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FROM KINETIC EQUATIONS TO MULTIDIMENSIONAL ISENTROPIC GAS DYNAMICS BEFORE SHOCKS*

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Abstract. This article is devoted to the proof of the hydrodynamical limit from kinetic equations (including BGK-like equations) to multidimensional isentropic gas dynamics. It is based on a relative entropy method; hence the derivation is valid only before shocks appear on the limit system solution. However, no a priori knowledge on high velocity distributions for kinetic functions is needed. The case of the Saint–Venant system with topography (where a source term is added) is included.

Key words. hydrodynamic limit, entropy method, BGK equation, isentropic gas dynamics, Saint–Venant system

AMS subject classifications. 35L65, 82C40, 76N, 35F20

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1. Introduction.

1.1. Context and results. This article is devoted to the study of the hydrodynamical limit of kinetic equations to the multidimensional system of isentropic gas dynamics:

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho u) = 0, \quad t \in \mathbb{R}^+,\ x \in \mathbb{R}^n, \\ \partial_t(\rho u) + \mathrm{div}_x(\rho u \otimes u + I\rho^\gamma) = \rho F, \quad t \in \mathbb{R}^+,\ x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

for $1 \leq \gamma \leq \frac{n+2}{n}$ and a given external force field F.

This is a simplified situation of the long-term problem concerning the compressible limit of the Boltzmann equation. In this case, the hydrodynamical limit has been performed by Caflisch [9] only for smooth data during a small time. The asymptotic limit of the Boltzmann equation in low Mach number towards incompressible Euler (or Navier–Stokes) systems has been achieved recently by Saint-Raymond [26] and Lions and Masmoudi [22] following the pioneering work of Bardos, Golse, and Levermore [1]. As for our work, they are still local time results, since it is valid in the lapse of time in which the limit solution remains smooth. However, at the kinetic level, no strong smoothness property is needed. Notice that in our case we deal with compressible gases, and even the existence of a solution to (1.1) after shocks appear is not known in the multidimensional situation.

At the kinetic level, we consider a Fokker–Planck equation for the isothermal case ($\gamma = 1$) and a BGK-like equation for the other values of γ. Originally, BGK equations have been introduced by Bathnagar, Gross, and Krook as a simplification of the Boltzmann equation. This model has been extended in order to construct kinetic equations associated with different hydrodynamical systems (see the book of Perthame [24] for a survey of this field). In our particular case, the BGK model we use has been introduced for the full range of γ by Bouchut [5]. Our main result is the following.

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THEOREM 1.1. *Let F be in $C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Let $(\rho^0, \rho^0 u^0) \in L^1(\mathbb{R}^n)$ be the initial data of a solution $U = (\rho, \rho u) \in C^1([0,T[\times\mathbb{R}^n) \cap L^1([0,T[\times\mathbb{R}^n)$ to (1.1) such that $\rho > 0$; ρ, u, $\partial_x \rho$, $\partial_x u$ are bounded with respect to (t,x); and ρu^2, $h(\rho)$ are integrable with respect to (t,x), where $h(\rho) = \rho^\gamma/(\gamma-1)$ for $\gamma > 1$ and $h(\rho) = \rho \ln \rho$ for $\gamma = 1$. Consider a family of kinetic initial values f^0_ε verifying $f^0_\varepsilon \in L^1(\mathbb{R}^{2n})$, $H(f^0_\varepsilon, v) \in L^1(\mathbb{R}^{2n})$ (where H is the kinetic entropy associated with the system (1.1); see section 4). We assume that it verifies*

$$\int_{\mathbb{R}^n} (f^0_\varepsilon, v f^0_\varepsilon, H(f^0_\varepsilon, v))\, dv \underset{\varepsilon\to 0}{\longrightarrow} (\rho^0, \rho^0 u^0, \rho^0 (u^0)^2/2 + h(\rho^0)) \quad \text{in } L^1(\mathbb{R}^n).$$

Let f_ε be the solution to the BGK equation (4.1) for $1 < \gamma \leq n/(n+2)$ or the solution to the Fokker–Planck equation (4.5) for the isotherm case ($\gamma = 1$). We denote

$$(\rho_\varepsilon, \rho_\varepsilon u_\varepsilon) = \left(\int_{\mathbb{R}^n} f_\varepsilon \, dv, \int_{\mathbb{R}^n} v f_\varepsilon \, dv \right).$$

Then ρ_ε converges strongly in $C^0(0,T; L^p_{\mathrm{loc}}(\mathbb{R}^n))$ to ρ for every $1 \leq p < \gamma$ and $\rho_\varepsilon u_\varepsilon$ converges strongly to ρu in $C^0(0,T; L^q_{\mathrm{loc}}(\mathbb{R}^n))$ for every $1 \leq q < 2\gamma/(\gamma+1)$.

In the monodimensional case ($n = 1$) a stronger result has been achieved by Berthelin and Bouchut [3] in the similar situation where we have only one entropy. This result is valid even when shocks appear. The simpler case dealing with the complete family of entropies has been performed by Berthelin and Bouchut [2] (see also Serre [27] for regular systems). However, notice that in our case, no a priori assumption on the support of f_ε in v is needed. Everything is controlled by the energy bound. (We also refer the reader to [28], [17] for the convergence of discrete kinetic models to the Lagrangian version of the p-system in the one-dimensional case but even after the appearance of shocks.)

The main tool is a relative entropy method. It relies on the “weak-strong” uniqueness principle, established by Dafermos for multidimensional systems of hyperbolic conservation laws admitting a convex entropy functional [10]. It is close to the concept of dissipative solutions for the Euler equations of Lions [21]. It has been frequently used for systems of particles and rarefied gas dynamics; see Yau [29] and Golse, Levermore, and Saint-Raymond [14] (see also Goudon, Jabin, and Vasseur [18]). For different asymptotic problems it is called the “modulated energy” method (Brenier [7], Masmoudi [23], and Brenier [8]).

1.2. Numerical motivation. The kinetic structure of hyperbolic conservation laws have been used for a long time to construct entropic numerical schemes (see Kaniel [20], Giga and Miyakawa [13], the “transport-collapse” method of Brenier [6], etc.). This method has been intensively developed by the group of Perthame (see [24] for a review). In this framework, study of hydrodynamical limits of BGK-like equations can give a first step for the proof of the convergence of those schemes.

Recently an intense activity has been produced to solve numerically hyperbolic conservation laws with source terms. As a test, the Saint–Venant system with bottom topography is often proposed:

$$(1.2) \qquad \begin{cases} \partial_t h + \mathrm{div}_x(hu) = 0, \quad t > 0,\ x \in \mathbb{R}^2, \\ \partial_t(hu) + \mathrm{div}_x(hu \otimes u) + \nabla_x h^2 + Z'(x) h = 0, \quad t > 0,\ x \in \mathbb{R}^2, \\ (h, hu)|_{t=0} = (h^0, h^0 u^0), \quad x \in \mathbb{R}^2, \end{cases}$$

where Z is the given bottom topography, h is the unknown depth of the water, and u is the unknown velocity of the water. This system models the evolution of a river.

Different numerical methods have been proposed to solve such problems (see Gosse [16], [15], Jin [19], Gallouët, Hérard, and Seguin [12], etc.). Botchorishvili, Perthame, and Vasseur [4] developed a kinetic procedure to construct numerical schemes and showed the convergence in the scalar case. This method has been successfully implemented by Perthame and Simeoni [25] for the Saint–Venant system. Notice that the Saint–Venant system corresponds exactly to the sytem (1.1) with $\gamma = 2$ and $Z' = F$. Our result can be seen as a first attempt to show the convergence of kinetic schemes in this framework.

1.3. Idea of the proof. As mentioned above, the proof relies on a relative entropy method. We consider the following abstract conservation law:

$$\partial_t U + \mathrm{div}_x\, A(U) = Q(U, x), \tag{1.3}$$

with $U(t, x) \in \mathcal{U} \subset \mathbb{R}^p$ for $t \in \mathbb{R}^+$, $x \in \mathbb{R}^n$, $A : \mathcal{U} \to \mathbb{R}^p$, and $Q : \mathcal{U} \times \mathbb{R}^n \to \mathbb{R}^p$. We assume that there exists an entropy, entropy flux couple (η, G) with $\eta \in C^2(\mathcal{U}, \mathbb{R})$ convex such that

$$\partial_i G_k(W) = \sum_j \partial_j \eta(W)\, \partial_i A_{jk}(W) \quad \forall k, i,\ \forall W. \tag{1.4}$$

For smooth solutions of this system, we have the entropy equality

$$\partial_t \eta(U) + \partial_x G(U) = \eta'(U) Q(U, x). \tag{1.5}$$

Following the notations of Dafermos, for every function $\Phi \in C^1(\mathbb{R}^p)$ of U we introduce the associated related quantity $\Phi(\cdot|\cdot) \in C^0(\mathbb{R}^p \times \mathbb{R}^p)$:

$$\Phi(U_1|U_2) = \Phi(U_1) - \Phi(U_2) - \nabla\Phi(U_2)(U_1 - U_2). \tag{1.6}$$

For example, the relative entropy is defined by

$$\eta(U_1|U_2) = \eta(U_1) - \eta(U_2) - \eta'(U_2) \cdot (U_1 - U_2), \tag{1.7}$$

and the entropy flux by

$$A(U_1|U_2) = A(U_1) - A(U_2) - A'(U_2) \cdot (U_1 - U_2). \tag{1.8}$$

We notice that if Φ is convex, then $\Phi(U_1|U_2) \geq 0$. Moreover, if it is strictly convex, then $\Phi(U_1|U_2) = 0$ if and only if $U_1 = U_2$.

We consider in the same way an abstract kinetic equation

$$\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + q(f_\varepsilon) = \frac{\mathcal{Q}(f_\varepsilon, v)}{\varepsilon}, \tag{1.9}$$

where $f_\varepsilon = f_\varepsilon(t, x, v) \in \mathbb{R}$ with $t \in \mathbb{R}^+$, $x, v \in \mathbb{R}^n$, and q a linear operator, $\mathcal{Q} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and with $a : \mathbb{R}^n \to \mathbb{R}^p$, where the collision term $\mathcal{Q}$ satisfies

$$\int_{\mathbb{R}^n} a(v) \mathcal{Q}(f, v)\, dv = 0 \quad \text{ for any } f \in \mathbb{R}. \tag{1.10}$$

We assume the existence of a kinetic entropy $H : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ which is well related to the entropy η of the hyperbolic system we want to relax. In more precise words, we need that the following nonincrease is checked for the solution of the kinetic equation,

$$\frac{d}{dt} \iint_{\mathbb{R}^{2n}} H(f_\varepsilon, v)\, dv\, dx \leq \int_{\mathbb{R}^n} \eta'(U_\varepsilon) Q(U_\varepsilon)\, dx, \tag{1.11}$$

where

$$U_\varepsilon(t,x) = \int_{\mathbb{R}^n} a(v) f_\varepsilon(t,x,v)\, dv, \tag{1.12}$$

and that the following compatibility between the entropy η and the kinetic entropy H is satisfied:

$$\eta(U_\varepsilon) \leq \int_{\mathbb{R}^n} H(f_\varepsilon, v)\, dv. \tag{1.13}$$

In addition we assume that

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} (|a(v)| + |a(v)\otimes v|) f_\varepsilon \, dv + \int_{\mathbb{R}^n} a(v) q(f_\varepsilon)\, dv \right| \\ &\qquad \leq \int_{\mathbb{R}^n} (f_\varepsilon + H(f_\varepsilon, v))\, dv. \end{aligned} \tag{1.14}$$

Then we have the following abstract theorem.

THEOREM 1.2. *Let $U \in [C^1([0,T]\times\mathbb{R}^n)]^p$ be a strong solution on $[0,T]$ of the multidimensional hyperbolic system* (1.3)*, a system with a convex, C^2 entropy, for an initial data U^0. We assume in addition that U, $\eta'(U)$, and $\partial_x \eta'(U)$ are bounded and that U and $\eta(U)$ are integrable with respect to x. Let f_ε be a solution to the kinetic equation* (1.9)*, satisfying* (1.10)–(1.14) *and $f_\varepsilon + H(f_\varepsilon, v)$ integrable with respect to x and v for every t. We set*

$$U_\varepsilon(t,x) = \int_{\mathbb{R}^n} a(v) f_\varepsilon(t,x,v)\, dv.$$

We assume the convergence of initial data

$$\int_{\mathbb{R}^n} \eta(U_\varepsilon^0 | U^0)\, dx \leq C_0 \sqrt{\varepsilon}, \tag{1.15}$$

and the following compatibility for the initial data

$$\left| \int_{\mathbb{R}^n} H(f_\varepsilon^0, v)\, dv - \eta(U_\varepsilon^0) \right| \leq C_0\sqrt{\varepsilon}. \tag{1.16}$$

If we have the control of the kinetic quantities

$$\int_0^T \int_{\mathbb{R}^n} \left| A(U_\varepsilon) - \int_{\mathbb{R}^n} v \otimes a(v) f_\varepsilon \, dv \right| dx\, dt \leq C_1 \sqrt{\varepsilon}, \tag{1.17}$$

$$\int_0^T \int_{\mathbb{R}^n} \left| Q(U_\varepsilon, x) + \int_{\mathbb{R}^n} a(v) q(f_\varepsilon)\, dv \right| dx\, dt \leq C_1 \sqrt{\varepsilon}, \tag{1.18}$$

and the control of the relative flux and the source terms by the relative entropy as

$$|A(U_\varepsilon | U)| \leq C_2\, \eta(U_\varepsilon | U), \tag{1.19}$$

$$|Q(U)\eta'(U_\varepsilon|U) + [Q(U_\varepsilon) - Q(U)](\eta'(U_\varepsilon) - \eta'(U))| \leq C_2\, \eta(U_\varepsilon|U), \tag{1.20}$$

where C_1 and C_2 are positive constants, then we get, for a constant C,

$$\int_{\mathbb{R}^n} \eta(U_\varepsilon | U)(t,x)\, dx \leq C\sqrt{\varepsilon} \quad \textit{for any } t \in [0,T]. \tag{1.21}$$

Hypothesis (1.15) and (1.16) are compatibility conditions on the initial data. Hypothesis (1.19) and (1.20) are structure conditions on the system (1.3). This theorem shows that if (1.17) and (1.18) are fulfilled (which will be derived from kinetic dissipation), and the system (1.3) has a good structure, then the total relative entropy of U_ε with respect to U converges to 0. This applies to the convergence of U_ε to U. Notice that this presentation splits nicely the kinetic dissipation effect from the control of the nonlinearities. The kinetic dissipation is needed in order to fulfill the consistency (1.17) and (1.18). The nonlinearity is driven by the relative entropy method which can be applied if (1.3) verifies (1.19) and (1.20). Notice that the method depends only on the structure of the system whatever the kinetic equation is.

Then we show that the isentropic system (1.1) verifies the structure compatibility and that the involved kinetic equations verify the dissipation properties needed. Notice that the full Euler system (with the added energy equation) does not verify (1.19). Hence this method cannot be applied directly to the convergence from the Boltzmann equation to the Euler system, for instance (see the appendix). The problem relies already on the structure itself of the system (the relative flux cannot be controlled by the relative entropy because of the high macroscopic velocities). Of course an additional difficulty lies in the kinetic level to control high velocities to obtain (1.17) and (1.18).

2. Study of the abstract problem. We consider the abstract equation (1.3) and abstract kinetic equation (1.9). This section is devoted to the proof of Theorem 1.2.

2.1. The key estimate. In the following proposition, we describe the evolution of the relative entropy using canonical quantities associated with the system (1.3) and entropy equation (1.5). We do not claim any originality in this result. It can be found in [10], except for the slight generalization concerning the source term Q. However, we give the proof for the sake of completeness.

PROPOSITION 2.1. *For the entropy $\eta \in C^2(\mathbb{R}^p)$ and for any $U, V \in [C^1(\mathbb{R}^n)]^p$, we have*

$$\begin{aligned}
\partial_t \eta(V|U) = {} & [\partial_t \eta(V) + \operatorname{div}_x G(V) - \eta'(V)Q(V)] \\
& - [\partial_t \eta(U) + \operatorname{div}_x G(U) - \eta'(U)Q(U)] \\
& - \eta''(U) \cdot [\partial_t U + \operatorname{div}_x A(U) - Q(U))] \cdot (V - U) \\
& - \eta'(U) \cdot [\partial_t V + \operatorname{div}_x A(V) - Q(V)] \\
& + \eta'(U) \cdot [\partial_t U + \operatorname{div}_x A(U) - Q(U))] \\
& + \operatorname{div}_x [G(U) - G(V)] + \sum_{ik} \partial_{x_k} [\partial_i G_k(U)(V_i - U_i)] \\
& + \sum_{jk} \partial_j \eta(U) \partial_{x_k} [A(V|U)] \\
& + Q(U)\eta'(V|U) + [Q(V) - Q(U)](\eta'(V) - \eta'(U)).
\end{aligned}$$

Remark 2.2. Notice that if U and V are regular solutions to (1.3), the first five lines vanish. The sixth line has a divergence form, hence its integral is vanishing. Finally, the two last terms are quadratic with respect to $V-U$ (at least when $|V-U| \leq R$) as η is. Hence, from this proposition, we can expect to have a good structure to use Gronwall's lemma on $\int \eta(V|U)\, dx$.

Proof. From the definition of relative quantity (1.6), we have

$$\begin{aligned}\partial_t\eta(V|U) &= \partial_t\eta(V) - \partial_t\eta(U) - \partial_t\left[\eta'(U)\right]\cdot(V-U) - \eta'(U)\cdot\partial_t(V-U)\\ &= \left[\partial_t\eta(V) + \operatorname{div}_x G(V) - \eta'(V)Q(V)\right]\\ &\quad - \left[\partial_t\eta(U) + \operatorname{div}_x G(U) - \eta'(U)Q(U)\right]\\ &\quad - \eta''(U)\cdot\left[\partial_t U + \operatorname{div}_x A(U) - Q(U)\right]\cdot(V-U)\\ &\quad - \eta'(U)\cdot\left[\partial_t V + \operatorname{div}_x A(V) - Q(V)\right]\\ &\quad + \eta'(U)\cdot\left[\partial_t U + \operatorname{div}_x A(U) - Q(U)\right] + R_1 + R_2,\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}R_1 &= \eta'(V)Q(V) - \eta'(U)Q(U) - \eta''(U)\cdot Q(U)\cdot(V-U)\\ &\quad - \eta'(U)\cdot Q(V) + \eta'(U)\cdot Q(U)\\ &= Q(U)\eta'(V|U) + \left[\eta'(U) - \eta'(V)\right]\cdot\left[Q(U) - Q(V)\right]\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}R_2 = \operatorname{div}_x &\left[G(U) - G(V)\right]\\ &+ \eta''(U)\cdot\operatorname{div}_x A(U)\cdot(V-U)\\ &+ \eta'(U)\cdot\operatorname{div}_x\left[A(V) - A(U)\right].\end{aligned}$$

The existence of the associated entropy flux G gives the relation (see (1.4))

$$\partial_i G_k(W) = \sum_j \partial_j\eta(W)\,\partial_i A_{jk}(W) \quad \forall k, i,\ \forall W.$$

A derivation of this relation with respect to W_l gives

$$\sum_j \partial_{lj}\eta(W)\partial_i A_{jk}(W) = \partial_{il}G_k(W) - \sum_j \partial_j\eta(W)\partial_{il}A_{jk}(W).$$

We use this relation with $W = U$ and get

$$\begin{aligned}&\eta''(U)\cdot\operatorname{div}_x A(U)\cdot(V-U)\\ &\quad= \sum \partial_{lj}\eta(U)\partial_{x_k}[A_{jk}(U)](V_l - U_l)\\ &\quad= \sum \partial_{lj}\eta(U)\partial_i A_{jk}(U)\partial_{x_k}U_i(V_l - U_l)\\ &\quad= \sum \partial_{il}G_k(U)\partial_{x_k}U_i(V_l - U_l) - \sum \partial_j\eta(U)\partial_{il}A_{jk}(U)\partial_{x_k}U_i(V_l - U_l),\end{aligned}$$

and now

$$\begin{aligned}&-\partial_j\eta(U)\partial_{il}A_{jk}(U)\partial_{x_k}U_i(V_l - U_l)\\ &\quad= \partial_j\eta(U)\left[-\partial_{x_k}\left[\partial_l A_{jk}(U)\right](V_l - U_l)\right]\\ &\quad= \partial_j\eta(U)\left[-\partial_{x_k}\left[\partial_l A_{jk}(U)(V_l - U_l)\right] + \partial_l A_{jk}(U)\partial_{x_k}(V_l - U_l)\right];\end{aligned}$$

therefore, we obtain

$$\begin{aligned} R_2 &= \operatorname{div}_x [G(U) - G(V)] + \sum \partial_{il} G_k(U) \partial_{x_k} U_i (V_l - U_l) \\ &\quad + \sum \partial_j \eta(U) \left[-\partial_{x_k} [\partial_l A_{jk}(U)(V_l - U_l)] + \partial_l A_{jk}(U) \partial_{x_k} (V_l - U_l) \right] \\ &\quad + \eta'(U) \cdot \operatorname{div}_x [a(V) - A(U)] \\ &= \operatorname{div}_x [G(U) - G(V)] + \sum \partial_{x_k} [\partial_l G_k(U)] (V_l - U_l) \\ &\quad - \sum \partial_j \eta(U) \partial_{x_k} [\partial_l A_{jk}(U)(V_l - U_l)] \\ &\quad + \sum \partial_j \eta(U) \partial_l A_{jk}(U) \partial_{x_k} (V_l - U_l) \\ &\quad + \sum \partial_j \eta(U) \partial_{x_k} [A_{jk}(V) - A_{jk}(U)]. \end{aligned}$$

Permuting indexes i and l, we can rewrite (1.4) in the following way:

$$\sum_j \partial_j \eta(U) \partial_l A_{jk}(U) = \partial_l G_k(U).$$

Thus we find

$$\begin{aligned} R_2 &= \operatorname{div}_x [G(U) - G(V)] + \sum \partial_{x_k} [\partial_i G_k(U)(V_i - U_i)] \\ &\quad + \sum \partial_j \eta(U) \partial_{x_k} [A(V|U)]. \end{aligned} \tag{2.3}$$

Equation (2.1), with (2.2) and (2.3), gives the desired relation. □

Remark 2.3. We notice that in particular, the term R_2 of the proof satisfies

$$\begin{aligned} \int_{\mathbb{R}^n} R_2 \, dx &= \int_{\mathbb{R}^n} \sum_{jk} \partial_j \eta(U) \partial_{x_k} [A(V|U)] \, dx \\ &= -\int_{\mathbb{R}^n} \sum_{jk} \partial_{x_k} [\partial_j \eta(U)] \, A_{jk}(V|U) \, dx. \end{aligned}$$

This result is now used to obtain information if one deals with weak, strong, or/and approximated solutions.

2.2. Weak and strong solutions. This subsection is completely imbedded in Dafermos [10] (except for the slight generalization of the source term). Moreover, it is completely independent of the remainder of the paper. We give it since it clarifies the structure conditions (1.19) and (1.20) needed to use the relative entropy method without a priori condition on V in L^∞ for instance. We assume here that U is a strong solution of (1.3) (and as a consequence (1.5) is satisfied), and that V is a weak solution of (1.3) satisfying the entropy inequality

$$\partial_t \eta(V) + \partial_x G(V) \le \eta'(V) Q(V). \tag{2.4}$$

Thus applying Proposition 2.1 (on a regularization of V and passing to the limit in the regularization), we get with the notations (2.2) and (2.3)

$$\partial_t \eta(V|U) \le R_1 + R_2, \tag{2.5}$$

and using Remark 2.3, it leads to the following result.

Corollary 2.4. *Let $U \in [C^1([0,T]\times\mathbb{R}^n)]^p$ be a strong solution of* (1.3) *such that U, $\eta'(U)$, $\partial_x\eta'(U)$ are bounded and U, $\eta(U)$ are integrable. Let V be an entropy weak solution of* (1.3)*. Then we have*

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^n}\eta(V|U)\,dx \le -\int_{\mathbb{R}^n}&\sum_{jk}\partial_{x_k}\left[\partial_j\eta(U)\right]A_{jk}(V|U)\,dx\\ &+\int_{\mathbb{R}^n}Q(U)\eta'(V|U)\,dx\\ &+\int_{\mathbb{R}^n}[Q(V)-Q(U)](\eta'(V)-\eta'(U))\,dx.\end{aligned} \tag{2.6}$$

This result clarifies necessary information needed on the structure of the system (1.3). If we have for every $V,U\in\mathbb{R}^p$

$$|A(V|U)| \le C\eta(V|U) \tag{2.7}$$

and

$$|Q(U)\eta'(V|U)+[Q(V)-Q(U)](\eta'(V)-\eta'(U))| \le C\eta(V|U), \tag{2.8}$$

then we get

$$\frac{d}{dt}\int_{\mathbb{R}^n}\eta(V|U)\,dx \le (C(U)+1)C\int_{\mathbb{R}^n}\eta(V|U)\,dx,$$

and by a Gronwall's argument, it gives

$$\int_{\mathbb{R}^n}\eta(V|U)(t,x)\,dx \le \int_{\mathbb{R}^n}\eta(V|U)(0,x)\,dx\;e^{(C(U)+1)Ct}.$$

Thus if $U^0=V^0$, then

$$\eta(V|U)(t,x)=0\quad\forall t\in[0,T],\ \text{a.e. } x\in\mathbb{R}^n.$$

It gives $V=U$ if η is strictly convex. We recover here part of the classical results for weak = strong solutions. In fact, estimates as (2.7)–(2.8) are the important point to perform our entropy method. If we do not have a source term, it says that we need a control of the relative flux of the system by the relative entropy. This was already the case in Brenier [8]. We want now to extend the possible applications by studying the link between a strong solution and some approximations of it.

2.3. Strong and approximated solutions. We now assume that U is a strong solution of (1.3), and U_ε is any approximation of a solution, coming for example from a kinetic model. We get the following corollary from Proposition 2.1.

Corollary 2.5. *Let $U \in [C^1([0,T]\times\mathbb{R}^n)]^p$ be a strong solution of* (1.3) *such that U, $\eta'(U)$, $\partial_x\eta'(U)$ are bounded and U, $\eta(U)$ are integrable. Then we have, for any function $U_\varepsilon\in[C^1([0,T]\times\mathbb{R}^n)]^p$,*

$$\begin{aligned}\partial_t\eta(U_\varepsilon|U) = {}&[\partial_t\eta(U_\varepsilon)+\mathrm{div}_x\,G(U_\varepsilon)-\eta'(U_\varepsilon)Q(U_\varepsilon)]\\ &-\eta'(U)\cdot[\partial_tU_\varepsilon+\mathrm{div}_x\,A(U_\varepsilon)-Q(U_\varepsilon)]\\ &+\mathrm{div}_x\,[G(U)-G(U_\varepsilon)]+\sum\partial_{x_k}[\partial_iG_k(U)((U_\varepsilon)_i-U_i)]\\ &+\sum_{jk}\partial_j\eta(U)\partial_{x_k}\left[A(U_\varepsilon|U)\right]\\ &+Q(U)\eta'(U_\varepsilon|U)+[Q(U_\varepsilon)-Q(U)](\eta'(U_\varepsilon)-\eta'(U)).\end{aligned}$$

In this situation we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^n}[\eta(U_\varepsilon|U)-\eta(U_\varepsilon)]\,dx = &-\int_{\mathbb{R}^n}\eta'(U_\varepsilon)Q(U_\varepsilon)\,dx\\ &-\int_{\mathbb{R}^n}\eta'(U)\cdot[\partial_t U_\varepsilon+\operatorname{div}_x A(U_\varepsilon)-Q(U_\varepsilon)]\;dx\\ &-\int_{\mathbb{R}^n}\sum_{jk}\partial_{x_k}[\partial_j\eta(U)]\,A_{jk}(U_\varepsilon|U)\,dx\\ &+\int_{\mathbb{R}^n}Q(U)\eta'(U_\varepsilon|U)\,dx\\ &+\int_{\mathbb{R}^n}[Q(U_\varepsilon)-Q(U)](\eta'(U_\varepsilon)-\eta'(U))\,dx.\end{aligned}$$

We use now this relation in the case where U_ε comes from a kinetic equation.

2.4. Approximation from a kinetic equation. We consider f_ε a solution to the kinetic model (1.9) which satisfies (1.10)–(1.14) with $f_\varepsilon+H(f_\varepsilon,v)$ integrable with respect to x and v for every t. Let U_ε be the moments of f_ε defined by (1.12). We set

$$\Delta_\varepsilon=\eta(U_\varepsilon|U)+\int_{\mathbb{R}^n}H(f_\varepsilon,v)\,dv-\eta(U_\varepsilon). \tag{2.9}$$

From (1.13) and the convexity of η, we have

$$\Delta_\varepsilon\geq 0. \tag{2.10}$$

Using (1.11) and the relation of the previous section, we obtain (again after a regularization)

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^n}\Delta_\varepsilon\,dx\leq &-\int_{\mathbb{R}^n}\eta'(U)\cdot[\partial_t U_\varepsilon+\operatorname{div}_x A(U_\varepsilon)-Q(U_\varepsilon)]\;dx\\ &-\int_{\mathbb{R}^n}\sum_{jk}\partial_{x_k}[\partial_j\eta(U)]\,A_{jk}(U_\varepsilon|U)\,dx\\ &+\int_{\mathbb{R}^n}Q(U)\eta'(U_\varepsilon|U)\,dx\\ &+\int_{\mathbb{R}^n}[Q(U_\varepsilon)-Q(U)](\eta'(U_\varepsilon)-\eta'(U))\,dx.\end{aligned}$$

Now multiplying the kinetic equation (1.9) by $a(v)$ and then integrating it with respect to v and using (1.10), we have

$$\partial_t U_\varepsilon+\operatorname{div}_x\int_{\mathbb{R}^n}v\otimes a(v)f_\varepsilon\,dv+\int_{\mathbb{R}^n}a(v)q(f_\varepsilon)\,dv=0.$$

It gives

$$\begin{aligned}\partial_t U_\varepsilon+\operatorname{div}_x A(U_\varepsilon)-Q(U_\varepsilon)=\operatorname{div}_x&\left(A(U_\varepsilon)-\int_{\mathbb{R}^n}v\otimes a(v)f_\varepsilon\,dv\right)\\ &-\left[\int_{\mathbb{R}^n}a(v)q(f_\varepsilon)\,dv+Q(U_\varepsilon)\right].\end{aligned}$$

Therefore we get the following result.

PROPOSITION 2.6. *We assume that the system* (1.3) *admits a strictly convex entropy* $\eta \in C^2(\mathbb{R}^p)$. *Let* $U \in [C^1([0,T] \times \mathbb{R}^n)]^p$ *be a strong solution of* (1.3) *such that* U, $\eta'(U)$, $\partial_x \eta'(U)$ *are bounded and* U, $\eta(U)$ *are integrable. Let* f_ε *be a solution of* (1.9) *such that* (1.10)–(1.14) *are satisfied and* $f_\varepsilon + H(f_\varepsilon, v)$ *are integrable with respect to* x *and* v *for every time* t. *We set*

$$U_\varepsilon(t,x) = \int_{\mathbb{R}^n} a(v) f_\varepsilon(t,x,v)\, dv.$$

Then there exists a constant $C(U)$ *such that*

$$\frac{d}{dt}\int_{\mathbb{R}^n}\left[\eta(U_\varepsilon|U) + \int_{\mathbb{R}^n} H(f_\varepsilon)\, dv - \eta(U_\varepsilon)\right] dx$$

$$\leq C(U)\left(\int_{\mathbb{R}^n}\left|A(U_\varepsilon) - \int_{\mathbb{R}^n} v\otimes a(v) f_\varepsilon\, dv\right| dx \right. \tag{2.11}$$

$$+ \int_{\mathbb{R}^n}\left|Q(U_\varepsilon) + \int_{\mathbb{R}^n} a(v) q(f_\varepsilon)\, dv\right| dx \tag{2.12}$$

$$+ \int_{\mathbb{R}^n} |\, Q(U)\eta'(U_\varepsilon|U) + [Q(U_\varepsilon) - Q(U)](\eta'(U_\varepsilon) - \eta'(U))\,|\, dx \tag{2.13}$$

$$\left. + \int_{\mathbb{R}^n} |A(U_\varepsilon|U)|\, dx\right). \tag{2.14}$$

Remark 2.7. This inequality uncouples the various structures which come into play. The term (2.11) is related to the kinetic approximation, the term (2.14) is related to the structure of the system, the term (2.12) is related to the kinetic structure of the source term, and the term (2.13) is related to the structure of the source term with respect to the hyperbolic system.

We use this majoration to get the convergence result from a solution of a kinetic equation to a strong solution of a multidimensional hyperbolic system, that is, Theorem 1.2.

2.5. Proof of Theorem 1.2. We use again the notation Δ_ε given by (2.9). From Proposition 2.6, we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^n}\Delta_\varepsilon(t,x)\, dx \leq C(U)\left(\int_{\mathbb{R}^n}\left|A(U_\varepsilon) - \int_{\mathbb{R}^n} v\otimes a(v) f_\varepsilon\, dv\right| dx\right. \\ + \int_{\mathbb{R}^n}\left|Q(U_\varepsilon) + \int_{\mathbb{R}^n} a(v) q(f_\varepsilon)\, dv\right| dx \\ \left. + 2C_2 \int_{\mathbb{R}^n}\eta(U_\varepsilon|U)\, dx\right),\end{aligned}$$

and thus, for $t \in [0,T]$,

$$\begin{aligned}&\int_{\mathbb{R}^n}\Delta_\varepsilon(t,x)\, dx \\ &\quad \leq \int_{\mathbb{R}^n}\Delta_\varepsilon(0,x)\, dx + 2C(U)C_1\sqrt{\varepsilon} + 2C(U)C_2\int_0^t\int_{\mathbb{R}^n}\Delta_\varepsilon(s,x)\, dx\, ds.\end{aligned}$$

By Gronwall's argument, it gives

$$\int_{\mathbb{R}^n}\Delta_\varepsilon(t,x)\, dx \leq \left(\int_{\mathbb{R}^n}\Delta_\varepsilon(0,x)\, dx + 2C(U)C_1\sqrt{\varepsilon}\right) e^{2C(U)C_2 t}.$$

From (1.15)–(1.16), we have

$$\left| \int_{\mathbb{R}^n} \Delta_\varepsilon(0,x)\,dx \right| \le C\sqrt{\varepsilon},$$

and consequently, since $0 \le \eta(U_\varepsilon|U) \le \Delta_\varepsilon$, we obtain the result. ∎

Two independant studies to apply this result for a given example are thus necessary: the study of the system structure and the study of the dissipation of the kinetic model.

3. Study of the system structure. We consider here the case of the multidimensional isentropic gas dynamics system (1.1) avoiding the appearance of the vacuum. The associated entropy is

$$\eta(\rho, \rho u) = \rho\frac{u^2}{2} + h(\rho), \tag{3.1}$$

where $h(\rho) = \frac{1}{\gamma-1}\rho^\gamma$ for $\gamma > 1$ and $h(\rho) = \rho\ln\rho$ for the isotherm case $\gamma = 1$. The existence of strong solution for this problem is related to the classical result for regular solution for hyperbolic systems endowed with a strong entropy (see for instance [11]). In order to apply the convergence result of the previous section, we require that the structure of the system and the source terms be controllable by the relative entropy. For the system (1.1), the relative entropy is given by

$$\eta(U_1|U_2) = \frac{\rho_1}{2}|u_1 - u_2|^2 + h(\rho_1|\rho_2). \tag{3.2}$$

The relative flux of the system is

$$A(U_1|U_2) = (0, \rho_1(u_1 - u_2) \otimes (u_1 - u_2) + h(\rho_1|\rho_2)I). \tag{3.3}$$

We clearly have the existence of a constant C such that

$$|A(U_1|U_2)| \le C\eta(U_1|U_2) \tag{3.4}$$

for every $U_1, U_2 \in \mathbb{R}^{n+1}$. This fulfills estimate (1.19).

For the system (1.1), the source terms reads

$$Q(\rho, \rho u, x) = (0, \rho F(x)). \tag{3.5}$$

This gives

$$Q(U_2)\eta'(U_1|U_2) = -(u_2 - u_1)(\rho_2 - \rho_1)F \tag{3.6}$$

and

$$[Q(U_1) - Q(U_2)](\eta'(U_1) - \eta'(U_2)) = (u_2 - u_1)(\rho_2 - \rho_1)F, \tag{3.7}$$

and finally

$$Q(U_2)\eta'(U_1|U_2) + [Q(U_1) - Q(U_2)](\eta'(U_1) - \eta'(U_2)) = 0. \tag{3.8}$$

Thus the term (1.20) associated with the system (1.1) does not appear in the system structure study.

We turn now to the study of the terms related to the kinetic structure.

4. Study of the kinetic structure. We begin introducing the kinetic model we are dealing with. For $\gamma > 1$ we consider the following BGK kinetic equation:

$$\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + F(x) \cdot \nabla_v f_\varepsilon = \frac{M f_\varepsilon - f_\varepsilon}{\varepsilon}, \tag{4.1}$$

where the unknown is $f_\varepsilon = f_\varepsilon(t, x, v) \in \mathbb{R}$ with $t \in \mathbb{R}^+$, $x, v \in \mathbb{R}^n$. The force term $F : \mathbb{R}^n \to \mathbb{R}^n$ is given. The equilibrium function $M f_\varepsilon$ is defined in the following way:

$$M f_\varepsilon(t, x, v) = M(\rho_\varepsilon(t, x), \rho_\varepsilon u_\varepsilon(t, x), v) \tag{4.2}$$

with

$$\begin{aligned} \rho_\varepsilon(t, x) &= \int_{\mathbb{R}^n} f_\varepsilon(t, x, v)\, dv, \\ \rho_\varepsilon u_\varepsilon(t, x) &= \int_{\mathbb{R}^n} v f_\varepsilon(t, x, v)\, dv, \end{aligned}$$

where the Maxwellian $M : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}$ is given by

$$M(\rho, \rho u, v) = \mathbf{1}_{|u-v|^n \le c_n \rho} \quad \text{for } \gamma = \frac{n+2}{n}, \tag{4.3}$$

$$M(\rho, \rho u, v) = c \left(\frac{2\gamma}{\gamma - 1} \rho^{\gamma - 1} - |v - u|^2 \right)_+^{d/2} \quad \text{else.} \tag{4.4}$$

The constants are given by

$$\begin{aligned} c_n &= n/|\mathbb{S}_n|, \\ d &= \frac{2}{\gamma - 1} - n, \\ c &= \left(\frac{2\gamma}{\gamma - 1} \right)^{-1/(\gamma-1)} \frac{\Gamma(\frac{\gamma}{\gamma-1})}{\pi^{n/2} \Gamma(d/2 + 1)}. \end{aligned}$$

In the isothermal case $\gamma = 1$ we consider the following Fokker–Planck equation:

$$\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + F(x) \cdot \nabla_v f_\varepsilon = \frac{1}{\varepsilon} \operatorname{div}_v((v - u_\varepsilon) f_\varepsilon + \nabla_v f_\varepsilon), \tag{4.5}$$

where

$$\rho_\varepsilon = \int_{\mathbb{R}^n} f_\varepsilon\, dv, \qquad \rho_\varepsilon u_\varepsilon = \int_{\mathbb{R}^n} v f_\varepsilon\, dv. \tag{4.6}$$

This section is devoted to the proof of the estimates needed to apply Theorem 1.2 for each model. For each case, we will first show that it verifies (1.10)–(1.14), and in the second step that it verifies the more difficult estimates (1.17) and (1.18).

4.1. BGK structure for isentropic gas with $1 < \gamma \le (n + 2)/n$. In this section, we start by the study of kinetic BGK equations whose hydrodynamic limit is the isentropic system with an external force field.

We denote

$$\begin{aligned} U_\varepsilon &= (\rho_\varepsilon, \rho_\varepsilon u_\varepsilon), \\ U &= (\rho, \rho u), \\ a(v) &= (1, v), \\ q(f) &= F(x) \cdot \nabla_v f, \\ \mathcal{Q}(f, v) &= Mf - f. \end{aligned}$$

The Maxwellian M satisfies (see Bouchut [5])

$$\int_{\mathbb{R}^n} a(v)M(U,v)\,dv = U \qquad \forall U \in \mathbb{R}^p, \tag{4.7}$$

$$\int_{\mathbb{R}^n} v \otimes a(v)M(U,v)\,dv = A(U) \qquad \forall U \in \mathbb{R}^p, \tag{4.8}$$

$$\int_{\mathbb{R}^n} a(v)q(M(U,v))\,dv = -Q(U,x) \quad \forall U \in \mathbb{R}^p, \quad x \in \mathbb{R}^n. \tag{4.9}$$

This is the classical compatibility conditions required for the kinetic equation to be related to the system. Notice that thanks to (4.7), we have (1.10).

The kinetic entropy is the following:

$$H(f,v) = \frac{|v|^2}{2}f \quad \text{for } \gamma = \frac{n+2}{n},$$
$$H(f,v) = \frac{|v|^2}{2}f + \frac{1}{2c^{2/d}}\frac{f^{1+2/d}}{1+2/d} \quad \text{else.}$$

We have (see Bouchut [5]) that, for any f satisfying $\int_{\mathbb{R}^n}(f + H(f,v))\,dv < \infty$, and denoting $U = \int_{\mathbb{R}^n} a(v)f(v)\,dv$, the following minimization principle holds:

$$\int_{\mathbb{R}^n} H(M(U,v),v)\,dv \le \int_{\mathbb{R}^n} H(f)\,dv, \tag{4.10}$$

and a compatibility between the entropy η and the kinetic entropy H is satisfied as

$$\int_{\mathbb{R}^n} H(M(U,v),v)\,dv = \eta(U) \quad \text{for any } U \in \mathbb{R}^p. \tag{4.11}$$

First notice that (1.14) is verified. As a consequence of (4.10) and (4.11), we get

$$\eta(U) = \int_{\mathbb{R}^n} H(M(U,v),v)\,dv \le \int_{\mathbb{R}^n} H(f,v)\,dv,$$

which in particular gives (1.13). We prove now the decrease (1.11). We give it for $\gamma = (n+2)/n$. The other case is similar. Multiplying (4.1) by $|v|^2/2$ and then integrating in (v,x), we get

$$\begin{aligned}\frac{d}{dt}\iint_{\mathbb{R}^{2n}} \frac{|v|^2}{2} f_\varepsilon\,dv\,dx &= \iint_{\mathbb{R}^{2n}} F(x)\cdot v f_\varepsilon\,dv\,dx \\ &\quad + \frac{1}{\varepsilon}\iint_{\mathbb{R}^{2n}} \frac{|v|^2}{2}(Mf_\varepsilon - f_\varepsilon)\,dv\,dx,\end{aligned} \tag{4.12}$$

since

$$\iint_{\mathbb{R}^{2n}} \frac{|v|^2}{2} F(x)\cdot\nabla_v f_\varepsilon\,dv\,dx = -\iint_{\mathbb{R}^{2n}} F(x)\cdot v f_\varepsilon\,dv\,dx.$$

In particular, from (4.10), it gives

$$\begin{aligned}\frac{d}{dt}\iint_{\mathbb{R}^{2n}} H(f_\varepsilon,v)\,dv\,dx &\le \int_{\mathbb{R}^n} F(x)\cdot\left(\int_{\mathbb{R}^n} v f_\varepsilon\,dv\right)dx \\ &\le \int_{\mathbb{R}^n} F(x)\cdot\rho_\varepsilon u_\varepsilon\,dx.\end{aligned} \tag{4.13}$$

Since

$$\int_{\mathbb{R}^n} \eta'(U_\varepsilon) Q(U_\varepsilon)\, dx = \int_{\mathbb{R}^n} F(x)\cdot \rho_\varepsilon u_\varepsilon\, dx,$$

this leads to (1.11).

We want now to prove (1.17)–(1.18). Since

$$A(U_\varepsilon) = \int_{\mathbb{R}^n} v\otimes a(v) M f_\varepsilon\, dv$$

and

$$Q(U_\varepsilon) = \sum_j \int_{\mathbb{R}^n} F_j(x)\partial_{v_j} a(v) M f_\varepsilon\, dv,$$

it gives

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^n} \left| A(U_\varepsilon) - \int_{\mathbb{R}^n} v\otimes a(v) f_\varepsilon\, dv\right| dx\, dt\\ &\quad = \int_0^T \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} v\otimes a(v)(M f_\varepsilon - f_\varepsilon)\, dv\right| dx\, dt \end{aligned} \tag{4.14}$$

and

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^n} \left| Q(U_\varepsilon, x) - \int_{\mathbb{R}^n} F(x)\nabla_v a(v) f_\varepsilon\, dv\right| dx\, dt\\ &\quad = \int_0^T \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} F(x)\nabla_v a(v)(f_\varepsilon - M f_\varepsilon)\, dv\right| dx\, dt. \end{aligned} \tag{4.15}$$

We have

$$\int_{\mathbb{R}^n} \partial_{v_i} a_j(v)(f_\varepsilon - M f_\varepsilon)\, dv = \delta_{i+1,j}\int_{\mathbb{R}^n} (f_\varepsilon - M f_\varepsilon)\, dv = 0;$$

thus the kinetic structure of the source term (1.18) vanishes. In order to apply the convergence result for this kinetic model, it only remains to control the entropy dissipation (1.17). It is the technical point of this example.

4.1.1. Control of the entropy dissipation. This subsection is devoted to the proof of the following proposition.

PROPOSITION 4.1. *Let f_ε be a solution to the BGK equation of the previous section with initial value f_ε^0 bounded in $L^1(\mathbb{R}^{2n})$ verifying (finite energy)*

$$\iint_{\mathbb{R}^{2n}} |v|^2 f_\varepsilon^0(x,v)\, dv\, dx \le C^0 < \infty, \tag{4.16}$$

and with $\gamma = (n+2)/n$. Then there exists C_n such that for every $\varepsilon < 1$, we have

$$\int_0^T \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} v\otimes a(v)(M f_\varepsilon - f_\varepsilon)\, dv\right| dx\, dt \le C_n\sqrt{\varepsilon}.$$

We define

$$D_\varepsilon(t,x) = \int_{\mathbb{R}^n} |v|^2 (f_\varepsilon(t,x,v) - Mf_\varepsilon(t,x,v))\,dv.$$

From (4.13), we have

$$\begin{aligned}\frac{d}{dt}\iint_{\mathbb{R}^{2n}} H(f_\varepsilon, v)\,dv\,dx &\le \|F\|_{L^\infty} \iint_{\mathbb{R}^{2n}} |v| f_\varepsilon\,dx\,dv\\ &\le \|F\|_{L^\infty} \iint_{\mathbb{R}^{2n}} (1+|v|^2) f_\varepsilon\,dx\,dv\\ &\le \|F\|_{L^\infty}\left(\|\rho_\varepsilon\|_{L^1} + 2\iint_{\mathbb{R}^{2n}} H(f_\varepsilon, v)\,dx\,dv\right);\end{aligned}$$

thus by Gronwall's argument, we obtain

$$\iint_{\mathbb{R}^{2n}} |v|^2 f_\varepsilon(t,x,v)\,dv\,dx \le C, \quad 0 \le t \le T. \tag{4.17}$$

Integrating now (4.13) with respect to t leads to

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^n} D_\varepsilon(t,x)\,dx\,dt\\ &\quad\le \varepsilon\left(\iint_{\mathbb{R}^{2n}} |v|^2 f^0(x,v)\,dv\,dx - 2\iiint_{[0,T]\times\mathbb{R}^{2n}} F(x) v f_\varepsilon\,dv\,dx\,dt\right)\\ &\quad\le \varepsilon\left(C^0 + 2\|F\|_{L^\infty}\iiint_{[0,T]\times\mathbb{R}^{2n}} |v| f_\varepsilon\,dv\,dx\,dt\right)\\ &\quad\le \varepsilon\left(C^0 + 2\|F\|_{L^\infty}\iiint_{[0,T]\times\mathbb{R}^{2n}} (1+|v|^2) f_\varepsilon\,dv\,dx\,dt\right)\\ &\quad\le \varepsilon(C^0 + 2\|F\|_{L^\infty} T \|f_\varepsilon^0\|_{L^1} + 2\|F\|_{L^\infty} C)\\ &\quad\le \varepsilon\,\tilde{C}. \end{aligned} \tag{4.18}$$

This gives a bound in ε for

$$\int_0^T \iint_{\mathbb{R}^{2n}} |v|^2 (f_\varepsilon - Mf_\varepsilon)\,dv\,dx\,dt,$$

but we need to control

$$\int_0^T \int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} v\otimes a(v)(Mf_\varepsilon - f_\varepsilon)\,dv\right|\,dx\,dt,$$

which is more delicate.

We set $a_1(v) = 1$ and $a_2(v) = v$ such that $a = (a_1, a_2)$. Similarly, we define $A_1(U) = \rho u$ and $A_2(U) = \rho u \otimes u + I\rho^\gamma$.

Since

$$A(U_\varepsilon) = \int_{\mathbb{R}^n} v \otimes a(v) M f_\varepsilon\,dv,$$

the first component of $\left|\int_{\mathbb{R}^n} v\otimes a(v)(Mf_\varepsilon - f_\varepsilon)\,dv\right|$ is still zero here. Now we have

$$\begin{aligned}\left|A_2(U_\varepsilon) - \int_{\mathbb{R}^n} v\otimes v f_\varepsilon\,dv\right| &= \left|\int_{\mathbb{R}^n} (v-u)\otimes(v-u)(Mf_\varepsilon - f_\varepsilon)\,dv\right| \\ &\le \int_{\mathbb{R}^n} |v-u|^2|Mf_\varepsilon - f_\varepsilon|\,dv.\end{aligned}$$

The first equality uses (4.7). Thus to control the second component, we want to show that

$$\int_{\mathbb{R}^n} |v-u|^2|Mf_\varepsilon - f_\varepsilon|\,dv$$

can be controlled (at least for bounded mass ρ) by the dissipation of entropy

$$\int_{\mathbb{R}^n} |v|^2(f_\varepsilon - Mf_\varepsilon)\,dv.$$

It is the aim of the following proposition.

PROPOSITION 4.2. *For every $f\in L^1(\mathbb{R}^n)$ verifying $0\le f\le 1$, and every $u\in\mathbb{R}^n$ we denote*

$$\begin{aligned}\rho &= \int_{\mathbb{R}^n} f(v)\,dv,\\ F &= \int_{\mathbb{R}^n} |v-u|^2|f(v) - M(\rho,u,v)|\,dv,\\ D &= \int_{\mathbb{R}^n} |v|^2(f(v) - M(\rho,u,v))\,dv.\end{aligned}$$

Then there exists a constant C_n such that, for every $f\in L^1(\mathbb{R}^n)$ verifying $0\le f\le 1$,

$$F \le C_n(\rho^{\frac{n+2}{2n}}\sqrt{D} + D).$$

To prove this result, we first introduce some notations and prove preliminary results. Notice that, thanks to (4.7),

$$D = \int_{\mathbb{R}^n} |v-u|^2(f(v) - M(\rho,u,v))\,dv.$$

Then changing v by $v+u$ if necessary, we see that we can restrict ourselves to the case $u=0$. We first reduce the problem to a one-dimensional problem. We introduce the following quantities:

$$\begin{aligned}\overline{f}(r) &= \frac{1}{|\mathbb{S}_n|}\int_{\mathbb{S}_n} f(r\sigma)\,d\sigma,\\ \overline{M}(r) &= \frac{1}{|\mathbb{S}_n|}\int_{\mathbb{S}_n} M(\rho,0,r\sigma)\,d\sigma = \mathbb{1}_{\{r^n\le c_n\rho\}}(r).\end{aligned}$$

Since the integral of f is equal to the integral of $M(\rho,0,\cdot)$, we have

$$\int_0^\infty r^{n-1}\overline{f}(r)\,dr = \int_0^\infty r^{n-1}\overline{M}(r)\,dr. \tag{4.19}$$

We denote $r_1 = (c_n \rho)^{\frac{1}{n}}$, and we have

$$\begin{aligned} F &= |\mathbb{S}_n| \int_0^\infty r^{n+1} |\overline{f}(r) - \overline{M}(r)| \, dr \\ &= |\mathbb{S}_n| \left(\int_0^{r_1} r^{n+1} (1 - \overline{f}(r)) \, dr + \int_{r_1}^\infty r^{n+1} \overline{f}(r) \, dr \right), \\ D &= |\mathbb{S}_n| \int_0^\infty r^{n+1} (\overline{f}(r) - \overline{M}(r)) \, dr \\ &= |\mathbb{S}_n| \left(- \int_0^{r_1} r^{n+1} (1 - \overline{f}(r)) \, dr + \int_{r_1}^\infty r^{n+1} \overline{f}(r) \, dr \right). \end{aligned}$$

We define in addition

$$M = \int_0^{r_1} r^{n-1} (1 - \overline{f}(r)) \, dr = \int_{r_1}^\infty r^{n-1} \overline{f}(r) \, dr;$$

the last equality comes from (4.19) and $\overline{M}(r) = \mathbb{1}_{\{r \le r_1\}}(r)$. We have to do a different treatment for values close to r_1 and far from this value. For this purpose we consider $r_2 > r_1$ a new number which will be fixed later on. Then we denote

$$\begin{aligned} M_1 &= \int_{r_1}^{r_2} r^{n-1} \overline{f}(r) \, dr, \\ M_2 &= \int_{r_2}^\infty r^{n-1} \overline{f}(r) \, dr. \end{aligned}$$

We have $M = M_1 + M_2$. Then we define $0 < r_0 < r_1$ (in a unique way when r_2 is chosen) in the following way:

$$M_1 = \int_{r_0}^{r_1} r^{n-1} (1 - \overline{f}(r)) \, dr.$$

Then, from the definition of M and since M is the sum of M_1 and M_2, we have

$$M_2 = \int_0^{r_0} r^{n-1} (1 - \overline{f}(r)) \, dr.$$

In the same way we define F_1, F_2, D_1, D_2 in the following way:

$$\begin{aligned} F_1 &= \int_{r_0}^{r_2} r^{n+1} |\overline{f}(r) - \overline{M}(r)| \, dr \\ &= \int_{r_0}^{r_1} r^{n+1} (1 - \overline{f}(r)) \, dr + \int_{r_1}^{r_2} r^{n+1} \overline{f}(r) \, dr, \\ F_2 &= \int_0^{r_0} r^{n+1} |\overline{f}(r) - \overline{M}(r)| \, dr + \int_{r_2}^\infty r^{n+1} |\overline{f}(r) - \overline{M}(r)| \, dr \\ &= \int_0^{r_0} r^{n+1} (1 - \overline{f}(r)) \, dr + \int_{r_2}^\infty r^{n+1} \overline{f}(r) \, dr, \end{aligned}$$

$$\begin{aligned}
D_1 &= \int_{r_0}^{r_2} r^{n+1}(\overline{f}(r) - \overline{M}(r))\,dr \\
&= -\int_{r_0}^{r_1} r^{n+1}(1-\overline{f}(r))\,dr + \int_{r_1}^{r_2} r^{n+1}\overline{f}(r)\,dr, \\
D_2 &= \int_0^{r_0} r^{n+1}(\overline{f}(r) - \overline{M}(r))\,dr + \int_{r_2}^{\infty} r^{n+1}(\overline{f}(r) - \overline{M}(r))\,dr \\
&= -\int_0^{r_0} r^{n+1}(1-\overline{f}(r))\,dr + \int_{r_2}^{\infty} r^{n+1}\overline{f}(r)\,dr.
\end{aligned}$$

Notice that F_1, F_2, M_1, M_2 are nonnegative (as integrals of nonnegative functions) and verify

$$\begin{aligned}
M &= M_1 + M_2, \\
F &= F_1 + F_2, \\
D &= D_1 + D_2.
\end{aligned}$$

We can show, in addition, that D_1 and D_2 are nonnegative too.

LEMMA 4.3. *We have*

$$D_1 \geq 0, \qquad D_2 \geq 0.$$

Proof. We show the result for D_1 (the proof is similar for D_2). We have

$$\begin{aligned}
\int_{r_1}^{r_2} r^{n+1}\overline{f}(r)\,dr &= \int_{r_1}^{r_2} r^2(r^{n-1}\overline{f}(r))\,dr \geq r_1^2 M_1, \\
\int_{r_0}^{r_1} r^{n+1}\overline{f}(r)\,dr &= \int_{r_0}^{r_1} r^2(r^{n-1}\overline{f}(r))\,dr \leq r_1^2 M_1.
\end{aligned}$$

Since D_1 is the difference of those two terms, we find that D_1 is nonnegative. □

We first consider the values far from r_1.

LEMMA 4.4. *We can dominate F_2 by D_2 in the following way:*

$$F_2 \leq D_2 \left(\frac{r_1^2 + r_2^2}{r_2^2 - r_1^2} \right).$$

Proof. We have

$$\begin{aligned}
\int_{r_2}^{\infty} r^{n+1}\overline{f}(r)\,dr &\geq r_2^2 M_2 \\
&\geq r_2^2 \frac{1}{r_0^2} \int_0^{r_0} r^{n+1}(1-\overline{f}(r))\,dr \\
&\geq \frac{r_2^2}{r_1^2} \int_0^{r_0} r^{n+1}(1-\overline{f}(r))\,dr.
\end{aligned}$$

Hence we have

$$D_2 \geq \left(\frac{r_2^2}{r_1^2} - 1 \right) \int_0^{r_0} r^{n+1}(1-\overline{f}(r))\,dr.$$

But F_2 can be expressed in the following way:

$$F_2 = D_2 + 2\int_0^{r_0} r^{n+1}(1-\overline{f}(r))\,dr.$$

Those two expressions lead to

$$F_2 \le D_2 \left(\frac{r_1^2 + r_2^2}{r_2^2 - r_1^2} \right). \qquad \square$$

We consider now the values close to r_1.

LEMMA 4.5. *There exist $a > 0$ and a constant C_n depending only on n such that if $|r_2 - r_1| \le a r_1$, then*

$$F_1 \le C_n a^2 \rho^{\frac{n-2}{2n}} \sqrt{D_1}.$$

Proof. We split the proof in several parts.

(i) *Minimization of the entropy dissipation.* We define α and β such that

$$M_1 = \int_{r_1}^{\beta} r^{n-1}\,dr = \int_{\alpha}^{r_1} r^{n-1}\,dr.$$

From the definition of M_1, notice that $\beta \le r_2$. In the same way we have $\alpha \ge r_0$. We want to show that

$$D_1 \ge \int_{r_1}^{\beta} r^{n+1}\,dr - \int_{\alpha}^{r_1} r^{n+1}\,dr.$$

First we calculate

$$\begin{aligned}
&\int_{r_1}^{r_2} r^{n+1}\overline{f}(r)\,dr - \int_{r_1}^{\beta} r^{n+1}\,dr \\
&\quad = \int_{r_1}^{\beta} r^2[r^{n-1}(\overline{f}(r) - 1)]\,dr + \int_{\beta}^{r_2} r^2[r^{n-1}\overline{f}(r)]\,dr \\
&\quad = \int_{\beta}^{r_2} r^2[r^{n-1}\overline{f}(r)]\,dr - \int_{r_1}^{\beta} r^2[r^{n-1}(1 - \overline{f}(r))]\,dr \\
&\quad \ge \beta^2 \left[\int_{\beta}^{r_2} r^{n-1}\overline{f}(r)\,dr - \int_{r_1}^{\beta} r^{n-1}(1 - \overline{f}(r))\,dr \right] \\
&\quad \ge \beta^2 (M_1 - M_1) = 0.
\end{aligned}$$

In the same way we calculate

$$\begin{aligned}
&\int_{r_0}^{r_1} r^{n+1}(\overline{f}(r) - 1)\,dr + \int_{\alpha}^{r_1} r^{n+1}\,dr \\
&\quad = \int_{r_0}^{\alpha} r^{n+1}(\overline{f}(r) - 1)\,dr + \int_{\alpha}^{r_1} r^{n+1}\overline{f}(r)\,dr \\
&\quad \ge \alpha^2 \left[\int_{r_0}^{\alpha} r^{n-1}(\overline{f}(r) - 1)\,dr + \int_{\alpha}^{r_1} r^{n-1}\overline{f}(r)\,dr \right] \\
&\quad \ge 0.
\end{aligned}$$

Summing those two last inequalities gives the desired result.

(ii) *Taylor expansion of the critical entropy dissipation.* We call critical entropy dissipation the function defined by

$$D^c = \left(\int_{r_1}^{\beta} r^{n+1}\,dr - \int_{\alpha}^{r_1} r^{n+1}\,dr \right),$$

where α and β are defined in (i). Then we have

$$\begin{aligned} nM_1 &= \beta^n - r_1^n, \\ nM_1 &= r_1^n - \alpha^n, \\ (n+2)D^c &= \beta^{n+2} - 2r_1^{n+2} + \alpha^{n+2}, \end{aligned}$$

and therefore

$$\begin{aligned} \frac{D^c}{r_1^{n+2}} = \frac{\alpha+\beta-2r_1}{r_1} &+ \frac{n+1}{2}\left(\left(\frac{\beta-r_1}{r_1}\right)^2 + \left(\frac{\alpha-r_1}{r_1}\right)^2\right) \\ &+ O\left(\left(\frac{\beta-r_1}{r_1}\right)^3 + \left(\frac{\alpha-r_1}{r_1}\right)^3\right). \end{aligned}$$

Now

$$\begin{aligned} \frac{M_1}{r_1^n} &= \frac{\beta-r_1}{r_1} + \frac{n-1}{2}\left(\frac{\beta-r_1}{r_1}\right)^2 + O\left(\frac{\beta-r_1}{r_1}\right)^3 \\ &= \frac{r_1-\alpha}{r_1} - \frac{n-1}{2}\left(\frac{r_1-\alpha}{r_1}\right)^2 + O\left(\frac{r_1-\alpha}{r_1}\right)^3, \end{aligned}$$

and hence

$$\begin{aligned} 0 = \frac{\beta+\alpha-2r_1}{r_1} &+ \frac{n-1}{2}\left[\left(\frac{\beta-r_1}{r_1}\right)^2 + \left(\frac{r_1-\alpha}{r_1}\right)^2\right] \\ &+ O\left(\left(\frac{\beta-r_1}{r_1}\right)^3 + \left(\frac{r_1-\alpha}{r_1}\right)^3\right). \end{aligned}$$

Finally, we obtain

$$\begin{aligned} \frac{D^c}{r_1^{n+2}} &= \left[\left(\frac{\beta-r_1}{r_1}\right)^2 + \left(\frac{r_1-\alpha}{r_1}\right)^2\right] + O\left(\left(\frac{\beta-r_1}{r_1}\right)^3 + \left(\frac{r_1-\alpha}{r_1}\right)^3\right) \\ &= 2\left(\frac{M_1}{r_1^n}\right)^2 + O\left(\left(\frac{\beta-r_1}{r_1}\right)^3 + \left(\frac{r_1-\alpha}{r_1}\right)^3\right). \end{aligned}$$

Hence, there exist $\eta > 0$ and $\delta > 0$ such that

$$D^c \geq \frac{\delta}{r_1^{n-2}} M_1^2$$

whenever

$$\left|\frac{\beta-r_1}{r_1}\right| + \left|\frac{r_1-\alpha}{r_1}\right| \leq \eta.$$

(iii) *Final estimation.* From the definition of α, there exists $a > 0$ such that $|\frac{r_1-\alpha}{r_1}| \leq \eta$ whenever $|\frac{\beta-r_1}{r_1}| \leq a$. Remember that $r_2 \leq \beta$. Hence if $|r_2 - r_1| \leq ar_1$, then

$$\left|\frac{\beta-r_1}{r_1}\right| + \left|\frac{r_1-\alpha}{r_1}\right| \leq \eta$$

and

$$F_1 \le r_2^2 M_1 \le a^2 \delta \sqrt{D^c} r_1^{\frac{n+2}{2}}$$
$$\le C_n a^2 \sqrt{D_1} \rho^{\frac{n+2}{2n}}.$$

The first inequality uses the definition of F_1, the second one uses the result of (ii), and the third one uses the definition of r_1 and the result of (i). ∎

Now we are able to prove the estimate of Proposition 4.2.

Proof of Proposition 4.2. We fix a and r_2 verifying the properties of Lemma 4.5. Thanks to Lemmas 4.4 and 4.5, we have

$$F \le F_1 + F_2 \le D_2 \left(\frac{1+a}{a}\right) + C_n \rho^{\frac{n+2}{2n}} \sqrt{D_1}$$
$$\le C_n'(D + \rho^{\frac{n+2}{2n}} \sqrt{D}). \quad ∎$$

We are now able to prove the announced result.

Proof of Proposition 4.1. Thanks to Proposition 4.2, we have

$$\int_0^T \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} v \otimes a(v)(Mf_\varepsilon - f_\varepsilon)\, dv \right| dx\, dt$$
$$\le C_n \sqrt{\left(\int_0^T \int_{\mathbb{R}^n} \rho_\varepsilon^{\frac{n+2}{n}}(t,x,v)\, dx\, dt\right)\left(\int_0^T \iint_{\mathbb{R}^{2n}} D_\varepsilon(t,x)\, dx\, dt\right)}$$
$$+ C_n \int_0^T \iint_{\mathbb{R}^{2n}} D_\varepsilon(t,x)\, dx\, dt.$$

From (4.17) and

$$\rho_\varepsilon |u_\varepsilon|^2 + n\rho_\varepsilon^{\frac{n+2}{n}} = \int_{\mathbb{R}^n} |v|^2 Mf_\varepsilon(t,x,v)\, dv$$
$$\le \int_{\mathbb{R}^n} |v|^2 f_\varepsilon(t,x,v)\, dv, \tag{4.20}$$

we have

$$\int_0^T \int_{\mathbb{R}^n} \rho_\varepsilon^{\frac{n+2}{n}}(t,x,v)\, dx\, dt \le \frac{T}{n} C.$$

Using (4.18), those lead to

$$\int_0^T \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} v \otimes a(v)(Mf_\varepsilon - f_\varepsilon)\, dv \right| dx\, dt$$
$$\le C_n \sqrt{\frac{\varepsilon T}{n}} \tilde{C}^{1/2} + C_n \varepsilon \tilde{C}. \quad ∎$$

We can then conclude the convergence result for $\gamma = (n+2)/n$.

Proof of Theorem 1.1. We apply Theorem 1.2 to get

$$\int_{\mathbb{R}^n} \eta(U_\varepsilon | U)(t,x)\, dx \to 0 \quad \text{for } t \in [0,T], \text{ as } \varepsilon \to 0. \tag{4.21}$$

Now

$$\eta(U_\varepsilon|U) = \int_0^1 \eta''(U + \vartheta(U_\varepsilon - U)) \cdot (U_\varepsilon - U)^2 \vartheta \, d\vartheta$$

with

$$\eta''(\rho, \rho u) \cdot (X_0, X_1)^2 = \gamma \rho^{\gamma-2} X_0^2 + \frac{1}{\rho}(X_1 - u X_0)^2,$$

and thus we have

$$\int_{\mathbb{R}^n} \int_0^1 \vartheta(\rho + \vartheta(\rho_\varepsilon - \rho))^{\gamma-2} (\rho_\varepsilon - \rho)^2 \, d\vartheta \, dx \to 0 \quad \text{for } t \in [0,T], \text{ as } \varepsilon \to 0 \tag{4.22}$$

and

$$\int_{\mathbb{R}^n} \int_0^1 \frac{\vartheta \rho^2}{(\rho + \vartheta(\rho_\varepsilon - \rho))^3} (\rho_\varepsilon(u_\varepsilon - u))^2 \, d\vartheta \, dx \to 0 \quad \text{for } t \in [0,T], \text{ as } \varepsilon \to 0. \tag{4.23}$$

For $\gamma \geq 2$, (4.22) gives that, up to a subsequence, $\rho_\varepsilon \to \rho$ a.e. as $\varepsilon \to 0$ since $\rho > 0$. For $\gamma < 2$, it gives this result except at the points where $\rho_\varepsilon \to +\infty$, but in this case, as ρ stays bounded,

$$(\rho + \vartheta(\rho_\varepsilon - \rho))^{\gamma-2} (\rho_\varepsilon - \rho)^2 \underset{\varepsilon \to 0}{\sim} \vartheta^{\gamma-2} \rho_\varepsilon^\gamma \to 0 \ \text{ a.e.},$$

and thus this case is impossible. Now (4.23) gives that, up to a subsequence, $\rho_\varepsilon(u_\varepsilon - u) \to 0$ a.e. as $\varepsilon \to 0$ and therefore $\rho_\varepsilon u_\varepsilon \to \rho u$ a.e. as $\varepsilon \to 0$. But from (4.17) and (4.20), we have that ρ_ε is bounded in $L^\infty(0,T; L^\gamma(\mathbb{R}^n))$ and that $\sqrt{\rho_\varepsilon} u_\varepsilon$ is bounded in $L^\infty(0,T; L^2(\mathbb{R}^n))$. Hence, the whole family ρ_ε converges strongly in $L^\infty(0,T; L^p(\mathbb{R}^n))$ to ρ for every $1 \leq p < \gamma$ and $\sqrt{\rho_\varepsilon} u_\varepsilon$ converges strongly to $\sqrt{\rho} u$ in $L^\infty(0,T; L^q(\mathbb{R}^n))$ for every $1 \leq q < 2$. In particular, $\rho_\varepsilon u_\varepsilon$ converges strongly to ρu in $L^\infty(0,T; L^q(\mathbb{R}^n))$ for every $1 \leq q < 2\gamma/(\gamma+1)$. ☐

4.1.2. Extension to every γ. The previous model works for $\gamma = (n+2)/n$. In order to deal with the values of $\gamma \in]1, (n+2)/n[$, we use an other model which was introduced in [5] and is written as follows.

We consider the BGK equation

$$\partial_t f_\varepsilon + \xi \cdot \nabla_x f_\varepsilon + F(x) \cdot \nabla_\xi f_\varepsilon = \frac{M f_\varepsilon - f_\varepsilon}{\varepsilon}, \tag{4.24}$$

where $f_\varepsilon = f_\varepsilon(t,x,v) \in \mathbb{R}$ with $t \in \mathbb{R}^+$, $x \in \mathbb{R}^n$, $v = (\xi, I) \in \mathbb{R}^n \times \mathbb{R}^+$, and $F : \mathbb{R}^n \to \mathbb{R}^n$, with

$$M f_\varepsilon(t,x,v) = M(U_\varepsilon(t,x), v), \qquad U_\varepsilon(t,x) = \int_{\mathbb{R}^{n+1}} a(v) f_\varepsilon(t,x,v) \, dv \tag{4.25}$$

with $a : \mathbb{R}^{n+1} \to \mathbb{R}^p$, $a(v) = (1, \xi)$, $p = n+1$, and

$$dv = b_1 I^{d-1} \, dI \, d\xi, \qquad b_1 = 2\,\pi^{d/2}/\Gamma(d/2),$$

where d is the number of degrees of freedom satisfying

$$n + d = \frac{2}{\gamma - 1}. \tag{4.26}$$

We notice that the function f_ε takes its values in $[0, b_2]$, and the Maxwellian M is defined by

$$M(U, v) = b_2 \mathbb{1}_{|\xi - u|^2 + I^2 < b_3 \rho^{\gamma-1}}, \quad U = (\rho, \rho u), \tag{4.27}$$

where

$$b_2 = \left(\frac{2\pi\gamma}{\gamma-1}\right)^{-1/(\gamma-1)} \Gamma\left(\frac{\gamma}{\gamma-1}\right), \qquad b_3 = \frac{2\gamma}{\gamma-1},$$

and satisfies (4.7)–(4.8). It satisfies also (4.9) with $\partial_{\xi_j} a(v)$ instead of $\partial_{v_j} a(v)$. The kinetic entropy is

$$H(f, v) = \frac{1}{2}|v|^2 f = \frac{1}{2}(|\xi|^2 + I^2) f, \tag{4.28}$$

and satisfies (4.10)–(4.11). We get, as in the previous kinetic model, (1.11) and (1.18).

We recover the BGK model introduced previously integrating (4.24) (and the function f_ε) with respect to I with the measure $b_1 I^{d-1} dI$ [5].

Now for the control of the dissipation, we set

$$\overline{f}(r) = \frac{1}{s_n} \int_{\mathbb{S}^+_{n+1}} f(r\sigma)(\cos\theta)^{d-1}\, d\sigma,$$
$$\overline{M}(r) = \frac{1}{s_n} \int_{\mathbb{S}^+_{n+1}} M(\rho, 0, r\sigma)(\cos\theta)^{d-1}\, d\sigma = b_2 \mathbb{1}_{\{r^2 \le b_3 \rho^{\gamma-1}\}}(r),$$

where $\mathbb{S}^+_{n+1} = \{(\xi, I) \in \mathbb{S}_{n+1};\, I \ge 0\}$, $I = r\cos\theta$ and $s_n = \int_{\mathbb{S}^+_{n+1}} (\cos\theta)^{d-1}\, d\sigma$. Then, we get from the mass conservation

$$\int_0^\infty r^{n+d-1} \overline{f}(r)\, dr = \int_0^\infty r^{n+d-1} \overline{M}(r)\, dr. \tag{4.29}$$

By similar techniques to those in the previous section, we get the following estimate.

PROPOSITION 4.6. *For every $f \in L^1_{dv}(\mathbb{R}^{n+1})$ verifying $0 \le f \le b_2$ and every $u \in \mathbb{R}^n$ we denote*

$$\rho = \int_{\mathbb{R}^{n+1}} f(v)\, dv,$$
$$F = \int_{\mathbb{R}^{n+1}} |v - u|^2 |f(v) - M(\rho, u, v)|\, dv,$$
$$D = \int_{\mathbb{R}^{n+1}} |v|^2 (f(v) - M(\rho, u, v))\, dv.$$

Then there exists a constant C_n such that, for every $f \in L^1_{dv}(\mathbb{R}^{n+1})$ verifying $0 \le f \le b_2$,

$$F \le C_n(\rho^{\frac{n+d+2}{2(n+d)}} \sqrt{D} + D).$$

Now since

$$\frac{n+d+2}{n+d} = \gamma,$$

we get a similar dissipation result to that in Proposition 4.1, and we can conclude the convergence in this case for every γ such that (4.26) is satisfied with $d > 0$; that is to say,

$$1 < \gamma < \frac{n+2}{n}.$$

We then obtain Theorem 1.1 in the same way as in the previous case.

We thank Bouchut for noticing that the proof of the case $\gamma = (n+2)/n$ (of the previous subsection) is also valid for every $1 < \gamma < (n+2)/n$ using this model.

4.2. Fokker–Planck. In this subsection, we study the convergence from the Fokker–Planck kinetic equation to the isothermal system, that is, the case $\gamma = 1$.

4.2.1. The kinetic model. The Fokker–Planck equation on $f_\varepsilon = f_\varepsilon(t, x, v) \in \mathbb{R}$, with $t \in \mathbb{R}^+$ and $x, v \in \mathbb{R}^n$, is given by

$$\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + F(x) \cdot \nabla_v f_\varepsilon = \frac{1}{\varepsilon} \operatorname{div}_v((v - u_\varepsilon) f_\varepsilon + \nabla_v f_\varepsilon), \tag{4.30}$$

where

$$\rho_\varepsilon = \int_{\mathbb{R}^n} f_\varepsilon \, dv, \qquad \rho_\varepsilon u_\varepsilon = \int_{\mathbb{R}^n} v f_\varepsilon \, dv, \tag{4.31}$$

with the kinetic entropy

$$H(f, v) = \left(\frac{1}{2}|v|^2 + \ln f\right) f. \tag{4.32}$$

Here, we have $q(f) = F(x) \cdot \nabla_v f$, $a(v) = (1, v)$, and

$$\mathcal{Q}(f, v) = \operatorname{div}_v((v - u) f + \nabla_v f), \qquad (\rho, \rho u) = \int_{\mathbb{R}^n} a(v) f \, dv.$$

The property (1.10) is clear, and the property (1.13) comes from the following majorations: for $f \in L^1(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} H(f, v)\, dv < \infty$, denoting $(\rho, \rho u) = \int_{\mathbb{R}^n} a(v) f(v)\, dv$, we have

$$\rho u^2 = \frac{(\int_{\mathbb{R}^n} v f \, dv)^2}{\int_{\mathbb{R}^n} f(v)\, dv} \le \int_{\mathbb{R}^n} |v|^2 f(v)\, dv,$$

$$\rho \ln \rho = \left(\int_{\mathbb{R}^n} f(v)\, dv\right) \ln \left(\int_{\mathbb{R}^n} f(v)\, dv\right) \le \int_{\mathbb{R}^n} f(v) \ln f(v)\, dv,$$

by Cauchy–Schwarz and by Jensen's inequality. As in the previous section we can check that

$$Q(\rho_\varepsilon, \rho_\varepsilon u_\varepsilon) = (0, F\rho_\varepsilon) = -\int_{\mathbb{R}^n} a(v) q(f_\varepsilon)\, dv,$$

so (1.18) is verified. It remains to verify (1.11) and (1.17).

4.2.2. Control of the entropy estimate and convergence result. We have the following estimate.

PROPOSITION 4.7. *Let f_ε be a solution to the kinetic equation* (4.5) *with initial value f_ε^0 bounded in $L^1(\mathbb{R}^{2n})$ verifying (finite energy)*

$$\iint_{\mathbb{R}^{2n}} H(f_\varepsilon^0(x,v),v)\,dv\,dx \le C^0 < \infty. \tag{4.33}$$

Then f_ε satisfies (1.11) *and*

$$\int_0^T \int_{\mathbb{R}^n} \left| A(U_\varepsilon) - \int_{\mathbb{R}^n} v \otimes a(v) f_\varepsilon\,dv \right| dx\,dt \le C\sqrt{\varepsilon}, \tag{4.34}$$

where A is the flux of the isothermal system.

Proof. We have

$$\partial_f H(f_\varepsilon, v) = \frac{|v|^2}{2} + 1 + \ln f_\varepsilon.$$

So

$$\begin{aligned}\int_{\mathbb{R}^n} \partial_f H(f_\varepsilon,v) F(x)\cdot\nabla_v f_\varepsilon\,dv &= -\rho_\varepsilon u_\varepsilon F(x)\\ &= -\eta'(\rho_\varepsilon,\rho_\varepsilon u_\varepsilon) Q(\rho_\varepsilon,\rho_\varepsilon u_\varepsilon)\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^n} \partial_f H(f_\varepsilon,v)\,\mathrm{div}_v((v-u_\varepsilon)f_\varepsilon + \nabla_v f_\varepsilon)\,dv\\ &\quad= -\int_{\mathbb{R}^n} (v(v-u_\varepsilon)f_\varepsilon + v\nabla_v f_\varepsilon)\,dv - \int_{\mathbb{R}^n} \left(\frac{\nabla_v f_\varepsilon}{f_\varepsilon}(v-u_\varepsilon)f_\varepsilon + \frac{(\nabla_v f_\varepsilon)^2}{f_\varepsilon}\right) dv\\ &\quad= -\int_{\mathbb{R}^n} \frac{((v-u_\varepsilon)f_\varepsilon + \nabla_v f_\varepsilon)^2}{f_\varepsilon}\,dv - \int_{\mathbb{R}^n} u_\varepsilon(\nabla_v f_\varepsilon + (v-u_\varepsilon)f_\varepsilon)\,dv\\ &\quad= -\int_{\mathbb{R}^n} \frac{((v-u_\varepsilon)f_\varepsilon + \nabla_v f_\varepsilon)^2}{f_\varepsilon}\,dv.\end{aligned}$$

That is to say,

$$\begin{aligned}&\partial_t \int_{\mathbb{R}^n} H(f_\varepsilon,v)\,dv + \int_{\mathbb{R}^n} v\cdot\nabla_x H(f_\varepsilon,v)\,dv\\ &\qquad= \int_{\mathbb{R}^n} \partial_f H(f_\varepsilon,v)(\partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon)\,dv\\ &\qquad= F(x)\rho_\varepsilon u_\varepsilon - \frac{1}{\varepsilon}\int_{\mathbb{R}^n} \frac{((v-u_\varepsilon)f_\varepsilon + \nabla_v f_\varepsilon)^2}{f_\varepsilon}\,dv.\end{aligned}$$

The first consequence of this relation is

$$\frac{d}{dt}\iint_{\mathbb{R}^{2n}} H(f_\varepsilon,v)\,dv\,dx \le \int_{\mathbb{R}^n} \eta'(\rho_\varepsilon,\rho_\varepsilon u_\varepsilon) Q(\rho_\varepsilon,\rho_\varepsilon u_\varepsilon)\,dx.$$

In particular this implies property (1.11). Moreover,

$$\begin{aligned}\frac{d}{dt}\iint_{\mathbb{R}^{2n}} H(f_\varepsilon, v)\,dv\,dx &\le F(x)\iint_{\mathbb{R}^{2n}} v f_\varepsilon\,dv\,dx\\ &\le \|F\|_{L^\infty}\iint_{\mathbb{R}^{2n}} (|v|^2+1) f_\varepsilon\,dv\,dx\\ &\le C_1\left(\|\rho_\varepsilon^0\|_{L^1} + \iint_{\mathbb{R}^{2n}} H(f_\varepsilon, v)\,dv\,dx\right),\end{aligned}$$

since the quantity $\iint |v|^2 f_\varepsilon\,dx\,dv$ is controlled in a classical way by $\iint H(f_\varepsilon, v)\,dx\,dv$. Using Gronwall's lemma, we deduce that there exists a constant C depending on T and f_ε^0 such that, for every $0 \le t \le T$,

$$\iint_{\mathbb{R}^{2n}} H(f_\varepsilon(t,x,v), v)\,dv\,dx \le C. \tag{4.35}$$

The second consequence is

$$\begin{aligned}&\left|\iiint_{[0,T]\times\mathbb{R}^{2n}} \frac{((v-u_\varepsilon)f_\varepsilon + \nabla_v f_\varepsilon)^2}{f_\varepsilon}\,dv\,dx\,dt\right|\\ &\le \varepsilon\left(\iint_{\mathbb{R}^{2n}} H(f_\varepsilon^0)\,dv\,dx + \iint_{[0,T]\times\mathbb{R}^n} F(x)\rho_\varepsilon u_\varepsilon\,dx\,dt\right)\\ &\le \varepsilon\left(C^0 + \|F\|_{L^\infty}\iiint_{[0,T]\times\mathbb{R}^{2n}} |v| f_\varepsilon\,dv\,dx\,dt\right)\\ &\le \varepsilon\left(C^0 + \|F\|_{L^\infty}\iiint_{[0,T]\times\mathbb{R}^{2n}} (|v|^2+1) f_\varepsilon\,dv\,dx\,dt\right)\\ &\le \varepsilon\left(C^0 + \|F\|_{L^\infty}\left(C' + T\iint_{[0,T]\times\mathbb{R}^n} f_\varepsilon^0\,dv\,dx\right)\right)\\ &\le \varepsilon\, C_2.\end{aligned}$$

We have to estimate

$$\int_{\mathbb{R}^n}\left|A(U_\varepsilon) - \int_{\mathbb{R}^n} v\otimes a(v) f_\varepsilon\,dv\right|\,dx.$$

The first componant is zero. The second one is

$$E_2 = \int_{\mathbb{R}^n}\left|\rho_\varepsilon u_\varepsilon\otimes u_\varepsilon + \rho_\varepsilon I - \int_{\mathbb{R}^n} v\otimes v f_\varepsilon\,dv\right|\,dx,$$

which can be rewritten as

$$E_2 = \iint_{\mathbb{R}^{2n}} |v\otimes[(u_\varepsilon - v)f_\varepsilon - \nabla_v f_\varepsilon]|\,dv\,dx,$$

since

$$\int_{\mathbb{R}^n} v\otimes\nabla_v f_\varepsilon\,dv = -\int_{\mathbb{R}^n} f_\varepsilon\,dv I.$$

Thus we get

$$\int_0^T\int_{\mathbb{R}^{2n}} \left| A_2(U_\varepsilon) - \int_{\mathbb{R}^n} v\otimes a_2(v) f_\varepsilon\, dv \right| dx\, dt$$
$$\leq \left(\int_0^T\iint_{\mathbb{R}^{2n}} |v|^2 f_\varepsilon\, dv\, dx\, dt \right)^{1/2} \left(\int_0^T\iint_{\mathbb{R}^{2n}} \frac{((v-u_\varepsilon)f_\varepsilon + \nabla_v f_\varepsilon)^2}{f_\varepsilon}\, dv\, dx\, dt \right)^{1/2},$$

which concludes the proof. ∎

We can then apply the convergence result (Theorem 1.2). As in the previous section, this leads to Theorem 1.1 for the isothermal case.

Appendix. Euler. In this appendix, we calculate the various quantities which appear in our study in the case of the Euler system in order to see what prevents us from applying the method. For the full gas dynamics of Euler, the conservative variables are

$$U = (\rho, q, E) = \left(\rho, \rho u, \rho\frac{|u|^2}{2} + \frac{n}{2}\rho T \right),$$

and the flux is

$$A(U) = \left(\rho u, \rho u\otimes u + \rho T I, \rho u \frac{|u|^2}{2} + \frac{n+2}{2}\rho T u \right).$$

The entropy is

$$\eta(U) = \rho \ln\left(\frac{\rho}{(2\pi T)^{n/2}} \right) - \frac{n}{2}\rho,$$

and the associated flux is $G(U) = \eta(U)u$. The expression of the flux A in conservative variables is

$$A(U) = \left(q,\ \frac{1}{\rho} q\otimes q + \frac{2E}{n} I - \frac{1}{n\rho}|q|^2 I,\ \frac{n+2}{n}\frac{q}{\rho}E - \frac{1}{n}\frac{q}{\rho^2}|q|^2 \right).$$

Then we get

$$\begin{aligned}
\partial_\rho A_q(U) &= -u\otimes u + \frac{1}{n}|u|^2 I,\\
\partial_{q_i}(A_q)_{jk}(U) &= \delta_{ij}u_k + \delta_{ik}u_j - \delta_{jk}\frac{2u_i}{n},\\
\partial_E A_q(U) &= \frac{2}{n} I,\\
\partial_\rho A_E(U) &= -\frac{n-2}{2}u\frac{|u|^2}{2} - \frac{n+2}{2}uT,\\
\partial_{q_i}(A_E)_j(U) &= \delta_{ij}\left(\frac{|u|^2}{2} + \frac{n+2}{2}T \right) - \frac{2}{n}u_i u_j,\\
\partial_E A_E(U) &= \frac{n+2}{n}u,
\end{aligned}$$

and the relative flux is

$$A_\rho(U_1|U_2) = 0, \tag{A.1}$$

$$A_q(U_1|U_2) = \rho_1(u_1 - u_2) \otimes (u_1 - u_2) - \frac{1}{n}\rho_1|u_1 - u_2|^2 I, \tag{A.2}$$

$$\begin{aligned} A_E(U_1|U_2) = \frac{1}{2}\rho_1(|u_1|^2 - |u_2|^2)(u_1 - u_2) + \frac{n+2}{2}\rho_1(u_1 - u_2)(T_1 - T_2) \\ -\frac{1}{n}\rho_1 u_2 |u_1 - u_2|^2. \end{aligned} \tag{A.3}$$

We compute now the relative entropy. Since the linear part in a function disappeared in any relative quantity, we have to compute the flux of

$$\tilde{\eta}(U) = \left(1 + \frac{n}{2}\right)\rho \ln \rho - \frac{n}{2}\rho \ln\left(\frac{2E}{n} - \frac{|q|^2}{n\rho}\right),$$

which satisfies

$$\partial_\rho \tilde{\eta}(U) = 1 + \ln \rho + \frac{n}{2} - \frac{n}{2}\ln T - \frac{|u|^2}{2T}, \qquad \partial_q \tilde{\eta} = \frac{u}{T}, \qquad \partial_E \tilde{\eta} = -\frac{1}{T},$$

and thus we get

$$\eta(U_1|U_2) = h(\rho_1|\rho_2) + \frac{n\rho_1}{2T_2}h(T_2|T_1) + \frac{\rho_1}{2T_2}|u_1 - u_2|^2, \tag{A.4}$$

where $h(x) = x \ln x$.

We see that we cannot apply our method in this case because of the cubic power in velocity in $A_E(U_1|U_2)$ since such a term does not appear in $\eta(U_1|U_2)$.

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