

# Nonlinear stability of viscous shock wave to one-dimensional compressible isentropic Navier-Stokes equations with density dependent viscous coefficient

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**Abstract:** We prove nonlinear stability of viscous shock wave of arbitrary amplitudes to a one-dimensional compressible isentropic Navier-Stokes equations with density dependent viscosity. Under the assumption that the viscous coefficient is given as a power function of density, any viscous shock wave is shown to be nonlinear stable for small initial perturbations with integral zero. In contrast to previous related results [20, 22], there is no restrictions on the power index of the viscous coefficient and the amplitudes of the viscous shock wave in our result.

**Key Words:** Nonlinear stability, compressible isentropic Navier-Stokes equations, viscous shock wave, energy estimate.

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## 1 Introduction

In this paper, we study the nonlinear stability of viscous shock wave(traveling wave solution) to the following compressible isentropic Navier-Stokes equations:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = \left( \frac{\mu(v)}{v} u_x \right)_x, \end{cases} \quad (1.1)$$

with initial data

$$(v, u)(x, 0) = (v_0, u_0)(x), \quad x \in \mathbb{R}^1, \quad (1.2)$$

and far field condition

$$(v, u)(x, t) \rightarrow (v_{\pm}, u_{\pm}), \quad \text{as } x \rightarrow \pm\infty, \quad (1.3)$$

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where  $v(> 0)$  is the specific volume,  $u$  is the fluid velocity,  $p(v) = av^{-\gamma}$  is the pressure,  $\gamma(\geq 1)$  is the adiabatic constant,  $\mu(v) = bv^{-\alpha}$  ( $\alpha \in \mathbb{R}^1$ ) is the viscosity coefficient, and  $a, b$  are giving positive constants, which will be normalized to be 1.

The viscosity coefficient is often assumed to be a positive constant. However, it is well-known that the viscosity of the flow is not constant and depends on the temperature. For example, we can get the viscosity is proportional to the square root of the temperature when we use the Chapman-Enskog expansion to derive Navier-Stokes equations from Boltzmann equation, cf. [4, 5]. Especially, for isentropic flow, this dependence of the viscosity is translated into the dependence on the density. For more physical background, please refer to [12, 22] and references therein.

About the compressible Navier-Stokes equations (1.1) in Euler coordinates, Matsumura-Nishida [18, 19] considered the global existence of the smooth solutions in multi-dimensional whole space and obtained the global solutions tended to its equilibrium state in large time. They obtained decay rates of Navier-Stokes equations as

$$\|(\rho - \bar{\rho}, u)(t)\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{4}}, \quad t \geq 0,$$

where  $\bar{\rho} > 0$  is a constant. We are interested in the asymptotic behavior of nonconstant equilibrium state about the model (1.1), such as the nonlinear stability of its viscous shock wave. It is easily seen that (1.1) admits a traveling wave solution with shock profile, which can be called viscous shock wave

$$(v, u) = (\tilde{v}, \tilde{u})(x - st), \quad (\tilde{v}, \tilde{u})(\pm\infty) = (v_{\pm}, u_{\pm}), \quad (1.4)$$

under Rankine-Hugoniot and entropy conditions(see (1.7) and (1.9)), here  $s$  is the shock speed and  $v_{\pm} > 0$ ,  $u_{\pm} \in \mathbb{R}^1$  are far field states, given in (1.3).

Now we look for the viscous shock wave solution  $(\tilde{v}, \tilde{u})(x - st)$  to system (1.1). Let  $\xi = x - st$ , substituting  $(v, u)(\xi)$  into (1.1) and (1.3), one has the following system of differential equations:

$$\begin{cases} -s\tilde{v}_{\xi} - \tilde{u}_{\xi} = 0, \\ -s\tilde{u}_{\xi} + p(\tilde{v})_{\xi} = (\frac{\tilde{u}_{\xi}}{\tilde{v}^{\alpha+1}})_{\xi}, \\ (\tilde{v}, \tilde{u})(\pm\infty) = (v_{\pm}, u_{\pm}). \end{cases} \quad (1.5)$$

Integrating (1.5) with respect to  $\xi$  over  $(\pm\infty, \xi)$ , and using the fact that  $\tilde{u}_{\xi} \rightarrow 0$ , as  $\xi \rightarrow \pm\infty$ , we have

$$\begin{cases} s\tilde{v} + \tilde{u} = sv_{\pm} + u_{\pm}, \\ -s\tilde{u} + p(\tilde{v}) - \frac{\tilde{u}_{\xi}}{\tilde{v}^{\alpha+1}} = -su_{\pm} + p(v_{\pm}), \end{cases} \quad (1.6)$$

and this implies the the Rankine-Hugoniot jump condition for shock waves

$$\begin{cases} -s(v_+ - v_-) = (u_+ - u_-), \\ -s(u_+ - u_-) = -(p(v_+) - p(v_-)). \end{cases} \quad (1.7)$$

From which, we can obtain

$$s = \pm \sqrt{-\frac{p(v_+) - p(v_-)}{v_+ - v_-}}. \quad (1.8)$$

We only consider the case  $s < 0$  (first shock), and the analysis for  $s > 0$  (second shock) is similar, which together with the Rankine-Hugoniot jump condition imply the entropy condition

$$v_- > v_+, \quad u_- > u_+. \quad (1.9)$$

Under condition (1.9), it is easily to check that the ODE. (1.6) has a global smooth solution  $(\tilde{v}, \tilde{u})(\xi)$  (see [10, 20]), which implies the existence of viscous shock wave solution of (1.1) which translates at shock speed  $s$  and interpolate the asymptotic values  $(v_{\pm}, u_{\pm})$  at  $x = \pm\infty$ .

The stability of the viscous shock wave for system (1.1) is a very important problem from both mathematical and physical points of view. When the viscosity coefficient  $\mu$  is a positive constant, Matsumura and Nishihara [20] showed the viscous shock wave are asymptotically stable, provided the initial disturbance is suitably small and of integral zero, but there are additional conditions: “ $(\gamma - 1)(\text{total variation of initial data})$  is small”, this amounted to the amplitude of viscous shock was small. For the results with nonzero integral, Mascia-Zumbrun [17] obtained the asymptotic stability of viscous shock wave with small amplitude for (1.1) and related physical systems; Liu-Zeng [16] proved a similar result in their treatment of systems with artificial viscosity. When the viscosity coefficient  $\mu$  depends on the density, i.e.,  $\mu = v^{-\alpha}$ , Matsumura-Wang [22] proved the stability of viscous shock wave for small initial perturbations with integral zero by a weighted energy method (cf. [6, 21]), in order to deal with the nonlinear terms, they needed extra assumption  $\alpha \geq \frac{1}{2}(\gamma - 1)$ .

But there are no relevant results for the nonlinear stability of viscous shock with large amplitudes and arbitrary power index of the viscous coefficient  $\mu (= v^{-\alpha})$ , i.e.,  $\alpha \in \mathbb{R}^1$  is arbitrary. The main aim of this paper is to extend Matsumura’s result [20, 22] in these two aspects. Here, it is worth to mention the recent result of Vasseur, et al, they used relative entropy method to study the stability of shock (or viscous shock) waves for the scalar or system of conservation laws, we can refer to the results in [11, 14, 15] and references cited therein.

In order to deal with the nonlinear term better, we introduce the new effective velocity  $h = u + \frac{1}{\alpha}(v^{-\alpha})_x$ , if  $\alpha \neq 0$ ;  $h = u - (\ln v)_x$ , if  $\alpha = 0$ , then  $(v, h)$  satisfies

$$\begin{cases} v_t - h_x = \left(\frac{v_x}{v^{\alpha+1}}\right)_x, \\ h_t + p(v)_x = 0, \end{cases} \quad (1.10)$$

with initial data

$$(v, h)(x, 0) = (v_0, h_0)(x), \quad x \in \mathbb{R}^1, \quad (1.11)$$

and far field condition

$$(v, h)(x, t) \rightarrow (v_{\pm}, u_{\pm}), \quad \text{as } x \rightarrow \pm\infty, \quad t \geq 0. \quad (1.12)$$

Let us mention that this change of unknown transforms the system (1.1) as a parabolic equation on the specific volume and a transport equation for the new effective velocity, this new system will help us to deal with the nonlinear terms  $G$  and  $p(v|\tilde{v})$  in (2.3) without any restriction on  $\alpha$ . It is worth to mention that Bresch-Desjardins [1, 2, 3] and Shelukhin [23] used the new velocity  $h$  (in Euler coordinates) to obtain the entropy estimates, for density

dependent viscosity in multi-dimensional and constant viscosity in one-dimensional respectively, which gives the estimate of derivative of the density. Recently, by introducing new effective velocity  $h$  (in Euler coordinates), Haspot gave a new formulation of the compressible Navier-Stokes to deal with the well-posedness of the compressible Navier-Stokes equations with density dependent viscosity, please refer to [7, 8, 9] and references therein.

Furthermore, let  $\tilde{h} = \tilde{u} + \frac{1}{\alpha}(\tilde{v}^{-\alpha})_\xi$ ,  $\alpha \neq 0$ ;  $\tilde{h} = \tilde{u} - (\ln \tilde{v})_\xi$ , if  $\alpha = 0$ , then by the existence result of (1.5), the following equivalent problem

$$\begin{cases} -s\tilde{v}_\xi - \tilde{h}_\xi = (\frac{\tilde{v}_\xi}{\tilde{v}^{\alpha+1}})_\xi, \\ -s\tilde{h}_\xi + p(\tilde{v})_\xi = 0, \\ (\tilde{v}, \tilde{h})(\pm\infty) = (v_\pm, u_\pm), \end{cases} \quad (1.13)$$

has a global smooth solution  $(\tilde{v}, \tilde{h})(\xi)$ , with

$$\tilde{v}_\xi = \frac{\tilde{v}^{\alpha+1}}{s} [s^2(v_- - \tilde{v}) + p(v_-) - p(\tilde{v})] \triangleq \frac{\tilde{v}^{\alpha+1}}{s} H(\tilde{v}) < 0, \quad (1.14)$$

and  $|\tilde{v}_\xi|, |\tilde{v}_{\xi\xi}|, |\tilde{v}_{\xi\xi\xi}|$  are bounded.

Assume that

$$\begin{cases} (v_0 - \tilde{v}, h_0 - \tilde{h}) \in H^1 \cap L^1, \quad \inf_{x \in \mathbb{R}^1} v_0(x) > 0, \\ \int (v_0 - \tilde{v})(x) dx = 0, \quad \int (h_0 - \tilde{h})(x) dx = 0, \end{cases} \quad (1.15)$$

and

$$V_0(x) = \int_{-\infty}^x [v_0(z) - \tilde{v}(z)] dz, \quad H_0(x) = \int_{-\infty}^x [h_0(z) - \tilde{h}(z)] dz. \quad (1.16)$$

We further assume that

$$(V_0, H_0) \in L^2. \quad (1.17)$$

The main theorem of this paper is as follows.

**Theorem 1.1.** *Under the assumptions of (1.15) and (1.17), for any  $\alpha \in \mathbb{R}^1$ , there exists a positive constant  $\delta_0$  such that if  $\|V_0, H_0\|_2 \leq \delta_0$ , then the Cauchy problem (1.1)-(1.3) has a unique global solution  $(v, u)$ , satisfies*

$$(v - \tilde{v}, h - \tilde{h}) \in C^0([0, +\infty); H^1), \quad v - \tilde{v} \in L^2([0, +\infty); H^2), \quad h - \tilde{h} \in L^2([0, +\infty); H^1), \quad (1.18)$$

and

$$\sup_{x \in \mathbb{R}^1} |v(x, t) - \tilde{v}(x - st)| \rightarrow 0, \quad \sup_{x \in \mathbb{R}^1} |u(x, t) - \tilde{u}(x - st)| \rightarrow 0, \quad \text{as } t \rightarrow \infty, \quad (1.19)$$

here  $h = u + \frac{1}{\alpha}(v^{-\alpha})_x$ ,  $\tilde{h} = \tilde{u} + \frac{1}{\alpha}(\tilde{v}^{-\alpha})_\xi$ , if  $\alpha \neq 0$ ;  $h = u - (\ln v)_x$ ,  $\tilde{h} = \tilde{u} - (\ln \tilde{v})_\xi$ , if  $\alpha = 0$ .

**Remark 1.1.** Compared to the results of [20, 22], there is no restrictions on  $\alpha$  and the amplitude of the viscous shock wave.

Theorem 1.1 is the direct consequence of the following result about the equivalent problem (1.10)-(1.12).

**Theorem 1.2.** Under the assumptions of (1.15) and (1.17), for any  $\alpha \in \mathbb{R}^1$ , there exists a positive constant  $\delta_0$  such that if  $\|V_0, H_0\|_2 \leq \delta_0$ , then the Cauchy problem (1.10)-(1.12) has a unique global solution  $(v, h)$ , satisfies

$$(v - \tilde{v}, h - \tilde{h}) \in C^0([0, +\infty); H^1), \quad v - \tilde{v} \in L^2([0, +\infty); H^2), \quad h - \tilde{h} \in L^2([0, +\infty); H^1), \quad (1.20)$$

and

$$\sup_{x \in \mathbb{R}^1} |(v, h)(x, t) - (\tilde{v}, \tilde{h})(x - st)| \rightarrow 0, \quad \text{as } t \rightarrow \infty. \quad (1.21)$$

The rest of this paper is organized as follows. In Section 2, we reformulate the problem in terms of the anti-derivatives of the deviation functions from the viscous shock wave. In Section 3, we establish the *a priori* estimates, and prove the Theorem 2.1. And we complete the proof of the main theorem in Section 4.

**Notations:** In the following,  $C$  from line to line denote the generic positive constants depending only on the initial data and the physical coefficients but independent of time  $T$ . We use the standard notations  $L^p$ ,  $H^s$  to denote the  $L^p$  and Sobolev space in  $\mathbb{R}^1$ , with norms  $\|\cdot\|_{L^p}$  and  $\|\cdot\|_s$  respectively. For simplicity,  $\|\cdot\| := \|\cdot\|_{L^2(\mathbb{R}^1)}$ .

## 2 Reformulation of the problem

For the convenience of the analysis, changing the variables  $(x, t) \rightarrow (\xi = x - st, t)$  in (1.10), we have

$$\begin{cases} v_t - sv_\xi - h_\xi = \left(\frac{v_\xi}{v^{\alpha+1}}\right)_\xi, \\ h_t - sh_\xi + p(v)_\xi = 0. \end{cases} \quad (2.1)$$

Next, define the new functions  $(V, H)(\xi, t)$  as follows

$$V(\xi, t) = \int_{-\infty}^{\xi} [v(z, t) - \tilde{v}(z)] dz, \quad H(\xi, t) = \int_{-\infty}^{\xi} [h(z, t) - \tilde{h}(z)] dz, \quad (2.2)$$

then with (1.13), we have

$$\begin{cases} V_t - sV_\xi - H_\xi - \frac{1}{\tilde{v}^{\alpha+1}} V_{\xi\xi} = \left(\frac{1}{v^{\alpha+1}} - \frac{1}{\tilde{v}^{\alpha+1}}\right) V_{\xi\xi} + \left(\frac{1}{v^{\alpha+1}} - \frac{1}{\tilde{v}^{\alpha+1}}\right) \tilde{v}_\xi \triangleq G, \\ H_t - sH_\xi + p'(\tilde{v})V_\xi = -[p(v) - p(\tilde{v}) - p'(\tilde{v})(v - \tilde{v})] \triangleq -p(v|\tilde{v}), \end{cases} \quad (2.3)$$

with the initial data

$$(V, H)(\xi, 0) = (V_0, H_0)(\xi) \in H^2, \quad (2.4)$$

where

$$V_0(\xi) = \int_{-\infty}^{\xi} [v_0(z) - \tilde{v}(z)]dz, \quad H_0(\xi) = \int_{-\infty}^{\xi} [h_0(z) - \tilde{h}(z)]dz.$$

We look for the global existence of the solution to the problem (2.3)-(2.4). At first, we define the solution space  $X(0, T)$  for any  $0 \leq T < +\infty$ , we define

$$X(0, T) = \{(V, H) \in C(0, T; H^2) | V_{\xi} \in L^2(0, T; H^2), H_{\xi} \in L^2(0, T; H^1), \\ \sup_{0 \leq t \leq T} \|(V, H)(t)\|_2 \leq \frac{1}{2}v_-\}.$$

By the previous arguments as in [13, 18, 19, 20], the global smooth solution in  $X(0, \infty)$  is constructed by the combination of the local existence and the *a priori* estimate. The proof of the existence of local solution is standard, and we mainly concern about the *a priori* estimate.

**Proposition 2.1.** *For any  $\alpha \in \mathbb{R}^1$ , there exists a positive constant  $\delta$ , such that if  $(V, H) \in X(0, T)$  is the solution of the Cauchy problem (2.3)-(2.4) and  $\sup_{t \in [0, T]} \|(V, H)(t)\|_2 \leq \delta$ , it holds for  $t \in [0, T]$  that*

$$\|(V, H)(t)\|_2^2 + \int_0^t (\|V_{\xi}(\tau)\|_2^2 + \|H_{\xi}(\tau)\|_1^2) d\tau \leq C\|(V_0, H_0)\|_2^2, \quad (2.5)$$

here  $C$  is a positive constant independent of  $T$ .

**Remark 2.1.** *In the proof of Proposition 2.1, we can't obtain  $\|(V, H)(t)\| \leq C\|(V_0, H_0)\|$  directly, it will be controlled by the another term  $C\delta \int_0^t \int V_{\xi\xi}^2 d\xi d\tau$ , see Lemma 3.3. Using the smallness of  $\delta$ , we enclose the one order derivative estimates in Lemma 3.4, and obtain  $\|(V, H)(t)\| \leq C\|(V_0, H_0)\|_1$ .*

Once the Proposition 2.1 is obtained, we can show the following global existence theorem, which implies Theorem 1.2 by defining  $v = \tilde{v} + V_{\xi}$ ,  $h = \tilde{h} + H_{\xi}$ .

**Theorem 2.1.** *Assume  $(V_0, H_0) \in H^2$ , then for any  $\alpha \in \mathbb{R}^1$ , there exists a positive constant  $\delta_0$  such that if  $\|(V_0, H_0)\|_2 \leq \delta_0$ , then the Cauchy problem (2.3)-(2.4) has a unique global solution  $(V, H) \in X(0, \infty)$ , satisfies*

$$\sup_{x \in \mathbb{R}^1} |(V, H)_{\xi}(\xi, t)| \rightarrow 0, \quad \text{as } t \rightarrow \infty. \quad (2.6)$$

### 3 Proof of Proposition 2.1

Throughout this section, we assume that the problem (2.3)-(2.4) has a solution  $(V, H) \in X(0, T)$  for some  $T > 0$ . We will derive the *a priori* energy estimates for the system (2.3)-(2.4). To begin with, we make the following *a priori* assumptions for sufficiently small  $\delta > 0$ ,

$$\sup_{t \in [0, T]} \|(V, H)(t)\|_2 \leq \delta, \quad (3.1)$$

where  $T \in (0, \infty]$ , which implies

$$\sup_{t \in [0, T]} \{ \|(V, H)(t)\|_{L^{\infty}} + \|(v - \tilde{v}, h - \tilde{h})(t)\|_{L^{\infty}} \} \leq \delta, \quad (3.2)$$

where  $T \in (0, \infty]$ .

Let us start with some useful inequalities and inequalities.

**Lemma 3.1.** *Under the assumption of (3.1), we have*

$$p(v|\tilde{v}) \leq CV_\xi^2, \quad (3.3)$$

$$|p(v|\tilde{v})_\xi| \leq C(|V_{\xi\xi}|V_\xi + |\tilde{v}_\xi|V_\xi^2), \quad (3.4)$$

$$|p(v|\tilde{v})_{\xi\xi}| \leq C(|V_{\xi\xi\xi}|V_\xi + V_\xi^2 + V_{\xi\xi}^2 + |\tilde{v}_\xi|V_\xi|V_{\xi\xi}|). \quad (3.5)$$

*Proof.* By the representation of  $p(v|\tilde{v}) = v^{-\gamma} - \tilde{v}^{-\gamma} + \gamma\tilde{v}^{-\gamma-1}(v - \tilde{v})$ , through the tedious calculations, we have

$$p(v|\tilde{v})_\xi = -\gamma V_{\xi\xi}(v^{-\gamma-1} - \tilde{v}^{-\gamma-1}) - \gamma\tilde{v}_\xi(v^{-\gamma-1} - \tilde{v}^{-\gamma-1} + (\gamma+1)\tilde{v}^{-\gamma-2}V_\xi), \quad (3.6)$$

and

$$\begin{aligned} p(v|\tilde{v})_{\xi\xi} &= -\gamma V_{\xi\xi\xi}(v^{-\gamma-1} - \tilde{v}^{-\gamma-1}) - \gamma\tilde{v}_{\xi\xi}(v^{-\gamma-1} - \tilde{v}^{-\gamma-1} + (\gamma+1)\tilde{v}^{-\gamma-2}V_\xi) \\ &\quad + \gamma(\gamma+1)v^{-\gamma-2}V_{\xi\xi}^2 + \gamma(\gamma+1)\tilde{v}_\xi^2[v^{-\gamma-2} - \tilde{v}^{-\gamma-2} + (\gamma+2)\tilde{v}^{-\gamma-3}V_\xi] \\ &\quad + 2\gamma(\gamma+1)V_{\xi\xi}\tilde{v}_\xi(v^{-\gamma-2} - \tilde{v}^{-\gamma-2}). \end{aligned} \quad (3.7)$$

Then by mean value theore, Taloy's formula and (3.2), we complete the proof of Lemma 3.1.  $\square$

**Lemma 3.2.** *Under the assumption of (3.1), we have*

$$|G| \leq C(|V_\xi|V_{\xi\xi} + |\tilde{v}_\xi|V_\xi), \quad (3.8)$$

$$|G_\xi| \leq C(V_{\xi\xi}^2 + |V_\xi|V_{\xi\xi} + |V_\xi|V_{\xi\xi\xi} + |V_{\xi\xi}| + |V_\xi|). \quad (3.9)$$

*Proof.* By the representation of  $G = (\frac{1}{v^{\alpha+1}} - \frac{1}{\tilde{v}^{\alpha+1}})V_{\xi\xi} + (\frac{1}{v^{\alpha+1}} - \frac{1}{\tilde{v}^{\alpha+1}})\tilde{v}_\xi$ , we have

$$\begin{aligned} G_\xi &= -(\alpha+1)\frac{V_{\xi\xi}\tilde{v}^{\alpha+2} + \tilde{v}_\xi[\tilde{v}^{\alpha+2} - (V_\xi + \tilde{v})^{\alpha+2}]}{v^{\alpha+2}\tilde{v}^{\alpha+2}}V_{\xi\xi} \\ &\quad + (\frac{1}{v^{\alpha+1}} - \frac{1}{\tilde{v}^{\alpha+1}})V_{\xi\xi\xi} \\ &\quad - (\alpha+1)\frac{V_{\xi\xi}\tilde{v}^{\alpha+2} + \tilde{v}_\xi[\tilde{v}^{\alpha+2} - (V_\xi + \tilde{v})^{\alpha+2}]}{v^{\alpha+2}\tilde{v}^{\alpha+2}}\tilde{v}_\xi \\ &\quad + (\frac{1}{v^{\alpha+1}} - \frac{1}{\tilde{v}^{\alpha+1}})\tilde{v}_{\xi\xi} \\ &= -(\alpha+1)\frac{V_{\xi\xi}\tilde{v}^{\alpha+2} - (\alpha+2)\tilde{v}_\xi(\tilde{v} + \theta V_\xi)^{\alpha+1}V_\xi}{v^{\alpha+2}\tilde{v}^{\alpha+2}}V_{\xi\xi} \\ &\quad + (\frac{1}{v^{\alpha+1}} - \frac{1}{\tilde{v}^{\alpha+1}})V_{\xi\xi\xi} \\ &\quad - (\alpha+1)\frac{V_{\xi\xi}\tilde{v}^{\alpha+2} - (\alpha+2)\tilde{v}_\xi(\tilde{v} + \theta V_\xi)^{\alpha+1}V_\xi}{v^{\alpha+2}\tilde{v}^{\alpha+2}}\tilde{v}_\xi \\ &\quad + (\frac{1}{v^{\alpha+1}} - \frac{1}{\tilde{v}^{\alpha+1}})\tilde{v}_{\xi\xi}, \quad \text{for } \theta \in (0, 1), \end{aligned} \quad (3.10)$$

and using the smallness of  $|V_\xi|$  by (3.2), the boundness of  $|\tilde{v}_\xi|$  and  $|\tilde{v}_{\xi\xi}|$ , the proof of Lemma 3.2 is completed.  $\square$

With the above useful inequalities at hand, we begin to give the *a priori* estimates, the first is the basic energy estimate.

**Lemma 3.3.** *Under the assumptions of Theorem 1.2, it holds that*

$$\begin{aligned} & \int (V^2 + H^2) d\xi - s \int_0^t \int \left(\frac{1}{p'(\tilde{v})}\right)_\xi H^2 d\xi d\tau + \int_0^t \int V_\xi^2 dx d\tau \\ & \leq C \|(V_0, H_0)\|^2 + C\delta \int_0^t \int V_{\xi\xi}^2 d\xi d\tau. \end{aligned} \quad (3.11)$$

*Proof.* Multiplying (2.3)<sub>1</sub> and (2.3)<sub>2</sub> by  $V$  and  $-\frac{H}{p'(\tilde{v})}$ , respectively, summing them up and then integrating the resulting equality over  $\mathbb{R}^1$  by parts, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \left(-\frac{H^2}{p'(\tilde{v})} + V^2\right) d\xi - \frac{s}{2} \int \left(\frac{1}{p'(\tilde{v})}\right)_\xi H^2 d\xi + \int \frac{V_\xi^2}{\tilde{v}^{\alpha+1}} d\xi \\ = & \int p(v|\tilde{v}) \frac{H}{p'(\tilde{v})} d\xi + \int GV d\xi + (\alpha+1) \int \frac{V_\xi}{\tilde{v}^{\alpha+2}} \tilde{v}_\xi V d\xi \\ \leq & C\delta \int V_\xi^2 d\xi + \int \left\{ \left(\frac{1}{v^{\alpha+1}} - \frac{1}{\tilde{v}^{\alpha+1}}\right) V_{\xi\xi} + \left(\frac{1}{v^{\alpha+1}} - \frac{1}{\tilde{v}^{\alpha+1}} + (\alpha+1) \frac{V_\xi}{\tilde{v}^{\alpha+2}}\right) \tilde{v}_\xi \right\} V d\xi \\ \leq & C\delta \int V_\xi^2 d\xi + C\delta \int V_{\xi\xi}^2 d\xi + C \int |\tilde{v}_\xi| V_\xi^2 |V| d\xi \\ \leq & C\delta \int V_\xi^2 d\xi + C\delta \int V_{\xi\xi}^2 d\xi, \end{aligned} \quad (3.12)$$

here we have used (3.2) and Lemma 3.1. Next, taking  $\delta$  sufficiently small, and integrating the above inequality with respect to  $t$ , we obtain (3.11).  $\square$

**Remark 3.1.** *Since we consider the first shock,  $s < 0$ , and  $\left(\frac{1}{p'(\tilde{v})}\right)_\xi = -\frac{\gamma+1}{\gamma} \tilde{v}^\gamma \tilde{v}_\xi > 0$ , then  $-s \int_0^t \int \left(\frac{1}{p'(\tilde{v})}\right)_\xi H^2 d\xi d\tau$  is a good term in Lemma 3.3.*

**Lemma 3.4.** *Under the assumptions of Theorem 1.2, it holds that*

$$\|(V, H)(t)\|_1^2 + \int_0^t \|V_\xi(\tau)\|_1^2 d\tau \leq C \|(V_0, H_0)\|_1^2. \quad (3.13)$$

*Proof.* Multiplying (2.3)<sub>1</sub> and (2.3)<sub>2</sub> by  $-V_{\xi\xi}$  and  $\frac{H_{\xi\xi}}{p'(\tilde{v})}$  respectively, summing up and then integrating the resulting equality over  $\mathbb{R}^1$  by parts, yields

$$\begin{aligned} & \frac{d}{dt} \int \left(-\frac{H_\xi^2}{2p'(\tilde{v})} + \frac{V_\xi^2}{2}\right) d\xi - \frac{s}{2} \int \left(\frac{1}{p'(\tilde{v})}\right)_\xi H_\xi^2 d\xi + \int \frac{V_{\xi\xi}^2}{\tilde{v}^{\alpha+1}} d\xi \\ = & - \int p(v|\tilde{v}) \frac{H_{\xi\xi}}{p'(\tilde{v})} d\xi - \int H_\xi (p(v) - p(\tilde{v})) \left(\frac{1}{p'(\tilde{v})}\right)_\xi d\xi - \int GV_{\xi\xi} d\xi \\ = & \int p(v|\tilde{v}) H_\xi \left(\frac{1}{p'(\tilde{v})}\right)_\xi d\xi - \int H_\xi (p(v) - p(\tilde{v})) \left(\frac{1}{p'(\tilde{v})}\right)_\xi d\xi + \int p(v|\tilde{v})_\xi \frac{H_\xi}{p'(\tilde{v})} d\xi - \int GV_{\xi\xi} d\xi \\ = & - \int p'(\tilde{v}) V_\xi H_\xi \left(\frac{1}{p'(\tilde{v})}\right)_\xi d\xi + \int p(v|\tilde{v})_\xi \frac{H_\xi}{p'(\tilde{v})} d\xi - \int GV_{\xi\xi} d\xi, \end{aligned} \quad (3.14)$$



then by Lemmas 3.1-3.2 and (3.1)-(3.2), we have

$$\begin{aligned}
& \frac{d}{dt} \int \left( -\frac{H_\xi^2}{2p'(\tilde{v})} + \frac{V_\xi^2}{2} \right) d\xi - \frac{s}{2} \int \left( \frac{1}{p'(\tilde{v})} \right)_\xi H_\xi^2 d\xi + \int \frac{V_{\xi\xi}^2}{\tilde{v}^{\alpha+1}} d\xi \\
& \leq -\frac{s}{4} \int \left( \frac{1}{p'(\tilde{v})} \right)_\xi H_\xi^2 d\xi + C \int V_\xi^2 d\xi + C \int (|V_{\xi\xi}| |V_\xi| + |\tilde{v}_\xi| V_\xi^2) |H_\xi| d\xi \\
& \quad + C \int (|V_\xi| |V_{\xi\xi}| + |\tilde{v}_\xi| |V_\xi|) |V_{\xi\xi}| d\xi \\
& \leq -\frac{s}{4} \int \left( \frac{1}{p'(\tilde{v})} \right)_\xi H_\xi^2 d\xi + C \int V_\xi^2 d\xi + C\delta \int V_{\xi\xi}^2 d\xi + C\delta \int V_\xi^2 d\xi \\
& \quad + \epsilon \int V_{\xi\xi}^2 d\xi + C(\epsilon) \int V_\xi^2 d\xi.
\end{aligned} \tag{3.15}$$

Integrating (3.15) with respect to  $t$ , and using Lemma 3.3, yields

$$\begin{aligned}
& \int \left( -\frac{H_\xi^2}{2p'(\tilde{v})} + \frac{V_\xi^2}{2} \right) d\xi - \frac{s}{4} \int_0^t \int \left( \frac{1}{p'(\tilde{v})} \right)_\xi H_\xi^2 d\xi d\tau + \int_0^t \int \frac{V_{\xi\xi}^2}{\tilde{v}^{\alpha+1}} d\xi d\tau \\
& \leq C \|(H_0, V_0)\|_1^2 + C\delta \int_0^t \int V_{\xi\xi}^2 d\xi d\tau + \epsilon \int_0^t \int V_{\xi\xi}^2 d\xi d\tau \\
& \quad + C(1 + \delta + C(\epsilon)) \int_0^t \int V_\xi^2 d\xi d\tau \\
& \leq C \|(V_0, H_0)\|_1^2 + C\delta \int_0^t \int V_{\xi\xi}^2 d\xi d\tau + \epsilon \int V_{\xi\xi}^2 d\xi \\
& \quad + C(1 + \delta + C(\epsilon)) \delta \int_0^t \int V_{\xi\xi}^2 d\xi d\tau.
\end{aligned} \tag{3.16}$$

Choosing  $\epsilon$  appropriately small and  $\delta$  sufficiently small, we obtain

$$\begin{aligned}
& \int (V_\xi^2 + H_\xi^2) d\xi + \int_0^t \int V_{\xi\xi}^2 d\xi d\tau \\
& \leq C \|(V_0, H_0)\|_1^2,
\end{aligned} \tag{3.17}$$

and this together with Lemma 3.3 complete the proof of Lemma 3.4.  $\square$

**Lemma 3.5.** *Under the assumptions of Theorem 1.2, it holds that*

$$\int_0^t \|H_\xi(\tau)\|^2 d\tau \leq C \|(V_0, H_0)\|_1^2. \tag{3.18}$$

*Proof.* Multiplying (2.3)<sub>1</sub> by  $H_\xi$ , and using (2.3)<sub>2</sub>, we have

$$\begin{aligned}
H_\xi^2 &= (VH_\xi)_t - V[sH_{\xi\xi} - (p(v) - p(\tilde{v}))_\xi] \\
&\quad - sV_\xi H_\xi - \frac{V_{\xi\xi}}{\tilde{v}^{\alpha+1}} H_\xi - GH_\xi.
\end{aligned} \tag{3.19}$$

Integrating the above equality with respect to  $\xi$  and  $t$ , and using the integration by parts, Lemma 3.4 and (3.1), we obtain

$$\begin{aligned}
& \int_0^t \int H_\xi^2 d\xi d\tau \\
= & - \int V_\xi H d\xi - \int V_0 H_{0,\xi} d\xi - \int_0^t \int V_\xi (p(v) - p(\tilde{v})) d\xi d\tau \\
& - \int_0^t \int \frac{V_{\xi\xi}}{\tilde{v}^{\alpha+1}} H_\xi d\xi d\tau - \int_0^t \int G H_\xi d\xi d\tau \\
\leq & C \|(V_0, H_0)\|_1^2 + \frac{1}{4} \int_0^t \int H_\xi^2 d\xi d\tau + \int_0^t \int (|V_\xi| |V_{\xi\xi}| + |\tilde{v}_\xi| |V_\xi|) |H_\xi| d\xi d\tau \\
\leq & C \|(V_0, H_0)\|_1^2 + \frac{1}{2} \int_0^t \int H_\xi^2 d\xi d\tau,
\end{aligned} \tag{3.20}$$

and this completes the proof of Lemma 3.5.  $\square$

**Lemma 3.6.** *Under the assumptions of Theorem 1.2, it holds that*

$$\|(V_{\xi\xi}, H_{\xi\xi})(t)\|^2 + \int_0^t \int V_{\xi\xi\xi}^2 d\xi d\tau \leq C \|(V_0, H_0)\|_2^2. \tag{3.21}$$

*Proof.* Multiplying (2.3)<sub>2</sub> by  $\frac{1}{-p'(\tilde{v})}$ , differentiating the resulting equation with respect to  $\xi$  twice, and differentiating (2.3)<sub>1</sub> with respect to  $\xi$  twice, then multiplying them by  $H_{\xi\xi}$  and  $V_{\xi\xi}$ , respectively, and adding them up, integrating the results with respect to  $\xi$ , we obtain by integration by parts

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int \left( -\frac{H_{\xi\xi}^2}{p'(\tilde{v})} + V_{\xi\xi}^2 \right) d\xi - \frac{s}{2} \int \left( \frac{1}{p'(\tilde{v})} \right)_\xi H_{\xi\xi}^2 d\xi + \int \frac{V_{\xi\xi\xi}^2}{\tilde{v}^{\alpha+1}} d\xi \\
= & -2 \int \left( \frac{1}{p'(\tilde{v})} \right)_\xi (p(v) - p(\tilde{v}))_\xi H_{\xi\xi} d\xi - 2 \int \frac{p(v) - p(\tilde{v})}{(p'(\tilde{v}))^3} (p'(\tilde{v}))_\xi^2 H_{\xi\xi} d\xi \\
& + \int \frac{p(v) - p(\tilde{v})}{(p'(\tilde{v}))^2} p'(\tilde{v})_{\xi\xi} H_{\xi\xi} d\xi + 2 \int p(v|\tilde{v})_\xi \left( \frac{1}{p'(\tilde{v})} \right)_\xi H_{\xi\xi} d\xi \\
& + \int p(v|\tilde{v}) \left\{ 2 \frac{((p'(\tilde{v}))_\xi)^2}{(p'(\tilde{v}))^3} - p'(\tilde{v})^{-2} p'(\tilde{v})_{\xi\xi} \right\} H_{\xi\xi} d\xi \\
& + \int p(v|\tilde{v})_{\xi\xi} \frac{1}{p'(\tilde{v})} H_{\xi\xi} d\xi + \int G_{\xi\xi} V_{\xi\xi} d\xi + (1 + \alpha) \int \frac{V_{\xi\xi}}{\tilde{v}^{2+\alpha}} V_{\xi\xi\xi} \tilde{v}_\xi d\xi \\
= & -2 \int (p'(\tilde{v}) V_\xi)_\xi \left( \frac{1}{p'(\tilde{v})} \right)_\xi H_{\xi\xi} d\xi - 2 \int V_\xi \frac{(p'(\tilde{v}))_\xi^2}{(p'(\tilde{v}))^2} H_{\xi\xi} d\xi \\
& + \int \frac{V_\xi}{p'(\tilde{v})} p'(\tilde{v})_{\xi\xi} H_{\xi\xi} d\xi + \int p(v|\tilde{v})_{\xi\xi} \frac{1}{p'(\tilde{v})} H_{\xi\xi} d\xi \\
& + \int G_{\xi\xi} V_{\xi\xi} d\xi - \frac{\alpha+1}{2} \int V_{\xi\xi}^2 \left( \frac{\tilde{v}_\xi}{\tilde{v}^{\alpha+2}} \right)_\xi d\xi \doteq \sum_{i=1}^6 I_i.
\end{aligned} \tag{3.22}$$

Using Lemmas 3.2, 3.4-3.5, (3.1) and the boundness of  $|\tilde{v}_\xi|, |\tilde{v}_{\xi\xi}|, |\tilde{v}_{\xi\xi\xi}|$ , we can estimate  $I_i$  ( $i =$

1  $\dots$  , 6) as follows

$$\begin{aligned}
I_1 &= -2 \int (p'(\tilde{v})V_{\xi\xi} + p''(\tilde{v})\tilde{v}_\xi V_\xi) \left(\frac{1}{p'(\tilde{v})}\right)_\xi H_{\xi\xi} d\xi \\
&\leq -\frac{s}{4} \int \left(\frac{1}{p'(\tilde{v})}\right)_\xi H_{\xi\xi}^2 d\xi + C \int V_{\xi\xi}^2 d\xi + C \int V_\xi^2 d\xi,
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
I_2 &= -2(\gamma+1)^2 \int V_\xi \left(\frac{\tilde{v}_\xi}{\tilde{v}}\right)^2 H_{\xi\xi} d\xi \\
&= 2(\gamma+1)^2 \int V_{\xi\xi} \left(\frac{\tilde{v}_\xi}{\tilde{v}}\right)^2 H_\xi d\xi + 2(\gamma+1)^2 \int V_\xi \left(\left(\frac{\tilde{v}_\xi}{\tilde{v}}\right)^2\right)_\xi H_\xi d\xi \\
&\leq C \int V_{\xi\xi}^2 d\xi + C \int H_\xi^2 d\xi + C \int V_\xi^2 d\xi.
\end{aligned} \tag{3.24}$$

Similarly

$$\begin{aligned}
I_3 &= \int [(\gamma+1)(\gamma+2)\tilde{v}^{-2}\tilde{v}_\xi^2 - (\gamma+1)\tilde{v}^{-1}\tilde{v}_{\xi\xi}] V_\xi H_{\xi\xi} d\xi \\
&= - \int [(\gamma+1)(\gamma+2)\tilde{v}^{-2}\tilde{v}_\xi^2 - (\gamma+1)\tilde{v}^{-1}\tilde{v}_{\xi\xi}] V_{\xi\xi} H_\xi d\xi \\
&\quad - \int [(\gamma+1)(\gamma+2)\tilde{v}^{-2}\tilde{v}_\xi^2 - (\gamma+1)\tilde{v}^{-1}\tilde{v}_{\xi\xi}]_\xi V_\xi H_\xi d\xi \\
&\leq C \int V_{\xi\xi}^2 d\xi + C \int H_\xi^2 d\xi + C \int V_\xi^2 d\xi.
\end{aligned} \tag{3.25}$$

And by Sobolev embedding theorem

$$\begin{aligned}
I_4 &\leq C \int (|V_{\xi\xi\xi}| |V_\xi| + V_\xi^2 + V_{\xi\xi}^2 + |V_\xi| |V_{\xi\xi}|) |H_{\xi\xi}| d\xi \\
&\leq C \|V_\xi\|_{L^\infty} \|V_{\xi\xi\xi}\| \|H_{\xi\xi}\| + C \|V_\xi\|_{L^\infty} \|V_\xi\| \|H_{\xi\xi}\| \\
&\quad + C \|V_{\xi\xi}\|_{L^\infty} \|V_{\xi\xi}\| \|H_{\xi\xi}\| + C \|V_\xi\|_{L^\infty} \|V_{\xi\xi}\| \|H_{\xi\xi}\| \\
&\leq C\delta(\|V_{\xi\xi}\| + \|V_\xi\|) \|V_{\xi\xi\xi}\| + C\delta(\|V_{\xi\xi}\| + \|V_\xi\|)(\|V_{\xi\xi}\| + \|V_\xi\|) \\
&\quad + C\delta(\|V_{\xi\xi}\| + \|V_{\xi\xi\xi}\|) \|V_{\xi\xi}\| \\
&\leq C\delta(\|V_\xi\|^2 + \|V_{\xi\xi}\|^2 + \|V_{\xi\xi\xi}\|^2).
\end{aligned} \tag{3.26}$$

Similarly

$$\begin{aligned}
I_5 &= - \int G_\xi V_{\xi\xi\xi} d\xi \\
&\leq C \int (V_{\xi\xi}^2 + |V_\xi| |V_{\xi\xi}| + |V_\xi| |V_{\xi\xi\xi}| + |V_{\xi\xi}| + |V_\xi|) |V_{\xi\xi\xi}| d\xi \\
&\leq C\delta(\|V_{\xi\xi}\|^2 + \|V_{\xi\xi\xi}\|^2) + \epsilon \|V_{\xi\xi\xi}\|^2 + C(\epsilon)(\|V_\xi\|^2 + \|V_{\xi\xi}\|^2).
\end{aligned} \tag{3.27}$$

And by the boundness of  $|\tilde{v}_\xi|$  and  $|\tilde{v}_{\xi\xi}|$ , we have

$$I_6 \leq C \|V_{\xi\xi}\|^2. \tag{3.28}$$

Substituting (3.23)-(3.28) into (3.22), taking  $\epsilon$  appropriately small and  $\delta$  sufficiently small, integrating the resulting inequality, by Lemmas 3.4-3.5, we can complete the proof of Lemma 3.6.  $\square$

**Lemma 3.7.** *Under the assumptions of Theorem 1.2, it holds that*

$$\int_0^t \|H_{\xi\xi}(\tau)\|^2 d\tau \leq C\|(V_0, H_0)\|_2^2. \quad (3.29)$$

*Proof.* Multiplying (2.3)<sub>1</sub> by  $H_{\xi\xi\xi\xi}$ , and using (2.3)<sub>2</sub>, we obtain

$$\begin{aligned} H_{\xi\xi}^2 &= -(VH_{\xi\xi\xi})_t + sV_{\xi}H_{\xi\xi\xi} + V[sH_{\xi\xi\xi\xi} - (p(v) - p(\tilde{v}))_{\xi\xi\xi}] \\ &\quad + \frac{V_{\xi\xi}}{\tilde{v}^{\alpha+1}}H_{\xi\xi\xi} + GH_{\xi\xi\xi} + (H_{\xi}H_{\xi\xi})_{\xi}, \end{aligned} \quad (3.30)$$

integrating the above equality with respect to  $\xi$  and  $t$ , and using the integration by parts, Lemmas 3.2, 3.4-3.6 and (3.1), yields

$$\begin{aligned} &\int_0^t \int H_{\xi\xi}^2 d\xi d\tau \\ &= \int V_{\xi}H_{\xi\xi} d\xi - \int V_{0,\xi}H_{0,\xi\xi} d\xi - \int_0^t \int V_{\xi\xi\xi}(p(v) - p(\tilde{v})) d\xi d\tau \\ &\quad - \int_0^t \int \frac{V_{\xi\xi\xi}}{\tilde{v}^{\alpha+1}}H_{\xi\xi} d\xi d\tau + (\alpha+1) \int_0^t \int \frac{\tilde{v}_{\xi}V_{\xi\xi}H_{\xi\xi}}{\tilde{v}^{\alpha+2}} d\xi d\tau - \int_0^t \int G_{\xi}H_{\xi\xi} d\xi d\tau \\ &\leq C\|(V_0, H_0)\|_2^2 + \frac{1}{4} \int_0^t \int H_{\xi\xi}^2 d\xi d\tau \\ &\quad + C \int_0^t \int (V_{\xi\xi}^2 + |V_{\xi}| |V_{\xi\xi}| + |V_{\xi}| |V_{\xi\xi\xi}| + |V_{\xi\xi}| + |V_{\xi}|) |H_{\xi\xi}| d\xi d\tau \\ &\leq C\|(V_0, H_0)\|_2^2 + (\frac{1}{2} + \delta) \int_0^t \int H_{\xi\xi}^2 d\xi d\tau, \end{aligned} \quad (3.31)$$

taking  $\delta$  small enough, we obtain (3.29), and this completes the proof of Lemma 3.7.  $\square$

From Lemmas 3.4-3.7, if we choose  $\|(V_0, H_0)(t)\|_2 \leq \delta_0$ , and let  $C\delta_0^2 \leq \delta^2$ , we can obtain

$$\|(V, H)(t)\|_2^2 + \int_0^t (\|V_{\xi}(\tau)\|_2^2 + \|H_{\xi}(\tau)\|_1^2) d\tau \leq C\|(V_0, H_0)\|_2^2 (\leq \delta), \quad (3.32)$$

this proves Proposition 2.1.

On the other hand, from the global estimate (3.32) above, we derive

$$\|(V_{\xi}(\cdot, t), H_{\xi}(\cdot, t))\|_1 \rightarrow 0, \quad \text{as } t \rightarrow +\infty.$$

Consequently, for all  $\xi \in \mathbb{R}^1$ ,

$$\begin{aligned} V_\xi^2(\xi, t) &= 2 \int_{-\infty}^{\xi} V_\xi V_{\xi\xi}(y, t) dy \\ &\leq 2 \|V_\xi(t)\| \|V_{\xi\xi}(t)\| \rightarrow 0, \quad \text{as } t \rightarrow +\infty. \end{aligned}$$

Applying the same argument to  $H_\xi$  leads, for all  $\xi \in \mathbb{R}^1$ , to

$$H_\xi(\xi, t) \rightarrow 0, \quad \text{as } t \rightarrow +\infty.$$

Hence (2.6) is proved. Thus the proof of Theorem 2.1 is completed.

## 4 Proof of the main theorem

At last, we prove (1.19), then everything is done. From (2.1), we have

$$u_t - su_\xi + p(v)_\xi = \left(\frac{u_\xi}{v^{\alpha+1}}\right)_\xi, \quad (4.1)$$

this together with the equation

$$\tilde{u}_t - s\tilde{u}_\xi + p(\tilde{v})_\xi = \left(\frac{\tilde{u}_\xi}{\tilde{v}^{\alpha+1}}\right)_\xi, \quad (4.2)$$

implies that  $w \triangleq u - \tilde{u}$  satisfies the following parabolic equation

$$w_t - \left(\frac{1}{v^{\alpha+1}} w_\xi\right)_\xi = sw_\xi - (p(v) - p(\tilde{v}))_\xi + (\tilde{u}_\xi \left(\frac{1}{v^{\alpha+1}} - \frac{1}{\tilde{v}^{\alpha+1}}\right))_\xi. \quad (4.3)$$

From the global estimate (3.32), we know that the right hand side of equation (4.3) is bounded in  $L^2(0, T; L^2(\mathbb{R}^1))$ , then by the classical regularity results for parabolic equation gives

$$w_t \text{ is bounded in } L^2(0, T; L^2(\mathbb{R}^1)), \quad (4.4)$$

and

$$w \text{ is bounded in } L^2(0, T; H^2(\mathbb{R}^1)), \quad (4.5)$$

which implies

$$\|w(\xi, t)\| \rightarrow 0, \quad \text{as } t \rightarrow +\infty.$$

By (4.5), there exists a  $t_0 > 0$ , such that

$$w_\xi(\cdot, t_0) \text{ is bounded in } L^2(\mathbb{R}^1). \quad (4.6)$$

Set  $\bar{w} = w_\xi$ , then  $\bar{w}$  satisfies the following parabolic equation

$$\bar{w}_t - \left(\frac{1}{v^{\alpha+1}} \bar{w}\right)_{\xi\xi} = s\bar{w}_\xi - (p(v) - p(\tilde{v}))_{\xi\xi} + (\tilde{u}_\xi \left(\frac{1}{v^{\alpha+1}} - \frac{1}{\tilde{v}^{\alpha+1}}\right))_{\xi\xi}. \quad (4.7)$$

Again, by (3.32) and (4.5), we know that the right hand side of equation (4.7) is bounded in  $L^2(0, T; L^2(\mathbb{R}^1))$ , then by the classical regularity results for parabolic equation (4.7) with initial data (4.6) gives

$$\bar{w}_t \text{ is bounded in } L^2(t_0, T; L^2(\mathbb{R}^1)), \quad (4.8)$$

and

$$\bar{w} \text{ is bounded in } L^2(t_0, T; H^2(\mathbb{R}^1)), \quad (4.9)$$

which implies

$$\|w_\xi(\xi, t)\| \rightarrow 0, \text{ as } t \rightarrow +\infty.$$

Then we have

$$\|w(\xi, t)\|_1 \rightarrow 0, \text{ as } t \rightarrow +\infty.$$

By using the Sobolev inequality, we have

$$\sup_{x \in \mathbb{R}^1} |w(\xi, t)| \rightarrow 0, \text{ as } t \rightarrow \infty, \quad (4.10)$$

which together with (2.6) implies

$$\sup_{x \in \mathbb{R}^1} |v(x, t) - \tilde{v}(x - st)| \rightarrow 0, \quad \sup_{x \in \mathbb{R}^1} |u(x, t) - \tilde{u}(x - st)| \rightarrow 0, \text{ as } t \rightarrow \infty, \quad (4.11)$$

this is (1.19), then the proof of Theorem 1.1 is completed.

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