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Strong traces for solutions to multidimensional scalar conservation laws

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Abstract

In this paper we consider multidimensional scalar conservation laws without BV estimates defined in a subset $\Omega \subset \mathbb{R}^+ \times \mathbb{R}^d$. We show that, with a non-degeneracy hypothesis on the flux, we can define a strong notion of trace at the boundary of Ω reached by L^1 convergence.

Key words. conservation law – trace theorem – kinetic formulation – boundary value problem – averaging lemma

1. Introduction

In this article we consider an open subset $\Omega \subset \mathbb{R}^+ \times \mathbb{R}^d$ and functions $u \in L^\infty(\Omega)$ solutions to the scalar conservation law:

$$\partial_t u + \mathrm{div}_x A(u) = 0, \quad (t,x) \in \Omega. \tag{1}$$

The flux function A is assumed to be regular from $\mathbb{R}$ to $\mathbb{R}^d$. As is usual, we deal only with entropy solutions, namely which fulfill in the sense of distribution

$$\partial_t \phi(u) + \mathrm{div}_x H(u) \leq 0, \quad (t,x) \in \Omega, \tag{2}$$

for every convex function ϕ and related entropy flux defined by $H' = A'\phi'$. The problem is to define in a strong way the value of u at the boundary of Ω. Following [3] we consider domains Ω which have a “regular deformable Lipschitz boundary” namely which verify:
(i): For each $\hat{z} = (\hat{t}, \hat{x}) \in \partial\Omega$, there exists $r_{\hat{z}} > 0$, a Lipschitz mapping

$\gamma_{\hat z} : \mathbb{R}^d \to \mathbb{R}$ and an isometry for the euclidean norm $\mathcal{R}_{\hat z} : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ such that:

$$\mathcal{R}_{\hat z}(\hat z) = 0,$$
$$\mathcal{R}_{\hat z}(\Omega)\cap] - r_{\hat z}, r_{\hat z}[^{d+1} = \{y \in] - r_{\hat z}, r_{\hat z}[^{d+1} \mid y_0 > \gamma_{\hat z}(y_1, ..., y_d)\}.$$

(ii): There exists at least one $\partial\Omega$-regular Lipschitz deformation,

where, for K open subset of $\partial\Omega$, we call K-regular Lipschitz deformation every function $\psi : [0,1] \times K \to \bar\Omega$ homeomorphism bi-Lipschitz over its image which verifies:
(iii): $\psi(0,\cdot) \equiv I_K$, where I_K is the identity map over K.
(iv):

$$\lim_{s\to 0+} \nabla\psi(s,\cdot) \circ \tilde\gamma_{\hat z} = \nabla\tilde\gamma_{\hat z} \text{ in } L^1(] - r_{\hat z}, r_{\hat z}[^d \cap \mathcal{R}_{\hat z}(K)),$$

where for every $\hat z \in \partial\Omega$ $\tilde\gamma_{\hat z}$ is the restriction to $]-r_{\hat z}, r_{\hat z}[^d \cap \mathcal{R}_{\hat z}(K)$ of the map $(y_1, ..., y_d) \to \mathcal{R}_{\hat z}^{-1}(\gamma_{\hat z}(y_1, ..., y_d), y_1, ..., y_d)$. Notice that the set of Ω with regular Lipschitz boundary includes star-shaped domains, the domains whose boundaries satisfy the cone property, and domains with $\mathcal{C}^2$ boundaries ([3]).

More precisely, we want to study the limit of $u(\psi(s,\cdot))$ when s goes to zero for such $\partial\Omega$-regular Lipschitz deformation.

It is possible to define weak traces using Young measures (as in [17] for instance). Notice that since the limit is weak, even if we can define the trace of u itself (denoted u^τ), we did not know if the trace of $G(u)$ is equal to $G(u^\tau)$ for a nonlinear function G. The aim of this paper is to show that, with an hypothesis of non-degeneracy of A, in fact the trace is reached by L^1_{loc} convergence.

Theorem 1. *Let $\Omega \subset \mathbb{R}^{d+1}$ have a regular deformable Lipschitz boundary, and assume that $A \in \left[C^3(\mathbb{R})\right]^d$ and verifies for every $(\tau,\zeta) \in \mathbb{R}\times\mathbb{R}^d$, $(\tau,\zeta) \neq (0,0)$:*

$$\mathcal{L}(\{\xi \mid \tau + \zeta.A'(\xi) = 0\}) = 0, \tag{3}$$

where $\mathcal{L}$ is the Lebesgue measure. Then for every function $u \in L^\infty(\Omega)$ which verifies (1)(2) in Ω, there exists $u^\tau \in L^\infty(\partial\Omega)$ such that, for every $\partial\Omega$-regular Lipschitz deformation ψ and every compact set $K \subset\subset \partial\Omega$:

$$\operatorname*{esslim}_{s\to 0} \int_K |u(\psi(s,\hat z)) - u^\tau(\hat z)|\, d\mathcal{H}^d(\hat z) = 0, \tag{4}$$

where $\mathcal{H}^d$ is the d-dimensional Haussdorf measure. Especially, for every smooth function G, we have

$$[G(u)]^\tau = G(u^\tau). \tag{5}$$

This can be seen as a regularization effect at the boundary induced by the non-degeneracy of A (see [5,14] for other results of this kind). Our method is based on the kinetic formulation of scalar conservation laws introduced by Lions Perthame Tadmor [11] which allows to use the so-called averaging lemmas [1,9,7,16] with the assumption (3). This can be seen as a non-degeneracy property since it avoids flux functions whose restriction to an open subset is linear. Hence we use a localization method first introduced in [18]. Let us mention two consequences to motivate this result.

Initial layers

Here, we fix the open set $\Omega=]0,+\infty[\times\mathbb{R}^d$. Let us consider an initial value $u_0\in L^\infty(\mathbb{R}^d)$. Kruzkov has shown in [10] the uniqueness of the solution of (1)(2) verifying for any $R>0$:

$$\frac{1}{T}\int_0^T\int_{|x|\leq R}|u(t,x)-u_0(x)|\,dx\,dt\longrightarrow 0, \tag{6}$$

when T tends to 0. Now, if we don't have this assumption but only the weak form:

$$\frac{1}{T}\int_0^T\int_{\mathbb{R}^d}(u(t,x)-u_0(x))\phi(x)\,dx\,dt\longrightarrow 0,\ \text{when } T\to 0$$

for every test function ϕ, the question is: Have we got still uniqueness? This question was first mentioned by Natalini and arises naturally in relaxation limit problems. Chen and Rascle have shown in [4] that the answer is yes in the monodimensional case with an assumption of non-degeneracy on the flux. With the hypothesis (3), Theorem 1 extends this result to the multidimensional case. Indeed, it shows that every uniformly bounded solution of (1)(2) in $]0,+\infty[\times\mathbb{R}^d$ verifies (6), and so, is the unique Kruzkov solution. Notice that this problem is still open for a general flux A.

Initial-boundary problem

Now we consider a regular open subset $\omega\in\mathbb{R}^d$, we denote ν the normal at the boundary and we take $\Omega=]0,+\infty[\times\omega$. We introduce an initial value $u_0\in L^\infty(\mathbb{R}^d)$ and a boundary value $u^b\in L^\infty(]0,+\infty[\times\partial\omega)$. The following strong boundary condition has been introduced by Bardos, Leroux and Nedelec in [2]:

$$\text{sign}(u(t,\hat{x})-u^b(t,\hat{x}))(A(u(t,\hat{x})).\nu(\hat{x})-A(k).\nu(\hat{x}))\geq 0 \tag{7}$$

$$\text{for all } k \text{ in the interval with endpoints } u(t,\hat{x}) \text{ and } u^b(t,\hat{x}) \tag{8}$$

for almost every boundary point $(t,\hat{x})\in]0,+\infty[\times\partial\omega$. They have shown that if u_0 and u^b are regular, then there exists a unique solution $u\in C(\mathbb{R}^+,L^1_{\text{loc}}(\omega))$ solution of (1)(2) in Ω, satisfying (7)(8) on $\partial\omega$ and $u(t=0)=u_0$. Moreover this is the physical relevant one since it can be obtained by the viscosity method. Some equivalent formulations have been proposed aftewards (see for example [8] [12]). But all those formulations need to restrict themselves to regular boundary values in order to have BV solutions which allows them to have strong traces reached in L^1. This was needed in order to give a meaning at (7). A more general formulation has been introduced by Szepessy in [17] which can be defined without strong traces, using DiPerna tools from [6]. But he shows the existence and uniqueness of the solution only in the framework of regular boundary values. Finally, Otto has introduced a quite complicated weak formulation [13] which allows him to show the existence and uniqueness of solution for general boundary values.

Now, if we consider a flux A verifying (3), then thanks to Theorem 1, the condition (7) is well defined whatever the solution $u \in L^\infty(\Omega)$ of (1)(2) is. Then Otto's condition is equivalent to (7) and we can restrict ourselves to this strong formulation even for non smooth boundary values.

2. Reformulation of the problem

We have:

$$\partial\Omega \subset \bigcup_{\hat{z}\in\partial\Omega} \mathcal{R}_{\hat{z}}^{-1}(]-r_{\hat{z}}, r_{\hat{z}}[^{d+1}),$$

where $r_{\hat{z}}$ and $\mathcal{R}_{\hat{z}}$ are defined by property (i) of regular deformable Lipschitz boundary. Since $\partial\Omega \cap B(0,n)$ is a compact subset for every integer n, there exists a finite set I_n such that:

$$(\partial\Omega \cap B(0,n)) \subset \bigcup_{\alpha\in I_n} \mathcal{R}_{\hat{z}_\alpha}^{-1}(]-r_{\hat{z}_\alpha}, r_{\hat{z}_\alpha}[^{d+1}).$$

So, $I = \cup I_n$ is a countable set such that:

$$\partial\Omega = \bigcup_{\alpha\in I} \Gamma_\alpha,$$

where Γ_α is defined (using the property (i) of regular deformable Lipschitz boundary) by:

$$\Gamma_\alpha = \mathcal{R}_\alpha^{-1}(\{y \in]-r_\alpha, r_\alpha[^{d+1} \mid y_0 = \gamma_\alpha(y_1, \cdots, y_d)\}).$$

In order to simplify the notation we write α instead of $\hat{z}_\alpha$ in the indices, and we denote in the same way

$$\Omega_\alpha = \{y \in]-r_\alpha, r_\alpha[^{d+1} \mid y_0 > \gamma_\alpha(y_1, \cdots, y_d)\}.$$

From now on we work in Ω_α and in the new y coordinates. We denote $A_\alpha(\xi) = \mathcal{R}_\alpha(\xi, A^1(\xi), ..., A^d(\xi))$ and $H_\alpha(\xi) = \mathcal{R}_\alpha(\phi(\xi), H^1(\xi), ..., H^d(\xi))$. We define $u_\alpha : \Omega_\alpha \to \mathbb{R}$ by $u_\alpha(y) = u(\mathcal{R}_\alpha^{-1}(y))$. In the y coordinates, (1)(2) correspond in Ω_α to:

$$\mathrm{div}_y A_\alpha(u_\alpha) = 0 \tag{9}$$

$$\mathrm{div}_y H_\alpha(u_\alpha) \leq 0. \tag{10}$$

We use for the proof the kinetic formulation due to Lions, Perthame and Tadmor. In order to do so we set $L = \|u\|_{L^\infty(\Omega)}$ and introduce a new variable $\xi \in]-L, L[$ and denote for every $v \in]-L, L[$:

$$\begin{aligned} \chi(v,\xi) &= \mathbf{1}_{\{0 \leq \xi \leq v\}} \text{ if } v \geq 0 \\ &= -\mathbf{1}_{\{v \leq \xi \leq 0\}} \text{ if } v < 0. \end{aligned}$$

Then we introduce new functions called microscopic functions which depend on ξ and on a variable z which can lie on Ω_α, on Γ_α or on a local space as we will see later. We will consider especially the following ones:

Definition 1. Let N be an integer, $\mathcal{O}$ be an open set of $\mathbb{R}^N$, and the microscopic function $f \in L^\infty(\mathcal{O}\times]-L, L[)$ be such that $0 \leq \mathrm{sgn}(\xi) f(z,\xi) \leq 1$ for almost every (z,ξ). We say that f is a χ-function if there exists $u \in L^\infty(\mathcal{O})$ such that for almost every $z \in \mathcal{O}$:

$$f(z,\cdot) = \chi(u(z),\cdot).$$

Notice that in this case $u(z) = \int_{-L}^{L} f(z,\xi)\, d\xi$.

Lions Perthame and Tadmor have shown in [11] the following theorem:

Theorem 2. *A function $u \in L^\infty(\Omega_\alpha)$ with $|u| \leq L$ is solution of (9)(10) in Ω_α if and only if there exists a nonnegative measure $m \in \mathcal{M}^+(\Omega_\alpha \times]-L, L[)$ such that the related χ-function f defined by $f(y,\cdot) = \chi(u(y),\cdot)$ for almost every $y \in \Omega_\alpha$ verifies:*

$$a(\xi).\nabla_y f = \partial_\xi m \tag{11}$$

in $\Omega_\alpha \times]-L, L[$ with $a(\xi) = {A_\alpha}'(\xi)$.

Notice that the hypothesis (3) is equivalent to:

$$\mathcal{L}(\{\xi \mid \zeta.a(\xi) = 0\}) = 0, \tag{12}$$

for every $\zeta \in \mathbb{R}^{d+1} \setminus \{0\}$.

The χ-functions have the following properties which are linked to Young measures.

Lemma 1. *Let $\mathcal{O}$ be an open set of $\mathbb{R}^N$ and $f_n \in L^\infty(\mathcal{O}\times]-L, L[)$ be a sequence of χ-functions converging weakly to $f \in L^\infty(\mathcal{O}\times]-L, L[)$. We denote $u_n(\cdot) = \int_{-L}^{L} f_n(\cdot,\xi)\, d\xi$ and $u(\cdot) = \int_{-L}^{L} f(\cdot,\xi)\, d\xi$. Then the three following propositions are equivalent:*

- *f_n converges strongly to f in $L^1_{\mathrm{loc}}(\mathcal{O}\times]-L, L[)$,*
- *u_n converges strongly to u in $L^1_{\mathrm{loc}}(\mathcal{O})$,*
- *f is a χ-function.*

Proof of Lemma 1: Notice that if we denote $\mu_n(\cdot,\xi)$ the Young measure related to $u_n(\cdot)=\int_{-L}^{L} f_n(\cdot,\xi)\,d\xi$ then, since f_n is a χ-function we have

$$\mu_n=\delta_0(\xi)-\partial_\xi f_n. \tag{13}$$

If f_n converges strongly, the same holds for u_n. If u_n converges strongly then passing to the limit in (13) we find that $\delta_0(\xi)-\partial_\xi f$ is a Dirac function with respect to ξ providing that f is a χ-function. Finally, if f is a χ-function, in particular $\mathrm{sgn}(\xi)f=f^2$ so $\|f_n\|_{L^2_{\mathrm{loc}}(\mathcal{O}\times]-L,L[)}$ converges to $\|f\|_{L^2_{\mathrm{loc}}(\mathcal{O}\times]-L,L[)}$. This provides the strong convergence of f_n in $L^2_{\mathrm{loc}}(\mathcal{O}\times]-L,L[)$ and then in $L^1_{\mathrm{loc}}(\mathcal{O}\times]-L,L[)$. □

In the following we will use the notation:

$$a(\xi)=(a^0(\xi),\hat{a}(\xi)) \text{ with } \hat{a}(\xi)=(a^1(\xi),...,a^d(\xi)) \tag{14}$$
$$y=(y_0,\hat{y}) \text{ with } \hat{y}=(y_1,...,y_d). \tag{15}$$

3. A criterion for the existence of strong traces

In this section we consider Ω_α for a fixed α and the χ-function f associated to u_α. Since Γ_α is parametrized by $\hat{y}$, for every Γ_α-regular Lipschitz deformation ψ and every $\hat{y}\in]-r_\alpha,r_\alpha[^d$ we set:

$$\tilde{\psi}(s,\hat{y})=\psi(s,\mathcal{R}_\alpha^{-1}(\gamma_\alpha(\hat{y}),\hat{y})),$$
$$f_\psi(s,\hat{y},\xi)=f(\tilde{\psi}(s,\hat{y}),\xi).$$

Proposition 1. *Let f be a solution of (11) in $\Omega_\alpha\times]-L,L[$ with $a(\xi)$ verifying (12). Then there exists a unique $f^\tau\in L^\infty(]-r_\alpha,r_\alpha[^d\times]-L,L[)$ such that for all ψ Γ_α-regular Lipschitz deformation:*

$$\operatorname*{esslim}_{s\to 0} f_\psi(s,\cdot,\cdot)=f^\tau \text{ in } H^{-1}(]-r_\alpha,r_\alpha[^d\times]-L,L[).$$

This shows the existence of a weak trace on Γ_α which does not depend on the way chosen to reach the boundary. This result is closely related to [13] and a closed version was used in [17]. We give here a proof in our framework of kinetic formulation.

Proof of Proposition 1: Since $\|f_\psi(s,\cdot,\cdot)\|_{L^\infty}\le 1$, by Weak compactness and Sobolev imbedding, for every regular Lipschitz deformation ψ and every sequence s^n which tends to 0 there exists a subsequence n_p and a function $g^\tau_\psi\in L^\infty(]-r_\alpha,r_\alpha[^d\times]-L,L[)$ such that

$$f_\psi(s^{n_p},\cdot,\cdot)\overset{H^{-1}\cap L^\infty W*}{\longrightarrow} g^\tau_\psi \text{ when } n_p\to+\infty. \tag{16}$$

Let us now show that g^{τ}_{ψ} does not depend on ψ, on the sequence s^n and s^{n_p}. In order to do so, let us first consider the entropy flux associated to entropy ϕ:

$$H_{\phi}(y) = \int_{-L}^{L} a(\xi)\phi'(\xi)f(y,\xi)\,d\xi. \tag{17}$$

Multiplying (11) by $\phi'(\xi)$ and integrating it with respect to ξ we find:

$$\mathrm{div}_y H_{\phi} = -\int_{-L}^{L} \phi''(\xi)m(y,d\xi) \in \mathcal{M}(]-r_\alpha, r_\alpha[^{d+1}).$$

We can now use the following Theorem proved by Chen and Grid in [3]:

Theorem 3. *Let Ω be an open set with regular deformable Lipschitz boundary and $F \in [L^\infty(\Omega)]^{d+1}$ be such that $\mathrm{div}_y F$ is a bounded measure. Then there exists $F.\nu \in L^\infty(\partial\Omega)$ such that for every ψ $\partial\Omega$-regular Lipschitz deformation:*

$$\operatorname*{esslim}_{s\to 0} F(\psi(s,\cdot)).\nu_s(\cdot) = F.\nu \text{ in } L^\infty(\partial\Omega)\ w*,$$

where ν_s is a unit outward normal field of $\psi(\{s\}\times\partial\Omega)$.

So, since Γ_α is a regular deformable Lipschitz boundary, this Theorem insures that there exists $H^{\tau}_{\phi}.\nu \in L^\infty(]-r_\alpha, r_\alpha[^d)$ which does not depend on ψ such that

$$H_{\phi}(\tilde{\psi}(s,\cdot)).\nu_s(\cdot) \overset{\mathcal{D}'(]-r_\alpha,r_\alpha[^d)}{\underset{s\to 0}{\longrightarrow}} H^{\tau}_{\phi}.\nu, \tag{18}$$

for every regular Lipschitz deformation ψ. Thanks to property (iv) of regular Lipschitz deformation, ν_s converges strongly in $L^1(]-r_\alpha, r_\alpha[^d)$ to ν, unit outward normal field of Γ_α. So, using (17) and (16), (18) leads to:

$$\int_{]-r_\alpha,r_\alpha[^d}\int_{-L}^{L} \varphi(\hat{y})\phi'(\xi)a(\xi).\nu(\hat{y})g^{\tau}_{\psi}(\hat{y},\xi)\,d\xi\,d\hat{y} = \int_{]-r_\alpha,r_\alpha[^d} H^{\tau}_{\phi}.\nu(\hat{y})\varphi(\hat{y})\,d\hat{y}$$

for every test functions $(\phi,\varphi) \in \mathcal{D}(]-L,L[)\times\mathcal{D}(]-r_\alpha, r_\alpha[^d)$. The right-hand side of this equation is independent of ψ, sequence s^n and subsequence s^{n_p} so $a(\xi).\nu(\hat{y})g^{\tau}_{\psi}(\hat{y},\xi)$ does not depend on those quantities too. Since (thanks to (12)):

$$\mathcal{L}(\{\xi \mid a(\xi).\nu(\hat{y}) = 0\}) = 0,$$

g^{τ}_{ψ} does not depend on those quantities too. We denote it f^τ and by uniqueness of the limit, $f_\psi(s,\cdot,\cdot)$ converges in $H^{-1}(]-r_\alpha, r_\alpha[^d\times]-L,L[)$ to $f^\tau(\cdot,\cdot)$ for every regular Lipschitz deformation ψ. □

The following description of strong traces, namely which are reached strongly by L^1 convergence, is a straightforward consequence of Lemma 1 with $N = d$, $\mathcal{O} =]-r_\alpha, r_\alpha[^d$, $f_n = f_\psi(s_n,\cdot,\cdot)$ and $f = f^\tau$ where $s_n \to 0$.

Proposition 2. *The function f^τ is a χ-function if and only if f^τ is a strong trace, namely for every ψ regular Lipschitz deformation:*

$$\operatorname*{esslim}_{s\to 0} f_\psi(s,\cdot,\cdot) = f^\tau \text{ in } L^1(]-r_\alpha, r_\alpha[^d \times]-L, L[).$$

4. The localization method

In this section Ω_α is still fixed and we prove that $f^\tau(\hat{y},\cdot)$ is a χ-function for almost every $\hat{y} \in]-r_\alpha, r_\alpha[^d$. We fix the particular Γ_α-Lipschitz deformation on Ω_α defined by: $\tilde{\psi}(s,\hat{y}) = (s+\gamma_\alpha(\hat{y}),\hat{y})$. Now we identify the variable $s = y_0$ and we denote

$$\tilde{f}(y,\xi) = f_\psi(y_0,\hat{y},\xi) = f(\tilde{\psi}(y),\xi).$$

Notice that $\tilde{\psi}(y) \in \Omega_\alpha$ if and only if $y \in]0, r_\alpha[\times]-r_\alpha, r_\alpha[^d$. This will be convenient for the localization method. From (11) we find that $\tilde{f}$ is solution of:

$$\tilde{a}^0(\hat{y},\xi)\partial_{y_0}\tilde{f} + \hat{a}(\xi).\nabla_{\hat{y}}\tilde{f} = \partial_\xi \tilde{m}, \tag{19}$$

where

$$\begin{aligned} \tilde{a}^0(\hat{y},\xi) &= a^0(\xi) - \nabla\gamma_\alpha(\hat{y}).\hat{a}(\xi) && (20)\\ &= \lambda(\hat{y})a(\xi).\nu(\hat{y}) && (21)\end{aligned}$$

with a $\lambda(\hat{y}) \neq 0$ and $\tilde{m}(y,\xi) = \tilde{m}(\tilde{\psi}(y),\xi)$.

Before introducing the notion of rescaled solution, let us state two lemmas.

Lemma 2. *There exists a sequence ϵ_n which converges to 0 and a set $\mathcal{E} \subset]-r_\alpha, r_\alpha[^d$ with $\mathcal{L}(]-r_\alpha, r_\alpha[^d \setminus \mathcal{E}) = 0$ such that for every $\hat{y} \in \mathcal{E}$ and every $R > 0$*

$$\lim_{n\to\infty} \frac{1}{\epsilon_n^d} \tilde{m}\left(]0, R\epsilon_n[\times(\hat{y}+]-R\epsilon_n, R\epsilon_n[^d)\times]-L,L[\right) = 0.$$

Proof of Lemma 2: For every integer N, we denote

$$M_\epsilon^N(\hat{y}) = \frac{1}{\epsilon^d}\tilde{m}\left(]0,N\epsilon[\times(\hat{y}+]-N\epsilon,N\epsilon[^d)\times]-L,L[\right).$$

Since M_ϵ^N is nonnegative, the L^1 norm of M_ϵ^N is :

$$\begin{aligned}&\int_{]-r_\alpha,r_\alpha[^d} M_\epsilon^N(\hat{y})\,d\hat{y}\\ &= \int_{]-r_\alpha,r_\alpha[^d} \frac{1}{\epsilon^d}\int_0^{N\epsilon}\int_{]-N\epsilon,N\epsilon[^d}\int_{-L}^{L} \tilde{m}(z_0,\hat{y}+\hat{z},\xi)\,d\xi\,d\hat{z}\,dz_0\,d\hat{y}\\ &\le \frac{1}{\epsilon^d}\int_{]-N\epsilon,N\epsilon[^d}\int_0^{N\epsilon}\int_{-L}^{L}\int_{]-r_\alpha-N\epsilon,r_\alpha+N\epsilon[^d} \tilde{m}(z_0,\hat{y},\xi)\,d\hat{y}\,d\xi\,dz_0\,d\hat{z}.\end{aligned}$$

We denote abusively $\tilde{m}(dz_0, d\hat{z}, d\xi) = \tilde{m}(z_0, \hat{z}, \xi)\, dz_0\, d\hat{z}\, d\xi$ in this computation as if it was a function. This calculation is still correct since we just use the Fubini Theorem and a linear change of variable which are valid for measures. The last inequality can be written as:

$$\begin{aligned}
&\int_{]-r_\alpha, r_\alpha[^d} M_\epsilon^N(\hat{y})\, d\hat{y} \\
&\leq \frac{1}{\epsilon^d} \int_{]-N\epsilon, N\epsilon[^d} \tilde{m}(]0, N\epsilon[\times] - r_\alpha - N\epsilon, r_\alpha + N\epsilon[^d \times] - L, L[)\, d\hat{z} \\
&\leq N^d \tilde{m}(]0, N\epsilon[\times] - r_\alpha - N\epsilon, r_\alpha + N\epsilon[^d \times] - L, L[).
\end{aligned}$$

By monotone convergence, since $\bigcap_{\epsilon>0}]0, N\epsilon[= \emptyset$, this converges to 0 when ϵ converges to 0. Finally the L^1 norm of M_ϵ^N converges to 0 so there exists a subsequence ϵ_n and a set $\mathcal{E}_N \subset]-r_\alpha, r_\alpha[^d$ with $\mathcal{L}(]-r_\alpha, r_\alpha[^d \backslash \mathcal{E}_N) = 0$ such that for every $\hat{y} \in \mathcal{E}_N$ $M_{\epsilon_n}^N(\hat{y})$ converges to 0 when ϵ_n goes to 0. By diagonal extraction, we can choose ϵ_n such that for every integer N and every $\hat{y} \in \mathcal{E}_N$, $M_{\epsilon_n}^N(\hat{y})$ converges to 0. This sequence ϵ_n with subset $\mathcal{E} = \cap_N \mathcal{E}_N$ verifies the required condition. □

Lemma 3. *There exists a subsequence still denoted ϵ_n and a subset $\mathcal{E}'$ of $]-r_\alpha, r_\alpha[^d$ such that $\mathcal{E}' \subset \mathcal{E}$, $\mathcal{L}(]-r_\alpha, r_\alpha[^d \setminus \mathcal{E}') = 0$, and for every $\hat{y} \in \mathcal{E}'$ and every $R > 0$:*

$$\lim_{\epsilon_n \to 0} \int_{-L}^{L} \int_{]-R,R[^d} |f^\tau(\hat{y}, \xi) - f^\tau(\hat{y} + \epsilon_n \underline{\hat{y}}, \xi)|\, d\underline{\hat{y}}\, d\xi = 0,$$

$$\lim_{\epsilon_n \to 0} \int_{-L}^{L} \int_{]-R,R[^d} |\tilde{a}^0(\hat{y}, \xi) - \tilde{a}^0(\hat{y} + \epsilon_n \underline{\hat{y}}, \xi)|\, d\underline{\hat{y}}\, d\xi = 0.$$

Proof of Lemma 3: For every integer N we denote:

$$F_{\epsilon_n}^N(\hat{y}) = \int_{-L}^{L} \int_{]-N,N[^d} |f^\tau(\hat{y}, \xi) - f^\tau(\hat{y} + \epsilon_n \underline{\hat{y}}, \xi)|\, d\underline{\hat{y}}\, d\xi.$$

Since $f^\tau \in L^\infty(]-r_\alpha, r_\alpha[^d \times]-L, L[)$, the L^1 norm of this function goes to zero as n tends to ∞ so there exists a subsequence still denoted ϵ_n and a subset $\mathcal{E}'_N \subset \mathcal{E}$ with $\mathcal{L}(]-r_\alpha, r_\alpha[^d \backslash \mathcal{E}'_N) = 0$ such that for every $\hat{y} \in \mathcal{E}'_N$, $F_{\epsilon_n}^N(\hat{y})$ converges to 0 when n tends to infinity. By diagonal extraction we can find a subsequence such that this holds true for every N. Then this subsequence and $\mathcal{E}' = \cap_N \mathcal{E}'_N$ fulfill the required condition for the first limit. We consider in the same way the term with $\tilde{a}^0$ noticing that

$$\tilde{a}^0(\hat{y}, \xi) - \tilde{a}^0(\hat{y} + \epsilon_n \underline{\hat{y}}, \xi) = \hat{a}(\xi).[\nabla\gamma_\alpha(\hat{y}) - \nabla\gamma_\alpha(\hat{y} + \epsilon_n \underline{\hat{y}})]$$

with $\nabla\gamma_\alpha \in L^\infty(]-r_\alpha, r_\alpha[^d)$. □

We are now able to introduce the localization method. We denote

$$\Omega_\alpha^\epsilon =]0, r_\alpha/\epsilon[\times]-r_\alpha/\epsilon, r_\alpha/\epsilon[^d.$$

We want to show that for every $\hat{y} \in \mathcal{E}'$, $f^\tau(\hat{y},\cdot)$ is a χ-function. From now on we fix such a $\hat{y} \in \mathcal{E}'$, and we denote $y = (0,\hat{y})$ the associated point on Γ_α. Then, we rescale the $\tilde{f}$ function by introducing a new function $\tilde{f}_\epsilon$ which depends on a new variable $\underline{y} \in \Omega_\alpha^\epsilon$ and which is defined by:

$$\tilde{f}_\epsilon(\underline{y},\xi) = \tilde{f}(y + \epsilon\underline{y}, \xi). \tag{22}$$

This function depends obviously on y but, since it is fixed all along this section, we skip it in the notation. Function $\tilde{f}_\epsilon$ is still a χ-function and we can notice that:

$$\tilde{f}_\epsilon(0,\underline{\hat{y}},\xi) = f^\tau(\hat{y} + \epsilon\underline{\hat{y}}, \xi). \tag{23}$$

Hence we expect to gain some knowledge on $f^\tau(\hat{y},\cdot)$ itself by studying the limit of $\tilde{f}_\epsilon$ when $\epsilon \to 0$. We define in the same way:

$$\tilde{a}_\epsilon^0(\underline{\hat{y}},\xi) = \tilde{a}^0(\hat{y} + \epsilon\underline{\hat{y}}, \xi),$$

and we get from (19):

$$\tilde{a}_\epsilon^0(\underline{\hat{y}},\xi)\partial_{\underline{y}_0}\tilde{f}_\epsilon + \hat{a}(\xi).\nabla_{\underline{\hat{y}}}\tilde{f}_\epsilon = \partial_\xi \tilde{m}_\epsilon, \tag{24}$$

where $\tilde{m}_\epsilon$ is the nonnegative measure defined for every real $R_1^i < R_2^i, L_1 < L_2$ by:

$$\tilde{m}_\epsilon\Big(\mathop{\Pi}_{0\le i\le d}[R_1^i, R_2^i]\times[L_1,L_2]\Big) = \frac{1}{\epsilon^d}\tilde{m}\Big(\mathop{\Pi}_{0\le i\le d}[y_i + \epsilon R_1^i, y_i + \epsilon R_2^i]\times[L_1,L_2]\Big). \tag{25}$$

We now pass to the limit when ϵ goes to 0 in the scaling.

Proposition 3. *There exist a sequence ϵ_n which goes to 0, and a χ-function $\tilde{f}_\infty \in L^\infty(\mathbb{R}^+\times\mathbb{R}^d\times]-L,L[)$ such that $\tilde{f}_{\epsilon_n}$ converges strongly to $\tilde{f}_\infty$ in $L^1_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R}^d\times]-L,L[)$ and:*

$$\tilde{a}^0(\hat{y},\xi)\partial_{\underline{y}_0}\tilde{f}_\infty + \hat{a}(\xi).\nabla_{\underline{\hat{y}}}\tilde{f}_\infty = 0. \tag{26}$$

Notice that $\tilde{a}^0(\hat{y},\xi)$ does not depend on the local variable $\underline{y}$. In fact, we have $\tilde{a}^0(\hat{y},\xi) = \lambda(\hat{y})a(\xi).\nu(\hat{y})$ where $\hat{y}$ is the fixed point of the localization.

Proof of Proposition 3: We consider the sequence ϵ_n of Lemma 3. By weak compactness, there exists a function $\tilde{f}_\infty \in L^\infty(\mathbb{R}^+\times\mathbb{R}^d\times]-L,L[)$ such that, up to extraction, $\tilde{f}_{\epsilon_n}$ converges weakly in $L^\infty *$ to $\tilde{f}_\infty$. Thanks to Lemma 2, $\tilde{m}_{\epsilon_n}$ converges to 0 in the sense of measure. Thanks to Lemma 3, $\tilde{a}^0_{\epsilon_n}$ converges strongly in $L^1_{\mathrm{loc}}(\mathbb{R}^d\times]-L,L[)$ to $\tilde{a}^0(\hat{y},\cdot)$, so passing to the limit in (24) gives (26). The strong convergence is an application of averaging lemmas. Here we use the following one which is a particular case of the version of Perthame and Souganidis (see [16]):

Theorem 4. *Let N be an integer, f_n be bounded in $L^\infty(\mathbb{R}^N)$ and $\{h_n^1, h_n^2\}$ be relatively compact in $\left[L^p(\mathbb{R}^N)\right]^{2N}$ with $1 < p < +\infty$ solutions of the transport equation:*

$$a(\xi).\nabla_y f_n = \partial_\xi(\nabla_y.h_n^1) + \nabla_y.h_n^2,$$

where $a \in \left[C^2(\mathbb{R})\right]^N$ verifies the non-degeneracy condition (12). Let $\phi \in \mathcal{D}(\mathbb{R})$, then the average $u_n^\phi(y) = \int_{\mathbb{R}} \phi(\xi) f_n(y,\xi)\,d\xi$ is relatively compact in $L^p(\mathbb{R}^N)$.

In order to do so we use the strategy of [15]. First we localize in $\underline{y}, \xi$. We introduce Φ_1, Φ_2 with values in $[0,1]$ such that $\Phi_1 \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R}^d)$, $\Phi_2 \in \mathcal{D}(\mathbb{R})$, and $\mathrm{Supp}(\Phi_1) \subset]1/(2R), 2R[\times]-2R, 2R[^d$, $\mathrm{Supp}(\Phi_2) \subset]-2L, 2L[$. Moreover $\phi_1(\underline{y}) = 1$ for $\underline{y} \in]1/R, R[\times]-R, R[^d$ and $\Phi_2(\xi) = 1$ for $\xi \in]-L, L[$. Hence for $\epsilon < r_\alpha/(2R)$, we can define on $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$:

$$\tilde{f}_\epsilon^R = \Phi_1 \Phi_2 \tilde{f}_\epsilon,$$

(where $\tilde{f}_\epsilon^R = 0$ if $\tilde{f}_\epsilon$ is not defined). On $]1/R, R[\times]-R, R[^d \times]-L, L[$ we have $\tilde{f}_\epsilon^R = \tilde{f}_\epsilon$. So, if we denote $a_{\hat{y}}(\xi) = (\tilde{a}^0(\hat{y},\xi), \hat{a}(\xi))$ (which depends only on ξ since $\hat{y}$ is fixed), from (24) we get:

$$a_{\hat{y}}(\xi).\nabla_{\underline{\hat{y}}} \tilde{f}_\epsilon^R = \partial_\xi(\Phi_1\Phi_2\tilde{m}_\epsilon) - \Phi_1\Phi_2'\tilde{m}_\epsilon + a_{\hat{y}}(\xi).\nabla_{\underline{\hat{y}}}\phi_1\phi_2\tilde{f}_\epsilon \tag{27}$$

$$+\partial y_0[(\tilde{a}^0(\hat{y},\xi) - \tilde{a}_\epsilon^0(\underline{\hat{y}},\xi))\tilde{f}_\epsilon^R] \tag{28}$$

$$= \partial_\xi \mu_{1,\epsilon} + \mu_{2,\epsilon} + \partial y_0[(\tilde{a}^0(\hat{y},\xi) - \tilde{a}_\epsilon^0(\underline{\hat{y}},\xi))\tilde{f}_\epsilon^R], \tag{29}$$

where μ_{1,ϵ_n} and μ_{2,ϵ_n} are measures uniformly bounded with respect to n. Moreover thanks to Lemma 3 $a_{\hat{y}}^0 - \tilde{a}_{\epsilon_n}^0(\underline{\hat{y}},\xi)$ converges to 0 in $L^1_{\mathrm{loc}}(\mathbb{R}^d \times]-L, L[)$. So it converges to 0 in L^p_{loc} for every $1 \le p < \infty$ since this functions are bounded in L^∞. Since the measures are compactly imbedding in $W^{-1,p}$ for $1 \le p < (d+2)/(d+1)$, we can apply Theorem 4 with $N = d+1$, $f_n = \tilde{f}_{\epsilon_n}^R$, $\phi(\xi) = \phi_2(\xi)$ and $a(\xi) = a_{\hat{y}}(\xi)$. It follows that $\int \tilde{f}_\epsilon^R \phi_2(\xi)\,d\xi$ is compact in L^p for $1 \le p < (d+2)/(d+1)$. And so by uniqueness of the limit, $\int \tilde{f}_{\epsilon_n}(\cdot,\xi)\,d\xi$ converges strongly to $\int \tilde{f}_\infty(\cdot,\xi)\,d\xi$ in $L^1_{\mathrm{loc}}(\mathbb{R}^{d+1})$. Lemma 1 ensures us that $\tilde{f}_{\epsilon_n}$ converges strongly to $\tilde{f}_\infty$ in $L^1_{\mathrm{loc}}(\mathbb{R}^{d+1} \times]-L, L[)$ and that $\tilde{f}_\infty$ is a χ-function. □

We now turn to characterize the limit function $\tilde{f}_\infty$. First notice that from (26) $\tilde{a}^0(\hat{y},\xi)\tilde{f}_\infty$ lies in $C^0(\mathbb{R}^+, W^{-1,\infty}(\mathbb{R}^d \times]-L, L[))$. From (12) $\tilde{a}^0(\hat{y},\xi) \neq 0$ for almost every ξ so we can define $\tilde{f}_\infty(0,\cdot,\cdot)$ from it.

Proposition 4. *For every $\hat{y} \in \mathcal{E}'$ we have the following equality:*

$$\tilde{f}_\infty(0, \underline{\hat{y}}, \xi) = f^\tau(\hat{y}, \xi)$$

for almost every $(\underline{\hat{y}}, \xi) \in \mathbb{R}^d \times]-L, L[$.

Notice that this implies that $\tilde{f}_\infty(0,\cdot,\cdot)$ does not depend on $\hat{\underline{y}}$.

Proof of Proposition 4: Let us introduce

$$h^\epsilon_\Phi(\underline{y}_0) = \int_{-L}^{L}\int_{\mathbb{R}^d}[\tilde{a}^0_\epsilon(\hat{\underline{y}},\xi)\tilde{f}_\epsilon(\underline{y}_0,\hat{\underline{y}},\xi) - \tilde{a}^0(\hat{y},\xi)\tilde{f}_\infty(\underline{y}_0,\hat{\underline{y}},\xi)]\Phi(\hat{\underline{y}},\xi)\,d\hat{\underline{y}}\,d\xi$$

for every test function $\Phi \in C^\infty_0(\mathbb{R}^d\times]-L,L[)$. We have:

$$|h^\epsilon_\Phi(0)| \leq C(\|h^\epsilon_\Phi\|_{L^1(]0,1[)} + \|\partial_{\underline{y}_0}h^\epsilon_\Phi\|_{\mathcal{M}(]0,1[)}),$$

and from (24) and (26) h^ϵ_Φ is a BV function and:

$$\begin{aligned}
&\|\partial_{\underline{y}_0}h^\epsilon_\Phi\|_{\mathcal{M}} \leq C_\Phi(\|\tilde{f}_\epsilon - \tilde{f}_\infty\|_{L^1_{\text{loc}}} + \|\tilde{m}_\epsilon\|_{\mathcal{M}})\\
&\|h^\epsilon_\Phi\|_{L^1(]0,1[)} \leq C_\Phi(\|\tilde{f}_\epsilon - \tilde{f}_\infty\|_{L^1_{\text{loc}}} + \|\tilde{a}^0_\epsilon - \tilde{a}^0\|_{L^1_{\text{loc}}}).
\end{aligned}$$

Lemma 2 and definition (25) ensure that $\|\tilde{m}_{\epsilon_n}\|_{\mathcal{M}}$ converges to 0, Lemma 3 that $\|\tilde{a}^0_{\epsilon_n} - \tilde{a}^0\|_{L^1_{\text{loc}}}$ converges to 0 and Proposition 3 that $\|\tilde{f}_{\epsilon_n} - \tilde{f}_\infty\|_{L^1_{\text{loc}}}$ converges to 0. So $h^{\epsilon_n}_\Phi(0)$ converges to 0 when n converges to $+\infty$. Remembering (23), thanks to Lemma 3, $\tilde{f}_{\epsilon_n}(0,\cdot,\cdot)$ converges strongly to f^τ, and so, $\tilde{a}^0_{\epsilon_n}(\cdot,\cdot)\tilde{f}_{\epsilon_n}(0,\cdot,\cdot)$ converges strongly to $\tilde{a}^0(\hat{y},\cdot)f^\tau$. Then

$$\tilde{a}^0(\hat{y},\xi)\tilde{f}_\infty(0,\hat{\underline{y}},\xi) = \tilde{a}^0(\hat{y},\xi)f^\tau(\hat{y},\xi)$$

for almost every $(\hat{\underline{y}},\xi) \in \mathbb{R}^d\times]-L,L[$ which leads to the desired result noticing that $\tilde{a}^0(\hat{y},\xi) \neq 0$ for almost every ξ. □

From (26) we deduce that:

$$\tilde{f}_\infty(\tilde{a}^0(\hat{y},\xi)\underline{y}_0, \hat{\underline{y}} + \hat{a}(\xi)\underline{y}_0, \xi) = f^\tau(\hat{y},\xi),$$

for almost every $\underline{y}_0 > 0$. But $\tilde{a}^0(\hat{y},\xi) \neq 0$ for almost every ξ so

$$\tilde{f}_\infty(\underline{y},\xi) = f^\tau(\hat{y},\xi)$$

for almost every $(\underline{y},\xi) \in \mathbb{R}^{d+1}\times]-L,L[$, which is constant with respect to $\underline{y}$. Finally since $\tilde{f}_\infty$ is a χ-function for almost every $\underline{y}$, we deduce that:

Proposition 5. *For every $\hat{y} \in \mathcal{E}'$ function $f^\tau(\hat{y},\cdot)$ is a χ-function.*

5. Proof of Theorem 1

For every α and every ψ $\partial\Omega$-regular Lipschitz deformation, the restriction of ψ to Γ_α is a Γ_α-regular Lipschitz deformation. From Proposition 5 and Proposition 2 it follows:

$$\operatorname*{esslim}_{y_0\to 0}\int_{-r_\alpha}^{r_\alpha}\int_{-L}^{L}|f_\psi(y_0,\hat{y},\xi)-f^\tau(\hat{y},\xi)|\,d\xi\,d\hat{y}=0. \tag{30}$$

We define $u^\tau\in L^\infty(\partial\Omega)$ by:

$$u^\tau(\hat{z})=\int_{-L}^{L}f^\tau(\hat{y},\xi)\,d\xi$$

if $(\gamma_\alpha(\hat{y}),\hat{y})=\mathcal{R}_\alpha(\hat{z})$. Notice that it is uniquely well defined thanks to the uniqueness of f^τ in Proposition 1. Finally, since $\|\nabla\gamma_\alpha\|_{L^\infty(]-r_\alpha,r_\alpha[)}\le C$, from (30) and remembering that $u(\psi(s,\hat{z}))=\int_{-L}^{L}f_\psi(s,\hat{y},\xi)\,d\xi$ if $\mathcal{R}_\alpha(\hat{z})=(\gamma_\alpha(\hat{y}),\hat{y})$, we deduce:

Proposition 6. *For every α and every ψ $\partial\Omega$-regular Lipschitz deformation,*

$$\operatorname*{esslim}_{s\to 0}\int_{\Gamma_\alpha}|u(\psi(s,\hat{z}))-u^\tau(\hat{z})|\,d\mathcal{H}^d(\hat{z})=0.$$

Finally for every compact set $K\subset\subset\partial\Omega$, $\{\Gamma_\alpha\}$ is a covering of K with open sets of $\partial\Omega$ so there exists I_0 finite set such that $K\subset\bigcup_{\alpha\in I_0}\Gamma_\alpha$ and so

$$\int_K|u(\psi(s,\hat{z}))-u^\tau(\hat{z})|\,d\mathcal{H}^d(\hat{z})\le\sum_{\alpha\in I_0}\int_{\Gamma_\alpha}|u(\psi(s,\hat{z}))-u^\tau(\hat{z})|\,d\mathcal{H}^d(\hat{z}).$$

Then Theorem 1 follows from Proposition 6. □

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