}^{1/2}\nabla_y f$ appeared, and to [4] where a connection with SDEs is established, with statements that do not require the a priori estimate on $\nabla_y f$ (an issue for uniqueness even when ellipticity is enforced). For our purposes, we directly use the uniqueness results in [18, Theorem 3.1], see also [16]. The analysis relies on renormalization techniques, in the spirit of the pioneering work [2] on transport equations. ■

The analysis performed on (12) can be reproduced when we do not assume that the length scale vanishes; namely, we replace (16)–(17) by

$$\tau\eta^2 = 1, \qquad \tau = \varepsilon^2 \to 0, \qquad \lambda > 0 \text{ fixed.}$$

In this regime the term involving

$$\frac{1}{\varepsilon^2}\mathbb{E}\int_{t-\varepsilon^2}^{t} \iint \frac{t-\sigma}{\lambda} (\nabla_x E)\Big(\frac{t}{\varepsilon^2}, \frac{x+(t-\sigma)v}{\lambda}\Big)\ E\Big(\frac{\sigma}{\varepsilon^2}, \frac{x}{\lambda}\Big) \\ \cdot (\nabla_v \varphi)(x+(t-\sigma)v, v) f_\varepsilon(\sigma, x, v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\sigma.$$

is of order ε^2 and it vanishes when $\varepsilon \to 0$. The asymptotic regime is still described by a Fokker-Planck, like in Theorem 3.2, but now the diffusion coefficient does not depend on the velocity variable: it becomes

$$D = \int_0^1 R(\theta, 0)\,\mathrm{d}\theta.$$

We refer the reader to GP2 [7, Theorem 2.8] for the analysis of a very similar situation. However, with this scaling we do not get the almost sure convergence. Indeed, the correlation matrix becomes

$$\mathfrak{R}\Big(\theta, \frac{X}{\lambda}, \frac{X+\varepsilon^2\theta V}{\lambda}\Big) = \begin{pmatrix} R(\theta, \frac{\varepsilon^2}{\lambda}\theta v) & R(\theta, \frac{x-y}{\lambda} - \frac{\varepsilon^2}{\lambda}\theta w) \\ R(\theta, \frac{y-x}{\lambda} - \frac{\varepsilon^2}{\lambda}\theta v) & R(\theta, \frac{\varepsilon^2}{\lambda}\theta w) \end{pmatrix},$$

and the off-diagonal terms do not disappear when $\varepsilon \to 0$. The limit equation for $F = \lim_{\varepsilon\to 0}\mathbb{E}F_\varepsilon$ is a Fokker-Planck equation with the (space-dependent) diffusion coefficient

$$\mathfrak{D}(X) = \int_0^1 \begin{pmatrix} R(\theta, 0) & R(\theta, \frac{x-y}{\lambda}) \\ R(\theta, \frac{y-x}{\lambda}) & R(\theta, 0) \end{pmatrix} \,\mathrm{d}\theta.$$

In particular $F(t,X,V) \neq f(t,x,v)f(t,y,w)$, and we cannot prove this way the almost sure convergence.

s:FE

4 Further examples

4.1 Very thin spray model

The discussion adapts readily to the following model for very thin sprays

$$\partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon + \nabla_v\cdot\big((\eta u(t/\varepsilon^2, x/\lambda) - v)f_\varepsilon\big) = 0.$$

Here f_ε is the distribution function of particles immersed in a "turbulent" fluid: the function $(t,x)\mapsto \eta u(t/\tau, x/\lambda) \in \mathbb{R}^N$ represents the local velocity of the carrier fluid and the particles' dynamics is governed by the drag force exerted by the fluid on the particles. With

$$\eta = \frac{1}{\varepsilon} \quad\text{and}\quad \frac{\varepsilon^2}{\lambda} \to \kappa \in [0,\infty)$$

the limiting equation has the form of a Fokker-Planck equation with friction

$$\partial_t f + v\cdot\nabla_x f = \nabla_v\cdot(vf + D(v)\nabla_v f) = 0,$$

see GP2 [7, Theorem 2.8]. When $\kappa > 0$ the convergence holds almost surely with respect to $\omega\in\Omega$. The proof is exactly the same as above.

4.2 Quantum transport

The starting point of the problem we address is the Schrödinger equations: for $n\in\mathbb{N}$,

$$i\varepsilon\partial_t\psi_{n,\varepsilon} + \frac{\varepsilon^2}{2}\Delta_x\psi_{n,\varepsilon} = \sqrt{\varepsilon}\,\underbrace{V(t/\varepsilon, x/\varepsilon)}_{V_\varepsilon(t,x)}\,\psi_{n,\varepsilon}$$

that describe the dynamics of quantum particles. The equations are completed by the initial data

$$\psi_{n,\varepsilon}\big|_{t=0} = \psi_{n,\varepsilon}^{\text{init}}.$$

The particles are subjected to a fluctuating environment described through the potential V. Here we adopt the same scaling as in PV [15] where the short time and length scales of variation of the potential have the same order of magnitude than the Planck constant: $0 < \varepsilon \ll 1$ (compared to the scales of observation). We assume that $\{\psi_{n,\varepsilon}^{\text{init}},\ n \in \mathbb{N}\}$ is an orthonormal system in $L^2(\mathbb{R}^N)$. It follows that, for any $t \geq 0$, $\{\psi_{n,\varepsilon}(t,\cdot),\ n \in \mathbb{N}\}$ is an orthonormal system in $L^2(\mathbb{R}^N)$, too. To each index n, we associate an occupation probability $\lambda_{n,\varepsilon} \geq 0$ such that

$$\sum_{n\in\mathbb{N}} \lambda_{n,\varepsilon} = 1, \qquad \sum_{n\in\mathbb{N}} |\lambda_{n,\varepsilon}|^2 \leq C_0 \varepsilon^N.$$

In order to investigate the semi-classical regime $\varepsilon \to 0$, we introduce the Wigner transform associated to the solutions of the Schrödinger equations, namely

$$f_\varepsilon(t,x,v) = \sum_{n\in\mathbb{N}} \lambda_{n,\varepsilon} \int e^{-iy\cdot v} \psi_{n,\varepsilon}(t, x+\varepsilon y/2) \overline{\psi_{n,\varepsilon}(t, x-\varepsilon y/2)}\, \mathrm{d}y.$$

It satisfies the following Wigner equation

$$(\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon)(t,x,v) = L_\varepsilon(t)[f_\varepsilon(t,\cdot)](x,v) \tag{20}$$ wig1

where, for a function f that depends on x, v, we denote

$$L_\varepsilon(t)[f](x,v) = \frac{i}{\sqrt{\varepsilon}} \int e^{i\xi\cdot v} \Big(V_\varepsilon(t, x+\varepsilon\xi/2) - V_\varepsilon(t, x-\varepsilon\xi/2) \Big) \widehat{f}(x,\xi) \mathrm{đ}\xi.$$

In this formula, $\widehat{\cdot}$ stands for the Fourier transform $v \to \xi$, that is

$$\widehat{f}(t,x,\xi) = \int e^{-i\xi\cdot v} f(t,x,v)\, \mathrm{d}v,$$

and $\mathrm{đ}\xi$ stands for the Lebesgue measure with the normalization arising in the inverse Fourier transform $\mathrm{đ}\xi = \frac{\mathrm{d}\xi}{(2\pi)^N}$. Note that, as usual in semi-classical analysis, we use the abuse of notation where the integral symbol has actually a distributional meaning. We refer the reader to LiPa [12] for the analysis of the semi-classical limit which leads from the Wigner equation to the Liouville equation, with a force field given by the gradient of the potential. For our purposes, we bear in mind the following properties.

Lemma 4.1 *The operator L_ε is a bounded operator on $L^2(\mathbb{R}^N \times \mathbb{R}^N)$, with*

$$|||L_\varepsilon||| \leq \frac{2}{\sqrt{\varepsilon}} \|V\|_{L^\infty(\mathbb{R}\times\mathbb{R}^N)}.$$

Lemma 4.2 *The functions f_ε are real-valued and the sequence $\big(f_\varepsilon\big)_{\varepsilon>0}$ is bounded in $L^\infty((0,\infty); L^2(\mathbb{R}^N \times \mathbb{R}^N))$. Moreover, if $f_\varepsilon \rightharpoonup f$ weakly$-\star$ in $L^\infty([0,\infty); L^2(\mathbb{R}^N \times \mathbb{R}^N))$, then $f \geq 0$.*

We keep the same assumptions as in PV [15] on the random potential.

h1) $V \in L^\infty(\mathbb{R} \times \mathbb{R}^N)$, h2) $\mathbb{E}V = 0$,

h3) $\mathbb{E}\big(V(t,x)V(s,y)\big) = R(t-s, x-y)$, where $R \in L^\infty(\mathbb{R}; L^1(\mathbb{R}^N))$ is such that $\operatorname{supp}(R) \subset [-r,r] \times \mathbb{R}^N$.

h4) The Fourier transform

$$Q(\tau, v) = \int e^{-ix\cdot v} R(\tau, x)\,\mathrm{d}x$$

lies in $L^\infty(\mathbb{R}; L^1(\mathbb{R}^N, (1+|v|)\,\mathrm{d}v))$.

Let us collect a few formulae and results form PV [15] that will be useful for our analysis. It is convenient to observe that

$$\begin{aligned}
&L_\varepsilon(t)[f](x,v)\\
&\quad= \frac{i}{\sqrt{\varepsilon}} \int f(x,w) \left(\int e^{i\xi\cdot(v-w)} \Big(V_\varepsilon(t, x+\varepsilon\xi/2) - V_\varepsilon(t, x-\varepsilon\xi/2) \Big) \text{đ}\xi \right) \mathrm{d}w\\
&\quad= \frac{i2^N}{\sqrt{\varepsilon}} \int f(x,w)\ \left(e^{2i(z-x/\varepsilon)\cdot(v-w)} - e^{-2i(z-x/\varepsilon)\cdot(v-w)} \right)\ V(t/\varepsilon, z)\text{đ}z\,\mathrm{d}w,\\
&\qquad\qquad \text{by using the change of variables } \tfrac{1}{\varepsilon}(x \pm \varepsilon\xi/2) \to z\\
&\quad= \frac{i2^N}{\sqrt{\varepsilon}} \int f(x,w) \left\{ e^{-2ix\cdot(v-w)/\varepsilon} \widehat{V}(t/\varepsilon, 2(w-v)) - e^{2ix\cdot(v-w)/\varepsilon} \widehat{V}(t/\varepsilon, 2(v-w)) \right\} \text{đ}w\\
&\quad= \frac{i}{\sqrt{\varepsilon}} \int \left(f(x, v+w'/2) - \int f(x, v-w'/2) \right)\ e^{iw'\cdot x/\varepsilon}\ \widehat{V}(t/\varepsilon, w')\text{đ}w',\\
&\qquad\qquad \text{by using the change of variables } \pm 2(v-w) \to w'.
\end{aligned} \tag{21} \quad \boxed{\texttt{nn}}$$

The analysis still relies on an intensive use of the Duhamel formula, based on the free-transport semi-group

$$f_\varepsilon(t,x,v) = f_\varepsilon(t-s, x-sv, v) + \int_{t-s}^{t} L_\varepsilon(\sigma)[f(\sigma,\cdot)](x-(t-\sigma)v, v)\,\mathrm{d}\sigma. \tag{22} \quad \boxed{\texttt{QDuha}}$$

As a matter of fact, (QDuha 22) with $s = t$ already shows that $f_\varepsilon(t,\cdot)$ only depends on the realizations of $V_\varepsilon(\sigma,\cdot)$ for $0 \le \sigma \le t$, Owing to the decorrelation assumption h3), and the initial data being deterministic, we deduce that $f_\varepsilon(t-s,\cdot)$ and $V_\varepsilon(t,\cdot)$ are independent random variables when $s \ge \varepsilon$, see PV [15, Lemma 4.6 & 4,7].

Therefore, the strategy consists in applying the Duhamel formula (QDuha 22) with $s = \varepsilon$ in order to compute $\mathbb{E}\big(L_\varepsilon(t)[f_\varepsilon(t,\cdot)](x,v)\big)$. It turns out that the singular term $\mathbb{E}\big(L_\varepsilon(t)[f_\varepsilon(t-\varepsilon,\cdot)](x,v)\big)$ vanishes, owing to the decorrelation property and the fact that V_ε is centered, see h2). Therefore, we are led to compute

$$\mathbb{E} \int_{t-\varepsilon}^{t} L_\varepsilon(t) S_{t-\sigma} L_\varepsilon(\sigma)[f(\sigma,\cdot)](x,v)\,\mathrm{d}\sigma$$

where S_t denotes the semi-group of the transport equation

$$S_t[f](x,v) = f(x-tv, v).$$

Since in the integration domain $0 \le t-\sigma \le \varepsilon$, we can replace, up to small error terms $f_\varepsilon(\sigma,\cdot)$ by $f_\varepsilon(t,\cdot)$ and even by $\mathbb{E} f_\varepsilon(t,\cdot)$, see [15, Lemma 4.8]. We are left with the task of finding the limit as $\varepsilon \to 0$ of (see [15, formula (58)]

$$
\begin{aligned}
&\mathbb{E} L_\varepsilon(t) S_{t-\sigma} L_\varepsilon(\sigma)[f](x,v)\\
&= -\frac{1}{\varepsilon}\mathbb{E}\int_{t-\varepsilon}^{t} \iint e^{iw\cdot x/\varepsilon}\ \widehat{V}(t/\varepsilon,w)\ \widehat{V}(\sigma/\varepsilon,w')\\
&\qquad\qquad \times\Big(e^{iw'\cdot(x-(t-\sigma)(v+w/2))/\varepsilon}\Big[f(x-(t-\sigma)(v+w/2),v+w/2+w'/2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad -f(x-(t-\sigma)(v+w/2),v+w/2-w'/2)\Big]\\
&\qquad\qquad\quad -e^{iw'\cdot(x-(t-\sigma)(v-w/2))/\varepsilon}\Big[f(x-(t-\sigma)(v-w/2),v-w/2+w'/2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad -f(x-(t-\sigma)(v-w/2),v-w/2-w'/2)\Big]\Big)\,\text{đ}w'\text{đ}w\,\mathrm{d}\sigma\\
&= -\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}\iint \mathbb{E}\big[\widehat{V}(t/\varepsilon,w)\ \widehat{V}(\sigma/\varepsilon,w')\big]e^{i(w+w')\cdot x/\varepsilon}\\
&\qquad\qquad \times\Big(e^{-i(t-\sigma)w'\cdot(v+w/2)/\varepsilon}\Big[f(x-(t-\sigma)(v+w/2),v+w/2+w'/2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad -f(x-(t-\sigma)(v+w/2),v+w/2-w'/2)\Big]\\
&\qquad\qquad\quad -e^{-i(t-\sigma)w'\cdot(v-w/2)/\varepsilon}\Big[f(x-(t-\sigma)(v-w/2),v-w/2+w'/2)\\
&\qquad\qquad\qquad\qquad\qquad\qquad -f(x-(t-\sigma)(v-w/2),v-w/2-w'/2)\Big]\Big)\,\text{đ}w'\text{đ}w\,\mathrm{d}\sigma.
\end{aligned}
\tag{23}
$$

nnnn

Note that the factor $1/\varepsilon$ is compensated by the fact that the time integral is taken over an interval of size ε, so that $\mathbb{E} L_\varepsilon(t) S_{t-\sigma} L_\varepsilon(\sigma)[f](x,v)$ remains of order $\mathscr{O}(1)$. By virtue of the decorrelation assumption

$$\mathbb{E}(V(t,x)V(s,y)) = R(t-s,x-y),$$

we obtain (see [15, formula (61)]

$$
\begin{aligned}
\mathbb{E}(\widehat{V}(t,w)\widehat{V}(s,w')) &= \mathbb{E}\iint e^{-ix\cdot w}e^{-iy\cdot w'}V(t,x)V(s,y)\,\mathrm{d}y\,\mathrm{d}x\\
&= \iint e^{-ix\cdot w}e^{-iy\cdot w'}R(t-s,x-y)\,\mathrm{d}y\,\mathrm{d}x\\
&= \int e^{-ix\cdot(w+w')}\underbrace{\left(\int e^{iz\cdot w'}R(t-s,z)\,\mathrm{d}z\right)}_{Q(t-s,w')}\mathrm{d}x\\
&= Q(t-s,w')\ \delta_0(w+w').
\end{aligned}
\tag{24}
$$

espVVq

Note that

$$R(-\tau,-z) = R(\tau,z)$$

implies that

$$Q(-\tau,-w) = Q(\tau,w).$$

In order to continue the computation, we go back to (23) with this information. As a matter of fact, we observe that the factor $e^{i(w+w')\cdot x/\varepsilon}$ in the integrand of (23) is always equal to $e^0=1$. We now make gain and loss terms appear (see [15, Lemma 4.9]), with the change of variables

$$\frac{t-\sigma}{\varepsilon} = \tau, \qquad w = \pm(v-w').$$

We are led to

$$\mathbb{E}L_\varepsilon(t)S_{t-\sigma}L_\varepsilon(\sigma)[f](x,v) = \mathrm{q}_\varepsilon^+ - \mathrm{q}_\varepsilon^-$$

with

$$\begin{aligned}\mathrm{q}_\varepsilon^+(t,x,v) &= \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}\int Q\Big(\frac{t-\sigma}{\varepsilon},w\Big)\Big[f(x-(t-\sigma)(v-w),v-w/2)e^{i(t-\sigma)(v-w/2)\cdot w/\varepsilon}\\ &\qquad +f(x-(t-\sigma)(v-w),v+w/2)e^{i(t-\sigma)(v+w/2)\cdot w/\varepsilon}\Big]\,\mathrm{đ}w'\,\mathrm{d}\sigma,\end{aligned}$$

and

$$\begin{aligned}\mathrm{q}_\varepsilon^-(t,x,v) &= \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}\int Q\Big(\frac{t-\sigma}{\varepsilon},w\Big)\Big[f(x-\varepsilon\tau(v+w/2),v)e^{i(t-\sigma)(v+w/2)\cdot w/\varepsilon}\\ &\qquad +f(x-\varepsilon\tau(v-w/2),v)e^{i(t-\sigma)(v-w/2)\cdot w/\varepsilon}\Big]\,\mathrm{đ}w\,\mathrm{d}\sigma.\end{aligned}$$

With the change of variables

$$\frac{t-\sigma}{\varepsilon} = \tau, \qquad w = \pm(v-v'),$$

and letting ε go to 0, we are led to

$$\mathrm{q}^+[f](x,v) = \int f(x,v')q(v,v')\,\mathrm{đ}v'$$

with

$$\begin{aligned}q(v,v') &= \int_0^1 Q(\tau,v-v')e^{i\tau(|v|^2-|v'|^2)/2}\,\mathrm{d}\tau + \int_0^1 Q(\tau,v'-v)e^{i\tau(|v'|^2-|v|^2)/2}\,\mathrm{d}\tau\\ &= \int_{-1}^1 Q(\tau,v-v')e^{i\tau(|v|^2-|v'|^2)/2}\,\mathrm{d}\tau\end{aligned}$$

since Q is an even function of its arguments. Similarly, we obtain

$$\mathrm{q}^-[f](x,v) = f(x,v)\int q(v',v)\,\mathrm{đ}v.$$

We refer the reader to PV [15, Proposition 4.5] for more precise estimates. In particular, we bear in mind the following property which makes the definition of $\mathrm{q}^\pm$ consistent with the usual form of the gain and loss terms of the linear Boltzmann operator.

Lemma 4.3 PV *[15, Lemma 4.5] The cross section*

$$q(v,v') = \int_{-1}^1 Q(\tau,v-v')e^{i\tau(|v|^2-|v'|^2)/2}\,\mathrm{d}\tau$$

is real-valued and in fact non-negative; it is also invariant by changing the role of v and v'.

We conclude this presentation with the following statement.

Theorem 4.4 PV *[15, Theorem 4.2] The expectation* $\mathbb{E} f_\varepsilon$ *converges to* f *in* $C^0([0,T]; L^2(\mathbb{R}^N \times \mathbb{R}^N) - \mathrm{weak})$*; the limit* f *is non negative, it lies in* $L^\infty(0,T; L^2(\mathbb{R}^N \times \mathbb{R}^N))$ *and it satisfies*

$$(\partial_t + v \cdot \nabla_x) f = C[f] \tag{25}$$

eqlimq

where C *is the collision operator*

$$C[f] = \mathrm{q}^+[f] - \mathrm{q}^-[f].$$

thPVq

We turn to prove that, with the same assumptions as in PV [15], the convergence holds pointwise with respect to the random variable.

th:nous_2

Theorem 4.5 *In Theorem* thPVq *4.4, extracting further subsequences if necessary,* f_ε^ω *converges in* $C^0([0,T]; L^2(\mathbb{R}^N \times \mathbb{R}^N) - \mathrm{weak})$ *to* f *for* $\mathrm{d}\mu$*-almost every* $\omega \in \Omega$*.*

We are going to apply the same analysis to the density obtained by doubling the variables. Namely, denoting

$$X = (x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N, \quad V = (v_1, v_2) \in \mathbb{R}^N \times \mathbb{R}^N$$

we set

$$F_\varepsilon(t, X, V) = f_\varepsilon(t, x_1, v_1) f_\varepsilon(t, x_2, v_2).$$

It satisfies

$$(\partial_t + V \cdot \nabla_X) F_\varepsilon(t, X, V) = \mathfrak{L}_\varepsilon(t)[F_\varepsilon(t, \cdot)](X, V) \tag{26}$$

wig2

where the extended Wigner operator is defined with the same formulae as above, in the extended variables and with the potential

$$\mathfrak{V}_\varepsilon(t, X) = V_\varepsilon(t, x_1) + V_\varepsilon(t, x_2) = \mathfrak{V}(t/\varepsilon, X/\varepsilon).$$

Namely, we have, see (mn 21)

$$\mathfrak{L}_\varepsilon(t)[F](X,V) = \frac{i}{\sqrt{\varepsilon}} \int e^{iW\cdot X/\varepsilon} (F(X, V + W/2) - F(X, V - W/2)) \underbrace{\widehat{\mathfrak{V}}(t/\varepsilon, W)}_{\delta(w_1)\widehat{V}(t/\varepsilon, w_2) + \delta(w_2)\widehat{V}(t/\varepsilon, w_1)} \, đW.$$

Indeed, we start by observing

$$\mathfrak{L}_\varepsilon(t)[F](X,V) = f(x_1, v_1) L_\varepsilon(t)[f](x_2, v_2) + f(x_2, v_2) L_\varepsilon(t)[f](x_1, v_1).$$

It yields the asserted expression by writing $f(x,v)$ as $\mathscr{F}^{-1}_{\xi \to v} \mathscr{F}_{y \to \xi} f(x,v)$. In order to identify the analog of (nnnn 23), by using (espVVq 24), we compute

$$\begin{aligned} &\mathbb{E}[\widehat{\mathfrak{V}}(t, W) \widehat{\mathfrak{V}}(s, W')] \\ &\quad = \underbrace{\delta(w_1)\delta(w_1')\delta(w_2 + w_2') Q(t-s, w_2) + \delta(w_2)\delta(w_2')\delta(w_1 + w_1') Q(t-s, w_1)}_{\mathrm{G}(t-s, W, W')} \\ &\qquad + \underbrace{\delta(w_1)\delta(w_2')\delta(w_1' + w_2) Q(t-s, w_2) + \delta(w_2)\delta(w_1')\delta(w_1 + w_2') Q(t-s, w_1)}_{\mathrm{B}(t-s, W, W')} \end{aligned}$$

The first line, denoted $\mathrm{G}(t-s,W,W')$, corresponds to the good terms that leads to the expected cross section. It can be cast as

$$\mathrm{G}(t-s,W,W') = \mathfrak{Q}(t-s,W)\delta(W+W')$$

with

$$\mathfrak{Q}(\tau,W) = \delta(w_1)Q(\tau,w_2) + \delta(w_2)Q(\tau,w_1). \tag{27}$$ $\boxed{\texttt{extkq}}$

The second line, denoted $\mathrm{B}(t-s,W,W')$, contains the correlation that are expected to go away in the limit $\varepsilon \to 0$.

Indeed, if $f(t,x,v)$ denotes the limit of $\mathbb{E}f_\varepsilon(t,x,v)$, which thus satisfies the kinetic equation ($\overset{\texttt{eqlimq}}{25}$), then let us set $\widetilde{F}(t,X,V) = f(t,x_1,v_1)f(t,x_2,v_2)$. It satisfies

$$(\partial_t + V\cdot\nabla_X)\widetilde{F}(t,X,V) = \underbrace{f(t,x_2,v_2)C[f](t,x_1,v_1) + f(t,x_1,v_1)C[f](t,x_2,v_2)}_{\mathfrak{C}[\widetilde{F}](t,X,V)}$$

where the extended collision operator $\mathfrak{C}$ is associated to the cross section

$$\begin{aligned}
\mathfrak{q}(V,V') &= \delta(v_1-v_1')q(v_2,v_2') + \delta(v_2-v_2')q(v_1,v_1')\\
&= \delta(v_1-v_1')\int_{-1}^{1} Q(\tau,v_2-v_2')e^{i\tau(|v_2|^2-|v_2'|^2)/2}\,\mathrm{d}\tau\\
&\quad +\delta(v_2-v_2')\int_{-1}^{1} Q(\tau,v_1-v_1')e^{i\tau(|v_1|^2-|v_1'|^2)/2}\,\mathrm{d}\tau\\
&= \int_{-1}^{1} \mathfrak{Q}(\tau,V-V')e^{i\tau(|V|^2-|V'|^2)/2}\,\mathrm{d}\tau
\end{aligned}$$

with $\mathfrak{Q}(\tau,V)$ given by ($\overset{\texttt{extkq}}{27}$). Going back to ($\overset{\texttt{mnnn}}{23}$) the kernel $\mathrm{G}(\tau,W,W')$ is multiplied by

$$e^{iX\cdot(W+W')/\varepsilon}e^{-i\tau W'\cdot(V\pm W/2)}.$$

In particular in this product the fast oscillation term $e^{iX\cdot(W+W')/\varepsilon}$ becomes $e^0 = 1$. Then, we can reproduce the same calculations as above, and this part leads to the collision operator $\mathfrak{C}$.

We are left with the task of investigating the contribution of $\mathrm{B}(\tau,W,W')$, which thus contains the correlations. In ($\overset{\texttt{mnnn}}{23}$) it is multiplied by

$$e^{2ix_1\cdot(w_1+w_1')/\varepsilon}e^{2ix_2\cdot(w_2+w_2')/\varepsilon} = e^{2iX\cdot(W+W')/\varepsilon} \tag{28}$$ $\boxed{\texttt{toto10}}$

and

$$\begin{aligned}
&F(X-(t-\sigma)(V+W/2), V+W/2+W'/2)e^{-i(t-\sigma)W'\cdot(V+W)/\varepsilon}\\
&\quad -F(X-(t-\sigma)(V-W/2), V-W/2+W'/2)e^{-i(t-\sigma)W'\cdot(V-W)/\varepsilon}\\
&\quad +F(X-(t-\sigma)(V+W/2), V+W/2-W'/2)e^{-i(t-\sigma)W'\cdot(V+W)/\varepsilon}\\
&\quad -F(X-(t-\sigma)(V-W/2), V-W/2-W'/2)e^{-i(t-\sigma)W'\cdot(V-W)/\varepsilon}.
\end{aligned}$$

Therefore, due to the form of the kernel B, in the corresponding expression, one always have $W\cdot W' = 0$ since either

$$\text{(a)} \qquad W' = (0,w_2'), \quad W = (w_1,0) = (-w_2',0)$$

or

$$\text{(b)} \qquad W' = (w_1', 0), \quad W = (0, w_2) = (0, -w_1').$$

Accordingly, the factor in (28) becomes

$$\text{either (a) } e^{2iw_1\cdot(x_1-x_2)/\varepsilon} \text{ or (b) } e^{2iw_2\cdot(x_2-x_1)/\varepsilon},$$

while

$$W'\cdot(V\pm W) = W'\cdot V = \begin{cases} w_2'\cdot v_2 = -w_1\cdot v_2 & \text{case (a)},\\ w_1'\cdot v_1 = -w_2\cdot v_1 & \text{case (b)}.\end{cases}$$

Therefore, we have to deal with terms which have the following form

$$\int_0^1\int Q(\tau,w_1)e^{iw_1(v_2+(x_2-x-1))/\varepsilon}F\left(X-\varepsilon\tau\left(V+\frac{1}{2}\begin{pmatrix}w_1\\-w_1\end{pmatrix}\right), V\pm\frac{1}{2}\begin{pmatrix}w_1\\-w_1\end{pmatrix}\right)\text{đ}w_1\,\mathrm{d}\tau. \tag{29}$$

In the approach of [15], the asymptotic regime is obtained by multiplying (20) by a trial function $\varphi\in C_c^\infty(\mathbb{R}^N\times\mathbb{R}^N)$. The linear part with the convection term clearly passes to the limit, the difficulty comes only from the Wigner term. The latter is treated by using the Duhamel formula, as explained above, and thus we only have to discuss the fact that

$$\mathbb{E}\left[\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}[L_\varepsilon(t)S_{t-\sigma}L_\varepsilon(\sigma)]^\star[\varphi](x,v)\,\mathrm{d}\sigma\right]$$

tends to $C^\star[\varphi](x,v)$ in $L^2((\mathbb{R}^N\times\mathbb{R}^N))$ as $\varepsilon\to 0$, see [15, Proposition 2.5]. We wish to apply the same strategy for (26). Since the structure of the operator $\left[\mathfrak{L}_\varepsilon(t)S_{t-\sigma}\mathfrak{L}_\varepsilon(\sigma)\right]$ and its adjoint $\left[\mathfrak{L}_\varepsilon(t)S_{t-\sigma}\mathfrak{L}_\varepsilon(\sigma)\right]^\star$ are the same, we are going to justify that

$$\mathbb{E}\left[\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t}\mathfrak{L}_\varepsilon(t)S_{t-\sigma}\mathfrak{L}_\varepsilon(\sigma)[\Phi](X,V)\,\mathrm{d}\sigma\right]$$

tends to $\mathfrak{C}[\Phi]$ as $\varepsilon\to 0$, for a given $\Phi\in C_c^\infty(\mathbb{R}^{2N}\times\mathbb{R}^{2N})$. With the analog of (23), the analysis of [15] applies directly to prove that the contribution with $\mathrm{G}(\tau,W,W')$ leads to $\mathfrak{C}[\Phi]$, as sketched above. We are just left with the task of proving that the contribution with $\mathrm{B}(\tau,W,W')$ tends to 0, which reduces to show that

$$\int_0^1\int Q(\tau,w_1)e^{iw_1\cdot(v_2+(x_2-x_1))/\varepsilon}\Phi\left(X-\varepsilon\tau\left(V+\frac{1}{2}\begin{pmatrix}w_1\\-w_1\end{pmatrix}\right), V\pm\frac{1}{2}\begin{pmatrix}w_1\\-w_1\end{pmatrix}\right)\text{đ}w_1\,\mathrm{d}\tau$$

tends to 0 in $L^2(\mathbb{R}^{2N}_X\times\mathbb{R}^{2N}_V)$ as $\varepsilon\to 0$. It is clear that we can safely remove the shift with respect to the space variable (at the price of an error of order $\mathscr{O}(\varepsilon)$), and we only address the question for

$$\mathrm{I}_\varepsilon(X,V) = \int_0^1\int Q(\tau,w_1)e^{iw_1\cdot(v_2+(x_2-x_1))/\varepsilon}\Phi\left(X, V\pm\frac{1}{2}\begin{pmatrix}w_1\\-w_1\end{pmatrix}\right)\text{đ}w_1\,\mathrm{d}\tau.$$

Let us introduce the following function, seen as a function of the variable $w_1\in\mathbb{R}^N$, parametrized by $(X,V)\in\mathbb{R}^{2N}\times\mathbb{R}^{2N}$:

$$\Psi_{X,V}(w_1) = \int_0^1 Q(\tau,w_1)\,\mathrm{d}\tau\times\Phi\left(X, V\pm\frac{1}{2}\begin{pmatrix}w_1\\-w_1\end{pmatrix}\right).$$

Then, we have

$$\mathrm{I}_\varepsilon(X,V) = \widehat{\Psi}_{X,V}\Big(v_2 + \frac{x_2 - x_1}{\varepsilon}\Big).$$

where $\widehat{\cdot}$ denotes here the Fourier transform with respect to the variable w_1. By virtue of h4), for any $(X,V) \in \mathbb{R}^{2N} \times \mathbb{R}^{2N}$, the function $w_1 \mapsto \Psi_{X,V}(w_1)$ is integrable. Hence the Riemann-Lebesgue implies that $\lim_{\varepsilon\to 0} \mathrm{I}_\varepsilon(X,V)$ for a. e. $(X,V) \in \mathbb{R}^{2N} \times \mathbb{R}^{2N}$ (precisely as far as (x_1,x_2) does not lie on the diagonal of $\mathbb{R}^{2N}$, which is a negligible set). Moreover for any $n \in \mathbb{N}$, there exists C_n such that

$$(1+|X|^2)^n(1+|V|^2)^n \left|\Phi\left(X, V \pm \frac{1}{2}\begin{pmatrix} w_1 \\ -w_1\end{pmatrix}\right)\right| \leq C_n$$

holds for every $(X,V) \in \mathbb{R}^{2N} \times \mathbb{R}^{2N}$ and $w_1 \in \mathbb{R}^N$. Thus, using h4) again, we can dominate

$$|\mathrm{I}_\varepsilon(X,V)| \leq \|Q\|_{L^\infty(\mathbb{R};L^1(\mathbb{R}^N))} \frac{C_n}{(1+|X|^2)^n(1+|V|^2)^n}$$

and we select n large enough so that the function on the right hand side lies in $L^2(\mathbb{R}^{2N} \times \mathbb{R}^{2N})$. Finally, we can conclude by applying the Lebesgue theorem.

We deduce that the limit $F(t,X,V)$ of $\mathbb{E}F_\varepsilon(t,X,v) = \mathbb{E}[f_\varepsilon(t,x_1,v_1)f_\varepsilon(t,x_2,v_2)]$ satisfies

$$(\partial_t + V\cdot\nabla_X)F = \mathfrak{C}(F),$$

with initial data $F(0,X,V) = f(0,x_1,v_1)f(0,x_2,v_2)$. By uniqueness of the solution of the Cauchy problem, see PV [15, Lemma 4.11], it follows that $F(t,X,V) = f(t,x_1,v_1)f(t,x_2,v_2)$, (denoted $\widetilde{F}(t,X,V)$ above) with f the limit of $\mathbb{E}f_\varepsilon$. Applying the same reasoning as in the previous section, it allows us to establish the asserted almost sure convergence.

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