Global weak solution to the viscous two-phase model with finite energy

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Abstract

In this paper, we prove the existence of global weak solutions to the compressible Navier-Stokes equations when the pressure law is in two variables. The method is based on the Lions argument and the Feireisl-Novotny-Petzeltova method. The main contribution of this paper is to develop a new argument for handling a nonlinear pressure law $P(\rho, n) = \rho^\gamma + n^\alpha$ where ρ, n satisfy the mass equations. This yields the strong convergence of the densities, and provides the existence of global solutions in time, for the compressible barotropic Navier-Stokes equations, with large data. The result holds in three space dimensions on condition that the adiabatic constants $\gamma > \frac{9}{5}$ and $\alpha \geq 1$. Our result is the first global existence theorem on the viscous compressible two-phase model with pressure law in two variables, for large initial data, in the multidimensional space.

keywords. two-phase model, compressible Navier-Stokes equations, global weak solutions.
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1 Introduction

We are particularly interested in the Dirichlet problem of a viscous two-phase model. The model is governed by the following compressible Navier-Stokes equations with a pressure law in two variables. Our paper is devoted to the proof of the existence of global weak solutions to the following system

$$\begin{cases} n_t + \mathrm{div}(nu) = 0, \\ \rho_t + \mathrm{div}(\rho u) = 0, \\ \big[(\rho+n)u\big]_t + \mathrm{div}\big[(\rho+n)u\otimes u\big] + \nabla P(n,\rho) = \mu\Delta u + (\mu+\lambda)\nabla \mathrm{div} u \quad \text{on } \Omega\times(0,\infty), \end{cases} \tag{1.1}$$

with the initial and boundary conditions

$$n(x,0) = n_0(x),\ \rho(x,0) = \rho_0(x),\ (\rho+n)u(x,0) = M_0(x) \ \text{ for } \ x\in\overline{\Omega}, \tag{1.2}$$

$$u|_{\partial\Omega} = 0 \quad \text{for } t\geq 0, \tag{1.3}$$

where $P(n,\rho) = \rho^\gamma + n^\alpha$ denotes the pressure for $\gamma > 1$ and $\alpha \geq 1$, u stands for the velocity of fluid, ρ and n are the densities of two phases, μ and λ are the viscosity coefficients. Here we assume that μ and λ are fixed constants, and

$$\mu > 0,\ \ 2\mu + 3\lambda \geq 0.$$

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The two-phase model was originally developed by Zuber and Findlay [35], Wallis [33], and Ishii [20, 21]. In the case $\alpha = 1$ in (1.1), this corresponds to the hydrodynamic equations in [6, 30]. It could be derived from the coupled system by the compressible Navier-Stokes equations and a kinetic equation, by using asymptotic analysis, see [6, 30]. In the case $\alpha = 2$ in (1.1), it is related to a similar but different model, i.e., a compressible Oldroyd-B type model with stress diffusion, see [1]. Compared to the classical compressible Navier-Stokes equations, here the pressure law $P(\rho, n) = \rho^\gamma + n^\alpha$ is in two variables. In this context, the existence of weak solutions to equations (1.1) remained open until now. We refer the reader to [2, 3, 4, 20, 21, 33, 35] for more physical background and discussion of numerical studies for such mathematical models.

Compared to incompressible flow, it is a tough problem to handle the issue of the vacuum in the study of the compressible flows. The first existence result on the compressible Navier-Stokes equations in one dimensional space was established by Kazhikhov and Shelukhin [23], where the initial density should be bounded away from zero due to the obstacle from the vacuum. This result has been extended by Hoff [15] and Serre [31] for the discontinuous initial data, and by Mellet-Vasseur [29] in the case of density dependent viscosity coefficient. For 2D or 3D case, the first global existence with the small initial data was proved by Matsumura and Nishida [26, 27, 28], and later by Hoff [16, 17, 18] for discontinuous initial data. Lions in [24] introduced the concept of renormalized solutions for the compressible Navier-Stokes equations which can control the possible oscillation of density. This removes the barrier of vacuum. Thus the global existence for $\gamma \geq \frac{9}{5}$ concerning large initial data was constructed by Lions [24] and then later improved by Jiang and Zhang [22] for spherically symmetric initial data for $\gamma > 1$ and by Feireisl-Novotný-Petzeltová [13] and Feireisl [14] for $\gamma > \frac{3}{2}$, and even to Navier-Stokes-Fourier system. It is crucial in [24, 22, 13] to obtain the higher integrability of the density due to the elliptic structure of the pressure $P = \rho^\gamma$. Relying on this structure, Lions deduced that the density ρ is uniformly bounded in $L^{\gamma+\frac{2\gamma}{3}-1}$. However, as $1 \leq \gamma \leq \frac{3}{2}$, the construction of the weak solutions for large data remains largely open, see [25]. The primary difficulty is the possible concentration of the convective term for this case. Very recently, Hu [19] studied the concentration phenomenon of the kinetic energy, $\rho|u|^2$, associated to isentropic compressible Navier-Stokes equations for $1 \leq \gamma \leq \frac{3}{2}$. Here, we have to mention the work of Bresch-Jabin [5], where the authors proved global existence of appropriate weak solutions for the compressible Navier-Stokes equations for more general pressure law in one variable.

The problem becomes even more challenging when pressure law is in two variables as follows

$$P(\rho, n) = \rho^\gamma + n^\alpha.$$

To the best of our knowledge, the only available results on global existence of the solutions to system (1.1) with large initial data concern the one-dimensional case, see [9, 11]. See also [8, 10, 34] for more relevant works. In [1], Barrett-Lu-Suli established the existence of weak solutions to a compressible Oldroyd-B type model with the pressure law $P = \rho^\gamma + n^2$, but the equation of n has a term $\epsilon\Delta n$. This provides higher regularity of n due to the parabolic structure. Our goal of this current paper is to establish the existence of global weak solution to (1.1) with large data. The main obstacles are from the nonlinear pressure law in two variables. One is the concentration phenomenon of convective term related to n, that is $nu \otimes u$ when $1 \leq \alpha \leq \frac{3}{2}$. This is reminiscent to improving the range of γ for the weak solution to the compressible Navier-Stokes equations, which is a long standing open problem. The other is oscillation of the pressure term $\rho^\gamma + n^\alpha$ because we have to handle the cross products like $\rho^\gamma n$ and $n^\alpha \rho$ for the compactness. To control these possible oscillation and concentration, a novel approach will be developed in the paper. Our approach relies on a commutator lemma. By the following energy inequality (1.4), we see that

$$u \in L^2(0,T;H^1(\Omega)).$$

To apply a commutator lemma for $(1.1)_1$ and$(1.1)_2$, the densities ρ and n are needed to be bounded in $L^2(0,T;L^2(\Omega))$. This requires $\gamma+\frac{2\gamma}{3}-1\geq 2$ because ρ is bounded in $L^{\gamma+\frac{2\gamma}{3}-1}(0,T;\Omega)$, that is, $\gamma\geq\frac{9}{5}$. Due to one more technical reason for (5.36), we need $\gamma>\frac{9}{5}$. The current work is the first result on global solutions to the viscous two-phase model (1.1) with large initial data in high dimensions. Meanwhile, it is an extension for the works of [24, 13] to a nonlinear pressure law $P(\rho,n)=\rho^\gamma+n^\alpha$ in two variables. The new argument developed in this paper allows us to handle the more general pressure law, and helps us to study the other challenging models in the complex fluids. This provides new insight for improving the range of γ for the weak solution to the compressible Navier-Stokes equations.

For any smooth solution of system (1.1), the following energy inequality holds for any time $0\leq t\leq T$:

$$\frac{d}{dt}\int_\Omega\Big[\frac{(\rho+n)|u|^2}{2}+G_\alpha(n)+\frac{1}{\gamma-1}\rho^\gamma\Big]\,dx+\int_\Omega\Big[\mu|\nabla u|^2+(\mu+\lambda)|\text{div}u|^2\Big]\,dx\leq 0, \tag{1.4}$$

where

$$G_\alpha(n)=\begin{cases} n\ln n-n+1, & \text{for}\quad \alpha=1,\\ \frac{n^\alpha}{\alpha-1}, & \text{for}\quad \alpha>1.\end{cases}$$

As usual, we assume that

$$\int_\Omega\Big[\frac{(\rho_0+n_0)|u_0|^2}{2}+G_\alpha(n_0)+\frac{1}{\gamma-1}\rho_0^\gamma\Big]\,dx<\infty$$

in the whole paper. Thus, we set the following restriction on the initial data

$$\inf_{x\in\Omega}\rho_0\geq 0,\quad \inf_{x\in\Omega}n_0\geq 0,\quad \rho_0\in L^\gamma(\Omega),\quad G_\alpha(n_0)\in L^1(\Omega), \tag{1.5}$$

and

$$\frac{M_0}{\sqrt{\rho_0+n_0}}\in L^2(\Omega)\ \text{ where }\ \frac{M_0}{\sqrt{\rho_0+n_0}}=0\ \text{ on }\ \{x\in\Omega|\rho_0(x)+n_0(x)=0\}. \tag{1.6}$$

The energy inequality allows us to give the following definition of weak solutions in the energy space.

Definition 1.1 *We call $(\rho,n,u):\Omega\times(0,\infty)\to\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^3$ a global weak solution of (1.1)-(1.3) if for any $0<T<+\infty$,*

- $\rho\in L^\infty\big(0,T;L^\gamma(\Omega)\big),\ G_\alpha(n)\in L^\infty\big(0,T;L^1(\Omega)\big),\ \sqrt{\rho+n}u\in L^\infty\big(0,T;L^2(\Omega)\big), u\in L^2\big(0,T;H^1_0(\Omega)\big)$,
- (ρ,n,u) solves the system (1.1) in $\mathcal{D}'(Q_T)$, where $\mathrm{Q_T}=\Omega\times(0,T)$,
- $\big(\rho,n,(\rho+n)u\big)(x,0)=\big(\rho_0(x),n_0(x),M_0(x)\big)$, for a.e. $x\in\Omega$,
- *The energy inequality* (1.4) *holds in* $\mathcal{D}'\big(\mathbb{R}^3\times(0,T)\big)$,
- $(1.1)_1$ and$(1.1)_2$ hold in $\mathcal{D}'\big(\mathbb{R}^3\times(0,T)\big)$ provided ρ,n,u are prolonged to be zero on $\mathbb{R}^3/\Omega$,

- *the equation (1.1)$_1$ and (1.1)$_2$ are satisfied in the sense of renormalized solutions, i.e.,*

$$\partial_t b(f) + \mathrm{div}\big(b(f)u\big) + [b'(f)f - b(f)]\mathrm{div}u = 0$$

holds in $\mathcal{D}'(Q_T)$, for any $b \in C^1(\mathbb{R})$ such that $b'(z) \equiv 0$ for all $z \in \mathbb{R}$ large enough, where $f = \rho, n$.

The weak solution in the sense of above definition is called a renormalized solution. The renormalized solution was first introduced by DiPerna-Lions [7] for the kinetic equation. It has been used in the compressible Navier-Stokes equations by Lions [24].

The main result of this paper reads as follows

Theorem 1.2 *Let the initial data be under the conditions* (1.5)-(1.6)*, and*

$$\begin{cases} \mathrm{n}_0 \le \mathrm{c}_0\rho_0 \text{ on } \Omega \quad \text{if } \alpha \in [1, \gamma+\tau), \\ \frac{1}{\mathrm{c}_0}\rho_0 \le \mathrm{n}_0 \le \mathrm{c}_0\rho_0 \text{ on } \Omega \quad \text{if } \alpha \in [\gamma+\tau, \infty) \end{cases}$$

where c_0 is a positive constant and $\tau = \min\left\{1, \frac{2\gamma}{3} - 1, \frac{\gamma}{3}\right\}$, then there exists a global weak solution (ρ, n, u) to (1.1)-(1.3) for any $\gamma > \frac{9}{5}$ and $\alpha \ge 1$.

The proof of Theorem 1.2 will be done by means of an error estimate (Theorem 2.2) and the following approximation system

$$\begin{cases} n_t + \mathrm{div}(nu) = \epsilon\Delta n, \\ \rho_t + \mathrm{div}(\rho u) = \epsilon\Delta\rho, \\ \big[(\rho+n)u\big]_t + \mathrm{div}\big[(\rho+n)u\otimes u\big] + \nabla(n^\alpha + \rho^\gamma) + \delta\nabla(\rho+n)^\beta + \epsilon\nabla u\cdot\nabla(\rho+n) \\ = \mu\Delta u + (\mu+\lambda)\nabla\mathrm{div}u \end{cases} \tag{1.7}$$

with suitable regularized initial and boundary conditions, where $\epsilon > 0$ and $\delta > 0$ are small, and $\beta > 0$ large enough. This approximation was motivated by the work of [13]. It can be solved by the Galerkin argument in the similar way as [13]. To recover a weak solution to (1.1), we have to develop new argument to handle the difficulty resulting from the pressure law in two variables. Our main tool is Theorem 2.2, which allows us to control the error of two variables in some sense. This yields the compactness of the sequence of solutions.

Our novelty is stated as follows.

In Section 2, we develop a new tool to handle the pressure law in two variables. In particular, for any $\rho_K \ge 0$ and $n_K \ge 0$ which are the solutions to the following equations respectively,

$$(\rho_K)_t + \mathrm{div}(\rho_K u_K) = \nu_K\Delta\rho_K, \qquad \rho_K|_{t=0} = \rho_0, \qquad \nu_K\frac{\partial\rho_K}{\partial\nu}|_{\partial\Omega} = 0,$$

and

$$(n_K)_t + \mathrm{div}(n_K u_K) = \nu_K\Delta n_K, \qquad n_K|_{t=0} = n_0, \qquad \nu_K\frac{\partial n_K}{\partial\nu}|_{\partial\Omega} = 0$$

with $\nu_K \ge 0$, and $\nu_K \to 0$ as $K \to +\infty$, then for any $s > 1$,

$$\lim_{K\to+\infty}\int_0^T\int_\Omega \rho_K|a_K - a|^s\,dx\,dt = 0, \tag{1.8}$$

where $a_K = \frac{n_K}{\rho_K}$ if $\rho_K \ne 0$, $a = \frac{n}{\rho}$ if $\rho \ne 0$, $0 \le a_K, a \le \mathcal{C}$ for some positive constant $\mathcal{C}$ independent of K, $a_K\rho_K = n_K$, $a\rho = n$, and (n, ρ) is the limit of (n_K, ρ_K) in a suitable weak topology. This helps us to study the pressure law in two variables. We are able to pass to the limits as $\epsilon \to 0$ and $\delta \to 0$ relying on (1.8) because it is true even for $\nu_K = 0$.

In section 3, we solve the approximation system by the Galerkin argument. For the Galerkin level, we point out that $n_k - c_0\rho_k$ is a solution of the following parabolic equation

$$\begin{cases} (n_k - c_0\rho_k)_t + \operatorname{div}\big[(n_k - c_0\rho_k)u_k\big] = \epsilon\Delta(n_k - c_0\rho_k), \\ (n_k - c_0\rho_k)|_{t=0} = n_{0,\delta} - c_0\rho_{0,\delta}, \\ \nabla(n_k - c_0\rho_k)\cdot\nu|_{\partial\Omega} = 0, \end{cases}$$

where $n_{0,\delta}$ and $\rho_{0,\delta}$ are given by (3.4). The maximum principle gives us that n_k could be bounded by ρ_k. This implies that n_k at least has the same integrability as ρ_k. It allows us to avoid concentration phenomenon of convective term $nu \otimes u$ for each level approximation, even if $1 \le \alpha \le \frac{9}{5}$.

In section 4, we study the limits as ϵ goes to zero. The focus of this section is to prove

$$\overline{n^\alpha + \rho^\gamma + \delta(\rho+n)^\beta} = n^\alpha + \rho^\gamma + \delta(\rho+n)^\beta.$$

Here and always, $\overline{f}$ is the weak limit of f_ϵ. One of the key step is to control the product of $n_\epsilon^\alpha + \rho_\epsilon^\gamma$ and $n_\epsilon + \rho_\epsilon$, we rewrite them as follows

$$\begin{aligned} n_\epsilon^\alpha + \rho_\epsilon^\gamma =& \rho_\epsilon^\gamma + a_\epsilon^\alpha\rho_\epsilon^\alpha = \rho_\epsilon^\gamma + a^\alpha\rho_\epsilon^\alpha + (a_\epsilon^\alpha - a^\alpha)\rho_\epsilon^\alpha, \\ n_\epsilon + \rho_\epsilon =& \rho_\epsilon + a_\epsilon\rho_\epsilon = \rho_\epsilon + a\rho_\epsilon + (a_\epsilon - a)\rho_\epsilon, \end{aligned}$$

where $a_\epsilon = \frac{n_\epsilon}{\rho_\epsilon}$ if $\rho_\epsilon \neq 0$, $a = \frac{n}{\rho}$ if $\rho \neq 0$, $0 \le a_\epsilon, a \le c_0$, and $a_\epsilon\rho_\epsilon = n_\epsilon, a\rho = n$, (ρ, n) is the limit of $(\rho_\epsilon, n_\epsilon)$ in a suitable weak topology. Here we want to show

$$(a_\epsilon^\alpha - a^\alpha)\rho_\epsilon^\alpha(n_\epsilon + \rho_\epsilon) \to 0 \quad \text{and} \quad (a_\epsilon - a)\rho_\epsilon(\rho_\epsilon^\gamma + a^\alpha\rho_\epsilon^\alpha) \to 0$$

in some sense as $\epsilon \to 0$. This can be done because ρ_ϵ and n_ϵ are bounded uniformly for ϵ in $L^{\beta+1}(Q_T)$ where $\beta > \max\{\alpha, \gamma, 4\}$, and

$$\lim_{\epsilon\to 0^+} \int_0^T \int_\Omega \rho_\epsilon |a_\epsilon - a|^2 \, dx\, dt = 0.$$

It is an application of (1.8) with $\nu_K = \epsilon$.

In section 5, we recover the weak solution by letting δ goes to zero. The highlight of this section is to show

$$\overline{n^\alpha + \rho^\gamma} = n^\alpha + \rho^\gamma,$$

which is similar to the limits in term of ϵ. However, we loose the regularity resulting from $\delta(\rho+n)^\beta$. we have to use a cut-off function which was used in [13] to show the strong convergence of ρ_δ and n_δ. With (1.8) for $\nu_K = 0$, the compactness could be done. At this level of approximation, we require $\gamma > \frac{9}{5}$ such that ρ_δ is bounded in $L^{\gamma+\theta}(Q_T)$ with $\gamma + \theta > 2$ for some θ satisfying $\theta < \frac{\gamma}{3}$ and $\theta \le \min\left\{1, \frac{2\gamma}{3} - 1\right\}$.

2 An error estimate

The main objective of this section is to prove the following Theorem 2.2. The key estimate (2.17) in Theorem 2.2 is crucial to obtain the weak stability of solutions to (1.1). We begin with the following Lemma 2.1.

Lemma 2.1 *Let* $\{(g_K, h_K)\}_{K=1}^{\infty}$ *be a sequence with the following properties*

$$(g_K, h_K) \to (g, h) \text{ weakly in } L^p(Q_T) \text{ as } K \to \infty, \tag{2.9}$$

for any given $p > 1$, $g_K, h_K \geq 0$, *and*

$$\lim_{K\to+\infty} \int_0^T \int_\Omega a_K h_K \, dx\, dt \leq \int_0^T \int_\Omega h a_h \, dx\, dt, \tag{2.10}$$

where $a_K = \frac{h_K}{g_K}$ *if* $g_K \neq 0$, $a_h = \frac{h}{g}$ *if* $g \neq 0$, $0 \leq a_K, a_h \leq \mathcal{C}$ *for some positive constant* $\mathcal{C}$ *independent of* K, *and* $a_K g_K = h_K$, $a_h g = h$, *then*

$$\lim_{K\to+\infty} \int_0^T \int_\Omega g_K |a_K - a_h|^2 \, dx\, dt = 0. \tag{2.11}$$

In particular,

$$\lim_{K\to+\infty} \int_0^T \int_\Omega g_K |a_K - a_h|^s \, dx\, dt = 0, \tag{2.12}$$

for any $s > 1$.

Proof. Note that

$$\int_0^T \int_\Omega g_K |a_K - a_h|^2 \, dx\, dt = \int_0^T \int_\Omega a_K h_K \, dx\, dt - 2 \int_0^T \int_\Omega h_K a_h \, dx\, dt + \int_0^T \int_\Omega g_K a_h^2 \, dx\, dt,$$

one obtains

$$\lim_{K\to+\infty} \int_0^T \int_\Omega g_K |a_K - a_h|^2 \, dx\, dt \leq \int_0^T \int_\Omega h a_h \, dx\, dt - 2 \int_0^T \int_\Omega h a_h \, dx\, dt + \int_0^T \int_\Omega h a_h \, dx\, dt$$
$$= 0,$$

where we have used $a_K g_K = h_K$, $a_h g = h$, (2.10) and the weak compactness of g_K and h_K in (2.9). This deduces (2.11).

By the Hölder inequality and (2.11), (2.12) follows for $s \in (1, 2)$. If $s \in [2, \infty)$, note that $(a_K - a_h)$ is bounded in $L^\infty\big(\Omega \times (0, T)\big)$. This allows us to have (2.12). □

Theorem 2.2 *Let* $\nu_K \to 0$ *as* $K \to +\infty$, *and* $\nu_K \geq 0$. *If* $\rho_K \geq 0$ *and* $n_K \geq 0$ *are the solutions to*

$$(\rho_K)_t + \operatorname{div}(\rho_K u_K) = \nu_K \Delta \rho_K, \quad \rho_K|_{t=0} = \rho_0, \quad \nu_K \frac{\partial \rho_K}{\partial \nu}|_{\partial\Omega} = 0, \tag{2.13}$$

and

$$(n_K)_t + \operatorname{div}(n_K u_K) = \nu_K \Delta n_K, \quad n_K|_{t=0} = n_0, \quad \nu_K \frac{\partial n_K}{\partial \nu}|_{\partial\Omega} = 0, \tag{2.14}$$

respectively, with $C_0 > 0$ *independent of* K *such that*

- $$n_K(x, t) \leq C_0 \rho_K(x, t) \quad \textit{for any } (x, t), \tag{2.15}$$

- $$\|\rho_K\|_{L^\infty(0,T;L^2(\Omega))} \leq C_0, \ \sqrt{\nu_K}\|\nabla \rho_K\|_{L^2(0,T;L^2(\Omega))} \leq C_0, \ \sqrt{\nu_K}\|\nabla n_K\|_{L^2(0,T;L^2(\Omega))} \leq C_0,$$

- $$\|u_K\|_{L^2(0,T;H_0^1(\Omega))} \le C_0,$$

- *for any $K>0$ and any $t>0$:*
$$\int_\Omega \frac{n_K^2}{\rho_K}\,dx \le \int_\Omega \frac{n_0^2}{\rho_0}\,dx. \tag{2.16}$$

Then, up to a subsequence, we have

$$n_K \to n, \quad \rho_K \to \rho \quad \text{weakly in } L^\infty(0,T;L^2(\Omega)),$$

$$u_K \to u \text{ weakly in } L^2(0,T;H_0^1(\Omega));$$

and for any $s>1$,

$$\lim_{K\to+\infty}\int_0^T\int_\Omega \rho_K|a_K-a|^s\,dx\,dt = 0, \tag{2.17}$$

where $a_K = \frac{n_K}{\rho_K}$ if $\rho_K \ne 0$, $a=\frac{n}{\rho}$ if $\rho\ne 0$, $0\le a_K, a\le C_0$ for some positive constant C_0, and $a_K\rho_K = n_K$, $a\rho = n$.

Remark 2.3 *If $\nu_K>0$, u_K is smooth enough and ρ_K is bounded by below, then (2.16) is verified. In fact, choosing $\varphi(\rho_K,n_K)=\frac{n_K^2}{\rho_K}$, one obtains*

$$\begin{aligned}
&\frac{\partial\varphi(\rho_K,n_K)}{\partial t} + \operatorname{div}(\varphi u_K) + [\frac{\partial\varphi}{\partial\rho_K}\rho_K + \frac{\partial\varphi}{\partial n_K}n_K - \varphi] \\
&+\nu_K(\frac{\partial^2\varphi}{\partial\rho_K^2}|\nabla\rho_K|^2 + \frac{\partial^2\varphi}{\partial\rho_K^2}|\nabla n_K|^2 + 2\frac{\partial^2\varphi}{\partial\rho_K\partial n_K}\nabla\rho_K\cdot\nabla n_K) - \nu_K\Delta\varphi = 0.
\end{aligned}$$

Note that
$$\frac{\partial\varphi}{\partial\rho_K}\rho_K + \frac{\partial\varphi}{\partial n_K}n_K - \varphi = 0$$

and φ is convex, thus we have
$$\frac{d}{dt}\int_\Omega \varphi(\rho_K,n_K)\,dx \le 0.$$

We will rely on the following lemma to prove our main result of this section.

Lemma 2.4 *Let $\beta:\mathbb{R}^N\to\mathbb{R}$ be a C^1 function with $|\nabla\beta(X)|\in L^\infty(\mathbb{R}^N)$, and $R\in\left(L^2(0,T;L^2(\Omega))\right)^N$, $u\in L^2(0,T;H_0^1(\Omega))$ satisfy*

$$\frac{\partial}{\partial_t}R + \operatorname{div}(u\otimes R) = 0, \;\; R|_{t=0} = R_0(x) \tag{2.18}$$

in the distribution sense. Then we have

$$(\beta(R))_t + \operatorname{div}(\beta(R)u) + [\nabla\beta(R)\cdot R - \beta(R)]\operatorname{div}u = 0 \tag{2.19}$$

in the distribution sense. Moreover, if $R\in L^\infty(0,T;L^\gamma(\Omega))$ for $\gamma>1$, then

$$R\in C([0,T];L^1(\Omega)),$$

and so
$$\int_\Omega \beta(R)\,dx(t) = \int_\Omega \beta(R_0)\,dx - \int_0^t\int_\Omega [\nabla\beta(R)\cdot R - \beta(R)]\operatorname{div}u\,dx\,dt.$$

Remark 2.5 *If $N = 1$, it is the result of Feireisl [14].*

To prove Lemma 2.4, we shall rely on the following lemma which was called the commutator lemma.

Lemma 2.6 *[24]. There exists $C > 0$ such that for any $\rho \in L^2(\mathbb{R}^d)$ and $u \in H^1(\mathbb{R}^d)$,*

$$\|\eta_\sigma * \mathrm{div}(\rho u) - \mathrm{div}(u(\rho * \eta_\sigma))\|_{L^1(\mathbb{R}^d)} \leq C\|u\|_{H^1(R^d)}\|\rho\|_{L^2(\mathbb{R}^d)}.$$

In addition,

$$\eta_\sigma * \mathrm{div}(\rho u) - \mathrm{div}(u(\rho * \eta_\sigma)) \to 0 \text{ in } L^1(\mathbb{R}^d), \text{ as } \sigma \to 0,$$

where $\eta_\sigma = \frac{1}{\sigma^d}\eta(\frac{x}{\sigma})$, and $\eta(x) \geq 0$ is a smooth even function compactly supported in the space ball of radius 1, and with integral equal to 1.

Proof of Lemma 2.4. Here, we devote the proof of Lemma 2.4. The first two steps are similar to the work of [13].

Step 1: Proof of (2.19).

Applying the regularizing operator $f \longmapsto f * \eta_\sigma$ to both sides of (2.18), we obtain

$$(R_\sigma)_t + \mathrm{div}(u \otimes R_\sigma) = S^\sigma, \tag{2.20}$$

almost everywhere on $O \subset \bar{O} \subset (0,T) \times \Omega$ provided $\sigma > 0$ small enough, where

$$S^\sigma = \mathrm{div}(u \otimes R_\sigma) - (\mathrm{div(u \otimes R)})_\sigma,$$

and $f_\sigma(x) = f * \eta_\sigma$. Thanks to Lemma 2.6, we conclude that

$$S^\sigma \to 0 \quad \text{in } L^1(O) \quad \text{as } \sigma \to 0.$$

Equation (2.20) multiplied by $\nabla\beta(R)$, where β is a C^1 function, gives us

$$[\beta(R_\sigma)]_t + \mathrm{div}[\beta(\mathrm{R}_\sigma)\mathrm{u}] + [\nabla\beta(\mathrm{R}_\sigma) \cdot \mathrm{R}_\sigma - \beta(\mathrm{R}_\sigma)]\mathrm{divu} = \nabla\beta(\mathrm{R}_\sigma) \cdot \mathrm{S}^\sigma.$$

This yields (2.19) by letting $\sigma \to 0$.

Step 2: Continuity of R in the strong topology.

By (2.19), we have

$$\frac{\partial}{\partial_t}T_K(R) + \mathrm{div}(T_K(R)u) + (\nabla T_K(R) \cdot R - T_K(R))\mathrm{div}u = 0 \quad \text{in } \mathfrak{D}'((0,T) \times \Omega), \tag{2.21}$$

where $T_k(R)$ is a cutoff function verifying

$$T_K(R) = \widetilde{T_K}(|R|), \quad \text{and } \widetilde{T_K}(z) = KT(\frac{z}{K}),$$

and $T(z) = z$ for any $z \in [0,1]$, and it is concave on $[0,\infty)$, $T(z) = 2$ for any$z \geq 3$, and T is a C^∞ function. We conclude that $T_K(R)$ is bounded in $C([0,T]; L^2_{weak}(\Omega))$ due to $R \in L^\infty(0,T; L^2(\Omega))$. Thanks to (2.21), we have

$$T_K(R) \quad \text{belong to } C([0,T]; L^\gamma_{\mathrm{weak}}(\Omega)) \tag{2.22}$$

for any $K \geq 1$.

Applying the same argument as in step 1 for (2.21), we have

$$\frac{\partial}{\partial_t}[T_K(R)]_\sigma + \mathrm{div}([T_K(R)]_\sigma u) = A_K^\sigma \quad \text{a.e. on } (0,T)\times U, \tag{2.23}$$

where U is a compact subset of Ω. Thanks to Lemma 2.6, A_K^σ is bounded in $L^2(0,T;L^1(\Omega))$. Meanwhile, using $2[T_K(R)]_\sigma$ to multiply (2.23), we have

$$\frac{\partial}{\partial_t}([T_K(R)]_\sigma)^2 + \mathrm{div}(([T_K(R)]_\sigma)^2 u) + ([T_K(R)]_\sigma)^2 \mathrm{div} u = 2[T_K(R)]_\sigma A_K^\sigma.$$

Thus, for any test function $\eta \in \mathcal{D}(\Omega)$, the family of functions with respect to σ for fixed K

$$t \longmapsto \int_\Omega ([T_K(R)]_\sigma)^2(t,x)\eta(x)\,dx,\ \sigma > 0 \ \text{ is precompact in } C[0,T].$$

Note that $[T_K(R)]_\sigma \to [T_K(R)]$ in $L^2(\Omega)$ for any $t \in [0,T]$ as $\sigma \to 0$, we obtain

$$t \longmapsto \int_\Omega ([T_K(R)])^2(t,x)\eta(x)\,dx \text{ is in } C[0,T]$$

for any fixed $\eta(x)$. Thus, $T_K(R) \in C([0,T];L^2(\Omega))$ for any fixed $K \geq 1$. It allows us to have

$$R \in (C([0,T];L^1(\Omega)))^N,$$

thanks to (2.22).

Step 3: Final inequality.

Taking integration on (2.19) with respect to t, we have

$$\int_\Omega \beta(R(t))\,dx = \int_\Omega \beta(R(s))\,dx - \int_s^t \int_\Omega [\nabla\beta(R)\cdot R - \beta(R)]\mathrm{div} u\,dx\,dt,$$

where $0 < s < t < T$. Thanks to

$$R \in (C([0,T];L^1(\Omega)))^N,$$

we are able to let $s \to 0$, thus

$$\int_\Omega \beta(R(t))\,dx = \int_\Omega \beta(R_0)\,dx - \int_0^t \int_\Omega [\nabla\beta(R)\cdot R - \beta(R)]\mathrm{div} u\,dx\,dt$$

for any $0 \leq t \leq T$. □

Now, we are going to prove our main result in this section-Theorem 2.2.

Proof of Theorem 2.2. Up to a subsequence,

$$\rho_K \to \rho,\ n_K \to n \ \text{ weakly in } L^\infty(0,T;L^2(\Omega)), \quad u_K \to u \text{ weakly in } L^2(0,T;H_0^1(\Omega)), \tag{2.24}$$

as $K \to \infty$. Passing to the limit as $K \to \infty$ in (2.13) and (2.14) respectively, we have

$$\rho_t + \mathrm{div}(\rho u) = 0, \qquad \rho|_{t=0} = \rho_0,$$

and

$$n_t + \mathrm{div}(nu) = 0, \qquad n|_{t=0} = n_0,$$

and

$$n \leq C_0 \rho \quad \text{a.e. on } \Omega \times (0,T).$$

Using Lemma 2.4 with $R = (n, \rho)$ and $u = u$, $\beta_\sigma(n,\rho) = \frac{n^2}{\rho+\sigma}$, note that $\nabla\beta_\sigma(R)\cdot R - \beta_\sigma(R) = \sigma\frac{n^2}{(\rho+\sigma)^2}$, thus we have, thanks to Lemma 2.4:

$$\int_\Omega \frac{n(t,x)^2}{\rho(t,x)+\sigma}\,dx = \int_\Omega \frac{n_0^2}{\rho_0+\sigma}\,dx - \sigma\int_0^t\int_\Omega \frac{n^2}{(\rho+\sigma)^2}\mathrm{div}u\,dx\,dt,$$

for almost everywhere $t \in [0,T]$.
Note that

$$\frac{n^2}{\rho+\sigma} \leq \frac{n^2}{\rho} \quad \text{and} \quad \frac{n_0^2}{\rho_0+\sigma} \leq \frac{n_0^2}{\rho_0},$$

by the dominated convergence theorem, we obtain the following equality by letting σ goes to zero,

$$\int_\Omega \frac{n(t,x)^2}{\rho(t,x)}\,dx = \int_\Omega \frac{n_0^2}{\rho_0}\,dx \tag{2.25}$$

for almost everywhere $t \in [0,T]$.

By (2.16) and (2.25), one obtains that

$$\int_\Omega \frac{n_K(t,x)^2}{\rho_K(t,x)}\,dx = \int_\Omega n_K a_K\,dx \leq \int_\Omega \frac{n(t,x)^2}{\rho(t,x)}\,dx = \int_\Omega na\,dx \tag{2.26}$$

for almost everywhere $t \in [0,T]$.

Thanks to (2.24), setting $(n_K, \rho_K) = (h_K, g_K)$ for Lemma 2.1, one obtains (2.17) for any $s > 1$.
□

3 Faedo-Galerkin approach

In this section, we construct a global weak solution (ρ, n, u) to the following approximation (3.1)-(3.3) with a finite energy. In addition, a particular connection between ρ and n is given in Lemma 3.1. It will be crucial in this paper.

Motivated by the work of [13], we propose the following approximation system

$$\begin{cases} n_t + \mathrm{div}(nu) = \epsilon\Delta n, \\ \rho_t + \mathrm{div}(\rho u) = \epsilon\Delta\rho, \\ \big[(\rho+n)u\big]_t + \mathrm{div}\big[(\rho+n)u\otimes u\big] + \nabla(n^\alpha + \rho^\gamma) + \delta\nabla(\rho+n)^\beta + \epsilon\nabla u\cdot\nabla(\rho+n) \\ = \mu\Delta u + (\mu+\lambda)\nabla\mathrm{div}u \end{cases} \tag{3.1}$$

on $\Omega \times (0,\infty)$, with initial and boundary condition

$$\big(\rho, n, (\rho+n)u\big)|_{t=0} = (\rho_{0,\delta}, n_{0,\delta}, M_{0,\delta}) \text{ on } \overline{\Omega}, \tag{3.2}$$

$$(\frac{\partial\rho}{\partial\nu}, \frac{\partial n}{\partial\nu}, u)|_{\partial\Omega} = 0, \tag{3.3}$$

where $\epsilon, \delta > 0$, $\beta > \max\{\alpha, \gamma\}$, $M_{0,\delta} = (\rho_{0,\delta} + n_{0,\delta})u_{0,\delta}$ and $n_{0,\delta}, \rho_{0,\delta} \in C^3(\overline{\Omega})$, $u_{0,\delta} \in C_0^3(\Omega)$ satisfying

$$\begin{cases} 0 < \delta \le \rho_{0,\delta} \le \delta^{-\frac{1}{2\beta}}, \quad 0 < \delta \le n_{0,\delta} \le c_0\rho_{0,\delta}, \quad (\frac{\partial n_{0,\delta}}{\partial\nu}, \frac{\partial\rho_{0,\delta}}{\partial\nu})|_{\partial\Omega} = 0, \\ \lim\limits_{\delta\to 0} \big(\|\rho_{0,\delta} - \rho_0\|_{L^\gamma(\Omega)} + \|n_{0,\delta} - n_0\|_{L^\gamma(\Omega)} + \|n_{0,\delta} - n_0\|_{L^\alpha(\Omega)}\big) = 0, \\ u_{0\delta} = \frac{\varphi_\delta}{\sqrt{\rho_{0,\delta}+n_{0,\delta}}}\eta_\delta * (\frac{M_0}{\sqrt{\rho_0+n_0}}), \\ \sqrt{\rho_{0,\delta} + n_{0,\delta}}u_{0,\delta} \to \frac{M_0}{\sqrt{\rho_0+n_0}} \quad \text{in } L^2(\Omega) \text{ as } \delta \to 0, \\ m_{0,\delta} \to M_0 \quad \text{in } L^1(\Omega) \text{ as } \delta \to 0, \end{cases} \tag{3.4}$$

where $\delta \in (0, 1)$, η is the standard mollifier, $\varphi_\delta \in C_0^\infty(\Omega)$, $0 \le \varphi_\delta \le 1$ on $\overline{\Omega}$ and $\varphi_\delta \equiv 1$ on $\{x \in \Omega | \text{dist}(x, \partial\Omega) > \delta\}$.

We are able to use Faedo-Galerkin approach to construct a global weak solution to (3.1), (3.2) and (3.3). To begin with, we consider a sequence of finite dimensional spaces

$$X_k = [span\{\psi_j\}_{j=1}^k]^3, \quad k \in \{1, 2, 3, \cdots\},$$

where $\{\psi_i\}_{i=1}^\infty$ is the set of the eigenfunctions of the Laplacian:

$$\begin{cases} -\Delta\psi_i = \lambda_i\psi_i \quad \text{on } \Omega, \\ \psi_i|_{\partial\Omega} = 0. \end{cases}$$

For any given $\epsilon, \delta > 0$, we shall look for the approximate solution $u_k \in C([0, T]; X_k)$ (for any fixed $T > 0$) given by the following form:

$$\begin{aligned} &\int_\Omega (\rho_k + n_k)u_k(t) \cdot \psi\, dx - \int_\Omega m_{0,\delta} \cdot \psi\, dx = \int_0^t \int_\Omega \big[\mu\Delta u_k + (\mu + \lambda)\nabla \text{div} u_k\big] \cdot \psi\, dx\, ds \\ &- \int_0^t \int_\Omega \Big[\text{div}\big[(\rho_k + n_k)u_k \otimes u_k\big] + \nabla(n_k^\alpha + \rho_k^\gamma) + \delta\nabla(\rho_k + n_k)^\beta + \epsilon\nabla u_k \cdot \nabla(\rho_k + n_k)\Big] \cdot \psi\, dx\, ds \end{aligned} \tag{3.5}$$

for $t \in [0, T]$ and $\psi \in X_k$, where $\rho_k = \rho_k(u_k)$ and $n_k = n_k(u_k)$ satisfying

$$\begin{cases} \partial_t n_k + \text{div}(n_k u_k) = \epsilon\Delta n_k, \\ \partial_t\rho_k + \text{div}(\rho_k u_k) = \epsilon\Delta\rho_k, \\ n_k|_{t=0} = n_{0,\delta}, \quad \rho_k|_{t=0} = \rho_{0,\delta}, \\ (\frac{\partial\rho_k}{\partial\nu}, \frac{\partial n_k}{\partial\nu})|_{\partial\Omega} = 0. \end{cases} \tag{3.6}$$

From Remark 2.3, we have

$$\int_\Omega \frac{n_k^2}{\rho_k}\, dx \le \int_\Omega \frac{n_0^2}{\rho_0}\, dx \quad \text{for every } t > 0.$$

Using the convexity of $\varphi(\rho, n) = \frac{n^2}{\rho}$ and letting $k \to \infty$, we have

$$\int_\Omega \frac{n^2}{\rho}\, dx \le \int_\Omega \frac{n_0^2}{\rho_0}\, dx, \quad \text{for every } t > 0. \tag{3.7}$$

This holds for the solutions of the each level approximation.

Due to Lemmas 2.1 and 2.2 in [13], the problem (3.5) can be solved on a short time interval $[0, T_k]$ for $T_k \leq T$ by a standard fixed point theorem on the Banach space $C([0, T_k]; X_k)$. To show that $T_k = T$, we need the uniform estimates resulting from the following energy equality

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \Big[\frac{(\rho_k+n_k)|u_k|^2}{2} + G_\alpha(n_k) + \frac{1}{\gamma-1}\rho_k^\gamma + \frac{\delta}{\beta-1}(\rho_k+n_k)^\beta\Big]\,dx \\
&+ \int_\Omega \Big[\mu|\nabla u_k|^2 + (\mu+\lambda)|\mathrm{div} u_k|^2\Big]\,dx \qquad (3.8)\\
&+ \int_\Omega \Big[\epsilon\alpha n_k^{\alpha-2}|\nabla n_k|^2 + \epsilon\gamma\rho_k^{\gamma-2}|\nabla\rho_k|^2 + \epsilon\beta\delta(\rho_k+n_k)^{\beta-2}|\nabla(\rho_k+n_k)|^2\Big]\,dx = 0, \ \text{ on } (0, T_k),
\end{aligned}$$

where

$$G_\alpha(n_k) = \begin{cases} n_k \ln n_k - n_k + 1, & \text{for} \quad \alpha = 1, \\ \frac{n_k^\alpha}{\alpha-1}, & \text{for} \quad \alpha > 1. \end{cases}$$

This could be done by differentiating (3.5) with respect to time, taking $\psi = u_k(t)$ and using (3.6). We refer the readers to [13] for more details. Thus, we obtain a solution (ρ_k, n_k, u_k) to (3.5)-(3.6) globally in time t.

The next step is to pass the limit of (ρ_k, n_k, u_k) as $k \to \infty$. Following the same arguments of Section 2.3 of [13], energy equality (3.8) gives us the following bounds

$$0 < \frac{1}{c_k} \leq \rho_k(x,t), n_k(x,t) \leq c_k \text{ for a.e.}(x,t) \in \Omega \times (0,T), \qquad (3.9)$$

$$\sup_{t\in[0,T]} \|\rho_k(t)\|_{L^\gamma(\Omega)}^\gamma \leq C(\rho_0, n_0, M_0), \qquad (3.10)$$

$$\sup_{t\in[0,T]} \|n_k(t)\|_{L^\alpha(\Omega)}^\alpha \leq C(\rho_0, n_0, M_0) \quad \text{for } \alpha \geq 1, \qquad (3.11)$$

$$\delta \sup_{t\in[0,T]} \|\rho_k(t) + n_k(t)\|_{L^\beta(\Omega)}^\beta \leq C(\rho_0, n_0, M_0), \qquad (3.12)$$

$$\sup_{t\in[0,T]} \|\sqrt{\rho_k + n_k}(t) u_k(t)\|_{L^2(\Omega)}^2 \leq C(\rho_0, n_0, M_0), \qquad (3.13)$$

$$\int_0^T \|u_k(t)\|_{H_0^1(\Omega)}^2\,dt \leq C(\rho_0, n_0, M_0), \qquad (3.14)$$

$$\epsilon \int_0^T \big(\|\nabla\rho_k(t)\|_{L^2(\Omega)}^2 + \|\nabla n_k(t)\|_{L^2(\Omega)}^2\big)\,dt \leq C(\beta, \delta, \rho_0, n_0, M_0), \qquad (3.15)$$

and

$$\|\rho_k + n_k\|_{L^{\beta+1}(Q_T)} \leq C(\epsilon, \beta, \delta, \rho_0, n_0, M_0), \qquad (3.16)$$

where $Q_T = \Omega \times (0,T)$ and $\beta \geq 4$.

To gain higher integrability estimate of n_k independent of k, ϵ, and δ when α is less than γ, we need the following lemma.

Lemma 3.1 *If* (ρ_k, n_k, u_k) *is a solution to (3.5) and (3.6), then the following inequality holds*

$$n_k(x,t) \le c_0 \rho_k(x,t) \tag{3.17}$$

for a.e. $(x,t) \in Q_T$.

Proof. It is easy to check that $n_k - c_0\rho_k$ is a solution of the following parabolic equation

$$\begin{cases} (n_k - c_0\rho_k)_t + \operatorname{div}\big[(n_k - c_0\rho_k)u_k\big] = \epsilon\Delta(n_k - c_0\rho_k), \\ (n_k - c_0\rho_k)|_{t=0} = n_{0,\delta} - c_0\rho_{0,\delta}, \\ \nabla(n_k - c_0\rho_k)\cdot\nu|_{\partial\Omega} = 0. \end{cases}$$

(3.17) can be obtained by applying the maximum principle on it.

□

With the help of (3.10), (3.11), and (3.17), it yields the following estimate on n_k

$$\sup_{t\in[0,T]} \|n_k(t)\|_{L^{\alpha_1}(\Omega)}^{\alpha_1} \le C(\rho_0, n_0, M_0), \tag{3.18}$$

where $\alpha_1 = \max\{\alpha, \gamma\}$.

Relying on the above uniform estimates, i.e., (3.10)-(3.17) and (3.18), and the Aubin-Lions lemma, we are able to recover the global solution to approximation system (3.1)-(3.3) by passing to the limit for (ρ_k, n_k, u_k) as $k \to \infty$. We have the following Proposition on the weak solutions of the approximation (3.1), (3.2) and (3.3).

Proposition 3.2 *Suppose* $\beta > \max\{4, \alpha, \gamma\}$. *For any given* $\epsilon, \delta > 0$, *there exists a global weak solution* (ρ, n, u) *to (3.1), (3.2) and (3.3) such that for any given* $T > 0$, *the following estimates*

$$\sup_{t\in[0,T]} \|(\rho(t), n(t))\|_{L^\gamma(\Omega)}^{\gamma} \le C(\rho_0, n_0, M_0), \tag{3.19}$$

$$\sup_{t\in[0,T]} \|n(t)\|_{L^\alpha(\Omega)}^{\alpha} \le C(\rho_0, n_0, M_0), \tag{3.20}$$

$$\delta \sup_{t\in[0,T]} \|(\rho(t), n(t))\|_{L^\beta(\Omega)}^{\beta} \le C(\rho_0, n_0, M_0), \tag{3.21}$$

$$\sup_{t\in[0,T]} \|\sqrt{\rho+n}(t)u(t)\|_{L^2(\Omega)}^2 \le C(\rho_0, n_0, M_0), \tag{3.22}$$

$$\int_0^T \|u(t)\|_{H_0^1(\Omega)}^2 \, dt \le C(\rho_0, n_0, M_0), \tag{3.23}$$

$$\epsilon \int_0^T \|(\nabla\rho(t), \nabla n(t))\|_{L^2(\Omega)}^2 \, dt \le C(\beta, \delta, \rho_0, n_0, M_0), \tag{3.24}$$

and

$$\|(\rho(t), n(t))\|_{L^{\beta+1}(Q_T)} \le C(\epsilon, \beta, \delta, \rho_0, n_0, M_0) \tag{3.25}$$

hold, where the norm $\|(\cdot,\cdot)\|$ denotes $\|\cdot\|+\|\cdot\|$, and $\rho, n \geq 0$ a.e. on Q_T.

In addition, ρ and n satisfy

$$n(x,t) \leq c_0\rho(x,t) \quad \text{for a.e. } (x,t) \in Q_T. \tag{3.26}$$

Finally, there exists $r > 1$ such that $\rho_t, n_t, \nabla^2\rho, \nabla^2 n \in L^r(Q_T)$ and the equations (3.1)$_1$ and (3.1)$_2$ are satisfied a.e. on Q_T.

Remark 3.3 *The solution (ρ, n, u) stated in Proposition 3.2 actually depends on ϵ and δ. We omit the dependence in the solution itself for brevity.*

4 The vanishing viscosity limit $\epsilon \to 0^+$

The goal of this section is to pass to the limit of $(\rho_\epsilon, n_\epsilon, u_\epsilon)$ as ϵ goes to zero. To vanish ϵ, the uniform estimates are needed. Compared to the work of [13], the pressure law here is in two variables, which bring new difficulty-possible oscillation of $\rho^\gamma + n^\alpha$. The uniform estimates resulting from the energy inequality in Proposition 3.2 and Lemma 4.1 are not enough to handle the weak limit of such a pressure. In Section 3.1, we pass to the limits for the weak solution constructed in Proposition 3.2 as ϵ goes to zero by standard compactness argument. In Section 3.2, we will focus on the weak limit of the pressure and the strong convergence of ρ_ϵ and n_ϵ. In this section, let C denote a generic positive constant depending on the initial data and δ but independent of ϵ.

4.1 Passing to the limit as $\epsilon \to 0^+$

The uniform estimates resulting from (3.19), (3.20), and (3.21) are not enough to obtain the convergence of the pressure term $\rho_\epsilon^\gamma + n_\epsilon^\alpha$. Thus we need to obtain higher integrability estimates of the pressure term uniformly for ϵ.

First, following the same argument in [13], we are able to get the following estimate in Lemma 4.1.

Lemma 4.1 *Let (ρ, n, u) be the solution stated in Proposition 3.2, then*

$$\int_0^T \int_\Omega (n^{\alpha+1} + \rho^{\gamma+1} + \delta\rho^{\beta+1} + \delta n^{\beta+1})\, dx\, dt \leq C$$

for $\beta > 4$.

With Lemma 4.1 and (3.26), one obtains the following bound on n, which is crucial to get the convergence of n^α.

Corollary 4.2 *Let (ρ, n, u) be the solution stated in Lemma 3.2, then*

$$\int_0^T \int_\Omega n^{\gamma+1}\, dx\, dt \leq C. \tag{4.1}$$

In this step, we fix $\delta > 0$ and shall let $\epsilon \to 0^+$. Then the solution (ρ, n, u) constructed in Proposition 3.2 is naturally dressed in the subscript "ϵ", i.e., $(\rho_\epsilon, n_\epsilon, u_\epsilon)$.

With (3.19)-(3.24), Lemma 4.1, and (4.1), letting $\epsilon \to 0^+$ (take the subsequence if necessary), we have

$$\begin{cases} (\rho_\epsilon, n_\epsilon) \to (\rho, n) \text{ in } C([0,T]; L^\beta_{weak}(\Omega)) \text{ and weakly in } L^{\beta+1}(Q_T) \text{ as } \epsilon \to 0^+, \\ (\epsilon\Delta\rho_\epsilon, \epsilon\Delta n_\epsilon) \to 0 \text{ weakly in } L^2(0,T; H^{-1}(\Omega)) \text{ as } \epsilon \to 0^+, \\ u_\epsilon \to u \text{ weakly in } L^2(0,T; H^1_0(\Omega)) \text{ as } \epsilon \to 0^+, \\ (\rho_\epsilon + n_\epsilon)u_\epsilon \to (\rho+n)u \text{ in } C([0,T]; L^{\frac{2\gamma}{\gamma+1}}_{weak}) \cap C([0,T]; H^{-1}(\Omega)) \text{ as } \epsilon \to 0^+, \\ (\rho_\epsilon u_\epsilon, n_\epsilon u_\epsilon) \to (\rho u, nu) \text{ in } \mathcal{D}'(Q_T) \text{ as } \epsilon \to 0^+, \\ (\rho_\epsilon + n_\epsilon)u_\epsilon \otimes u_\epsilon \to (\rho+n)u \otimes u \text{ in } \mathcal{D}'(Q_T) \text{ as } \epsilon \to 0^+, \\ n_\epsilon^\alpha + \rho_\epsilon^\gamma + \delta(\rho_\epsilon + n_\epsilon)^\beta \to \overline{n^\alpha + \rho^\gamma + \delta(\rho+n)^\beta} \text{ weakly in } L^{\frac{\beta+1}{\beta}}(Q_T) \text{ as } \epsilon \to 0^+, \\ \epsilon\nabla u_\epsilon \cdot \nabla(\rho_\epsilon + n_\epsilon) \to 0 \text{ in } L^1(Q_T) \text{ as } \epsilon \to 0^+. \end{cases} \tag{4.2}$$

By virtue of (3.26) and $(4.2)_1$, we have

$$n_\epsilon(x,t) \le c_0\rho_\epsilon(x,t) \text{ and } n(x,t) \le c_0\rho(x,t) \quad \text{for a.e. } (x,t) \in Q_T. \tag{4.3}$$

With $(4.2)_1$ and $(4.2)_4$, we get

$$\big(\rho, n, (\rho+n)u\big)|_{t=0} = (\rho_{0,\delta}, n_{0,\delta}, M_{0,\delta}).$$

In summary, the limit (ρ, n, u) solves the following system in the sense of distribution on Q_T for any $T > 0$:

$$\begin{cases} n_t + \operatorname{div}(nu) = 0, \\ \rho_t + \operatorname{div}(\rho u) = 0, \\ \big[(\rho+n)u\big]_t + \operatorname{div}\big[(\rho+n)u \otimes u\big] + \nabla\big(\overline{n^\alpha + \rho^\gamma + \delta(\rho+n)^\beta}\big) = \mu\Delta u + (\mu+\lambda)\nabla\operatorname{div}u \end{cases} \tag{4.4}$$

with initial and boundary condition

$$\big(\rho, n, (\rho+n)u\big)|_{t=0} = (\rho_{0,\delta}, n_{0,\delta}, M_{0,\delta}), \tag{4.5}$$

$$u|_{\partial\Omega} = 0, \tag{4.6}$$

where $\overline{f(t,x)}$ denotes the weak limit of $f_\epsilon(t,x)$ as $\epsilon \to 0$.

To this end, we have to show that $\overline{n^\alpha + \rho^\gamma + \delta(\rho+n)^\beta} = n^\alpha + \rho^\gamma + \delta(\rho+n)^\beta$, which is a nonlinear two-variables function in term of ρ and n. It seems that the argument in [13] fails here due to the difficulty resulting from the new variable n. New ideas are necessary to handle this weak limit. We are going to focus on this issue next subsection.

4.2 The weak limit of pressure

The main task of this subsection is to develop new argument to handle the possible oscillation for the pressure $n_\epsilon^\alpha + \rho_\epsilon^\gamma + \delta(\rho_\epsilon + n_\epsilon)^\beta$. To achieve this goal, we have to show the strong convergence of ρ_ϵ and n_ϵ. It allows us to have the following Proposition on its weak limit.

Proposition 4.3 $$\overline{n^\alpha + \rho^\gamma + \delta(\rho+n)^\beta} = n^\alpha + \rho^\gamma + \delta(\rho+n)^\beta$$

a.e. on Q_T.

To prove this proposition, we shall rely on the following lemmas. The first one is on the effective viscous flux for two-variables pressure function. In particular, let

$$\begin{aligned} H_\epsilon :=& n_\epsilon^\alpha + \rho_\epsilon^\gamma + \delta(\rho_\epsilon + n_\epsilon)^\beta - (2\mu+\lambda)\text{div}u_\epsilon, \\ H :=& \overline{n^\alpha + \rho^\gamma + \delta(\rho+n)^\beta} - (2\mu+\lambda)\text{div}u, \end{aligned}$$

then we will have the following lemma. The proof is very similar to the work of [13].

Lemma 4.4 *Let* $(\rho_\epsilon, n_\epsilon, u_\epsilon)$ *be the solution stated in Lemma 3.2, and* (ρ, n, u) *be the limit in the sense of (4.2), then*

$$\lim_{\epsilon\to 0^+} \int_0^T \psi \int_\Omega \phi H_\epsilon(\rho_\epsilon + n_\epsilon)\, dx\, dt = \int_0^T \psi \int_\Omega \phi H(\rho+n)\, dx\, dt, \tag{4.7}$$

for any $\psi \in C_0^\infty(0,T)$ *and* $\phi \in C_0^\infty(\Omega)$.

The key idea of proving Proposition 4.3 is to rewrite the terms related pressure as follows

$$\begin{aligned} n_\epsilon^\alpha + \rho_\epsilon^\gamma =& \rho_\epsilon^\gamma + a_\epsilon^\alpha \rho_\epsilon^\alpha = \rho_\epsilon^\gamma + a^\alpha \rho_\epsilon^\alpha + (a_\epsilon^\alpha - a^\alpha)\rho_\epsilon^\alpha, \\ n_\epsilon + \rho_\epsilon =& \rho_\epsilon + a_\epsilon \rho_\epsilon = \rho_\epsilon + a\rho_\epsilon + (a_\epsilon - a)\rho_\epsilon, \end{aligned}$$

where $a_\epsilon = \frac{n_\epsilon}{\rho_\epsilon}$ if $\rho_\epsilon \neq 0$, $a = \frac{n}{\rho}$ if $\rho \neq 0$, $0 \le a_\epsilon, a \le c_0$, and $a_\epsilon \rho_\epsilon = n_\epsilon, a\rho = n$, (ρ, n) is the limit of $(\rho_\epsilon, n_\epsilon)$ in a suitable weak topology. We are able to apply the ideas in [13] to handle the product $\rho_\epsilon^\gamma + a^\alpha \rho_\epsilon^\alpha$ and $\rho_\epsilon + a\rho_\epsilon$, because a is bounded in $L^\infty(0,T; L^\infty(\Omega))$. What we expect is to show the terms $(a_\epsilon^\alpha - a^\alpha)\rho_\epsilon^\alpha(\rho_\epsilon + n_\epsilon)$ and $(a_\epsilon - a)\rho_\epsilon(\rho_\epsilon^\gamma + a^\alpha \rho_\epsilon^\alpha)$ approach to zero as ϵ goes to zero.

We divide the proof of Proposition 4.3 into several steps as follows:

Step 1: Control ρ_ϵ and n_ϵ in $L\log L$.

The current step of our proof is to control ρ_ϵ and n_ϵ in the space of $L\log L$, which is a little better than L^1. It will help us to obtain the strong convergence of ρ_ϵ and n_ϵ. We give our control in the following lemma.

Lemma 4.5 *Let* $(\rho_\epsilon, n_\epsilon)$ *be the solution stated in Proposition 3.2, and* (ρ, n) *be the limit in the sense of (4.2), then*

$$\begin{aligned} &\int_\Omega \big[\rho_\epsilon \log\rho_\epsilon - \rho\log\rho + n_\epsilon \log n_\epsilon - n\log n\big](t)\, dx \\ \le& \int_0^t \int_\Omega (\rho+n)\text{div}u\, dx\, ds - \int_0^t \int_\Omega (\rho_\epsilon + n_\epsilon)\text{div}u_\epsilon\, dx\, ds \end{aligned} \tag{4.8}$$

for a.e. $t \in (0,T)$.

Proof. Since n_ϵ and ρ_ϵ solve $(3.1)_1$ and $(3.1)_2$ a.e. on Q_T, respectively, we have

$$[b(f_\epsilon)]_t + \text{div}\big(b(f_\epsilon)u_\epsilon\big) + \big[b'(f_\epsilon)f_\epsilon - b(f_\epsilon)\big]\text{div}u_\epsilon = \epsilon\Delta b(f_\epsilon) - \epsilon b''(f_\epsilon)|\nabla f_\epsilon|^2 \text{ on } \mathrm{Q_T}, \tag{4.9}$$

where $f_\epsilon = \rho_\epsilon, n_\epsilon$, and $b \in C^2[0,\infty)$.

Taking $b(z) = (z+\frac{1}{j})\log(z+\frac{1}{j})$ in (4.9), and integrating it over $\Omega\times(0,t)$ for $t\in[0,T]$, we have

$$\begin{aligned}&\int_\Omega (f_\epsilon+\frac{1}{j})\log(f_\epsilon+\frac{1}{j})(t)\,dx+\int_0^t\int_\Omega\big[f_\epsilon-\frac{1}{j}\log(f_\epsilon+\frac{1}{j})\big]\mathrm{div}u_\epsilon\,dx\,ds\\ \le&\int_\Omega (f_{0,\epsilon}+\frac{1}{j})\log(f_{0,\epsilon}+\frac{1}{j})\,dx,\end{aligned} \tag{4.10}$$

where we have used the convexity of b and the boundary condition (3.3). Letting $j\to\infty$ in (4.10), one obtains

$$\int_\Omega (f_\epsilon\log f_\epsilon)(t)\,dx+\int_0^t\int_\Omega f_\epsilon\mathrm{div}u_\epsilon\,dx\,ds\le\int_\Omega f_{0,\delta}\log f_{0,\delta}\,dx, \tag{4.11}$$

where $f_\epsilon = \rho_\epsilon, n_\epsilon$ and $f_{0,\delta}=\rho_{0,\delta}, n_{0,\delta}$.

Since the limit (n,u) and (ρ,u) solve $(4.4)_1$ and $(4.4)_2$ in the sense of renormalized solutions, we can take $\beta(z)=z\log z$ in accordance with Remark 1.1 in [13] or by approximating the function $z\log z$ by a sequence of such the $\beta(z)$ stated in Lemma 2.4 and then passing to the limit. This allows us to have

$$\int_\Omega (f\log f)(t)\,dx+\int_0^t\int_\Omega f\mathrm{div}u\,dx\,ds=\int_\Omega f_{0,\delta}\log f_{0,\delta}\,dx, \tag{4.12}$$

where $f=\rho,n$ and $f_{0,\delta}=\rho_{0,\delta},n_{0,\delta}$. By (4.11) and (4.12), (4.8) follows.

Step 2: Control the right hand side of (4.8)

With the help of Theorem 2.2, we are able to control the right hand side of (4.8). The following two lemmas of this step shows us how to control it.

Lemma 4.6 *Let $(\rho_\epsilon, n_\epsilon)$ be the solution stated in Proposition 3.2, and (ρ,n) be the limit in the sense of (4.2), then*

$$\int_0^t\psi\int_\Omega\phi(\rho+n)\overline{n^\alpha+\rho^\gamma}\,dx\,ds\le\int_0^t\psi\int_\Omega\phi\overline{(\rho+n)(n^\alpha+\rho^\gamma)}\,dx\,ds$$

for any $t\in[0,T]$ and any $\psi\in C[0,t]$, $\phi\in C(\overline{\Omega})$ where $\psi,\phi\ge0$.

Proof. Note that

$$\begin{aligned}n_\epsilon^\alpha+\rho_\epsilon^\gamma=&\rho_\epsilon^\gamma+a_\epsilon^\alpha\rho_\epsilon^\alpha=\rho_\epsilon^\gamma+a^\alpha\rho_\epsilon^\alpha+(a_\epsilon^\alpha-a^\alpha)\rho_\epsilon^\alpha,\\ n_\epsilon+\rho_\epsilon=&\rho_\epsilon+a_\epsilon\rho_\epsilon=\rho_\epsilon+a\rho_\epsilon+(a_\epsilon-a)\rho_\epsilon,\end{aligned}$$

where $a_\epsilon=\frac{n_\epsilon}{\rho_\epsilon}$ if $\rho_\epsilon\ne0$, $a=\frac{n}{\rho}$ if $\rho\ne0$, $0\le a_\epsilon, a\le c_0$, and $a_\epsilon\rho_\epsilon=n_\epsilon, a\rho=n$, (ρ,n) is the limit of $(\rho_\epsilon,n_\epsilon)$ in a suitable weak topology.

For any $\psi \in C[0,t]$, $\phi \in C(\overline{\Omega})$ where $\psi, \phi \geq 0$, we have

$$
\begin{aligned}
&\int_0^t \psi \int_\Omega \phi(n_\epsilon^\alpha + \rho_\epsilon^\gamma)(\rho_\epsilon + n_\epsilon)\,dx\,ds \\
=&\int_0^t \psi \int_\Omega \phi(\rho_\epsilon^\gamma + a^\alpha \rho_\epsilon^\alpha)\big(\rho_\epsilon + a\rho_\epsilon + (a_\epsilon - a)\rho_\epsilon\big)\,dx\,ds \\
&+ \int_0^t \psi \int_\Omega \phi(a_\epsilon^\alpha - a^\alpha)\rho_\epsilon^\alpha\big(\rho_\epsilon + a\rho_\epsilon + (a_\epsilon - a)\rho_\epsilon\big)\,dx\,ds \\
=&\int_0^t \psi \int_\Omega \phi(\rho_\epsilon^\gamma + a^\alpha \rho_\epsilon^\alpha)(\rho_\epsilon + a\rho_\epsilon)\,dx\,ds + \int_0^t \psi \int_\Omega \phi(\rho_\epsilon^\gamma + a^\alpha\rho_\epsilon^\alpha)(a_\epsilon - a)\rho_\epsilon\,dx\,ds \\
&+ \int_0^t \psi \int_\Omega \phi(a_\epsilon^\alpha - a^\alpha)\rho_\epsilon^\alpha(\rho_\epsilon + n_\epsilon)\,dx\,ds \\
=&\sum_{i=1}^3 II_i.
\end{aligned}
\tag{4.13}
$$

For II_2, there exists a positive integer k_0 large enough such that

$$
\max\{\frac{k_0\gamma}{k_0 - 1}, \frac{k_0\alpha}{k_0 - 1}\} \leq \beta \tag{4.14}
$$

due to the assumption that $\max\{\alpha, \gamma\} < \beta$.

Using the Hölder inequality, Lemma 4.1 and (4.14), we have

$$
\begin{aligned}
II_2 \leq& C\left(\int_0^T \int_\Omega \rho_\epsilon |a_\epsilon - a|^{k_0}\,dx\,dt\right)^{\frac{1}{k_0}} \left(\int_0^T \int_\Omega \rho_\epsilon |\rho_\epsilon^\gamma + a^\alpha \rho_\epsilon^\alpha|^{\frac{k_0}{k_0-1}}\,dx\,dt\right)^{\frac{k_0-1}{k_0}} \\
\leq& C\left(\int_0^T \int_\Omega \rho_\epsilon |a_\epsilon - a|^{k_0}\,dx\,dt\right)^{\frac{1}{k_0}}.
\end{aligned}
\tag{4.15}
$$

Choosing $\nu_k = \epsilon$ for Theorem 2.2, we conclude that

$$
\left(\int_0^T \int_\Omega \rho_\epsilon |a_\epsilon - a|^{k_0}\,dx\,dt\right)^{\frac{1}{k_0}} \to 0
$$

as ϵ goes to zero. Thus, we have $II_2 \to 0$ as $\epsilon \to 0$.

For II_3, there exists a positive integer k_1 large enough such that

$$
(\alpha + 1 - \frac{1}{k_1})\frac{k_1}{k_1 - 1} < \beta + 1 \tag{4.16}
$$

due to the assumption $\alpha < \beta$. We employ the Hölder inequality to have

$$
\begin{aligned}
|II_3| \leq& C\left(\int_0^T \int_\Omega \rho_\epsilon^{(\alpha+1-\frac{1}{k_1})\frac{k_1}{k_1-1}}\,dx\,dt\right)^{\frac{k_1-1}{k_1}} \left(\int_0^T \int_\Omega \rho_\epsilon \big|a_\epsilon^\alpha - a^\alpha\big|^{k_1}\,dx\,dt\right)^{\frac{1}{k_1}} \\
\leq& C\left(\int_0^T \int_\Omega \rho_\epsilon \big|a_\epsilon^\alpha - a^\alpha\big|^{k_1}\,dx\,dt\right)^{\frac{1}{k_1}} \to 0
\end{aligned}
\tag{4.17}
$$

as $\epsilon \to 0^+$, where we have used (3.26), (4.16), Lemma 4.1, and the fact that

$$\begin{aligned}\int_0^T \int_\Omega \rho_\epsilon \big|a_\epsilon^\alpha - a^\alpha\big|^{k_1}\,dx\,dt \le& \alpha^{k_1}\int_0^T \int_\Omega \rho_\epsilon \big(\max\{a_\epsilon, a\}\big)^{\alpha-1}\big|a_\epsilon - a\big|^{k_1}\,dx\,dt \\ \le& C\int_0^T \int_\Omega \rho_\epsilon \big|a_\epsilon - a\big|^{k_1}\,dx\,dt \to 0\end{aligned} \tag{4.18}$$

as $\epsilon \to 0^+$, due to Theorem 2.2 with $\nu_K = \epsilon$. In fact, $\rho_\epsilon \in L^\infty(0,T;L^\beta(\Omega))$ for $\beta > 4$, and $u_\epsilon \in L^2(0,T;H_0^1(\Omega))$, and

$$\sqrt{\epsilon}\|\nabla\rho_\epsilon\|_{L^2(0,T;L^2(\Omega))} \le C_0,\ \sqrt{\epsilon}\|\nabla n_\epsilon\|_{L^2(0,T;L^2(\Omega))} \le C_0,$$

and for any $\epsilon > 0$ and any $t > 0$:

$$\int_\Omega \frac{n_\epsilon^2}{\rho_\epsilon}\,dx \le \int_\Omega \frac{n_0^2}{\rho_0}\,dx.$$

Thus, we are able to apply Theorem 2.2 to control (4.18).

By virtue of (4.13), (4.15) and (4.17), one deduces that

$$\begin{aligned}\lim_{\epsilon\to0^+}\int_0^t \psi\int_\Omega \phi(n_\epsilon^\alpha + \rho_\epsilon^\gamma)(\rho_\epsilon + n_\epsilon)\,dx\,ds =& \int_0^t \psi\int_\Omega \phi\overline{(\rho^\gamma + a^\alpha\rho^\alpha)(\rho + a\rho)}\,dx\,ds \\ \ge& \int_0^t \psi\int_\Omega \phi\overline{\rho^\gamma + a^\alpha\rho^\alpha}(\rho + a\rho)\,dx\,ds \\ =& \int_0^t \psi\int_\Omega \phi\overline{\rho^\gamma + a^\alpha\rho^\alpha}(\rho + n)\,dx\,ds\end{aligned} \tag{4.19}$$

where we have used $\overline{(\rho^\gamma + a^\alpha\rho^\alpha)(\rho + a\rho)} \ge \overline{\rho^\gamma + a^\alpha\rho^\alpha}(\rho + a\rho)$ because the functions $z \mapsto z^\gamma + a^\alpha z^\alpha$ and $z \mapsto z + az$ are increasing functions.

On the other hand,

$$\begin{aligned}\int_0^t \psi\int_\Omega \phi(\rho + n)\overline{n^\alpha + \rho^\gamma}\,dx\,ds =& \lim_{\epsilon\to0^+}\int_0^t \psi\int_\Omega \phi(\rho + n)(n_\epsilon^\alpha + \rho_\epsilon^\gamma)\,dx\,ds \\ =& \lim_{\epsilon\to0^+}\int_0^t \psi\int_\Omega \phi(\rho + n)(a^\alpha\rho_\epsilon^\alpha + \rho_\epsilon^\gamma)\,dx\,ds \\ &+ \lim_{\epsilon\to0^+}\int_0^t \psi\int_\Omega \phi(\rho + n)\rho_\epsilon^\alpha(a_\epsilon^\alpha - a^\alpha)\,dx\,ds \\ =& \int_0^t \psi\int_\Omega \phi(\rho + n)\overline{a^\alpha\rho^\alpha + \rho^\gamma}\,dx\,ds,\end{aligned} \tag{4.20}$$

thanks to

$$\lim_{\epsilon\to0^+}\int_0^t \psi\int_\Omega \phi(\rho + n)\rho_\epsilon^\alpha(a_\epsilon^\alpha - a^\alpha)\,dx\,ds \to 0$$

as $\epsilon \to 0^+$.

By (4.19) and (4.20), we complete the proof of the lemma.

□

Since ψ and ϕ are arbitrary, we immediately get

Corollary 4.7 *Let $(\rho_\epsilon, n_\epsilon)$ be the solution stated in Proposition 3.2, and (ρ, n) be the limit in the sense of (4.2), then*

$$(\rho + n)\overline{n^\alpha + \rho^\gamma} \le \overline{(\rho + n)(n^\alpha + \rho^\gamma)} \tag{4.21}$$

a.e. on $\Omega \times (0,T)$.

Now we control the right hand side of (4.8) in the following lemma.

Lemma 4.8 *Let $(\rho_\epsilon, n_\epsilon)$ be the solution stated in Lemma 3.2, and (ρ, n) be the limit in the sense of (4.2), then*

$$\int_0^t \int_\Omega (\rho+n)\mathrm{div}u \, dx\, ds \le \lim_{\epsilon\to 0^+} \int_0^t \int_\Omega (\rho_\epsilon+n_\epsilon)\mathrm{div}u_\epsilon \, dx\, ds \tag{4.22}$$

for a.e. $t \in (0,T)$.

Proof. For $\psi_j \in C_0^\infty(0,t)$, $\phi_j \in C_0^\infty(\Omega)$ given by

$$\psi_j \in C_0^\infty(0,T), \quad \psi_j(t) \equiv 1 \text{ for any } t \in [\frac{1}{j}, T-\frac{1}{j}],\ 0 \le \psi_j \le 1,\ \psi_j \to 1, \tag{4.23}$$

as $j \to \infty$, and

$$\phi_j \in C_0^\infty(\Omega), \quad \phi_j(x) \equiv 1 \text{ for any } x \in \big\{x \in \Omega \big| \mathrm{dist(x,\partial\Omega)} \ge \frac{1}{\mathrm{j}}\big\},\ 0 \le \phi_j \le 1,\ \phi_j \to 1, \tag{4.24}$$

as $j \to \infty$, respectively, then

$$\begin{aligned}
&\int_0^t \int_\Omega (\rho+n)\mathrm{div}u \, dx\, ds \\
=&\int_0^t \psi_j \int_\Omega \phi_j(\rho+n)\mathrm{div}u\, dx\, ds + \int_0^t \int_\Omega (1-\psi_j\phi_j)(\rho+n)\mathrm{div}u\, dx\, ds \\
=&\frac{1}{2\mu+\lambda}\int_0^t \psi_j \int_\Omega \phi_j(\rho+n)\overline{n^\alpha+\rho^\gamma}\, dx\, ds + \frac{1}{2\mu+\lambda}\int_0^t \psi_j \int_\Omega \phi_j(\rho+n)\overline{\delta(\rho+n)^\beta}\, dx\, ds \\
&-\frac{1}{2\mu+\lambda}\int_0^t \psi_j \int_\Omega \phi_j(\rho+n)H\, dx\, ds + \int_0^t \int_\Omega (1-\psi_j\phi_j)(\rho+n)\mathrm{div}u\, dx\, ds \\
=&RHS_1+RHS_2+RHS_3+RHS_4,
\end{aligned} \tag{4.25}$$

where we have used

$$(2\mu+\lambda)\mathrm{div}u = \overline{n^\alpha+\rho^\gamma+\delta(\rho+n)^\beta} - H,$$

and

$$\overline{n^\alpha+\rho^\gamma+\delta(\rho+n)^\beta} = \overline{n^\alpha+\rho^\gamma} + \overline{\delta(\rho+n)^\beta}.$$

For RHS_2, we have

$$\begin{aligned}
RHS_2 =&\frac{1}{2\mu+\lambda}\int_0^t \psi_j \int_\Omega \phi_j(\rho+n)\overline{\delta(\rho+n)^\beta}\, dx\, ds \\
\le&\frac{1}{2\mu+\lambda}\liminf_{\epsilon\to 0^+}\int_0^t \psi_j \int_\Omega \phi_j\delta(\rho_\epsilon+n_\epsilon)(\rho_\epsilon+n_\epsilon)^\beta\, dx\, ds,
\end{aligned} \tag{4.26}$$

becuase $z \mapsto z$ and $z \mapsto z^\beta$ are increasing functions.

By virtue of (4.25), (4.21), (4.26), and (4.7), we have

$$
\begin{aligned}
&\int_0^t \int_\Omega (\rho+n)\mathrm{div}u \, dx\, ds \\
\leq & \frac{1}{2\mu+\lambda}\lim_{\epsilon\to 0^+}\int_0^t \psi_j \int_\Omega \phi_j (n_\epsilon^\alpha+\rho_\epsilon^\gamma)(\rho_\epsilon+n_\epsilon)\, dx\, ds \\
&+\frac{1}{2\mu+\lambda}\liminf_{\epsilon\to 0^+}\int_0^t \psi_j \int_\Omega \phi_j (\rho_\epsilon+n_\epsilon)\delta(\rho_\epsilon+n_\epsilon)^\beta\, dx\, ds \\
&-\frac{1}{2\mu+\lambda}\lim_{\epsilon\to 0^+}\int_0^t \psi_j \int_\Omega \phi_j (\rho_\epsilon+n_\epsilon)H_\epsilon\, dx\, ds+\int_0^t\int_\Omega (1-\psi_j\phi_j)(\rho+n)\mathrm{div}u\, dx\, ds \\
\leq & \lim_{\epsilon\to 0^+}\int_0^t\int_\Omega (\rho_\epsilon+n_\epsilon)\mathrm{div}u_\epsilon\, dx\, ds+\lim_{\epsilon\to 0^+}\int_0^t\int_\Omega (\psi_j\phi_j-1)(\rho_\epsilon+n_\epsilon)\mathrm{div}u_\epsilon\, dx\, ds \\
&+\int_0^t\int_\Omega (1-\psi_j\phi_j)(\rho+n)\mathrm{div}u\, dx\, ds.
\end{aligned}
\tag{4.27}
$$

Letting $j\to\infty$ in (4.27), we complete the proof of the lemma. □

Step 3: Strong convergence of ρ_ϵ and n_ϵ

Here our main task is to show the strong convergence of ρ_ϵ and n_ϵ. This yields Proposition 4.3. In particular, With (4.22), letting $\epsilon\to 0^+$ in (4.8), we deduce that

$$\int_\Omega \left[\overline{\rho\log\rho}-\rho\log\rho+\overline{n\log n}-n\log n\right](t)\, dx\leq 0.$$

Thanks to the convexity of $z\mapsto z\log z$, we have

$$\overline{\rho\log\rho}\geq\rho\log\rho \quad \text{and} \quad \overline{n\log n}\geq n\log n$$

a.e. on Q_T. This turns out that

$$\int_\Omega \left[\overline{\rho\log\rho}-\rho\log\rho+\overline{n\log n}-n\log n\right](t)\, dx=0.$$

Hence we get

$$\overline{\rho\log\rho}=\rho\log\rho \quad \text{and} \quad \overline{n\log n}=n\log n$$

a.e. on Q_T, which combined with Lemma 4.1 implies strong convergence of $\rho_\epsilon, n_\epsilon$ in $L^\beta(Q_T)$. Thus we complete the proof.

With Proposition 4.3, we recover a global weak solution to the system (4.4), (4.5) and (4.6) with $\overline{n^\alpha+\rho^\gamma+\delta(\rho+n)^\beta}$ replaced by $n^\alpha+\rho^\gamma+\delta(\rho+n)^\beta$.

Proposition 4.9 *Suppose $\beta>\max\{4,\alpha,\gamma\}$. For any given $\delta>0$, there exists a global weak solution $(\rho_\delta, n_\delta, u_\delta)$ to the following system over $\Omega\times(0,\infty)$:*

$$
\begin{cases}
n_t+\mathrm{div}(nu)=0,\\
\rho_t+\mathrm{div}(\rho u)=0,\\
\big[(\rho+n)u\big]_t+\mathrm{div}\big[(\rho+n)u\otimes u\big]+\nabla(n^\alpha+\rho^\gamma+\delta(\rho+n)^\beta)=\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}u,
\end{cases}
\tag{4.28}
$$

with initial and boundary condition

$$(\rho, n, (\rho+n)u)|_{t=0} = (\rho_{0,\delta}, n_{0,\delta}, M_{0,\delta}) \text{ on } \overline{\Omega}, \tag{4.29}$$

$$u|_{\partial\Omega} = 0 \text{ for } t \geq 0, \tag{4.30}$$

such that for any given $T > 0$, the following estimates

$$\sup_{t\in[0,T]} \|(\rho_\delta(t), n_\delta(t))\|^\gamma_{L^\gamma(\Omega)} \leq C(\rho_0, n_0, M_0), \tag{4.31}$$

$$\sup_{t\in[0,T]} \|n_\delta(t)\|^\alpha_{L^\alpha(\Omega)} \leq C(\rho_0, n_0, M_0), \tag{4.32}$$

$$\delta \sup_{t\in[0,T]} \|(\rho_\delta(t), n_\delta(t))\|^\beta_{L^\beta(\Omega)} \leq C(\rho_0, n_0, M_0), \tag{4.33}$$

$$\sup_{t\in[0,T]} \|\sqrt{\rho_\delta + n_\delta}(t) u_\delta(t)\|^2_{L^2(\Omega)} \leq C(\rho_0, n_0, M_0), \tag{4.34}$$

$$\int_0^T \|u_\delta(t)\|^2_{H^1_0(\Omega)}\, dt \leq C(\rho_0, n_0, M_0), \tag{4.35}$$

and

$$\|(\rho_\delta(t), n_\delta(t))\|_{L^{\beta+1}(Q_T)} \leq C(\beta, \delta, \rho_0, n_0, M_0) \tag{4.36}$$

hold, where the norm $\|(\cdot,\cdot)\|$ denotes $\|\cdot\| + \|\cdot\|$. Besides, we have

$$n_\delta(x,t) \leq c_0 \rho_\delta(x,t) \quad \text{for a.e. } (x,t) \in Q_T. \tag{4.37}$$

5 Passing to the limit in the artificial pressure term as $\delta \to 0^+$

In this section, we shall recover the weak solution to (1.1)-(1.3) by passing to the limit of $(\rho_\delta, n_\delta, u_\delta)$ as $\delta \to 0$. Note that the estimate (4.36) depends on δ. Thus to begin with, we shall get the higher integrability estimates of the pressure term uniformly for δ. Let C be a generic constant independent of δ which will be used throughout this section.

5.1 Passing to the limit as $\delta \to 0^+$

We can follow the similar argument as in [13] to have the higher integrability estimates of ρ and n in the following lemma. We only need to modify the proof a little bit on n.

Lemma 5.1 *Let $(\rho_\delta, n_\delta, u_\delta)$ be the solution stated in Proposition 4.9, then*

$$\int_0^T \int_\Omega (n_\delta^{\alpha+\theta} + \rho_\delta^{\gamma+\theta} + \delta\rho_\delta^{\beta+\theta} + \delta n_\delta^{\beta+\theta})\, dxdt \leq C \tag{5.1}$$

for any positive constant θ satisfying $\theta < \frac{\gamma}{3}$ and $\theta \leq \min\{1, \frac{2\gamma}{3} - 1\}$ for $\gamma \in (\frac{3}{2}, \infty)$.

In view of (5.1) and (4.37), we have the following corollary.

Corollary 5.2 *Let $(\rho_\delta, n_\delta, u_\delta)$ be the solution stated in Proposition 4.9, then*

$$\int_0^T \int_\Omega n_\delta^{\gamma+\theta}\, dx\, dt \le C. \tag{5.2}$$

With (4.31), (4.32), (4.34), (4.35), (4.37), (5.1), and (5.2), letting $\delta \to 0^+$ (take the subsequence if necessary), we have

$$\begin{cases} (\rho_\delta, n_\delta) \to (\rho, n) \text{ in } C([0,T]; L^\gamma_{weak}(\Omega)) \text{ and weakly in } L^{\gamma+\theta}(Q_T) \text{ as } \delta \to 0^+, \\ n_\delta \to n \text{ in } C([0,T]; L^\alpha_{weak}(\Omega)) \ \text{ as } \delta \to 0^+ \text{ for } \alpha > 1, \\ u_\delta \to u \text{ weakly in } L^2(0,T; H^1_0(\Omega)) \text{ as } \delta \to 0^+, \\ (\rho_\delta + n_\delta)u_\delta \to (\rho+n)u \text{ in } C([0,T]; L^{\frac{2\gamma}{\gamma+1}}_{weak}) \cap C([0,T]; H^{-1}(\Omega)) \text{ as } \delta \to 0^+, \\ (\rho_\delta u_\delta, n_\delta u_\delta) \to (\rho u, nu) \text{ in } \mathcal{D}'(Q_T) \text{ as } \delta \to 0^+, \\ (\rho_\delta + n_\delta)u_\delta \otimes u_\delta \to (\rho+n)u \otimes u \text{ in } \mathcal{D}'(Q_T) \text{ as } \delta \to 0^+, \\ n_\delta^\alpha + \rho_\delta^\gamma \to \overline{n^\alpha + \rho^\gamma} \text{ weakly in } L^{\min\{\frac{\gamma+\theta}{\gamma}, \frac{\alpha+\theta}{\alpha}\}}(Q_T) \text{ as } \delta \to 0^+, \\ \delta(\rho_\delta + n_\delta)^\beta \to 0 \text{ in } L^1(Q_T) \text{ as } \delta \to 0^+, \\ n(x,t) \le c_0 \rho(x,t) \quad \text{for a.e. } (x,t) \in Q_T. \end{cases} \tag{5.3}$$

In summary, the limit (ρ, n, u) solves the following system in the sense of distribution over $\Omega \times [0,T]$ for any given $T > 0$:

$$\begin{cases} n_t + \operatorname{div}(nu) = 0, \\ \rho_t + \operatorname{div}(\rho u) = 0, \\ \big[(\rho+n)u\big]_t + \operatorname{div}\big[(\rho+n)u \otimes u\big] + \nabla(\overline{\rho^\gamma + n^\alpha}) = \mu\Delta u + (\mu+\lambda)\nabla \operatorname{div} u, \end{cases} \tag{5.4}$$

with initial and boundary condition

$$(\rho, n, (\rho+n)u)|_{t=0} = (\rho_0, n_0, M_0) \quad \text{on } \overline{\Omega}, \tag{5.5}$$

$$u|_{\partial\Omega} = 0 \quad \text{for } t \ge 0, \tag{5.6}$$

where the convergence of the approximate initial data in (4.29) is due to (3.4).

To recover a weak solution to (1.1)-(1.3), we only need to show the following claim:

- **Claim.** $\overline{\rho^\gamma + n^\alpha} = \rho^\gamma + n^\alpha$**.**

5.2 The weak limit of pressure

The objective of this subsection is to show the strong convergence of ρ_δ and n_δ as δ goes to zero. This allows us to prove $\overline{\rho^\gamma + n^\alpha} = \rho^\gamma + n^\alpha$ as $\delta \to 0$. From now, we need that ρ_δ is bounded in $L^q(Q_T)$ for some q a little bit bigger than 2. By Lemma 5.1, this leads us to the restriction $\gamma > \frac{9}{5}$.

We consider a family of cut-off functions introduced in [13] and references therein, i.e.,

$$T_k(z) = kT(\frac{z}{k}),\ z \in \mathbb{R},\ k = 1, 2, \cdots$$

where $T \in C^\infty(\mathbb{R})$ satisfying

$$T(z) = z\ for\ z \le 1, \quad T(z) = 2\ for\ z \ge 3. \quad T\ is\ concave.$$

The cut-off functions will be used in particular to handle the cross terms due to the two-variable pressure, see the proof of Lemma 5.5. Since $\rho_\delta \in L^2(Q_T), u_\delta \in L^2(0, T; H_0^1(\Omega))$, Lemma 2.4 suggests that (ρ_δ, u_δ) is a renormalized solution of $(5.4)_2$. Thus we have

$$[T_k(\rho_\delta)]_t + \mathrm{div}[T_k(\rho_\delta)u_\delta] + [T_k'(\rho_\delta)\rho_\delta - T_k(\rho_\delta)]\mathrm{div}u_\delta = 0 \quad \text{in } \mathcal{D}'(Q_T). \tag{5.7}$$

For any given k, $T_k(\rho_\delta)$ is bounded in $L^\infty(Q_T)$. Passing to the limit as $\delta \to 0^+$ (taking the subsequence if necessary), we have

$$\begin{aligned}
&T_k(\rho_\delta) \to \overline{T_k(\rho)} \text{ weak}* \text{ in } L^\infty(Q_T),\\
&T_k(\rho_\delta) \to \overline{T_k(\rho)} \text{ in } C([0, T]; L^p_{weak}(\Omega)),\ \text{for any } p \in [1, \infty),\\
&T_k(\rho_\delta) \to \overline{T_k(\rho)} \text{ in } C([0, T]; H^{-1}(\Omega)),\\
&[T_k'(\rho_\delta)\rho_\delta - T_k(\rho_\delta)]\mathrm{div}u_\delta \to \overline{[T_k'(\rho)\rho - T_k(\rho)]\mathrm{div}u} \text{ weakly in } L^2(Q_T).
\end{aligned}$$

This yields

$$[\overline{T_k(\rho)}]_t + \mathrm{div}[\overline{T_k(\rho)}u] + \overline{[T_k'(\rho)\rho - T_k(\rho)]\mathrm{div}u} = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3 \times (0, T)). \tag{5.8}$$

Similarly, we have

$$[T_k(n_\delta)]_t + \mathrm{div}[T_k(n_\delta)u_\delta] + [T_k'(n_\delta)n_\delta - T_k(n_\delta)]\mathrm{div}u_\delta = 0 \quad \text{in } \mathcal{D}'(Q_T), \tag{5.9}$$

and

$$[\overline{T_k(n)}]_t + \mathrm{div}[\overline{T_k(n)}u] + \overline{[T_k'(n)n - T_k(n)]\mathrm{div}u} = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3 \times (0, T)). \tag{5.10}$$

Denote

$$\begin{aligned}
H_\delta :=& \rho_\delta^\gamma + n_\delta^\alpha - (2\mu + \lambda)\mathrm{div}u_\delta,\\
\overline{H} :=& \overline{\rho^\gamma + n^\alpha} - (2\mu + \lambda)\mathrm{div}u.
\end{aligned}$$

We will have the following Lemma on H_δ and $\overline{H}$.

Lemma 5.3 *Let $(\rho_\delta, n_\delta, u_\delta)$ be the solution stated in Proposition 4.9 and (ρ, n, u) be the limit, then*

$$\lim_{\delta \to 0^+} \int_0^T \psi \int_\Omega \phi H_\delta \big[T_k(\rho_\delta) + T_k(n_\delta)\big]\, dx\, dt = \int_0^T \psi \int_\Omega \phi \overline{H}\ \big[\overline{T_k(\rho)} + \overline{T_k(n)}\big]\, dx\, dt, \tag{5.11}$$

for any $\psi \in C_0^\infty(0, T)$ and $\phi \in C_0^\infty(\Omega)$.

Proof. The proof is similar to work of [13].

□

Now let us to devote our proof to the following Proposition.

Proposition 5.4 *For any $\gamma > \frac{9}{5}$ and $\alpha \geq 1$, then*

$$\overline{n^\alpha + \rho^\gamma} = n^\alpha + \rho^\gamma \tag{5.12}$$

a.e. on Q_T.

There are two steps to prove it.

Step 1: Study for the weak limit of $\rho_\delta^\gamma + n_\delta^\alpha$

Relying on Theorem 2.2 with $\nu_K = 0$, we are able to show the following lemma. It is crucial to obtain Proposition 5.4.

Lemma 5.5 *Let (ρ_δ, n_δ) be the solutions constructed in Propsotion 4.9, and (ρ, n) be the limit, then*

$$\int_0^t \psi \int_\Omega \phi \Big[\overline{T_k(\rho)} + \overline{T_k(n)}\Big] \big(\overline{\rho^\gamma + n^\alpha}\big)\, dx\, ds \leq \int_0^t \psi \int_\Omega \phi \overline{\big[T_k(\rho) + T_k(n)\big]\big(\rho^\gamma + n^\alpha\big)}\, dx\, ds. \tag{5.13}$$

for any $t \in [0, T]$ and any $\psi \in C[0, t]$, $\phi \in C(\overline{\Omega})$ where $\psi, \phi \geq 0$, and

$$\begin{cases} \mathrm{n_0 \leq c_0\rho_0 \text{ on } Q_T}, & \text{if } \alpha \in [1, \gamma + \tau), \\ \mathrm{\frac{1}{c_0}\rho_0 \leq n_0 \leq c_0\rho_0 \text{ on } Q_T}, & \text{if } \alpha \in [\gamma + \tau, \infty). \end{cases}$$

Proof.

$$\begin{aligned} &\lim_{\delta\to 0^+} \int_0^t \psi \int_\Omega \phi \big[T_k(\rho_\delta) + T_k(n_\delta)\big]\big(\rho_\delta^\gamma + n_\delta^\alpha\big)\, dx\, ds \\ =&\lim_{\delta\to 0^+} \int_0^t \psi \int_\Omega \phi T_k(\rho_\delta)\rho_\delta^\gamma\, dx\, ds + \lim_{\delta\to 0^+} \int_0^t \psi \int_\Omega \phi T_k(\rho_\delta) n_\delta^\alpha\, dx\, ds \\ &+ \lim_{\delta\to 0^+} \int_0^t \psi \int_\Omega \phi T_k(n_\delta)\rho_\delta^\gamma\, dx\, ds + \lim_{\delta\to 0^+} \int_0^t \psi \int_\Omega \phi T_k(n_\delta) n_\delta^\alpha\, dx\, ds \\ =&\sum_{i=1}^4 IV_i. \end{aligned} \tag{5.14}$$

For IV_1, since $z \mapsto T_k(z)$ and $z \mapsto z^\gamma$ are increasing functions, we have

$$\begin{aligned} 0 \leq& \lim_{\delta\to 0^+} \int_0^t \psi \int_\Omega \phi \big[T_k(\rho_\delta) - T_k(\rho)\big]\big[\rho_\delta^\gamma - \rho^\gamma\big]\, dx\, ds \\ =& \int_0^t \psi \int_\Omega \phi \overline{T_k(\rho)\rho^\gamma}\, dx\, ds - \int_0^t \psi \int_\Omega \phi \overline{T_k(\rho)}\rho^\gamma\, dx\, ds - \int_0^t \psi \int_\Omega \phi T_k(\rho)\overline{\rho^\gamma}\, dx\, ds \\ &+ \int_0^t \psi \int_\Omega \phi T_k(\rho)\rho^\gamma\, dx\, ds \\ =& \int_0^t \psi \int_\Omega \phi \overline{T_k(\rho)\rho^\gamma}\, dx\, ds - \int_0^t \psi \int_\Omega \phi \overline{T_k(\rho)}\,\overline{\rho^\gamma}\, dx\, ds \\ &+ \int_0^t \psi \int_\Omega \phi \big[\overline{T_k(\rho)} - T_k(\rho)\big]\big(\overline{\rho^\gamma} - \rho^\gamma\big)\, dx\, ds \\ \leq& \int_0^t \psi \int_\Omega \phi \overline{T_k(\rho)\rho^\gamma}\, dx\, ds - \int_0^t \psi \int_\Omega \phi \overline{T_k(\rho)}\,\overline{\rho^\gamma}\, dx\, ds \end{aligned} \tag{5.15}$$

where we have used the fact $\overline{\rho^\gamma} \geq \rho^\gamma$ and $\overline{T_k(\rho)} \leq T_k(\rho)$, which could be done by the convexity of $z \mapsto z^\gamma$ and the concavity of $z \mapsto T_k(z)$.

Thanks to (5.15), we have

$$\int_0^t \psi \int_\Omega \phi \overline{T_k(\rho)}\,\overline{\rho^\gamma}\, dx\, ds \leq \int_0^t \psi \int_\Omega \phi \overline{T_k(\rho)\rho^\gamma}\, dx\, ds = \lim_{\delta\to 0^+} \int_0^t \psi \int_\Omega \phi T_k(\rho_\delta)\rho_\delta^\gamma\, dx\, ds = IV_1. \quad (5.16)$$

Similar to (5.16), we have

$$\int_0^t \psi \int_\Omega \phi \overline{T_k(n)}\,\overline{n^\alpha}\, dx\, ds \leq \lim_{\delta\to 0^+} \int_0^t \psi \int_\Omega \phi T_k(n_\delta) n_\delta^\alpha\, dx\, ds = IV_4. \quad (5.17)$$

For IV_2, we need to discuss the size of α in order to guarantee the boundedness of ρ_δ^α in $L^q(Q_T)$ for some $q > 1$.

For $\alpha \in [1, \gamma+\tau)$**,** there exists a positive $\theta > 0$ and a positive integer k_2 large enough such that

$$\frac{\alpha k_2}{k_2 - 1} - \frac{1}{k_2 - 1} < \gamma + \theta < \gamma + \tau. \quad (5.18)$$

In this case, ρ_δ^α is bounded in $L^{\frac{\gamma+\theta}{\alpha}}(Q_T)$ for $\frac{\gamma+\theta}{\alpha} > 1$. Then

$$\begin{aligned}
IV_2 =& \lim_{\delta\to 0^+} \int_0^t \psi \int_\Omega \phi T_k(\rho_\delta)\rho_\delta^\alpha (a_\delta^\alpha - a^\alpha)\, dx\, ds + \lim_{\delta\to 0^+} \int_0^t \psi \int_\Omega \phi T_k(\rho_\delta)\rho_\delta^\alpha a^\alpha\, dx\, ds \\
\geq & -2kC \lim_{\delta\to 0^+} \Big(\int_0^t \int_\Omega \rho_\delta \big| a_\delta^\alpha - a^\alpha \big|^{k_2}\, dx\, ds \Big)^{\frac{1}{k_2}} \Big(\int_0^t \int_\Omega \rho_\delta^{\frac{\alpha k_2}{k_2-1} - \frac{1}{k_2-1}}\, dx\, ds \Big)^{\frac{k_2-1}{k_2}} \\
& + \int_0^t \psi \int_\Omega \phi \overline{T_k(\rho)\rho^\alpha} a^\alpha\, dx\, ds \\
\geq & -2kC\alpha \lim_{\delta\to 0^+} \Big(\int_0^t \int_\Omega \rho_\delta \Big| \big(\max\{a_\delta, a\}\big)^{\alpha-1} \big| a_\delta - a \big| \Big|^{k_2}\, dx\, dt \Big)^{\frac{1}{k_2}} \\
& + \int_0^t \psi \int_\Omega \phi \overline{T_k(\rho)\rho^\alpha} a^\alpha\, dx\, ds \\
\geq & -2kC\alpha \lim_{\delta\to 0^+} \Big(\int_0^t \int_\Omega \rho_\delta \big| a_\delta - a \big|^{k_2}\, dx\, dt \Big)^{\frac{1}{k_2}} + \int_0^t \psi \int_\Omega \phi \overline{T_k(\rho)\rho^\alpha} a^\alpha\, dx\, ds.
\end{aligned} \quad (5.19)$$

In view of Theorem 2.2 with $\nu_K = 0$, (5.19), and the arguments similar to (5.15), we have

$$\begin{aligned}
IV_2 \geq & \int_0^t \psi \int_\Omega \phi \overline{T_k(\rho)}\,\overline{\rho^\alpha} a^\alpha\, dx\, ds \\
= & \int_0^t \psi \int_\Omega \phi \overline{T_k(\rho)}\,\overline{n^\alpha}\, dx\, ds + \int_0^t \psi \int_\Omega \phi \overline{T_k(\rho)} \Big[\overline{\rho^\alpha} a^\alpha - \overline{n^\alpha} \Big]\, dx\, ds \\
= & \int_0^t \psi \int_\Omega \phi \overline{T_k(\rho)}\,\overline{n^\alpha}\, dx\, ds + \lim_{\delta\to 0^+} \int_0^t \psi \int_\Omega \phi \overline{T_k(\rho)} \rho_\delta^\alpha \Big[a^\alpha - a_\delta^\alpha \Big]\, dx\, ds \\
= & \int_0^t \psi \int_\Omega \phi \overline{T_k(\rho)}\,\overline{n^\alpha}\, dx\, ds.
\end{aligned} \quad (5.20)$$

For $\alpha \in [\gamma+\tau, \infty)$**,** we need the initial assumption $\frac{1}{c_0}\rho_0 \leq n_0$ in addition such that

$$\frac{1}{c_0}\rho_\delta \leq n_\delta \quad \text{on } Q_T. \quad (5.21)$$

In fact, the proof of (5.21) is similar to (4.37). By virtue of (5.1) and (5.21), we have

$$\int_{Q_T} \rho_\delta^{\alpha+\theta}\, dx\, dt \le C.$$

In this case, ρ_δ^α is bounded in $L^{\frac{\alpha+\theta}{\alpha}}(Q_T)$ for $\frac{\alpha+\theta}{\alpha} > 1$. Then we can choose the integer k_2 large enough such that

$$\frac{\alpha k_2}{k_2-1} - \frac{1}{k_2-1} < \alpha+\theta.$$

Repeating the arguments in (5.19) and (5.20), we get

$$IV_2 \ge \int_0^t \psi \int_\Omega \phi \overline{T_k(\rho)}\,\overline{n^\alpha}\, dx\, ds. \tag{5.22}$$

For IV_3, we have

$$\begin{aligned} IV_3 &= \lim_{\delta\to 0^+} \int_0^t \psi \int_\Omega \phi T_k(n_\delta)\rho_\delta^\gamma\, dx\, ds \\ &= \lim_{\delta\to 0^+} \int_0^t \psi \int_\Omega \phi T_k\big(a\rho_\delta\big)\rho_\delta^\gamma\, dx\, ds + \lim_{\delta\to 0^+} \int_0^t \psi \int_\Omega \phi \Big[T_k(n_\delta) - T_k\big(a\rho_\delta\big)\Big]\rho_\delta^\gamma\, dx\, ds. \end{aligned}$$

Similar to the proof of (5.16), we have

$$\begin{aligned} &\lim_{\delta\to 0^+} \int_0^t \psi \int_\Omega \phi T_k\big(a\rho_\delta\big)\rho_\delta^\gamma\, dx\, ds \\ &\ge \int_0^t \psi \int_\Omega \phi \overline{T_k\big(a\rho\big)}\,\overline{\rho^\gamma}\, dx\, ds \\ &= \lim_{\delta\to 0^+} \int_0^t \psi \int_\Omega \phi \big[T_k\big(a\rho_\delta\big) - T_k\big(n_\delta\big)\big]\overline{\rho^\gamma}\, dx\, ds + \int_0^t \psi \int_\Omega \phi \overline{T_k\big(n\big)}\,\overline{\rho^\gamma}\, dx\, ds \end{aligned}$$

In view of Theorem 2.2 with $\nu_K = 0$, in particular, of (2.17), we have

$$n_\delta - a\rho_\delta \to 0 \quad \text{a.e. in } Q_T$$

as $\delta \to 0^+$ (take the subsequence if necessary). This implies that

$$T_k\big(a\rho_\delta\big) - T_k\big(n_\delta\big) \to 0 \quad \text{a.e. in } Q_T \tag{5.23}$$

as $\delta \to 0^+$ (take the subsequence if necessary).

This, with the help of the Egrov theorem and $\big[T_k\big(a\rho_\delta\big) - T_k\big(n_\delta\big)\big]\overline{\rho^\gamma} \in L^{\frac{\gamma+\theta}{\gamma}}(Q_T)$ for any given k, yields

$$\lim_{\delta\to 0^+} \int_0^t \psi \int_\Omega \phi \big[T_k\big(a\rho_\delta\big) - T_k\big(n_\delta\big)\big]\overline{\rho^\gamma}\, dx\, ds = 0.$$

Similarly, we have

$$\lim_{\delta\to 0^+} \int_0^t \psi \int_\Omega \phi \Big[T_k(n_\delta) - T_k\big(a\rho_\delta\big)\Big]\rho_\delta^\gamma\, dx\, ds = 0.$$

Thus

$$IV_3 \geq \int_0^t \psi \int_\Omega \phi \overline{T_k(n)}\,\overline{\rho^\gamma}\, dx\, ds. \tag{5.24}$$

(5.14) combined with the estimates of IV_i, i=1,2,3,4, i.e., (5.16), (5.17), (5.20), (5.22), and (5.24), we have

$$\begin{aligned} &\int_0^t \psi \int_\Omega \phi \Big[\overline{T_k(\rho)} + \overline{T_k(n)}\Big]\big(\overline{\rho^\gamma + n^\alpha}\big)\, dx\, ds \\ &\leq \lim_{\delta\to 0^+} \int_0^t \psi \int_\Omega \phi \big[T_k(\rho_\delta) + T_k(n_\delta)\big]\big(\rho_\delta^\gamma + n_\delta^\alpha\big)\, dx\, ds. \end{aligned} \tag{5.25}$$

where we have used

$$\overline{\rho^\gamma} + \overline{n^\alpha} = \overline{\rho^\gamma + n^\alpha}.$$

(5.25) implies (5.13). The proof of the lemma is complete.

□

Since ψ and ϕ are arbitrary, we immediately get

Corollary 5.6 *Let (ρ_δ, n_δ) be the solutions constructed in Proposition 4.9, and (ρ, n) be the limit, then*

$$\Big[\overline{T_k(\rho)} + \overline{T_k(n)}\Big]\big(\overline{\rho^\gamma + n^\alpha}\big) \leq \overline{\big[T_k(\rho) + T_k(n)\big]\big(\rho^\gamma + n^\alpha\big)}$$

a.e. on $\Omega \times (0,T)$.

Step 2: Strong convergence of ρ_δ and n_δ

Here, we want to show the strong convergence of ρ_δ and n_δ. This allows us to have Proposition 5.4. As in [13], we define

$$L_k(z) = \begin{cases} z\log z, \quad 0 \leq z \leq k, \\ z\log k + z\displaystyle\int_k^z \frac{T_k(s)}{s^2}\, ds, \quad z \geq k, \end{cases}$$

satisfying

$$L_k(z) = \beta_k z - 2k \ for\ all\ z \geq 3k,$$

where

$$\beta_k = \log k + \int_k^{3k} \frac{T_k(s)}{s^2}\, ds + \frac{2}{3}.$$

We denote $b_k(z) := L_k(z) - \beta_k z$ where $b_k'(z) = 0$ for all large z, and

$$b_k'(z)z - b_k(z) = T_k(z). \tag{5.26}$$

Note that $\rho_\delta, n_\delta \in L^{\beta+1}(Q_T), \rho, n \in L^{\gamma+\theta}(Q_T)$, and $u_\delta, u \in L^2(0,T;H_0^1(\Omega))$ where $\beta \geq 4$, $\theta < \frac{\gamma}{3}$, $\theta \leq \min\{1, \frac{2\gamma}{3} - 1\}$ and $\gamma > \frac{9}{5}$. By Lemma 2.4, we conclude that (n_δ, u_δ), (ρ_δ, u_δ), (n, u) and (ρ, u) are the renormalized solutions of $(4.28)_i$ and $(5.4)_i$ for $i = 1, 2$, respectively. Thus we have

$$\begin{cases} [b_k(f_\delta)]_t + \operatorname{div}\big[b_k(f_\delta)u_\delta\big] + \big[b_k'(f_\delta)f_\delta - b_k(f_\delta)\big]\operatorname{div}u_\delta = 0 \quad \text{in } \mathcal{D}'(Q_T), \\ [b_k(f)]_t + \operatorname{div}\big[b_k(f)u\big] + \big[b_k'(f)f - b_k(f)\big]\operatorname{div}u = 0 \quad \text{in } \mathcal{D}'(Q_T), \end{cases}$$

where $f_\delta = \rho_\delta, n_\delta$ and $f = \rho, n$. Thanks to (5.26) and $b_k(z) = L_k(z) - \beta_k z$, we arrive at

$$\begin{cases}[L_k(\rho_\delta) + L_k(n_\delta)]_t + \mathrm{div}\big[\big(L_k(\rho_\delta) + L_k(n_\delta)\big)u_\delta\big] + \big[T_k(\rho_\delta) + T_k(n_\delta)\big]\mathrm{div}u_\delta = 0 \quad \text{in } \mathcal{D}'(Q_T),\\ [L_k(\rho) + L_k(n)]_t + \mathrm{div}\big[\big(L_k(\rho) + L_k(n)\big)u\big] + \big[T_k(\rho) + T_k(n)\big]\mathrm{div}u = 0 \quad \text{in } \mathcal{D}'(Q_T).\end{cases}$$

This gives

$$\begin{aligned}&[L_k(\rho_\delta) - L_k(\rho) + L_k(n_\delta) - L_k(n)]_t + \mathrm{div}\big[\big(L_k(\rho_\delta) + L_k(n_\delta)\big)u_\delta - \big(L_k(\rho) + L_k(n)\big)u\big]\\ &+ \big[T_k(\rho_\delta) + T_k(n_\delta)\big]\mathrm{div}u_\delta - \big[T_k(\rho) + T_k(n)\big]\mathrm{div}u = 0.\end{aligned} \tag{5.27}$$

Taking ϕ_j as the test function of (5.27), and letting $\delta \to \infty$, we have

$$\begin{aligned}&\int_\Omega [\overline{L_k(\rho)} - L_k(\rho) + \overline{L_k(n)} - L_k(n)]\phi_j \, dx\\ &- \lim_{\delta\to 0^+}\int_0^t \int_\Omega \big[\big(L_k(\rho_\delta) + L_k(n_\delta)\big)u_\delta - \big(L_k(\rho) + L_k(n)\big)u\big] \cdot \nabla\phi_j \, dx\, ds\\ &+ \lim_{\delta\to 0^+}\int_0^t \int_\Omega \Big(\big[T_k(\rho_\delta) + T_k(n_\delta)\big]\mathrm{div}u_\delta - \big[T_k(\rho) + T_k(n)\big]\mathrm{div}u\Big)\phi_j \, dx\, ds = 0,\end{aligned} \tag{5.28}$$

where

$$\begin{aligned}&\phi_j \in C_0^\infty(\Omega), \quad \phi_j(x) \equiv 1 \text{ for any } x \in \big\{x \in \Omega \big| \mathrm{dist}(\mathrm{x}, \partial\Omega) \geq \frac{1}{\mathrm{j}}\big\},\ 0 \leq \phi_j \leq 1,\\ &|\nabla\phi_j| \leq c_0 j,\ \phi_j \to 1 \quad as\ m \to \infty\end{aligned} \tag{5.29}$$

for some positive c_0 independent of j.

Letting $j \to \infty$ in (5.28), we gain

$$\begin{aligned}&\int_\Omega [\overline{L_k(\rho)} - L_k(\rho) + \overline{L_k(n)} - L_k(n)] \, dx\\ &= -\lim_{\delta\to 0^+}\int_0^t \int_\Omega \Big(\big[T_k(\rho_\delta) + T_k(n_\delta)\big]\mathrm{div}u_\delta - \big[T_k(\rho) + T_k(n)\big]\mathrm{div}u\Big) \, dx\, ds.\end{aligned} \tag{5.30}$$

In view of Lemma 5.3, we have

$$\begin{aligned}&-\lim_{\delta\to 0^+}\int_0^t \int_\Omega \big[T_k(\rho_\delta) + T_k(n_\delta)\big]\mathrm{div}u_\delta \, dx\, ds\\ =&-\frac{1}{2\mu+\lambda}\lim_{\delta\to 0^+}\int_0^t \int_\Omega \big[T_k(\rho_\delta) + T_k(n_\delta)\big]\big[(2\mu+\lambda)\mathrm{div}u_\delta - \rho_\delta^\gamma - n_\delta^\alpha\big] \, dx\, ds\\ &-\frac{1}{2\mu+\lambda}\lim_{\delta\to 0^+}\int_0^t \int_\Omega \big[T_k(\rho_\delta) + T_k(n_\delta)\big]\big[\rho_\delta^\gamma + n_\delta^\alpha\big] \, dx\, ds\\ =&-\frac{1}{2\mu+\lambda}\int_0^t \int_\Omega \psi_j\phi_j\big[\overline{T_k(\rho)} + \overline{T_k(n)}\big]\big[(2\mu+\lambda)\mathrm{div}u - \overline{\rho^\gamma + n^\alpha}\big] \, dx\, ds\\ &-\frac{1}{2\mu+\lambda}\lim_{\delta\to 0^+}\int_0^t \int_\Omega (1-\psi_j\phi_j)\big[T_k(\rho_\delta) + T_k(n_\delta)\big]\big[(2\mu+\lambda)\mathrm{div}u_\delta - \rho_\delta^\gamma - n_\delta^\alpha\big] \, dx\, ds\\ &-\frac{1}{2\mu+\lambda}\lim_{\delta\to 0^+}\int_0^t \int_\Omega \big[T_k(\rho_\delta) + T_k(n_\delta)\big]\big[\rho_\delta^\gamma + n_\delta^\alpha\big] \, dx\, ds,\end{aligned} \tag{5.31}$$

where ψ_j and ϕ_j are given by (4.23) and (4.24) respectively. Letting $j \to \infty$ in (5.31), we have

$$\begin{aligned}
&-\lim_{\delta\to 0^+}\int_0^t\int_\Omega \big[T_k(\rho_\delta)+T_k(n_\delta)\big]\mathrm{div}u_\delta\,dx\,ds\\
=&-\frac{1}{2\mu+\lambda}\int_0^t\int_\Omega \big[\overline{T_k(\rho)}+\overline{T_k(n)}\big]\big[(2\mu+\lambda)\mathrm{div}u-\overline{\rho^\gamma+n^\alpha}\big]\,dx\,ds\\
&-\frac{1}{2\mu+\lambda}\lim_{\delta\to 0^+}\int_0^t\int_\Omega \big[T_k(\rho_\delta)+T_k(n_\delta)\big]\big[\rho_\delta^\gamma+n_\delta^\alpha\big]\,dx\,ds.
\end{aligned}\tag{5.32}$$

In view of (5.30) and (5.32), we have

$$\begin{aligned}
&\int_\Omega[\overline{L_k(\rho)}-L_k(\rho)+\overline{L_k(n)}-L_k(n)]\,dx\\
=&\frac{1}{2\mu+\lambda}\int_0^t\int_\Omega\big(\overline{T_k(\rho)}+\overline{T_k(n)}\big)\big(\overline{\rho^\gamma+n^\alpha}\big)\,dx\,ds\\
&-\frac{1}{2\mu+\lambda}\lim_{\delta\to 0^+}\int_0^t\int_\Omega\big[T_k(\rho_\delta)+T_k(n_\delta)\big]\big(\rho_\delta^\gamma+n_\delta^\alpha\big)\,dx\,ds\\
&+\int_0^t\int_\Omega[T_k(\rho)-\overline{T_k(\rho)}+T_k(n)-\overline{T_k(n)}]\mathrm{div}u\,dx\,ds,
\end{aligned}$$

with Corollary 5.6, which gives

$$\int_\Omega[\overline{L_k(\rho)}-L_k(\rho)+\overline{L_k(n)}-L_k(n)]\,dx\le\int_0^t\int_\Omega[T_k(\rho)-\overline{T_k(\rho)}+T_k(n)-\overline{T_k(n)}]\mathrm{div}u\,dx\,ds.\tag{5.33}$$

Here we are able to control the right-hand side of (5.33) as in the following lemma.

Lemma 5.7

$$\lim_{k\to\infty}\int_0^t\int_\Omega[T_k(\rho)-\overline{T_k(\rho)}+T_k(n)-\overline{T_k(n)}]\mathrm{div}u\,dx\,ds=0.\tag{5.34}$$

Proof. Recalling that $T(z)\le z$ for all z, we have

$$\begin{aligned}
\|T_k(\rho)-\overline{T_k(\rho)}\|_{L^2(Q_T)}\le&\liminf_{\delta\to 0^+}\|T_k(\rho)-T_k(\rho_\delta)\|_{L^2(Q_T)}\\
\le&C\liminf_{\delta\to 0^+}\|\rho+\rho_\delta\|_{L^{\gamma+\theta}(Q_T)}\\
\le&C,
\end{aligned}$$

where we have used the Hölder inequality, $\gamma+\theta\ge 2$, (5.1) and (5.3). With the help of this estimate and (5.3), one deduces

$$\begin{aligned}
&\Big|\int_0^t\int_\Omega[T_k(\rho)-\overline{T_k(\rho)}]\mathrm{div}u\,dx\,ds\Big|\\
\le&\int_{Q_t\cap\{\rho\ge k\}}|T_k(\rho)-\overline{T_k(\rho)}|\,|\mathrm{div}u|\,dx\,ds+\int_{Q_t\cap\{\rho\le k\}}|T_k(\rho)-\overline{T_k(\rho)}|\,|\mathrm{div}u|\,dx\,ds\\
\le&\|T_k(\rho)-\overline{T_k(\rho)}\|_{L^2(Q_T)}\|\mathrm{div}u\|_{L^2\left(Q_t\cap\{\rho\ge k\}\right)}+\|T_k(\rho)-\overline{T_k(\rho)}\|_{L^2\left(Q_t\cap\{\rho\le k\}\right)}\|\mathrm{div}u\|_{L^2(Q_T)}\\
\le&C\|\mathrm{div}u\|_{L^2\left(Q_t\cap\{\rho\ge k\}\right)}+C\|T_k(\rho)-\overline{T_k(\rho)}\|_{L^2\left(Q_t\cap\{\rho\le k\}\right)}.
\end{aligned}\tag{5.35}$$

Note that $T_k(z) = z$ if $z \le k$, we have

$$\begin{aligned}
\|T_k(\rho) - \overline{T_k(\rho)}\|_{L^2\left(Q_t \cap \{\rho \le k\}\right)} =& \|\rho - \overline{T_k(\rho)}\|_{L^2\left(Q_t \cap \{\rho \le k\}\right)} \\
\le& \liminf_{\delta \to 0^+} \|\rho_\delta - T_k(\rho_\delta)\|_{L^2(Q_T)} \\
=& \liminf_{\delta \to 0^+} \|\rho_\delta - T_k(\rho_\delta)\|_{L^2(Q_T \cap \{\rho_\delta > k\})} \\
\le& 2 \liminf_{\delta \to 0^+} \|\rho_\delta\|_{L^2(Q_T \cap \{\rho_\delta > k\})} \\
\le& 2k^{1 - \frac{\gamma+\theta}{2}} \liminf_{\delta \to 0^+} \|\rho_\delta\|^{\frac{\gamma+\theta}{2}}_{L^{\gamma+\theta}(Q_T)} \to 0
\end{aligned} \tag{5.36}$$

as $k \to \infty$, due to (5.1) and the assumption $\gamma > \frac{9}{5}$ such that $\gamma + \theta > 2$.

By (5.35) and (5.36), we conclude

$$\lim_{k \to \infty} \int_0^t \int_\Omega [T_k(\rho) - \overline{T_k(\rho)}] \mathrm{div} u \, dx \, ds = 0. \tag{5.37}$$

Similarly, we have

$$\lim_{k \to \infty} \int_0^t \int_\Omega [T_k(n) - \overline{T_k(n)}] \mathrm{div} u \, dx \, ds = 0. \tag{5.38}$$

With (5.37) and (5.38), (5.34) follows. □

Note that (5.33) and (5.34), we have

$$\limsup_{k \to \infty} \int_\Omega [\overline{L_k(\rho)} - L_k(\rho) + \overline{L_k(n)} - L_k(n)] \, dx \le 0. \tag{5.39}$$

By the definition of $L(\cdot)$, it is not difficult to justify that

$$\begin{cases}
\lim\limits_{k \to \infty} \left[\|L_k(\rho) - \rho \log \rho\|_{L^1(\Omega)} + \|L_k(n) - n \log n\|_{L^1(\Omega)} \right] = 0, \\
\lim\limits_{k \to \infty} \left[\|\overline{L_k(\rho)} - \overline{\rho \log \rho}\|_{L^1(\Omega)} + \|\overline{L_k(n)} - \overline{n \log n}\|_{L^1(\Omega)} \right] = 0.
\end{cases} \tag{5.40}$$

Since $\rho \log \rho \le \overline{\rho \log \rho}$ and $n \log n \le \overline{n \log n}$ due to the convexity of $z \mapsto z \log z$, we have

$$0 \le \int_\Omega [\overline{\rho \log \rho} - \rho \log \rho + \overline{n \log n} - n \log n] \, dx \le 0, \tag{5.41}$$

where we have used (5.39) and (5.40). Thus we obtain

$$\overline{\rho \log \rho} = \rho \log \rho \quad \text{and} \quad \overline{n \log n} = n \log n.$$

It allows us to have the strong convergence of ρ_δ and n_δ in $L^\gamma(Q_T)$ and in $L^\alpha(Q_T)$ respectively. Therefore we proved (5.12).

□

With Proposition 5.4, the proof of Theorem 1.2 can be done.

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