Relative entropy applied to the stability of viscous shocks up to a translation for scalar conservation laws

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Abstract

We consider inviscid limits to shocks for viscous scalar conservation laws in one space dimension, with strict convex fluxes. We show that we can obtain sharp estimates in L^2, for a class of large perturbations. Those perturbations can be chosen big enough to destroy the viscous layer. This shows that the fast convergence to the shock does not depend on the fine structure of the viscous layers. This is the first application of the relative entropy method developed in [19], [20] to the study of an asymptotic limit to a shock.

1 Introduction and the main result

For any strictly convex flux function $A \in C^2(\mathbb{R})$, we consider the family of viscous scalar conservation laws in one space dimension:

$$\begin{cases} \partial_t U + \partial_x A(U) = \varepsilon \partial^2_{xx} U & \text{for } t > 0, x \in \mathbb{R}, \\ U(0,x) = U_0(x) \text{ for } x \in \mathbb{R}, \end{cases} \tag{1}$$

for any $\varepsilon > 0$ and $U_0 \in L^\infty$. Global unique solutions to (1) have been constructed by Hopf [14] and Oleĭnik [24]. The inviscid case, $\varepsilon = 0$, is covered by the theory of Kruzkov [17]. Kuznetsov showed in [18] that, for fixed initial values U_0, the solutions of (1) converges, when ε goes to zero, to the solution of the inviscid Burgers equation (equation (1) with $\varepsilon = 0$). He showed also that, for general initial data, the optimal rate of convergence in L^1 is $\sqrt{\varepsilon}$. In the case of the convergence to a shock, however, the rate is better. This is, usually, linked to the formation of layers.

In this paper we consider the asymptotic limit for general initial values. We are particularly interested in the cases where the initial values carry too much

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entropy for the structure of the layer to be preserved asymptotically. The shocks solutions of the inviscid case ($\varepsilon = 0$) can be described as follows. Consider two constants $C_L > C_R$, and the associated function defined by

$$S_0(x) = \begin{cases} C_L \text{ if } x < 0, \\ C_R \text{ if } x \geq 0. \end{cases} \tag{2}$$

Then, the Rankine-Hugoniot conditions ensures that the function

$$S_0(x - \sigma t), \qquad \sigma = \frac{A(U_L) - A(U_R)}{U_l - U_R}, \tag{3}$$

is solution to the inviscid equation (1) with $\varepsilon = 0$. The condition $C_L > C_R$ implies that they verify the entropy conditions, that is:

$$\partial_t \eta(U) + \partial_x G(U) \leq 0, \qquad t > 0,\ x \in \mathbb{R},$$

for any convex functions η, and

$$G' = \eta' A'. \tag{4}$$

Our main result is the following.

Theorem 1.1. *Let $C_L > C_R$ and $U_0 \in L^\infty(\mathbb{R}) \cap BV_{loc}(\mathbb{R})$ be such that*

$$(U_0 - S_0) \in L^2(\mathbb{R}) \qquad \text{and} \qquad (\frac{d}{dx} U_0)_+ \in L^2(\mathbb{R}).$$

Then, for any $T > 0$, there exist $\varepsilon_0 > 0$ and $C^* > 0$ such that the following holds true.

I. For any U solution to (1) with $0 < \varepsilon \leq \varepsilon_0$, there exists a curve $X \in L^\infty(0, T)$ such that $X(0) = 0$ and for any $0 < t < T$:

$$\|U(t) - S(t)\|^2_{L^2(\mathbb{R})} \leq \|U_0 - S_0\|^2_{L^2(\mathbb{R})} + C^* [\log(1/\varepsilon)]^{1+p} \varepsilon, \tag{5}$$

where $S(t, x) := S_0(x - X(t))$, and S_0 is defined by (2).

II. Moreover, this curve satisfies

$$|\dot{X}(t)| \leq C^* \quad \textit{and} \tag{6}$$

$$|X(t) - \sigma t|^2 \leq C^* t \left(\|U_0 - S_0\|^2_{L^2(\mathbb{R})} + [\log(1/\varepsilon)]^{1+p} \varepsilon \right). \tag{7}$$

III. The constant ε_0 depends only on $\|(\frac{d}{dx} U_0)_+\|_{L^2}$, while C^* depends on p, C_L, C_R, $\|U_0\|_{L^\infty}$, T, and the flux function A.

Remark 1.1. Note that our estimates do not depend on any local BV norms of U_0. The assumption $U_0 \in BV_{loc}$ ensures that $\frac{d}{dx} U_0$ is a Radon measure. Hence, $(\frac{d}{dx} U_0)_+$ is also a Radon measure, and the condition $(\frac{d}{dx} U_0)_+ \in L^2$ makes sense.

This result shows a rate of convergence slightly worst than ε (to the $\log^p$), for the inviscid limit to a shock, measured via the L^2 norm (squared). In the case of the limit to a regular solution of the inviscid case, the rate of convergence is $\sqrt{\varepsilon}$ (see [29], for instance).

An easy layer study shows that ε is the optimal rate for shocks with special initial data. Indeed, one can construct an associated steady viscous layer (see for example Oleinik [15]) S_1 solution to

$$\begin{cases} A(S_1) - A(C_L) - \sigma(S_1 - C_L) = S_1', & x \in \mathbb{R}, \\ \lim_{x\to-\infty} S_1 = C_L, \qquad \lim_{x\to+\infty} S_1 = C_R. \end{cases} \tag{8}$$

It is easy to show that $S_1(x/\varepsilon - \sigma t)$ is solution to (1) with initial data $S_1(x/\varepsilon)$. In this case, the rate of convergence is of order ε since:

$$\int_{\mathbb{R}} |S_1(x/\varepsilon - \sigma t) - S_0(x - \sigma t)|^2 \, dx \leq \varepsilon \int_{\mathbb{R}} |S_1(x) - S_0(x)|^2 \, dx = C\varepsilon.$$

This layer study can be extended to the case of small initial perturbation where:

$$\int_{\mathbb{R}} |U_0(x) - S_0(x)|^p \, dx \leq C\varepsilon,$$

for a $1 \geq p < \infty$. In this case, we can consider

$$V(t,x) = U(t/\varepsilon, x/\varepsilon),$$

and study the asymptotic for large time. The function V is solution to the equation

$$\begin{aligned} &\partial_t V + \partial_x A(V) - \partial^2_{xx} V = 0, \\ &V(0,x) = U(0, x/\varepsilon). \end{aligned}$$

The convergence to S_1, up to a (constant) drift, in this setting, has been extensively studied (see for instance Oleĭnik [15], Freistühler and Serre [12], Kenig and Merle [16]). In this situation of small perturbation of the initial shock, those results show that the convergence with rate ε for the system (1) is due to the asymptotic limit in large time of the layer function $U(\cdot/\varepsilon)$ to $S_1(\cdot/\varepsilon - \sigma t)$.

This layer study, however, collapses when

$$\int_{\mathbb{R}} |U_0(x) - S_0(x)|^2 \, dx \gg \varepsilon.$$

In this situation, there is too much entropy for the asymptotic limit of the layer structure to be true. The physical layer may be destroyed. Theorem 1.1 shows that, nevertheless, the sharp convergence (up to the $\log^p$) still holds.

Taking $\varepsilon = 0$ in Theorem 1.1, we recover the L^2 stability of shocks (up to a drift) first showed by Leger in [19]. Note that the stability result has to be up to a drift which depends on the solution itself (and may be not unique). This feature is also true for our result. The drift cannot be taken constant, as in the case of the layer problem.

Our result is based on the relative entropy method first used by Dafermos and DiPerna to show L^2 stability and uniqueness of Lipschitzian solutions to conservation laws [9, 10, 11]. They showed, in particular, that if $\overline{U}$ is a Lipschitzian solution of a suitable conservation law on a lapse of time $[0,T]$, then for any bounded weak entropic solution U it holds:

$$\int_{\mathbb{R}} |U(t)-\overline{U}(t)|^2\,dx \leq C\int_{\mathbb{R}} |U(0)-\overline{U}(0)|^2\,dx, \tag{9}$$

for a constant C depending on $\overline{U}$ and T.

The relative entropy method is also an important tool in the study of asymptotic limits. The main idea is that convergence holds thanks to the strong stability of the solutions of the limit equations. Roughly speaking, if we have good consistency of ε models, with respect to the limit one, then non linearities are driven by the strong stability of the solution of the limit equation. Applications of the relative entropy method in this context began with the work of Yau [30] and have been studied by many others. For incompressible limits, see Bardos, Golse, Levermore [1, 2], Lions and Masmoudi [21], Saint Raymond et al. [13, 26, 22, 25]. For compressible models, see Tzavaras [28] in the context of relaxation and [4, 3, 23] in the context of hydrodynamical limits. However, in all those cases, the method works as long as the limit solution is Lipschitz. This is due to the fact that strong stability as (9) is not true when $\overline{U}$ has a discontinuity. It has been proven in [19, 20], however, that some shocks are strongly stable up to a shift (see also related work from Chen and Frid [5, 6] and Chen Frid and Li [7]). This article is the first extension of those results of stability, to the study of asymptotic limits to a shock. This is part of the program initiated in [29].

The result can be extended to any entropy in the following way. Fix any strictly convex function $\eta \in C^2(\mathbb{R})$ as an entropy. We define the associated relative entropy functional $\eta(\cdot|\cdot)$ as

$$\eta(x|y) := \eta(x) - \eta(y) - \eta'(y)(x-y).$$

We then have the following extension.

Theorem 1.2. *Consider a strictly convex entropy functional η. Let $C_L > C_R$ and $U_0 \in L^\infty(\mathbb{R}) \cap BV_{loc}(\mathbb{R})$ be such that*

$$(U_0 - S_0) \in L^2(\mathbb{R}) \qquad \text{and} \qquad (\frac{d}{dx}U_0)_+ \in L^2(\mathbb{R}).$$

Then, for every $T>0$, there exist $\varepsilon_0 > 0$ and $C^* > 0$ such that the following holds true.

I. For any U solution to (1) *with $0 < \varepsilon \leq \varepsilon_0$, there exists a curve $X \in L^\infty(0,T)$ such that $X(0) = 0$ and for any $0 < t < T$:*

$$\int_{\mathbb{R}} \eta(U(t,x)|S(t,x))\,dx \leq \int_{\mathbb{R}} \eta(U_0(x)|S_0(x))\,dx + C^*[\log(1/\varepsilon)]^{1+p}\varepsilon, \quad (10)$$

where $S(t,x) := S_0(x - X(t))$, and S_0 is defined by (2).

II. Moreover, this curve satisfies

$$|\dot{X}(t)| \leq C^* \quad \textit{and} \quad (11)$$

$$|X(t) - \sigma t|^2 \leq C^* t \left(\int_{\mathbb{R}} \eta(U_0(x)|S_0(x))\,dx + [\log(1/\varepsilon)]^{1+p}\varepsilon \right). \quad (12)$$

III. The constant ε_0 depends only on $\|(\frac{d}{dx}U_0)_+\|_{L^2}$, while C^* depends on p, C_L, C_R, $\|U_0\|_{L^\infty}$, T, η, and the flux function A.

Actually, Theorem 1.1 is a direct application of Theorem 1.2 with $\eta(x) := x^2$. Indeed, in this case we have $\eta(x|y) = (x-y)^2$.

2 The proof of Theorem of 1.2

Proof of Theorem 1.2. We begin with some preliminaries, which can be found in the paper [19] with proofs :

2.1 Preliminaries with lemmas

If U is a solution to (1) and if C is any constant state, then we have

$$\partial_t \Big(\eta(U|C) \Big) + \partial_x \Big(F(U,C) \Big) = \varepsilon(\partial^2_{xx} U) \cdot (\eta'(U) - \eta'(C)) \quad (13)$$

where the flux $F(\cdot,\cdot)$ of the relative entropy $\eta(\cdot|\cdot)$ is defined by

$$F(x,y) := G(x) - G(y) - \eta'(y)(A(x) - A(y)). \quad (14)$$

We define the normalized relative netropy flux $f(\cdot,\cdot)$ by $f(x,y) := \frac{F(x,y)}{\eta(x|y)}$. In the following lemma, we collect some properties.

Lemma 2.1. *There exist constants $\lambda, \Lambda, B_0, B_1$ and B_2 such that for any x, y with $|x|, |y| \leq M_1$, we have*

.

$$\begin{aligned} & 0 < \lambda \leq \eta''(x) \leq \Lambda, \\ & \frac{1}{2}\lambda \cdot (x-y)^2 \leq \eta(x|y) \leq \frac{1}{2}\Lambda \cdot (x-y)^2 \quad \textit{and} \\ & |F(x,y)| \leq B_0 \cdot (x-y)^2. \end{aligned} \quad (15)$$

In addtion, $f \in C^1$ in both variables, and we have

$$\begin{aligned} &0 \le (\partial_1 f)(x,y) \le B_1 \quad \textit{and} \\ &0 < B_2 \le (\partial_2 f)(x,y). \end{aligned} \tag{16}$$

Proof. It is simple to prove (16). For (17), we refer to the section 2.1 of [19]. □

Remark 2.1. The above preliminaries imply that it is enough to show all conclusions of Theorem 1.2 for Burger's equation with L^2 entropy ($\eta(x) := x^2$) in order to get the same things for general scalar conservation laws with general entropies. However we will give all the details for completeness.

The following lemma is a sort of a weak form of Oleĭnik's principle. It says that L^2-norm of the positive part of the derivative of a solution is decreasing.

Lemma 2.2. $\|(\partial_x U(t))_+\|_{L^2(\mathbb{R})} \le \|(\frac{d}{dx}U_0)_+\|_{L^2(\mathbb{R})}$ *for any $t > 0$.*

Proof. We differentiate (1) w.r.t. x, multiply $(\partial_x U)_+$ and integrate in x to get

$$\begin{aligned} 0 &= \int (\partial_x U)_+ \cdot \Big[\partial_t\partial_x U + A''(U)\cdot|\partial_x U|^2 + A'(U)\cdot \partial^2_{xx}U - \varepsilon\cdot\partial^3_{xxx}U\Big]dx \\ &= \int \Big[\frac{1}{2}\partial_t([(\partial_x U)_+]^2) + A''(U)\cdot(\partial_x U)^3_+ \\ &\qquad + A'(U)\cdot\partial_x\Big(\frac{[(\partial_x U)_+]^2}{2}\Big) + \varepsilon\cdot|\partial_x((\partial_x U)_+)|^2\Big]dx. \end{aligned}$$

Then, we use the integration by parts to get

$$\begin{aligned} &= \int \Big[\frac{1}{2}\partial_t([(\partial_x U)_+]^2) + \frac{1}{2}A''(U)\cdot(\partial_x U)^3_+ + \varepsilon\cdot|\partial_x((\partial_x U)_+)|^2\Big]dx \\ &\ge \frac{1}{2}\frac{d}{dt}\int [(\partial_x U)_+]^2 dx. \end{aligned}$$

□

2.2 Definition of the curve $X(\cdot)$ with the proof of the Lipshitz estimate (11)

From now on, without loss of generality, we assume $U_0 \in C^1$.

Remark 2.2. Indeed, if U_0 is not C^1, then we regularize it first. This procedure will be stated in Appendix.

We define $X(\cdot)$ by solving the O.D.E.

$$\begin{cases} \dot X(t) = f\Big(U(t,X(t)), \frac{C_L + C_R}{2}\Big) \\ X(0) = 0 \end{cases}. \tag{17}$$

Then we get (11) easily:

$$\begin{aligned} |\dot{X}(t)| &\leq \Big| f\Big(U(t,X(t)), \frac{C_L+C_R}{2}\Big)\Big| \leq \frac{\Big|F\Big(U(t,X(t)), \frac{C_L+C_R}{2}\Big)\Big|}{\eta\Big(U(t,X(t))\Big|\frac{C_L+C_R}{2}\Big)} \\ &\leq \frac{B_0 \cdot \Big|(U(t,X(t)) - \frac{C_L+C_R}{2}\Big|^2}{(1/2)\lambda \cdot \Big|U(t,X(t)) - \frac{C_L+C_R}{2}\Big|^2} = \frac{2\cdot B_0}{\lambda} := D_1 \end{aligned} \tag{18}$$

2.3 The main proposition

Proposition 2.3. *Let $\delta > 0$ satisfy*

$$\delta\varepsilon \leq \Big(\frac{B_2}{2\cdot B_1 \cdot \|(\frac{d}{dx}U_0)_+\|_{L^2(\mathbb{R})}} \cdot \Big(\frac{C_L - C_R}{2}\Big)\Big)^2. \tag{19}$$

Let $\phi(\cdot)$ be a continuous increasing function satisfying $\begin{cases} \phi(x) = 0 \text{ if } x \leq 0 \\ \phi(x) = 1 \text{ if } x \geq \delta \end{cases}$.
Suppose $\phi\Big|_{(0,\delta)} \in C^1$. Then, we have, for $t > 0$,

$$\begin{aligned} &\frac{d}{dt}\int_{-\infty}^{\infty}\Big[\phi\Big(\frac{|x - X(t)|}{\varepsilon}\Big)\Big]^2 \eta\Big(U(t,x)|S(t,x)\Big)dx \\ &\leq 4\cdot {M_1}^2 \cdot \Big(\frac{\Lambda^2}{\lambda}\Big)\cdot \int_0^{\delta}\Big[\phi'(z)\Big]^2 \cdot \chi_{\{\phi'(\cdot) > E\cdot\phi(\cdot)\}}(z)dz \end{aligned} \tag{20}$$

where $E := \Big(\frac{(C_L - C_R)}{2}\cdot B_2 \cdot \frac{\lambda^2}{\Lambda^2}\Big)$.

Proof. In this proof, we denote $d := C_L - C_R > 0$. First we split the above integral into the two parts:

$$\begin{aligned} &\int_{-\infty}^{\infty}\Big[\phi\Big(\frac{|x - X(t)|}{\varepsilon}\Big)\Big]^2 \eta(U|S)dx \\ &= \int_{-\infty}^{\infty}\Big(\Big[\phi\Big(\frac{-x + X(t)}{\varepsilon}\Big)\Big]^2 + \Big[\phi\Big(\frac{x - X(t)}{\varepsilon}\Big)\Big]^2\Big)\eta(U|S)dx \\ &= \underbrace{\int_{-\infty}^{\infty}\Big[\phi\Big(\frac{-x + X(t)}{\varepsilon}\Big)\Big]^2 \eta(U|C_L)dx}_{(I)} + \underbrace{\int_{-\infty}^{\infty}\Big[\phi\Big(\frac{x - X(t)}{\varepsilon}\Big)\Big]^2 \eta(U|C_R)dx}_{(II)}. \end{aligned}$$

To estimate $\frac{d}{dt}(I)dx$, we put $C = C_L$ in (14), multiply $\Big[\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)\Big]^2$ and integrate in x. Then we have

$$\begin{aligned}\frac{d}{dt}(I) &= \int_{-\infty}^{\infty} \partial_t\Big(\Big[\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)\Big]^2\Big)\cdot \eta(U|C_L)dx \\ &\quad + \int_{-\infty}^{\infty} \partial_x\Big(\Big[\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)\Big]^2\Big)\cdot F(U,C_L)dx \\ &\quad + \varepsilon\cdot\int_{-\infty}^{\infty} \Big[\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)\Big]^2\cdot \partial_{xx}^2 U\cdot(\eta'(U)-\eta'(C_L))dx\end{aligned}$$

$$= \underbrace{\int_{X(t)-\delta\varepsilon}^{X(t)} (\frac{2}{\varepsilon})\cdot\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)\cdot\phi'\Big(\frac{-x+X(t)}{\varepsilon}\Big)\cdot\Big[\dot{X}(t)\eta(U|C_L)-F(U,C_L)\Big]dx}_{(I)_{\text{Hyp}}}$$

$$\underbrace{+\,\varepsilon\cdot\int_{-\infty}^{X(t)}\Big[\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)\Big]^2\cdot\partial_{xx}^2U\cdot(\eta'(U)-\eta'(C_L))dx}_{(I)_{\text{Dif}}}.$$

For $(I)_{\text{Hyp}}$(hyperbolic part), we use the definition of $X(t)$ to get

$$(I)_{\text{Hyp}} = \int_{X(t)-\delta\varepsilon}^{X(t)} (\frac{2}{\varepsilon})\cdot\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)\cdot\phi'\Big(\frac{-x+X(t)}{\varepsilon}\Big)\cdot\eta(U|C_L)\cdot h(t,x)dx$$

where $h(t,x) := \Big[f\Big(U(t,X(t)),\frac{C_L+C_R}{2}\Big)-f(U(t,x),C_L)\Big]$.

In order to make the function $h(t,x)$ strictly negative over the domain of the above integral, we use the condition $(\frac{d}{dx}U_0)_+\in L^2(\mathbb{R})$. We observe that for any $x\in[X(t)-\delta\varepsilon, X(t)]$,

$$\begin{aligned}U(t,X(t))-U(t,x) &= \int_x^{X(t)}(\partial_xU)(t,y)dy \le \int_x^{X(t)}(\partial_xU)_+(t,y)dy \\ &\le \|(\partial_xU(t))_+\|_{L^2(\mathbb{R})}\cdot\sqrt{|X(t)-x|}\le \|(\frac{d}{dx}U_0)_+\|_{L^2(\mathbb{R})}\cdot\sqrt{\delta\varepsilon}\end{aligned}$$

where we used non-increasing of $\|(\partial_xU(t))_+\|_{L^2}$ (see Lemma 2.2).

we have, by using

$$\begin{aligned}h(t,x) &= f\Big(U(t,X(t)),\frac{C_L+C_R}{2}\Big)-f\Big(U(t,x),\frac{C_L+C_R}{2}\Big) \\ &\quad + f\Big(U(t,x),\frac{C_L+C_R}{2}\Big)-f(U(t,x),C_L)\end{aligned}$$

Since f is increasing with respect to the first variable, we have

$$\begin{aligned}&\le f\Big(U(t,x)+\|(\frac{d}{dx}U_0)_+\|_{L^2(\mathbb{R})}\cdot\sqrt{\delta\varepsilon},\frac{C_L+C_R}{2}\Big)-f\Big(U(t,x),\frac{C_L+C_R}{2}\Big) \\ &\quad + f\Big(U(t,x),\frac{C_L+C_R}{2}\Big)-f(U(t,x),C_L)\end{aligned}$$

Then, thanks to the property (17) with the assumption (20), we get

$$\leq \|(\frac{d}{dx}U_0)_+\|_{L^2(\mathbb{R})} \cdot \sqrt{\delta\varepsilon} \cdot B_1 - \frac{C_L - C_R}{2} \cdot B_2 \leq -\frac{d}{2} \cdot \frac{B_2}{2} < 0$$

Thus we we use the fact $\phi, \phi' \geq 0$ to get

$$(I)_{\text{Hyp}} \leq \int_{X(t)-\delta\varepsilon}^{X(t)} \frac{2}{\varepsilon}\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)\phi'\Big(\frac{-x+X(t)}{\varepsilon}\Big)\eta(U|C_L)\Big(-\frac{d}{2}\cdot\frac{B_2}{2}\Big)dx$$

Then, by (16), we have

$$\leq \int_{X(t)-\delta\varepsilon}^{X(t)} (\frac{2}{\varepsilon})\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)\phi'\Big(\frac{-x+X(t)}{\varepsilon}\Big)\frac{\lambda}{2}(U-C_L)^2\Big(-\frac{d}{2}\cdot\frac{B_2}{2}\Big)dx.$$

On the other hand, for $(I)_{\text{Dif}}$(the diffusion part), we use the integration by parts to obtain

$$\begin{aligned}(I)_{\text{Dif}} = &\int_{-\infty}^{X(t)} 2\cdot\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)\cdot\phi'\Big(\frac{-x+X(t)}{\varepsilon}\Big)\cdot\partial_x U\cdot(\eta'(U)-\eta'(C_L))dx\\ &-2\varepsilon\cdot\int_{-\infty}^{X(t)}\Big[\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)\Big]^2\cdot\eta''(U)\cdot|\partial_x U|^2dx\end{aligned}$$

Then, by Hölder's inequality and by (16), we get

$$\begin{aligned}&\leq 2\varepsilon\lambda\cdot\int_{-\infty}^{X(t)}\Big[\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)\Big]^2\cdot|\partial_x U|^2dx\\ &+\frac{1}{8\varepsilon\lambda}\int_{\infty}^{X(t)}\Big[2\phi'\Big(\frac{-x+X(t)}{\varepsilon}\Big)(\eta'(U)-\eta'(C_L))\Big]^2dx\\ &-2\varepsilon\cdot\int_{-\infty}^{X(t)}\Big[\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)\Big]^2\cdot\lambda\cdot|\partial_x U|^2dx\end{aligned}$$

Then, the only middle term survives, and we estimate the middle part:

$$\leq \frac{2}{\varepsilon}\int_{X(t)-\delta\varepsilon}^{X(t)}\Big[\phi'\Big(\frac{-x+X(t)}{\varepsilon}\Big)\Big]^2\cdot\frac{\Lambda^2}{4\lambda}\cdot|U-C_L|^2dx$$

Now we combine the hyperbolic part $(I)_{\text{Hyp}}$ with the diffusion part $(I)_{\text{Dif}}$

and use the change of variables $\frac{-x+X(t)}{\varepsilon} = z$ to get

$$\begin{aligned}
&\frac{d}{dt}(I)dx = (I)_{\text{Hyp}} + (I)_{\text{Dif}} \\
&\leq \int_{X(t)-\delta\varepsilon}^{X(t)} (\frac{2}{\varepsilon}) \cdot \phi'\Big(\frac{-x+X(t)}{\varepsilon}\Big) \cdot (U(t,x) - C_L)^2 \\
&\qquad \cdot \Big[-\Big(\frac{\lambda}{2} \cdot \frac{d}{2} \cdot \frac{B_2}{2}\Big) \cdot \phi\Big(\frac{-x+X(t)}{\varepsilon}\Big) + \Big(\frac{\Lambda^2}{4\lambda}\Big) \cdot \phi'\Big(\frac{-x+X(t)}{\varepsilon}\Big)\Big] dx \\
&= \int_0^\delta 2\phi'(z)(U(t,X(t)-\varepsilon z) - C_L)^2 \Big[-\Big(\frac{\lambda}{2} \cdot \frac{d}{2} \cdot \frac{B_2}{2}\Big)\phi(z) + \frac{\Lambda^2}{4\lambda} \cdot \phi'(z)\Big] dz \\
&\leq 2 \cdot M_1^2 \cdot \Big(\frac{\Lambda^2}{\lambda}\Big) \cdot \int_0^\delta \Big[\phi'(z)\Big]^2 \cdot \chi_{\{\phi'(\cdot) > \big(\frac{d}{2} \cdot B_2 \cdot \frac{\lambda^2}{\Lambda^2}\big) \cdot \phi(\cdot)\}}(z) dz
\end{aligned} \tag{21}$$

Similarly, in order to estimate $\frac{d}{dt}(II)dx$, we put $C = C_R$ in (14), multiply $\Big[\phi\Big(\frac{x-X(t)}{\varepsilon}\Big)\Big]^2$, and repeat the above argument to get

$$\frac{d}{dt}(II)dx \leq 2 \cdot M_1^2 \cdot \Big(\frac{\Lambda^2}{\lambda}\Big) \cdot \int_0^\delta \Big[\phi'(z)\Big]^2 \cdot \chi_{\{\phi'(\cdot) > \big(\frac{d}{2} \cdot B_2 \cdot \frac{\lambda^2}{\Lambda^2}\big) \cdot \phi(\cdot)\}}(z) dz$$

Therefore, we conclude

$$\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{\infty} \Big[\phi\Big(\frac{|x-X(t)|}{\varepsilon}\Big)\Big]^2 \eta\Big(U(t,x)|S_0(x-X(t))\Big)dx = \frac{d}{dt}(I) + \frac{d}{dt}(II) \\
&\leq 4 \cdot {M_1}^2 \cdot \Big(\frac{\Lambda^2}{\lambda}\Big) \cdot \int_0^\delta \Big[\phi'(z)\Big]^2 \cdot \chi_{\{\phi'(\cdot) > E \cdot \phi(\cdot)\}}(z) dz
\end{aligned} \tag{22}$$

where $E := \Big(\frac{d}{2} \cdot B_2 \cdot \frac{\lambda^2}{\Lambda^2}\Big)$. □

2.4 A technical lemma

In this subsection, we estimate the last integral in (23) by choosing a particular function ϕ. The following lemma will be useful to obtain the main *log* estimate (10). We take a particular function ϕ, which is convex on $[0, \delta]$, to control the integral part $(*)$ in (24). More precisely, the function is chosen to make $(*)$ go to zero as fast as we want as δ goes to the infinity.

Lemma 2.4. *Let $q \in [1, \infty)$ and $\delta > 0$. Let $E > 0$ be a constant.*

Define ϕ by $\begin{cases} \phi(x) = 0 \text{ for } x \leq 0 \\ \phi(x) = e^{\frac{q+1}{q}} \cdot e^{-\frac{q+1}{q} \cdot \frac{\delta^q}{x^q}} \text{ for } 0 < x < \delta \cdot \\ \phi(x) = 1 \text{ for } x \geq \delta \end{cases}$

Then the function ϕ is increasing on $[0, \infty)$, and there exist two constants $E_i =$

$E_i(E,q) > 0$ for $i = 1,2$ such that

$$\underbrace{\int_0^\delta \Big[\phi'(x)\Big]^2 \cdot \chi_{\{\phi'(\cdot) > E\cdot\phi(\cdot)\}}(x)dx}_{(*)} \le E_1 \cdot \delta^{\frac{q}{q+1}} \cdot e^{-E_2\cdot\delta^{q/(q+1)}} \tag{23}$$

as long as $\delta \ge \left(\frac{q+1}{E}\right)$.

Remark 2.3. For example, for $q = 1$, we choose ϕ by the following way: we note that $e^{-1/x}$ is convex on $[0,1/2]$. Then we rescale it to have $\phi(\delta) = 1$.

Proof. Let $\delta \ge \left(\frac{q+1}{E}\right)$. Since $\phi'(x) = e^{\frac{q+1}{q}} \cdot e^{-\frac{q+1}{q}\cdot\frac{\delta^q}{x^q}} \cdot \left(\frac{(q+1)\delta^q}{x^{q+1}}\right) \cdot \chi_{\{0<x<\delta\}}$, we have

$$\begin{aligned}\{x \in (0,\delta) \quad | \quad \phi'(x) > E\cdot\phi(x)\} &= \{x \in (0,\delta) \quad | \quad \frac{(q+1)\delta^q}{x^{q+1}} > E\} \\ &= \{0 < x < \left(\frac{q+1}{E}\right)^{1/(q+1)} \cdot \delta^{\frac{q}{q+1}}\}.\end{aligned}$$

Thus we integrate not on $[0,\delta]$ but on $[0, \left(\frac{q+1}{E}\right)^{1/(q+1)} \cdot \delta^{\frac{q}{q+1}}]$ to get

$$\begin{aligned}(*) &= \int_0^{\left(\frac{q+1}{E}\right)^{1/(q+1)}\cdot\delta^{\frac{q}{q+1}}} \Big[\phi'(x)\Big]^2 dx \\ &= e^{2\cdot\frac{q+1}{q}} \cdot \Big((q+1)\delta^q\Big)^2 \cdot \int_0^{\left(\frac{q+1}{E}\right)^{1/(q+1)}\cdot\delta^{\frac{q}{q+1}}} e^{-2\cdot\frac{q+1}{q}\cdot\frac{\delta^q}{x^q}} \cdot \left(\frac{1}{x^{2q+2}}\right)dx\end{aligned}$$

Then we use the change of variables $x = \delta z$ to get

$$= e^{2\cdot\frac{q+1}{q}} \cdot \Big((q+1)\delta^q\Big)^2 \cdot \delta^{-2q-1} \cdot \int_0^{\left(\frac{q+1}{E\cdot\delta}\right)^{1/(q+1)}} e^{-2\cdot\frac{q+1}{q}\cdot\frac{1}{z^q}} \cdot \left(\frac{1}{z^{2q+2}}\right)dz$$

Since the integrand of the above integral is increasing on the interval $[0,1]$, which contains the domain of the integral, we have

$$\begin{aligned}&\le e^{2\cdot\frac{q+1}{q}} \cdot \Big((q+1)\delta^q\Big)^2 \cdot \delta^{-2q-1} \cdot \left(\frac{E\cdot\delta}{q+1}\right)^{(2q+1)/(q+1)} \cdot e^{-2\cdot\frac{q+1}{q}\cdot\left(\frac{E\cdot\delta}{q+1}\right)^{q/(q+1)}} \\ &\le E_1 \cdot \delta^{\frac{q}{q+1}} \cdot e^{-E_2\cdot\delta^{q/(q+1)}}.\end{aligned}$$

□

Remark 2.4. If we assume only $\delta > 0$ without the positive lower bound $\left(\frac{q+1}{E}\right)$ of δ, then the charateristic function in (24) does not help to control the integral any more so that we have to integrate the full interval $[0,\delta]$ so that we could have only

$$(*) \le C\cdot\delta^{-1}, \tag{24}$$

which is certainly weaker than (24) as δ increases. In Remark 2.6, we will see that the above weak estimate (25) could give us only $\varepsilon^{1/2}$ instead of $\Big(\Big[\log(\frac{1}{\varepsilon})\Big]^{1+p}\cdot\varepsilon\Big)$ in (10). Since $\varepsilon^{1/2}$ is worse than the other as ε goes to zero, the lower bound $\Big(\frac{q+1}{E}\Big)$ of δ in this lemma is crucial.

2.5 The Proof of the main estimate (10)

We are ready to finish the proof of the theorem 1.2. Let $q\ge 1$ fixed. For any $\varepsilon>0$, we define $\delta=\delta(\varepsilon):={E_2}^{-(q+1)/q}\cdot\Big[\log(\frac{1}{\varepsilon})\Big]^{(q+1)/q}$. Then we find $\varepsilon_0\in(0,e^{-1}]$ such that if $\varepsilon\in(0,\varepsilon_0]$, then

$$\begin{cases} \delta\ge\Big(\frac{q+1}{E}\Big) \text{ and} \\ \delta\varepsilon\le\Big(\frac{B_2}{2\cdot B_1\cdot\|(\frac{d}{dx}U_0)_+\|_{L^2(\mathbb{R})}}\cdot\Big(\frac{C_L-C_R}{2}\Big)\Big)^2. \end{cases} \tag{25}$$

This process is always possible because $\log x$ is increasing to ∞ and $(\log x)/x$ is decreasing to 0 as x grows to ∞.

Now take any $\varepsilon\in(0,\varepsilon_0]$. We split the integral in the left-hand side of (10) into two parts:

$$\begin{aligned}
&\int\eta\Big(U(t,x)|S(t,x)\Big)dx\\
&\quad=\int\Big[\phi\Big(\frac{|x-X(t)|}{\varepsilon}\Big)\Big]^2\eta\Big(U(t,x)|S(t,x))\Big)dx\\
&\qquad+\int\Big(1-\Big[\phi\Big(\frac{|x-X(t)|}{\varepsilon}\Big)\Big]^2\Big)\eta\Big(U(t,x)|S(t,x))\Big)dx\\
&\quad\le\int\Big[\phi\Big(\frac{|x|}{\varepsilon}\Big)\Big]^2\eta\Big(U_0(x)|S_0(x)\Big)dx\\
&\qquad+\int_0^t\frac{d}{ds}\int\Big[\phi\Big(\frac{|x-X(s)|}{\varepsilon}\Big)\Big]^2\eta\Big(U(s,x)|S(t,x)\Big)dxdt\\
&\qquad+\frac{\Lambda}{2}\cdot\int\Big(1-\Big[\phi\Big(\frac{|x-X(t)|}{\varepsilon}\Big)\Big]^2\Big)|U(t,x)-S(t,x)|^2dx
\end{aligned}$$

Then we use Proposition 2.3 to get

$$\begin{aligned}
&\leq \int \eta\Big(U_0(x)|S_0(x)\Big)dx \\
&\quad + t\cdot\Big(4\cdot {M_1}^2\cdot\Big(\frac{\Lambda^2}{\lambda}\Big)\cdot\int_0^\delta \Big[\phi'(z)\Big]^2\cdot\chi_{\{\phi'(\cdot)>E\cdot\phi(\cdot)\}}(z)dz\Big) \\
&\quad + \frac{\Lambda}{2}\cdot\int_{X(t)-\delta\varepsilon}^{X(t)+\delta\varepsilon}|U(t,x)-S(t,x))|^2dx
\end{aligned}$$

Then we use Lemma 2.4 to get

$$\begin{aligned}
&\leq \int \eta\Big(U_0(x)|S_0(x)\Big)dx \\
&\quad + t\cdot\Big(4\cdot {M_1}^2\cdot\Big(\frac{\Lambda^2}{\lambda}\Big)\cdot E_1\cdot\delta^{\frac{q}{q+1}}\cdot e^{-E_2\cdot\delta^{q/(q+1)}}\Big) \\
&\quad + 4\cdot\Lambda\cdot {M_1}^2\delta\varepsilon \\
&\leq \int \eta\Big(U_0(x)|S_0(x)\Big)dx + E_3\underbrace{\Big(\delta\varepsilon + t\cdot\delta^{\frac{q}{q+1}}\cdot e^{-E_2\cdot\delta^{q/(q+1)}}\Big)}_{(*)}
\end{aligned}$$

where E_3 is some constant depending only on $\{\lambda, \Lambda, M_1, E_1(E,q)\}$.

Since we take $\delta := {E_2}^{-(q+1)/q}\cdot\Big[\log(\frac{1}{\varepsilon})\Big]^{(q+1)/q}$ with $\varepsilon \leq e^{-1}$, we have

$$\begin{aligned}
(*) &\leq \Big({E_2}^{-(q+1)/q}\cdot\Big[\log(\frac{1}{\varepsilon})\Big]^{(q+1)/q}\cdot\varepsilon + t\cdot E_2^{-1}\cdot\log(\frac{1}{\varepsilon})\cdot\varepsilon\Big) \\
&\leq E_4\cdot\Big[\log(\frac{1}{\varepsilon})\Big]^{(q+1)/q}\cdot\varepsilon\cdot(1+t)
\end{aligned}$$

where E_4 is some constant depending only on $\{E_2(E,q), q\}$.

In conclusion,

$$\begin{aligned}
\int \eta\Big(U(t,x)|S(t,x)\Big)dx &\leq \int \eta\Big(U_0(x)|S_0(x)\Big)dx \\
&\quad + E_3\cdot E_4\cdot\Big[\log(\frac{1}{\varepsilon})\Big]^{(q+1)/q}\cdot\varepsilon\cdot(1+t)
\end{aligned} \tag{26}$$

for any $\varepsilon \leq \varepsilon_0$. By taking $p := 1/q$ and $D_1 := E_3\cdot E_4$, we proved (10) of Theorem 1.2.

Remark 2.5. Note that D_1 is independent of $\|(\frac{d}{dx}U_0)_+\|_{L^2(\mathbb{R})}$ while ε_0 depends on it.

Remark 2.6. If we use (25) instead of (24), then we take $\delta := \sqrt{\varepsilon}$ to minimize $\delta\varepsilon + t\cdot\delta^{-1}$. As a result, we could obtain only $\sqrt{\varepsilon}$ instead of $\Big(\Big[\log(\frac{1}{\varepsilon})\Big]^{(q+1)/q}\cdot\varepsilon\Big)$ in (27), which is certainly weaker as ε goes to zero. This is the reason we kept the characteristic function in (22) in the proof of Propostion 2.3 .

2.6 The Proof of the estimate (12) for the curve $X(\cdot)$

To prove (12), we define first ψ by $\psi(x) = \begin{cases} 0 \text{ if } |x| > 2, \\ 1 \text{ if } |x| \leq 1 \\ 2 - |x| \text{ if } 1 < |x| \leq 2 \end{cases}$. Let $s, R > 0$. We multiply $\Psi_R(s,x) := \psi(\frac{x-X(s)}{R})$ to the equation (1) and integrate in x to get

$$
\begin{aligned}
0 &= -\frac{d}{ds}\int \Psi_R \cdot U dx + \int \partial_x(\Psi_R) A(U) dx + \int \partial_t(\Psi_R) U dx + \varepsilon \int \Psi_R \cdot \partial_{xx}^2 U dx \\
&= \underbrace{-\frac{d}{ds}\int \psi(\frac{x-X(s)}{R}) \cdot U(s,x) dx}_{(I)} \\
&\quad \underbrace{+\frac{1}{R}\int \psi'(\frac{x-X(s)}{R}) \cdot \Big(A(U(s,x)) - \dot{X}(s) U(s,x)\Big) dx}_{(II)} \\
&\quad \underbrace{-\varepsilon \frac{1}{R}\int \psi'(\frac{x-X(s)}{R}) \cdot \partial_x U(s,x) dx}_{(III)}.
\end{aligned}
$$

By using the above observation $0 = -(II) + (I) + (III)$, we have

$$
\begin{aligned}
(\sigma - \dot{X}(s)) &= \frac{1}{C_L - C_R}\Big(A(C_L) - A(C_R) - (C_L - C_R)\dot{X}(s)\Big) \\
&= \frac{1}{C_L - C_R}\Big(A(C_L) - A(C_R) - (C_L - C_R)\dot{X}(s) - (II) + (I) + (III)\Big).
\end{aligned}
$$

Then we integrate the above equation in time on $[0,t]$ and take the absolute sign to get:

$$
\begin{aligned}
|\sigma t - X(t)| \leq C\Big(t \cdot \max_{s\in(0,t)} &\underbrace{\Big|A(C_L) - A(C_R) - (C_L - C_R)\dot{X}(s) - (II)\Big|}_{(*)} \\
&+ \Big|\int_0^t (I) ds\Big| + t \cdot \max_{s\in(0,t)} \Big|(III)\Big|\Big).
\end{aligned}
\tag{27}
$$

We observe

$$
\begin{aligned}
(*) \leq &\underbrace{\Big|A(C_L) - A(C_R) - \frac{1}{R}\int \psi'(\frac{x-X(s)}{R}) \cdot A(U) dx\Big|}_{(**)} \\
&+ \underbrace{\Big| - (C_L - C_R)\dot{X}(s) + \frac{1}{R}\int \psi'(\frac{x-X(s)}{R}) \cdot \Big(\dot{X}(s) U(s,x)\Big) dx\Big|}_{(***)}.
\end{aligned}
$$

For the first term $(**)$, we compute

$$\begin{aligned}(**) &= \Big|A(C_L) - \frac{1}{R}\int_{-2R+X(s)}^{-R+X(s)} A(U)dx - A(C_R) + \frac{1}{R}\int_{R+X(s)}^{2R+X(s)} A(U)dx\Big| \\ &\leq \frac{1}{R}\Big[\int_{-2R+X(s)}^{-R+X(s)} |A(C_L) - A(U)|dx + \int_{R+X(s)}^{2R+X(s)} |A(U) - A(C_R)|dx\Big].\end{aligned}$$

We use $|A(y) - A(z)| \leq C|y - z|$ for $|y|, |z| \leq M_1$ to get

$$\leq \frac{C}{R}\int_{-2R+X(s)}^{2R+X(s)} |U - S|dx.$$

We use Hölder's inequality to get

$$\leq \frac{C\sqrt{4R}}{R}\|U(s) - S(s)\|_{L^2(\mathbb{R})} \leq \frac{C}{\sqrt{R}} \cdot \|U(s) - S(s)\|_{L^2(\mathbb{R})}.$$

Likewise, for the second term $(***)$, we have

$$\begin{aligned}(***) &= |\dot{X}(s)| \cdot \Big| - (C_L - C_R) + \frac{1}{R}\int \psi'(\frac{x - X(s)}{R}) \cdot U(s, x)dx\Big| \\ &\leq \frac{C}{R}\int_{-2R+X(s)}^{2R+X(s)} |U - S|dx \leq \frac{C}{\sqrt{R}} \cdot \|U(s) - S(s)\|_{L^2(\mathbb{R})}\end{aligned}$$

where we used $|\dot{X}(s)| \leq C$. Thus we have

$$(*) \leq \frac{C}{\sqrt{R}} \cdot \|U(s) - S(s)\|_{L^2(\mathbb{R})}. \tag{28}$$

On the other hand, we compute

$$\begin{aligned}&\Big|\int_0^t (I)ds\Big| = \Big|\int \psi(\frac{x - X(t)}{R}) \cdot U(t, x)dx - \int \psi(\frac{x}{R}) \cdot U_0(x)dx\Big| \\ &= \Big|\int \psi(\frac{x - X(t)}{R}) \cdot \Big(U(t, x) - S(t, x)\Big)dx + \int \psi(\frac{x - X(t)}{R}) \cdot S(t, x)dx \\ &- \int \psi(\frac{x}{R}) \cdot S_0(x)dx - \int \psi(\frac{x}{R}) \cdot \Big(U_0(x) - S_0(x)\Big)dx\Big|.\end{aligned}$$

Note that $\int \psi(\frac{x-X(t)}{R}) \cdot S(t, x)dx = \int \psi(\frac{x}{R}) \cdot S_0(x)dx$. Thus, we have

$$\leq \Big|\int \psi(\frac{x - X(t)}{R}) \cdot \Big(U(t, x) - S(t, x)\Big)dx\Big| + \Big|\int \psi(\frac{x}{R}) \cdot \Big(U_0(x) - S_0(x)\Big)dx\Big|.$$

We use Hölder's to get

$$\leq C\sqrt{R}\Big(\|U(t) - S(t)\|_{L^2(\mathbb{R})} + \|U_0 - S_0\|_{L^2(\mathbb{R})}\Big). \tag{29}$$

Also, we have

$$\begin{aligned}\Big|(III)\Big| &= \frac{\varepsilon}{R}\Big|\int \psi'(\frac{x-X(s)}{R})\cdot\partial_x U(s,x)dx\Big|\\ &= \frac{\varepsilon}{R}\Big|\int_{-2R+X(s)}^{-R+X(s)}\partial_x U(s,x)dx - \int_{R+X(s)}^{2R+X(s)}\partial_x U(s,x)dx\Big| \\ &\le \frac{\varepsilon}{R}\cdot 4\cdot\|U(s)\|_{L^\infty} \le \frac{C\cdot\varepsilon}{R}.\end{aligned}\tag{30}$$

Finally, by using (10), we combine (29), (30) and (31) with (28) to get, for any $R, t \in (0,\infty)$,

$$\begin{aligned}|\sigma t - X(t)| \le{}& \frac{C\cdot t}{\sqrt{R}}\cdot\Big(\|U_0-S_0\|_{L^2(\mathbb{R})} + \sqrt{D_0\cdot\Big[\log(\frac{1}{\varepsilon})\Big]^{1+p}\cdot\varepsilon\cdot(t+1)}\Big)\\ &+ C\sqrt{R}\cdot\Big(\|U_0-S_0\|_{L^2(\mathbb{R})} + \sqrt{D_0\cdot\Big[\log(\frac{1}{\varepsilon})\Big]^{1+p}\cdot\varepsilon\cdot(t+1)}\Big)\\ &+ \frac{C\cdot\varepsilon\cdot t}{R}.\end{aligned}$$

Then, by taking $R := t$ and by using $\varepsilon \le e^{-1}$, the estimate (12) follows. □

3 Appendix

3.1 The proof of Corollary 1.3

Proof of Corollary 1.3. For given $U_0 \in L^\infty(\mathbb{R})$ such that $U_0 - S_0 \in L^2$, we define U_0^N for integers $N \ge 1$ by

$$U_0^N = U_0 * \varphi_{1/N}$$

where $\varphi \in C_c^\infty$ with $\int_{\mathbb{R}}\varphi = 1$, $supp(\varphi) \subset B_1(0)$ and we define $\varphi_r(\cdot) := (1/r)\varphi(\cdot/r)$.

From $U_0 * \varphi_{1/N} = (U_0 - S_0) * \varphi_{1/N} + S_0 * \varphi_{1/N}$ and $|S_0 * \varphi_{1/N} - S_0| \le \frac{C_L - C_R}{2}\cdot\chi_{|x|\le\frac{1}{N}}$, we observe that

$$\begin{aligned}\|U_0^N - S_0\|_{L^2} &\le \|(U_0-S_0)*\varphi_{1/N}\|_{L^2} + \|S_0*\varphi_{1/N} - S_0\|_{L^2}\\ &\le \|U_0-S_0\|_{L^2}\cdot\|\varphi_{1/N}\|_{L^1} + \|\frac{C_L-C_R}{2}\cdot\chi_{|x|\le\frac{1}{N}}\|_{L^2}\\ &\le \|U_0-S_0\|_{L^2} + \frac{C_L-C_R}{2}\cdot\sqrt{\frac{2}{N}}.\end{aligned}$$

From $(U_0*\varphi_{1/N})' = (U_0-S_0)*(\varphi_{1/N})' + (S_0)'*\varphi_{1/N}$ and $(S_0)' = -(C_L - C_R)\cdot\delta_0$

$$\begin{aligned}\|(\frac{d}{dx}U_0^N)_+\|_{L^2} &\le \|(U_0-S_0)*(\varphi_{1/N})'\|_{L^2} + \|(S_0)'*\varphi_{1/N}\|_{L^2}\\ &\le \|U_0-S_0\|_{L^2}\cdot\|(\varphi_{1/N})'\|_{L^1} + (C_L-C_R)\cdot\|\varphi_{1/N}\|_{L^2}\\ &\le CN\Big(\|U_0-S_0\|_{L^2} + (C_L-C_R)\Big).\end{aligned}$$

In sum, we have

$$\begin{cases} \|U_0^N\|_{L^\infty(\mathbb{R})} \leq \|U_0\|_{L^\infty(\mathbb{R})}, \\ \|U_0^N - S_0\|_{L^2(\mathbb{R})} \leq \|U_0 - S_0\|_{L^2(\mathbb{R})} + C \cdot \sqrt{\frac{1}{N}} \leq \|U_0 - S_0\|_{L^2(\mathbb{R})} + C \text{ and} \\ \|(\frac{d}{dx}U_0^N)_+\|_{L^2(\mathbb{R})} \leq C \cdot N \cdot \Big(\|U_0 - S_0\|_{L^2(\mathbb{R})} + 1\Big). \end{cases}$$

Let $p = 1$ fixed. For each $N \geq 1$, we can find $\varepsilon_0(N) > 0$, D_1, D_2, D_3 satisfying all conclusions of Theorem 1.2 where D_1, D_2, D_3 are independent of N. Then we extract $\varepsilon_N \in (0, \varepsilon_0(N))$ such that $\varepsilon_N \searrow 0$.

The rest follows the standard theory for scalar conservation laws via vanishing viscosity method. Indeed, for each $N \geq 1$, we can construct the solution U^N of (1) for $\varepsilon = \varepsilon_0(N)$ with the initial data U_0^N. Then, Theorem 1.2 gives us a Lipshitz curve $X^N(\cdot)$ for each N. Then we take a limit to get a weak entropy solution U of (13) and a Lipshitz curve $X(\cdot)$ satisfying all the conclusion of this corollary. □

3.2 About Remark 2.2: from C^1 to BV_{loc}

We suppose that Theorem 1.2 is true by assuming $U_0 \in C^1$. Let $U_0 \in BV_{loc}$. We define $U_0^N := U_0 * \varphi_{1/N}$ for integer $N \geq 1$. Then for each N, we can apply Theorem 1.2 to U^N, which is the solution for U_0^N. As we did in the proof of Corollary 1.3, we have

$$\begin{cases} \|U_0^N\|_{L^\infty(\mathbb{R})} \leq \|U_0\|_{L^\infty(\mathbb{R})} \text{ and} \\ \|U_0^N - S_0\|_{L^2(\mathbb{R})} \leq \|U_0 - S_0\|_{L^2(\mathbb{R})} + C \cdot \sqrt{\frac{1}{N}} \leq \|U_0 - S_0\|_{L^2(\mathbb{R})} + C. \end{cases}$$

Moreover, thanks to the relation: $0 \leq (\frac{d}{dx}U_0^N)_+ = ((\frac{d}{dx}U_0)*\varphi_{1/N})_+ \leq (\frac{d}{dx}U_0)_+ * \varphi_{1/N}$, we have $\|(\frac{d}{dx}U_0^N)_+\|_{L^2(\mathbb{R})} \leq \|(\frac{d}{dx}U_0)_+\|_{L^2(\mathbb{R})}$. Finally, we can take a limit $N \to \infty$ to get the same conclusion to U, which is the solution for U_0.

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