

GLOBAL WELL-POSEDNESS OF LARGE PERTURBATIONS OF TRAVELING WAVES IN A HYPERBOLIC-PARABOLIC SYSTEM ARISING FROM A CHEMOTAXIS MODEL

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ABSTRACT. We consider a one-dimensional system arising from a chemotaxis model in tumour angiogenesis, which is described by a Keller-Segel equation with singular sensitivity. This hyperbolic-parabolic system is known to allow viscous shocks (so-called traveling waves), and in literature, their nonlinear stabilities have been considered in the class of certain mean-zero small perturbations. We show the global existence of the solution without assuming the mean-zero condition for any initial data as arbitrarily large perturbations around traveling waves in the Sobolev space H^1 while the shock strength is assumed to be small enough. The main novelty of this paper is to develop the global well-posedness of any large H^1 -perturbations of traveling wave connecting two different end states. The discrepancy of the end states is linked to the complexity of the corresponding flux, which requires a new type of an energy estimate. To overcome, we use the *a priori* contraction estimate of a weighted relative entropy functional up to a translation, which was proved by Choi-Kang-Kwon-Vasseur [4]. The boundedness of the shift implies *a priori* bound of the relative entropy functional without a shift on any time interval of existence, which produces a H^1 -estimate thanks to a De Giorgi type lemma. Moreover, to remove possibility of vacuum appearance, we use the lemma again.

Résumé: Nous considérons un système mono-dimensionnel de chimiotaxie utilisé pour l'étude d'angiogenèse tumorale. Il est dérivé d'une équation de Keller-Segel avec sensibilité singulière. Il est bien connu que ce système Hyperbolique-parabolique génère une famille d'ondes progressives, appelées chocs visqueux. Dans cet article, nous prouvons l'existence globale de solutions pour ce système, et pour toutes données initiales qui peuvent être décrites comme perturbations dans H^1 d'un choc visqueux assez petit. Cela nous permet d'obtenir des solutions globales, sans hypothèse de petitesse dans H^1 , pour des données initiales connectant des états différents à gauche et à droite. A cause de la complexité de la fonctionnelle de flux, cette différence empêche l'utilisation des méthodes d'entropie classiques. Pour remédier à ce problème, nous utilisons une estimation a priori de contraction obtenue dans Choi-Kang-Kwon-Vasseur [4]. La régularité H^1 s'obtient alors grâce à un argument de type De Giorgi.

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1. INTRODUCTION AND MAIN THEOREM

We consider the following one dimensional system:

$$(1.1) \quad \begin{aligned} \partial_t n - \partial_x(nq) &= \nu \partial_{xx} n, \\ \partial_t q - \partial_x n &= 0 \quad \text{for } x \in \mathbb{R} \quad \text{and for } t > 0 \end{aligned}$$

where $\nu > 0$ is a positive constant. This hyperbolic-parabolic system is closely related to a certain Keller-Segel system (see Subsection 1.4). We are interested in the global-in-time existence issue of large perturbations of traveling waves (or viscous shocks) of the above system (1.1).

1.1. Traveling waves of (1.1). By [34] (also see [25], or see [4, Lemma 2.1]), it has been known that for any $\nu > 0$, (1.1) admits a smooth monotone traveling wave $\tilde{U}(x - \sigma t) = \begin{pmatrix} \tilde{n}(x - \sigma t) \\ \tilde{q}(x - \sigma t) \end{pmatrix}$ connecting two end-states $(n_-, q_-), (n_+, q_+) \in \mathbb{R}^+ \times \mathbb{R}$, *i.e.*,

$$(1.2) \quad \tilde{n}(-\infty) = n_- > 0, \quad \tilde{n}(+\infty) = n_+ > 0, \quad \tilde{q}(-\infty) = q_-, \quad \tilde{q}(+\infty) = q_+$$

(we denote $\lim_{x \rightarrow \pm\infty} f(x)$ by $f(\pm\infty)$ in short), provided the two end-states satisfy the Rankine-Hugoniot condition and the Lax entropy condition:

$$(1.3) \quad \exists \sigma \in \mathbb{R} \text{ such that } \begin{cases} -\sigma(n_+ - n_-) - (n_+q_+ - n_-q_-) = 0, \\ -\sigma(q_+ - q_-) - (n_+ - n_-) = 0, \end{cases}$$

and either $n_- > n_+$ and $q_- < q_+$ or $n_- < n_+$ and $q_- < q_+$ holds.

For notational convenience, we denote $\tilde{U}(x - \sigma t)$ by $\hat{U} = (\hat{n}, \hat{q}) := \tilde{U}(x - \sigma t)$ whenever there is no confusion about the wave \tilde{U} with its fixed boundary condition.

In short, for any $\nu > 0$, for any $n_- > 0$, for any $n_+ > 0$ with $n_+ \neq n_-$ and for any $q_- \in \mathbb{R}$, there exists a smooth monotone traveling wave $\hat{U}(t, x) = \tilde{U}(x - \sigma t)$ of (1.1) satisfying (1.2) where the constants σ and q_+ are determined by

$$(1.4) \quad \sigma := \begin{cases} \frac{-q_- + \sqrt{q_-^2 + 4n_+}}{2} > 0 & \text{if } n_- > n_+ > 0, \\ \frac{-q_- - \sqrt{q_-^2 + 4n_+}}{2} < 0 & \text{if } 0 < n_- < n_+ \end{cases}$$

and

$$(1.5) \quad q_+ := q_- + \frac{(n_- - n_+)}{\sigma}$$

Our motivation of this work is to answer the question how stable traveling waves are in the system. The paper [25] showed that waves are stable if the anti-derivative of a perturbation $(n - \tilde{n}, q - \tilde{q})$ is sufficiently small in the Sobolev space $H^2(\mathbb{R})$. Note that the initial perturbation should have the mean-zero condition:

$$\exists x_0 \in \mathbb{R} \text{ such that } \int_{\mathbb{R}} \begin{pmatrix} n_0(x) - \tilde{n}(x - x_0) \\ q_0(x) - \tilde{q}(x - x_0) \end{pmatrix} dx = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This restriction for the initial data is commonly assumed in studying stability of viscous shocks since the work of [11] and [19]. The main novelty of this paper is to remove both the mean-zero condition and the smallness condition of the initial perturbation.

In this paper, we frequently use the following facts (*e.g.* see [4, Lemma 2.1]):

$$\tilde{n} > 0, \quad \tilde{n}, \tilde{q}, \frac{1}{\tilde{n}} \in L^\infty(\mathbb{R}), \quad \text{and} \quad \tilde{n}', \tilde{n}'', \tilde{q}' \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}).$$

1.2. Global existence around waves and their contraction.

To state the contraction property, we need the following notion:

For $U_i = \begin{pmatrix} n_i \\ q_i \end{pmatrix}$ with $n_i > 0$ for $i = 1, 2$, we consider the relative entropy

$$\eta(U_1|U_2) := \frac{|q_1 - q_2|^2}{2} + \Pi(n_1|n_2),$$

where

$$(1.6) \quad \Pi(n_1|n_2) := \Pi(n_1) - \Pi(n_2) - \nabla \Pi(n_2)(n_1 - n_2), \quad \Pi(n) := n \log n - n.$$

Since $\Pi(n)$ is strictly convex in n , its relative functional $\Pi(\cdot|\cdot)$ above is positive definite, and so is $\eta(\cdot|\cdot)$. That is, $\eta(U_1|U_2) \geq 0$ for any U_1 and U_2 , and $\eta(U_1|U_2) = 0$ if and only if $U_1 = U_2$.

We present our main result for the fixed viscosity $\nu = 1$ case:

$$(1.7) \quad \begin{aligned} \partial_t n - \partial_x(nq) &= \partial_{xx} n, \\ \partial_t q - \partial_x n &= 0 \quad \text{for } x \in \mathbb{R} \quad \text{and for } t > 0, \end{aligned}$$

assuming the case of $n_- > n_+ > 0$. Then, in Remark 1.3 and 1.2, we illustrate that the main result still holds for any $\nu > 0$ and/or for $n_+ > n_- > 0$.

For a given wave \tilde{n} and for a given constant $\lambda > 0$, we define the weight function $a(\cdot)$ by

$$(1.8) \quad a := 1 + \frac{\lambda}{\epsilon}(n_- - \tilde{n})$$

where $\epsilon := (n_- - n_+) > 0$. Then we have $a(-\infty) = 1, a(+\infty) = 1 + \lambda$, and $a'(x) = \left(-\frac{\lambda}{\epsilon}\right)\tilde{n}'(x) > 0$ for $x \in \mathbb{R}$. Here is the main result:

Theorem 1.1. *For a given constant state $(n_-, q_-) \in \mathbb{R}^+ \times \mathbb{R}$, there exist constants $\kappa \in (0, \min\{n_-/(15), 1/8\})$ and $C > 0$ such that the following is true:*

For any $(n_+, q_+) \in \mathbb{R}^+ \times \mathbb{R}$ satisfying (1.5) with $0 < \epsilon := (n_- - n_+) < \kappa$, consider the traveling wave $\tilde{U} := \begin{pmatrix} \tilde{n} \\ \tilde{q} \end{pmatrix}$ of (1.7) with the boundary condition (1.2) and with the speed σ from (1.4). Take any constant λ between $\frac{\epsilon}{\sqrt{\kappa}}$ and $\sqrt{\kappa}$. Let $U_0(x) := \begin{pmatrix} n_0(x) \\ q_0(x) \end{pmatrix}$ satisfy

$$U_0 - \tilde{U} \in H^1(\mathbb{R}), \quad 0 < \frac{1}{n_0} \in L^\infty(\mathbb{R})$$

(i) Global existence : *Then there exists the unique global-in-time solution $U(t, x) := \begin{pmatrix} n(t, x) \\ q(t, x) \end{pmatrix}$ to (1.7) for $U|_{t=0} = U_0$ such that*

$$\begin{aligned} (n - \hat{n}, q - \hat{q}) &\in (C([0, T]; H^1(\mathbb{R})) \cap L^2(0, T; H^2(\mathbb{R}))) \times C([0, T]; H^1(\mathbb{R})), \\ 0 < \frac{1}{n} &\in L^\infty(0, T; L^\infty(\mathbb{R})) \end{aligned}$$

for any $T > 0$.

(ii) Contraction : *Moreover, there exists an absolutely continuous shift function $X : [0, \infty) \rightarrow \mathbb{R}$ with $X \in W_{loc}^{1,1}$ and $X(0) = 0$ such that*

$$(1.9) \quad \begin{aligned} &\int_{-\infty}^{\infty} a(x - \sigma t) \eta(U(t, x - X(t)) | \tilde{U}(x - \sigma t)) dx \\ &\quad + \sqrt{\kappa} \int_0^t \int_{-\infty}^{\infty} a(x - \sigma \tau) n(\tau, x - X(\tau)) \left| \partial_x \left(\log \frac{n(\tau, x - X(\tau))}{\tilde{n}(x - \sigma \tau)} \right) \right|^2 dx d\tau \\ &\leq \int_{-\infty}^{\infty} a(x) \eta(U_0(x) | \tilde{U}(x)) dx, \end{aligned}$$

where a is the monotone function defined by (1.8)

and

$$(1.10) \quad |\dot{X}(t) - \sigma| \leq \frac{1}{\varepsilon^2} \left(f(t) + C \int_{-\infty}^{\infty} \eta(U_0|\tilde{U})dx + 1 \right) \quad \text{for a.e. } t \in [0, \infty)$$

where f is some positive function satisfying $\|f\|_{L^1(0, \infty)} \leq C \frac{\lambda}{\varepsilon} \int_{-\infty}^{\infty} \eta(U_0|\tilde{U})dx$.

The proof is presented in Section 3.

Remark 1.2. The result for $n_+ > n_- > 0$ can be obtained by the change of variables $x \mapsto -x$ with $\sigma \mapsto -\sigma$. Therefore, from now on, we always assume $n_- > n_+ > 0$ and thus

$$0 < \sigma = \frac{-q_- + \sqrt{q_-^2 + 4n_+}}{2}.$$

Remark 1.3. For general $\nu > 0$ of (1.1), we have the global existence and the contraction by the following scaling:

If U^ν and \tilde{U}^ν are a solution and traveling wave to (1.1) for a fixed $\nu > 0$ with initial data U_0 , respectively, then $U(t, x) := U^\nu(\nu t, \nu x)$ (resp. $\tilde{U}(x) := \tilde{U}^\nu(\nu x)$) is a solution (resp. traveling wave) to (1.7) (e.g. also see [4, Remark 1.5]).

Remark 1.4. For $n_- > 0$, there exists a constant $C > 0$ such that for any $n_1 > 0$ and for any $n_2 \in (n_-/2, n_-)$,

$$(1.11) \quad \Pi(n_1|n_2) \leq C|n_1 - n_2|^2$$

by (2.1) and (2.3) in Lemma 2.1 (or see [4, Lemma 2.8]). If we take $n_+ \geq n_-/2$, it implies $n_-/2 < \tilde{n} < n_-$. Thus we have

$$\int_{\mathbb{R}} \eta(U(x)|\tilde{U}(x))dx \leq C\|U - \tilde{U}\|_{L^2(\mathbb{R})}^2$$

for any function U with $U - \tilde{U} \in L^2$. Therefore, the initial condition $U_0 - \tilde{U} \in H^1$ implies $\int_{-\infty}^{\infty} \eta(U_0|\tilde{U}) < \infty$. However, the reversed inequality is false because $\Pi \sim n_1 \log n_1$ when n_1 is large (see (1.6) and (2.2) in Lemma 2.1).

Remark 1.5. Since the weight function a satisfies that $|a(x) - 1| \leq \lambda < \sqrt{\kappa} < 1/2$ for all $x \in \mathbb{R}$, the contraction estimate (1.9) yields

$$\int_{-\infty}^{\infty} \eta(U(t, x - X(t))|\tilde{U}(x - \sigma t))dx \leq 4 \int_{-\infty}^{\infty} \eta(U_0(x)|\tilde{U}(x))dx.$$

In the previous work [4], it was turned out that both the smallness of the shock strength and the strict positivity of n_- and n_+ in (1.2) are technically important for our result even though the traveling waves exist even in the case of the large shock strength (or/and) $\min(n_-, n_+) = 0$. In particular, as explained in [26], the case of $\min(n_-, n_+) = 0$ is more relevant to the original modeling. The problem of the extension of our result seems to be beyond reach of current known methods. With the mean-zero condition, the stability for the case of $\min(n_-, n_+) = 0$ case were shown in a weighted Sobolev space in [14] and [24]. For planar waves on a cylinder, we refer to [3] and [2].

For the Cauchy problem of (1.1), we refer to [12, 23, 27]. For multi-dimensional cases, see [22] and references therein.

1.3. Ideas of Proof. In order to construct a global-in-time solution as a large H^1 -perturbation of the traveling wave \tilde{U} , we may first find the usual relative entropy inequality for the system (1.7). For that, we need to observe the evolution of the relative entropy, based on the relative entropy method [6, 8]. More precisely, using the computations in the proof of [4, Lemma 2.3] (or see [15, 16, 17, 18, 31]), we find that

$$(1.12) \quad \partial_t \eta(U|\tilde{U}) = -\partial_\xi \left(G(U; \tilde{U}) + (\partial_\xi n) \log(n/\tilde{n}) \right) - \frac{|\partial_\xi n|^2}{n} + \frac{\partial_\xi n \tilde{n}'}{\tilde{n}} - \frac{n - \tilde{n}}{\tilde{n}} \tilde{n}'' + \frac{\tilde{n}'}{\tilde{n}} (n - \tilde{n})(q - \tilde{q}),$$

where $\xi := x - \sigma t$, and $G(U; \tilde{U})$ denotes the flux of the relative entropy.

If $\tilde{n}(\xi)$ were constant in ξ like the case of $n_- = n_+$, then the above equality would become

$$\partial_t \eta(U|\tilde{U}) = -\partial_\xi \left(G(U; \tilde{U}) + (\partial_\xi n) \log(n/\tilde{n}) \right) - \frac{|\partial_\xi n|^2}{n},$$

which gives the dissipation of the (total) relative entropy :

$$(1.13) \quad \frac{d}{dt} \int_{\mathbb{R}} \eta(U|\hat{U}) dx + \int_{\mathbb{R}} \frac{|\partial_x n|^2}{n} dx \leq 0.$$

Note that the above inequality (in fact, contraction of the relative entropy) holds regardless of $q_- \neq q_+$ or $q_- = q_+$, i.e., discrepancy of the end states of \hat{q} .

However, we consider the traveling wave connecting two different states, that is, \tilde{n} is not constant. Therefore, it is not obvious to get such a simple relative entropy functional inequality (1.13) from (1.12). In fact, it turns out in [4] that that is a far complicated issue. There, it was proven that the weighted relative entropy is dissipative (or contractive) up to a time-dependent shift $X(t)$ (see Proposition 2.4). Therefore, Proposition 2.4 on the contraction property of the relative entropy will be importantly used in Proposition 3.2 to extend the life span of a local-in-time solution for all time.

We sketch the proof. Recall that Proposition 2.4 holds during $n > 0$ i.e. $1/n \in L^\infty$ (see the definition of the space (2.8)). Thus, we first show a local existence theorem (Proposition 3.1) guaranteeing that n does not vanish up to a certain time interval $[0, T]$. Then we apply Proposition 2.4 for the time interval in order to get the contraction of the weighted relative entropy functional (2.10) up to some shift $X(t)$ satisfying (2.11). In short, we have

$$(1.14) \quad \frac{d}{dt} \int_{\mathbb{R}} \hat{a}^X \eta(U|\hat{U}^X) dx + \sqrt{\kappa} \int_{\mathbb{R}} \hat{a}^X n \left| \partial_x \left(\log \frac{n}{\hat{n}^X} \right) \right|^2 dx \leq 0,$$

where $\hat{a}(t, x) = a(x - \sigma t)$ with (1.8) and the superscript X is defined by the translation in x -variable by the given shift $X(t)$ as in (4.2).

After the process, it remains to solve two main issues. First we obtain finiteness (see (4.10)) of the functional without a shift X and without a weight \hat{a} :

$$\sup_{[0, T]} \int_{\mathbb{R}} \eta(U|\hat{U}) dx \leq C(T),$$

thanks to boundedness of the shift (2.11). In this step, the estimate is little delicate due to the Log structure of the relative entropy at infinity (see (1.6) and (2.2)).

Second, we obtain $q \in L^\infty$ by using the particular structure (4.23) satisfied by $(n - \partial_x q)$. Here we take advantage of (4.24) from positivity of n . Since the dissipation term in (1.14) give the estimate of $\partial_x \sqrt{n} \in L^2$ (see (4.19)), we obtain $q \in L^\infty$ by decomposing each function into $L^1 + L^\infty$. Then the estimate $n, 1/n \in L^\infty$ follows from De Giorgi type Lemma 2.2. By having $n, q \in L^\infty$, the standard energy method gives all higher order estimates.

As a result, we get a priori bound in H^1 -norm up to any arbitrarily large time, which guarantee a L^∞ -bound of $1/n$ up to the life span of any solution due to De Giorgi type Lemma 2.2. It implies no finite-time blow-up happens. In other words, there is a global-in-time solution.

1.4. A chemotaxis model describing tumour angiogenesis. The system (1.1) can be derived from the following system of Keller-Segel type [20]:

$$(1.15) \quad \begin{aligned} \partial_t n - \nu \Delta n &= -\nabla \cdot (n \chi(c) \nabla c), \\ \partial_t c &= -c^m n \quad \text{for } \mathbf{x} \in \mathbb{R}^d \quad \text{and for } t > 0. \end{aligned}$$

This system has been used to describe chemotaxis phenomena including angiogenesis that is the formation of new blood vessels from pre-existing vessels. We may consider the formation as the mechanism for tumour progression and metastasis (*e.g.* see [9, 10, 21, 28, 29, 30], and references therein). In this interpretation, we consider $n(\mathbf{x}, t) > 0$ the density of endothelial cells and $c(\mathbf{x}, t)$ the concentration of the protein known as the vascular endothelial growth factor (VEGF) or just tumour angiogenesis factor (TAF). The given sensitivity function $\chi(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is usually assumed to be decreasing to reflect that the chemosensitivity becomes lower as the concentration of the chemical does higher. The positive exponent m of the chemical concentration represent the consumption rate of c (see the introduction in [4] for more details).

For the Cauchy problem of (1.15), we see [5, 10] and references therein. We refer to the study on traveling wave solutions of a Keller-Segel model in [20] and many other works including [13] (also see the survey paper [33]).

To derive our system (1.1), we just take $\chi(c) = c^{-1}$ and $m = 1$ and $d = 1$, into (1.15) to get

$$\begin{aligned} \partial_t n - \nu \partial_{xx} n &= -\partial_x \left(n \frac{\partial_x c}{c} \right), \\ \partial_t c &= -cn. \end{aligned}$$

Thanks to the restriction $m = 1$, we can treat the singularity in c of the sensitivity by the Cole-Hopf transformation

$$q := -\partial_x [\ln c] = -\frac{\partial_x c}{c}.$$

After the transform, we have (1.1) as in [34]. *cf*) For the case $m \neq 1$, we refer to the recent work [1] and references therein.

2. PRELIMINARIES

In this section, we present some lemmas that will be used throughout the paper.

2.1. Useful inequalities. We here present some useful inequalities on $\Pi(\cdot|\cdot)$, which were proved in [4, Lemma 2.8].

Lemma 2.1. ([4, Lemma 2.8]) *For given constants $\delta \in (0, \frac{1}{2}]$ and $n_- > 0$, there exist positive constants $C_1 = C_1(n_-)$, $C_2 = C_2(n_-, \delta)$ and $C_3 = C_3(n_-, \delta)$ such that the following inequalities hold:*

1) *For any $n_1 > 0$ and any $n_2 > 0$ with $\frac{n_-}{2} < n_2 < n_-$,*

$$(2.1) \quad \frac{1}{C_1} |n_1 - n_2|^2 \leq \Pi(n_1|n_2) \leq C_1 |n_1 - n_2|^2 \quad \text{whenever } \left| \frac{n_1}{n_2} - 1 \right| \leq \delta,$$

$$(2.2) \quad \frac{1}{C_2} (1 + n_1 \log^+ \frac{n_1}{n_2}) \leq \Pi(n_1|n_2) \leq C_2 (1 + n_1 \log^+ \frac{n_1}{n_2}) \quad \text{whenever } \left| \frac{n_1}{n_2} - 1 \right| \geq \delta,$$

$$(2.3) \quad \frac{1}{C_3} |n_1 - n_2| \leq \Pi(n_1|n_2) \leq C_3 |n_1 - n_2|^2 \quad \text{whenever } \left| \frac{n_1}{n_2} - 1 \right| \geq \delta,$$

where $\log^+(y)$ is the positive part of $\log(y)$.

2) *For any $n_1, n_2, m > 0$ satisfying $m \leq n_2 \leq n_1$ or $n_1 \leq n_2 \leq m$,*

$$(2.4) \quad \Pi(n_1|m) \geq \Pi(n_2|m).$$

2.2. De Giorgi type lemma.

We here present the following technical lemma, which may not be optimal but is enough for our purpose. This lemma might be classical, but we present its proof in Appendix A for completeness. The proof is based on the De Giorgi method [7].

Lemma 2.2. *Let $T_0 > 0$ and $R > 0$. Then there exists a constant $M = M(T_0, R) > 0$ with the following property:*

Let $T \in (0, T_0]$ and let p_1, p_2, p_3 be functions such that

$$(2.5) \quad p_1, p_2, p_3 \in L^\infty((0, T) \times \mathbb{R}), \quad p_2, \partial_x p_2, \partial_x p_3 \in L^2(0, T; L^2(\mathbb{R})).$$

Let $m \in L^\infty((0, T) \times \mathbb{R}) \cap C([0, T] \times \mathbb{R})$ be a non-negative function such that

$$(2.6) \quad \begin{cases} \partial_x m, \partial_{xx} m, \partial_t m \in L^2(0, T; L^2(\mathbb{R})), \\ \partial_t m - \partial_{xx} m + p_1 \partial_x m + m \partial_x (p_2 + p_3) \leq 0, \\ m = m_1 + m_2 \text{ with } m_1 \in L^\infty(0, T; L^2(\mathbb{R})) \text{ and } m_2 \in L^\infty(0, T; L^\infty(\mathbb{R})). \end{cases}$$

Assume

$$(2.7)$$

$$\|m|_{t=0}\|_{L^\infty(\mathbb{R})} + \| |p_1| + |p_2| + |p_3| + |m_2| \|_{L^\infty((0, T) \times \mathbb{R})} + \| |p_2| + |\partial_x p_3| \|_{L^2((0, T) \times \mathbb{R})} \leq R.$$

Then

$$\|m\|_{L^\infty((0,T)\times\mathbb{R})} \leq M.$$

Remark 2.3. We do not ask any quantitative bound but only finiteness for the norms of

$$\partial_x m, \partial_{xx} m, \partial_t m, \partial_x p_2 \in L^2(0, T; L^2(\mathbb{R})), m_1 \in L^\infty(0, T; L^2(\mathbb{R}))$$

to ensure that all computations in the proof make sense.

2.3. A priori contraction estimate. As in [4], we define the space

(2.8)

$$\mathcal{X}_T := \left\{ \begin{pmatrix} n \\ q \end{pmatrix} \in L^\infty((0, T) \times \mathbb{R})^2 \mid n > 0, n^{-1} \in L^\infty((0, T) \times \mathbb{R}), \partial_x n \in L^2((0, T) \times \mathbb{R}) \right\}$$

for each $T > 0$.

The following proposition on the contraction property is the main result of [4].

Proposition 2.4. [4, Theorem 1.2] *For a given constant state $(n_-, q_-) \in \mathbb{R}^+ \times \mathbb{R}$, there exist constants $\delta_0 \in (0, 1/2)$ and $\hat{C} > 0$ such that the following is true:*

For any $\varepsilon, \lambda > 0$ with $\varepsilon \in (0, n_-)$ and $\delta_0^{-1}\varepsilon < \lambda < \delta_0$, and for any $(n_+, q_+) \in \mathbb{R}^+ \times \mathbb{R}$ satisfying (1.5) with $|n_- - n_+| = \varepsilon$, there exists a smooth monotone function $a : \mathbb{R} \rightarrow \mathbb{R}^+$ with $\lim_{x \rightarrow \pm\infty} a(x) = 1 + a_\pm$ for some constants a_-, a_+ with $|a_+ - a_-| = \lambda$ such that the following holds:

Let $\tilde{U} := \begin{pmatrix} \tilde{n} \\ \tilde{q} \end{pmatrix}$ be a traveling wave of (1.7) with the boundary condition (1.2) and with the speed σ from (1.4). For a given $T > 0$, let $U(t, x) := \begin{pmatrix} n(t, x) \\ q(t, x) \end{pmatrix}$ be a solution to (1.7)

belonging to \mathcal{X}_T with initial data $U_0(x) := \begin{pmatrix} n_0(x) \\ q_0(x) \end{pmatrix}$ satisfying

$$(2.9) \quad \int_{-\infty}^{\infty} \eta(U_0 | \tilde{U}) dx < \infty.$$

Then there exists an absolutely continuous shift function $X : [0, T] \rightarrow \mathbb{R}$ with $X \in W_{loc}^{1,1}$ and $X(0) = 0$ such that

$$(2.10) \quad \begin{aligned} & \int_{-\infty}^{\infty} a(x - \sigma t) \eta(U(t, x - X(t)) | \tilde{U}(x - \sigma t)) dx \\ & + \delta_0 \int_0^t \int_{-\infty}^{\infty} a(x - \sigma \tau) n(\tau, x - X(\tau)) \left| \partial_x \left(\log \frac{n(\tau, x - X(\tau))}{\tilde{n}(x - \sigma \tau)} \right) \right|^2 dx d\tau \\ & \leq \int_{-\infty}^{\infty} a(x) \eta(U_0(x) | \tilde{U}(x)) dx, \end{aligned}$$

and

$$(2.11) \quad |\dot{X}(t) - \sigma| \leq \frac{1}{\varepsilon^2} \left(f(t) + \hat{C} \int_{-\infty}^{\infty} \eta(U_0 | \tilde{U}) dx + 1 \right) \quad \text{for a.e. } t \in [0, T]$$

where f is some positive function satisfying $\|f\|_{L^1(0,T)} \leq \hat{C} \frac{\lambda}{\varepsilon} \int_{-\infty}^{\infty} \eta(U_0 | \tilde{U}) dx$.

Remark 2.5. The diffusion term in (2.10) makes sense for solutions U of (1.1) in the class \mathcal{X}_T . Indeed, we find

$$\partial_x \left(\log \frac{n(t, x + Y(t))}{\tilde{n}} \right) \in L^2((0, T) \times \mathbb{R})$$

for any continuous and bounded function $Y : [0, T] \rightarrow \mathbb{R}$. It follows from $\partial_x n \in L^2((0, T) \times \mathbb{R})$, $n^{-1} \in L^\infty((0, T) \times \mathbb{R})$, $\tilde{n} \in L^\infty(\mathbb{R})$, and $\tilde{n}' \in L^2(\mathbb{R})$.

Remark 2.6. The estimate (2.11) implies

$$|X(t)| \leq \check{C} \cdot \left(\int_{\mathbb{R}} \eta(U_0 |\tilde{U}) dx + 1 \right) \cdot (t + 1)$$

for any $t \in [0, T]$ where the constant \check{C} depends only on the initial parameters n_-, q_-, ε , and λ . In particular, the constant \check{C} is independent of T .

3. PROOF OF THEOREM 1.1

In this section, we present the proof of Theorem 1.1.

3.1. Local existence in H^1 . We first present the local-in-time existence.

Proposition 3.1. *Let two given constant states $(n_-, q_-) \in \mathbb{R}^+ \times \mathbb{R}$ and $(n_+, q_+) \in \mathbb{R}^+ \times \mathbb{R}$ satisfy $n_- \neq n_+$ and (1.5). Consider the traveling wave $\tilde{U} = \begin{pmatrix} \tilde{n} \\ \tilde{q} \end{pmatrix}$ of (1.7) with the boundary condition (1.2) and with the speed σ from (1.4). For any $M_0 > 0$ and any $r_0 > 0$, there exists $\hat{T} > 0$ such that the following is true:*

For any initial datum $U_0 = \begin{pmatrix} n_0 \\ q_0 \end{pmatrix}$ satisfying

$$(3.1) \quad \|U_0 - \tilde{U}\|_{H^1(\mathbb{R})} \leq M_0 \quad \text{and} \quad \inf_{\mathbb{R}} n_0 \geq r_0,$$

there exists the unique solution $U = \begin{pmatrix} n \\ q \end{pmatrix}$ to (1.7) on $[0, \hat{T}]$ with the initial datum (n_0, q_0) such that

$$(3.2) \quad (n - \hat{n}, q - \hat{q}) \in \left(C([0, \hat{T}]; H^1(\mathbb{R})) \cap L^2(0, \hat{T}; H^2(\mathbb{R})) \right) \times C([0, \hat{T}]; H^1(\mathbb{R})),$$

$$(3.3) \quad \sup_{t \in [0, \hat{T}]} \|U(t) - \hat{U}(t)\|_{H^1(\mathbb{R})} \leq 2M_0 \quad \text{and} \quad \inf_{t \in [0, \hat{T}]} \inf_{x \in \mathbb{R}} n(x, t) \geq \frac{r_0}{2}.$$

Proof. The proof for local existence of strong solutions to the 1D hyperbolic-parabolic system such as (1.7) follows quite standard methods. For completeness, we present the proof in Appendix B. \square

3.2. Proposition 3.2 : a priori uniform estimates. To get the global-in-time existence, we present the main proposition on *a priori* uniform estimates:

Proposition 3.2. *Under the same hypotheses as in Theorem 1.1, if U is a solution of (1.7) on $[0, T_0)$ for some $T_0 > 0$ such that*

$$(3.4) \quad \begin{aligned} & (n - \tilde{n}, q - \tilde{q}) \in (C([0, T]; H^1(\mathbb{R})) \cap L^2(0, T; H^2(\mathbb{R}))) \times C([0, T]; H^1(\mathbb{R})), \\ & \text{and } 0 < \frac{1}{n} \in L^\infty(0, T; L^\infty(\mathbb{R})), \quad \forall T \in (0, T_0). \end{aligned}$$

Then there exists a constant $C(T_0)$ such that

$$\sup_{t \in [0, T_0)} \|U(t) - \hat{U}(t)\|_{H^1(\mathbb{R})} \leq C(T_0) \quad \text{and} \quad \sup_{t \in [0, T_0)} \|1/n\|_{L^\infty(\mathbb{R})} \leq C(T_0).$$

The proof of the main Proposition 3.2 will be handled in Section 4. Based on this Proposition, we here complete the proof of Theorem 1.1.

3.3. Proof of Theorem 1.1. For a given constant state $(n_-, q_-) \in \mathbb{R}^+ \times \mathbb{R}$, let us take the constants $\delta_0 \in (0, 1/2)$ and $\hat{C} > 0$ from Proposition 2.4. Then, choose any constant $\kappa > 0$ so that $\kappa < \min\{(\delta_0)^2/2, n_-/(15)\}$. Consider any $(n_+, q_+) \in \mathbb{R}^+ \times \mathbb{R}$ satisfying (1.5) with $0 < |n_- - n_+| < \kappa$.

Let $\varepsilon := |n_- - n_+|$ and take any λ between $\frac{\varepsilon}{\sqrt{\kappa}}$ and $\sqrt{\kappa}$. Note that these constants $\varepsilon, \lambda > 0$ satisfy the conditions $\varepsilon \in (0, n_-)$ and $\delta_0^{-1}\varepsilon < \lambda < \delta_0$ in Proposition 2.4. Then, we take the constant $\check{C} > 0$ from Remark 2.6.

Consider the traveling wave $\tilde{U} := \begin{pmatrix} \tilde{n} \\ \tilde{q} \end{pmatrix}$ of (1.7) with the boundary condition (1.2) and with the speed σ from (1.4). Let $U_0(x) := \begin{pmatrix} n_0(x) \\ q_0(x) \end{pmatrix}$ satisfy

$$U_0 - \tilde{U} \in H^1(\mathbb{R}), \quad n_0 > 0 \text{ on } \mathbb{R} \quad \text{and} \quad \frac{1}{n_0} \in L^\infty(\mathbb{R}).$$

We observe that Proposition 3.1 together with Remark 1.4 ensures the (local) existence of a solution U of (1.7) on $[0, \hat{T}]$ for some $\hat{T} > 0$ for $U|_{t=0} = U_0$ such that

$$\int_{-\infty}^{\infty} \eta(U_0|\tilde{U})dx < \infty,$$

$$(n - \hat{n}, q - \hat{q}) \in (C([0, \hat{T}]; H^1(\mathbb{R})) \cap L^2(0, \hat{T}; H^2(\mathbb{R}))) \times C([0, \hat{T}]; H^1(\mathbb{R})),$$

$$n > 0 \text{ on } [0, \hat{T}] \times \mathbb{R} \quad \text{and} \quad \frac{1}{n} \in L^\infty(0, \hat{T}; L^\infty(\mathbb{R})).$$

Now, in order to extend the solution U for all time,

$$(3.5) \quad \text{suppose that there is no global-in-time solution.}$$

Then there exists the finite maximal time interval $[0, T_0)$ for some $T_0 \in (\hat{T}, \infty)$ for the existence *i.e.*, there exists a solution U on $[0, T_0)$ such that

$$(3.6) \quad \begin{aligned} & (n - \tilde{n}, q - \tilde{q}) \in (C([0, T]; H^1(\mathbb{R})) \cap L^2(0, T; H^2(\mathbb{R}))) \times C([0, T]; H^1(\mathbb{R})), \\ & \text{and } 0 < \frac{1}{n} \in L^\infty(0, T; L^\infty(\mathbb{R})), \quad \forall T \in (0, T_0), \end{aligned}$$

but

$$\text{either } \sup_{t \in [0, T_0)} \|U(t) - \hat{U}(t)\|_{H^1(\mathbb{R})} = \infty \quad \text{or} \quad \inf_{t \in [0, T_0)} \inf_{x \in \mathbb{R}} n(x, t) = 0 \quad \text{holds.}$$

However, Proposition 3.2 and (3.6) implies

$$\sup_{t \in [0, T_0)} \|U(t) - \hat{U}(t)\|_{H^1(\mathbb{R})} \leq C(T_0) \quad \text{and} \quad \sup_{t \in [0, T_0)} \|1/n\|_{L^\infty(\mathbb{R})} \leq C(T_0),$$

where the constant $C(T_0)$ is independent of $T < T_0$. Therefore,

$$\sup_{t \in [0, T_0)} \|U(t) - \hat{U}(t)\|_{H^1(\mathbb{R})} < \infty \quad \text{and} \quad \inf_{t \in [0, T_0)} \inf_{x \in \mathbb{R}} n(x, t) > 0,$$

which produces a contradiction to the assumption (3.5). Therefore, we have a global solution. The proof of uniqueness follows the same standard energy method such as Step 5 in Appendix B. It proves the part (i).

For the part (ii), we first notice that the global solution U belongs to the class \mathcal{X}_T (see (2.8)) for any $T > 0$. Indeed, since $U - \hat{U} \in L^\infty(0, T; H^1(\mathbb{R}))$ and $\partial_x \hat{U} \in L^\infty(0, T; L^2(\mathbb{R}))$, we have $\partial_x n \in L^2((0, T) \times \mathbb{R})$, which implies $U \in \mathcal{X}_T$. Thus we apply Proposition 2.4 (or [4, Theorem 1.2]) for any arbitrarily large time interval. We recall how the shift is constructed in the proof of [4, Theorem 1.2], on which it is defined in a certain constructive way solving the given O.D.E. defined in [4, (3.2)] uniquely (see the explanation in Section 3.1 and Appendix A in [4]). Since the right-hand side of (3.2) in [4] is well defined uniquely for any time, we can construct a shift $X : [0, \infty) \rightarrow \mathbb{R}$ with the desired estimates (1.9) and (1.10).

Therefore, it only remains to prove Proposition 3.2.

4. PROOF OF PROPOSITION 3.2

First we note that for any $T \in (0, T_0)$, the local solution U we are considering belongs to the class \mathcal{X}_T (see (2.8)) thanks to (3.4). In this section, C denotes a positive constant which may change from line to line, and depends on the initial data (such as $\|U_0 - \tilde{U}\|_{H^1}$, $\|1/n_0\|_{L^\infty}$) and the parameters (such as n_\pm, q_-) and T_0 , but is independent of $T \in (0, T_0)$.

4.1. Uniform bound of the relative entropy. We will use Proposition 2.4 to show that

$$(4.1) \quad \sup_{t \in [0, T]} \int_{\mathbb{R}} \eta(U(t)|\hat{U}(t)) dx \leq C.$$

For simplicity, we here use the following notation:

for any function $f : \mathbb{R}_{\geq 0} \times \mathbb{R} \rightarrow \mathbb{R}$ and any shift $X : [0, \infty) \rightarrow \mathbb{R}$,

$$(4.2) \quad f^{\pm X}(t, x) := f(t, x \pm X(t)).$$

First of all, since Remark 1.4 together with $1/2 \leq a \leq 3/2$ yields

$$(4.3) \quad \int_{\mathbb{R}} a\eta(U_0|\tilde{U})dx \leq \int_{\mathbb{R}} \eta(U_0|\tilde{U})dx \leq C \int_{\mathbb{R}} |U_0 - \tilde{U}|^2 dx \leq C \|U_0 - \tilde{U}\|_{H^1(\mathbb{R})}^2,$$

Proposition 2.4 and Remark 2.6 imply that there exists a function X on $[0, T]$ such that

$$(4.4) \quad \begin{aligned} & \sup_{t \in [0, T]} \int_{\mathbb{R}} a^{-\sigma t} \eta([U(t)]^{-X(t)}|\tilde{U}^{-\sigma t})dx \\ & + \int_0^T \int_{-\infty}^{\infty} a^{-\sigma \tau} [n(\tau)]^{-X(\tau)} \cdot \left| \partial_x \left(\log \frac{[n(\tau)]^{-X(\tau)}}{\tilde{n}^{-\sigma \tau}} \right) \right|^2 dx d\tau \leq C \end{aligned}$$

and

$$\sup_{t \in [0, T]} |X(t)| \leq C.$$

For any $t \in [0, T]$, we have

$$(4.5) \quad \int_{\mathbb{R}} \eta(U(t)|\hat{U}(t))dx = \int_{\mathbb{R}} \eta(U(t)|\tilde{U}^{-\sigma t})dx = \int_{\mathbb{R}} \Pi(n(t)|\tilde{n}^{-\sigma t})dx + \frac{1}{2} \int_{\mathbb{R}} |q(t) - \tilde{q}^{-\sigma t}|^2 dx.$$

For the second term in (4.5), we have

$$(4.6) \quad \begin{aligned} \int_{\mathbb{R}} |q(t) - \tilde{q}^{-\sigma t}|^2 dx & \leq 2 \int_{\mathbb{R}} |q(t) - \tilde{q}^{X(t)-\sigma t}|^2 dx + 2 \int_{\mathbb{R}} |\tilde{q}^{X(t)-\sigma t} - \tilde{q}^{-\sigma t}|^2 dx \\ & = 2 \int_{\mathbb{R}} |[q(t)]^{-X(t)} - \tilde{q}^{-\sigma t}|^2 dx + 2 \int_{\mathbb{R}} |\tilde{q}^{X(t)} - \tilde{q}|^2 dx \\ & \leq C \int_{\mathbb{R}} a^{-\sigma t} |[q(t)]^{-X(t)} - \tilde{q}^{-\sigma t}|^2 dx + 2|q_+ - q_-| \cdot \int_{\mathbb{R}} |\tilde{q}^{X(t)} - \tilde{q}| dx \\ & \leq C \sup_{t \in [0, T]} \int_{\mathbb{R}} a^{-\sigma t} \eta([U(t)]^{-X(t)}|\tilde{U}^{-\sigma t})dx + 2|q_+ - q_-|^2 |X(t)| \\ & \leq C(T_0 + 1) \leq C. \end{aligned}$$

For the first term in (4.5), we have

$$\begin{aligned} & \int_{\mathbb{R}} \Pi(n(t)|\tilde{n}^{-\sigma t})dx \\ & = \int_{\{x \in \mathbb{R} \mid |\frac{n(t)}{\tilde{n}^{X(t)-\sigma t}} - 1| < \frac{1}{2}\}} \Pi(n(t)|\tilde{n}^{-\sigma t})dx + \int_{\{x \in \mathbb{R} \mid |\frac{n(t)}{\tilde{n}^{X(t)-\sigma t}} - 1| \geq \frac{1}{2}\}} \Pi(n(t)|\tilde{n}^{-\sigma t})dx =: I_1 + I_2. \end{aligned}$$

For I_1 , we use (1.11) to have

$$I_1 \leq C \int_{\{|\frac{n(t)}{\tilde{n}^{X(t)-\sigma t}} - 1| < \frac{1}{2}\}} |n(t) - \tilde{n}^{-\sigma t}|^2 dx.$$

Then, as in (4.6), we get

$$\begin{aligned}
(4.7) \quad I_1 &\leq C \int_{\{|\frac{n(t)}{\tilde{n}^{X(t)-\sigma t}} - 1| < \frac{1}{2}\}} |n(t) - \tilde{n}^{X(t)-\sigma t}|^2 dx + C \int_{\{|\frac{n(t)}{\tilde{n}^{X(t)-\sigma t}} - 1| < \frac{1}{2}\}} |\tilde{n}^{X(t)-\sigma t} - \tilde{n}^{-\sigma t}|^2 dx \\
&\leq C \int_{\{|\frac{[n(t)]^{-X(t)}}{\tilde{n}^{-\sigma t}} - 1| < \frac{1}{2}\}} |[n(t)]^{-X(t)} - \tilde{n}^{-\sigma t}|^2 dx + C \int_{\mathbb{R}} |\tilde{n}^{X(t)-\sigma t} - \tilde{n}^{-\sigma t}|^2 dx \\
&\leq C \int_{\{|\frac{[n(t)]^{-X(t)}}{\tilde{n}^{-\sigma t}} - 1| < \frac{1}{2}\}} \Pi([n(t)]^{-X(t)}|\tilde{n}^{-\sigma t}) dx + C \int_{\mathbb{R}} |\tilde{n}^{X(t)} - \tilde{n}|^2 dx \\
&\leq C \int a^{-\sigma t} \Pi([n(t)]^{-X(t)}|\tilde{n}^{-\sigma t}) dx + |n_- - n_+|^2 \cdot |X(t)| \leq C(T_0 + 1) \leq C,
\end{aligned}$$

where we used (2.1) for the third inequality.

For I_2 , we recall $0 < (n_- - n_+) < \kappa < n_-/(15) < n_-/4$, and so $n_- < \frac{4}{3}n_+$. Since $n_+ < \tilde{n} < n_-$, we find that for any $Y \in \mathbb{R}$,

$$\hat{n}^Y \leq \frac{4}{3}\hat{n}.$$

Thus,

$$\frac{n}{\hat{n}^Y} - 1 \geq \frac{1}{2} \Rightarrow \frac{n}{\hat{n}} - 1 \geq \frac{1}{8},$$

and

$$\frac{n}{\hat{n}^Y} - 1 \leq -\frac{1}{2} \Rightarrow \frac{n}{\hat{n}} - 1 \leq -\frac{1}{3},$$

which yield

$$\left\{ \left| \frac{n(t)}{\tilde{n}^{X(t)-\sigma t}} - 1 \right| \geq \frac{1}{2} \right\} = \left\{ \left| \frac{n(t)}{[\hat{n}(t)]^{X(t)}} - 1 \right| \geq \frac{1}{2} \right\} \subset \left\{ \left| \frac{n(t)}{\hat{n}(t)} - 1 \right| \geq \frac{1}{8} \right\}.$$

Thus we get

$$I_2 = \int_{\{|\frac{n(t)}{[\hat{n}(t)]^{X(t)}} - 1| \geq \frac{1}{2}\}} \Pi(n(t)|\hat{n}(t)) dx \leq \int_{\{|\frac{n(t)}{\hat{n}(t)} - 1| \geq \frac{1}{8}\}} \Pi(n(t)|\hat{n}(t)) dx.$$

We drop the t index for simplicity. Then, by (2.2), we get

$$I_2 \leq \int_{\{|\frac{n}{\hat{n}} - 1| \geq \frac{1}{8}\}} \Pi(n|\hat{n}) dx \leq C \int_{\{|\frac{n}{\hat{n}} - 1| \geq \frac{1}{8}\}} (1 + n \log^+ \frac{n}{\hat{n}}) dx,$$

Since the assumption $0 < (n_- - n_+) < \kappa < n_-/(15)$ implies

$$\hat{n}^Y \leq \frac{15}{14}\hat{n},$$

we have that for any Y ,

$$\left\{ \left| \frac{n}{\hat{n}} - 1 \right| \geq \frac{1}{8} \right\} \subset \left\{ \left| \frac{n}{\hat{n}^Y} - 1 \right| \geq \frac{1}{20} \right\}.$$

Observe that for any point on $\{|\frac{n}{\hat{n}} - 1| \geq \frac{1}{8}\}$ and for any $Y \in \mathbb{R}$, we have

$$(4.8) \quad (1 + n \log^+ \frac{n}{\hat{n}}) \leq C(1 + n \log^+ \frac{n}{\hat{n}^Y}).$$

Indeed, if $\frac{n}{\hat{n}} - 1 < -1/8$, then the estimate (4.8) is trivial due to $n < \hat{n}$. If $\frac{n}{\hat{n}} - 1 > 1/8$ i.e. $n > \frac{9}{8}\hat{n}$, then we have $\frac{n}{\hat{n}} \leq \frac{15}{14} \cdot \frac{n}{\hat{n}^Y}$ and $n > \frac{21}{20}\hat{n}^Y > \hat{n}^Y$ from (4.1), so we get

$$\begin{aligned} (1 + n \log^+ \frac{n}{\hat{n}}) &= (1 + n \log \frac{n}{\hat{n}}) \leq (1 + n \log \frac{15}{14} + n \log \frac{n}{\hat{n}^Y}) \\ &\leq (1 + n_- \cdot \log \frac{15}{14} + n \log \frac{n}{\hat{n}^Y}) \leq C(1 + n \log \frac{n}{\hat{n}^Y}) = C(1 + n \log^+ \frac{n}{\hat{n}^Y}). \end{aligned}$$

Thus, by (2.2), we get

$$\begin{aligned} (4.9) \quad I_2 &\leq C \int_{\{|\frac{n}{\hat{n}^X} - 1| \geq \frac{1}{20}\}} (1 + n \log^+ \frac{n}{\hat{n}^X}) dx \\ &\leq C \int_{\{|\frac{n}{\hat{n}^X} - 1| \geq \frac{1}{20}\}} \Pi(n|\hat{n}^X) dx \leq C \int_{\mathbb{R}} \Pi(n|\hat{n}^X) dx = C \int_{\mathbb{R}} \Pi(n^{-X}|\hat{n}) dx \\ &\leq C \int_{\mathbb{R}} a^{-\sigma t} \Pi([n(t)]^{-X(t)}|\tilde{n}^{-\sigma t}) dx \leq C. \end{aligned}$$

Thus from (4.6), (4.7), and (4.9), we have

$$(4.10) \quad \sup_{t \in [0, T]} \int_{\mathbb{R}} \eta(U(t)|\hat{U}(t)) dx \leq C(T_0 + 1) \leq C,$$

which gives (4.1).

4.2. Uniform bounds on $\|q - \hat{q}\|_{L^2}$ and $\|n - \hat{n}\|_{L^1 + L^2}$. We will use (4.1) to show that

$$(4.11) \quad \|q - \hat{q}\|_{L^\infty(0, T; L^2(\mathbb{R}))}^2 \leq C,$$

and there exists functions m_1, m_2 such that

$$(4.12) \quad n - \hat{n} = m_1 + m_2, \quad \|m_1\|_{L^\infty(0, T; L^1(\mathbb{R}))} + \|m_2\|_{L^\infty(0, T; L^2(\mathbb{R}))}^2 \leq C.$$

First of all, the definition of η and (4.10) implies that

$$\|q - \hat{q}\|_{L^\infty(0, T; L^2)}^2 \leq C(T_0 + 1) \leq C.$$

We define

$$(4.13) \quad m_1 := (n - \hat{n}) \mathbf{1}_{\{|\frac{n}{\hat{n}} - 1| \geq \frac{1}{2}\}} \quad \text{and} \quad m_2 := (n - \hat{n}) \mathbf{1}_{\{|\frac{n}{\hat{n}} - 1| < \frac{1}{2}\}},$$

which yields $n - \hat{n} = m_1 + m_2$.

We use (2.3) to have

$$\begin{aligned} (4.14) \quad \|m_1\|_{L^\infty(0, T; L^1(\mathbb{R}))} &= \|(n - \hat{n}) \mathbf{1}_{\{|\frac{n}{\hat{n}} - 1| \geq \frac{1}{2}\}}\|_{L^\infty(0, T; L^1(\mathbb{R}))} = \sup_{t \in [0, T]} \int_{\{|\frac{n}{\hat{n}} - 1| \geq \frac{1}{2}\}} |n(t) - \hat{n}(t)| dx \\ &\leq C \sup_{t \in [0, T]} \int_{\{|\frac{n}{\hat{n}} - 1| \geq \frac{1}{2}\}} \Pi(n(t)|\hat{n}(t)) dx \leq C \sup_{t \in [0, T]} \int_{\mathbb{R}} \Pi(n(t)|\hat{n}(t)) dx \leq C(T_0 + 1) \leq C. \end{aligned}$$

Using (2.1), we have

(4.15)

$$\begin{aligned} \|m_2\|_{L^\infty(0,T;L^2(\mathbb{R}))}^2 &= \|(n - \hat{n})\mathbf{1}_{\{|\frac{n}{\hat{n}}-1|<\frac{1}{2}\}}\|_{L^\infty(0,T;L^2(\mathbb{R}))}^2 = \sup_{t \in [0,T]} \int_{\{|\frac{n}{\hat{n}}-1|<\frac{1}{2}\}} |n(t) - \hat{n}(t)|^2 dx \\ &\leq C \sup_{t \in [0,T]} \int_{\{|\frac{n}{\hat{n}}-1|<\frac{1}{2}\}} \Pi(n(t)|\hat{n}(t)) dx \leq C \sup_{t \in [0,T]} \int_{\mathbb{R}} \Pi(n(t)|\hat{n}(t)) dx \leq C(T_0 + 1) \leq C. \end{aligned}$$

Therefore, we have (4.12).

4.3. **Uniform bound on** $\|\partial_x \sqrt{n}\|_{L^2}$. We will use (4.12) and (4.4) to get that

$$(4.16) \quad \int_0^T \int_{-\infty}^{\infty} |\partial_x \sqrt{n}|^2 dx d\tau \leq C.$$

First, we find from (4.4) that

$$\int_0^T \int_{-\infty}^{\infty} a^{X(\tau)-\sigma\tau} n \left| \partial_x \left(\log \frac{n}{\tilde{n}^{X(\tau)-\sigma\tau}} \right) \right|^2 dx d\tau \leq C.$$

Observe that for any $Y \in \mathbb{R}$,

$$\begin{aligned} n \left| \partial_x \left(\log \frac{n}{\tilde{n}^Y} \right) \right|^2 &= \frac{1}{(\tilde{n}^Y)^2} \cdot \frac{|(\partial_x n) \tilde{n}^Y - n \partial_x \tilde{n}^Y|^2}{n} = \frac{4}{\tilde{n}^Y} \cdot \frac{1}{4(\tilde{n}^Y)^3} \cdot \frac{|(\partial_x n) \tilde{n}^Y - n \partial_x \tilde{n}^Y|^2}{n} \\ &= \frac{4}{\tilde{n}^Y} \cdot \left| \partial_x \sqrt{\frac{n}{\tilde{n}^Y}} \right|^2. \end{aligned}$$

Then, using the fact that a and \tilde{n} are bounded from below and above by a positive constant, we have

$$(4.17) \quad \int_0^T \int_{-\infty}^{\infty} \left| \partial_x \sqrt{\frac{n}{\tilde{n}^{X(\tau)-\sigma\tau}}} \right|^2 dx d\tau \leq C.$$

Note, for any $Y \in \mathbb{R}$,

$$\begin{aligned} \left| \partial_x \sqrt{\frac{n}{\tilde{n}^Y}} \right| &= \left| \frac{((\partial_x n) \tilde{n}^Y - n(\tilde{n}^Y)') / (\tilde{n}^Y)^2}{2\sqrt{n/\tilde{n}^Y}} \right| \geq C^{-1} \left| \frac{(\partial_x n) \tilde{n}^Y - n(\tilde{n}^Y)'}{\sqrt{n}} \right| \\ &\geq C^{-1} (2\tilde{n}^Y |\partial_x \sqrt{n}| - |(\tilde{n}^Y)'| \sqrt{n}), \end{aligned}$$

and thus,

$$|\partial_x \sqrt{n}| \leq C \left| \partial_x \sqrt{\frac{n}{\tilde{n}^Y}} \right| + C |(\tilde{n}^Y)'| \sqrt{n}.$$

Thus we have

$$\begin{aligned} &\int_0^T \int_{-\infty}^{\infty} |\partial_x \sqrt{n}|^2 dx d\tau \\ &\leq C \int_0^T \int_{-\infty}^{\infty} \left| \partial_x \sqrt{\frac{n}{\tilde{n}^{X(\tau)-\sigma\tau}}} \right|^2 dx d\tau + C \underbrace{\int_0^T \int_{-\infty}^{\infty} |(\tilde{n}')^{X(\tau)-\sigma\tau}|^2 n dx d\tau}_{=: J}. \end{aligned}$$

To control J , using

$$(4.18) \quad |n| \leq |n - \hat{n}| + |\hat{n}| \leq |m_1| + |m_2| + |\hat{n}|,$$

and $\tilde{n}' \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$, together with (4.12), we have

$$J \leq C \cdot T_0 \cdot \left(\|m_1\|_{L^\infty(0,T;L^1(\mathbb{R}))} + \|m_2\|_{L^\infty(0,T;L^2(\mathbb{R}))} + 1 \right) \leq CT_0(T_0 + 1).$$

This and (4.17) yields

$$(4.19) \quad \int_0^T \int_{-\infty}^{\infty} |\partial_x \sqrt{n}|^2 dx d\tau \leq C(T_0 + 1)^2 \leq C.$$

4.4. Uniform bound on $\|q\|_{L^\infty}$. In order to get the uniform bounds for $\|n\|_{L^\infty(0,T;L^\infty(\mathbb{R}))}$ and $\|1/n\|_{L^\infty(0,T;L^\infty(\mathbb{R}))}$, we may first get $\|q\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \leq C$ and then apply Lemma 2.2. So we will here show

$$(4.20) \quad \|q\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \leq C.$$

For that, we first use (4.18) to find that for any $x \in \mathbb{R}$ and $t \in [0, T]$,

$$\begin{aligned} \|n(t)\|_{L^1([x-1,x+1])} &\leq \|m_1\|_{L^\infty(0,T;L^1(\mathbb{R}))} + \|m_2\|_{L^\infty(0,T;L^1([x-1,x+1]))} + 2\|\tilde{n}\|_{L^\infty(\mathbb{R})} \\ &\leq \|m_1\|_{L^\infty(0,T;L^1(\mathbb{R}))} + \sqrt{2}\|m_2\|_{L^\infty(0,T;L^2(\mathbb{R}))} + 2\|\tilde{n}\|_{L^\infty(\mathbb{R})}. \end{aligned}$$

So we have

$$(4.21) \quad \sup_{t \in [0, T]} \sup_{x \in \mathbb{R}} \|n(t)\|_{L^1([x-1,x+1])} \leq C(T_0 + 1) \leq C.$$

Since

$$n(t, x) = n(t, y) + \int_y^x (\partial_x n)(t, z) dz$$

and

$$(\partial_x n) = 2\sqrt{n}\partial_x \sqrt{n},$$

we have

$$n(t, x) = \frac{1}{2} \int_{x-1}^{x+1} n(t, y) dy + \int_{x-1}^{x+1} \int_y^x \sqrt{n} \partial_x \sqrt{n} dz dy.$$

Then, we use (4.16) and (4.21) to have

$$\begin{aligned} n(t, x) &\leq \frac{1}{2} \int_{x-1}^{x+1} n(t, y) dy + \int_{x-1}^{x+1} \int_{x-1}^{x+1} \sqrt{n} |\partial_x \sqrt{n}| dz dy \\ &\leq \frac{1}{2} \|n(t)\|_{L^1([x-1,x+1])} + 2 \sqrt{\int_{x-1}^{x+1} |n| dz} \sqrt{\int_{x-1}^{x+1} |\partial_x \sqrt{n}|^2 dz} \\ &\leq \frac{3}{2} \sup_{x \in \mathbb{R}} \|n(t)\|_{L^1([x-1,x+1])} + \|\partial_x \sqrt{n(t)}\|_{L^2(\mathbb{R})}^2, \end{aligned}$$

and thus,

$$(4.22) \quad \begin{aligned} \|n\|_{L^1(0,T;L^\infty(\mathbb{R}))} &\leq \frac{3}{2} T_0 \sup_{t \in [0, T]} \sup_{x \in \mathbb{R}} \|n(t)\|_{L^1([x-1,x+1])} + \|\partial_x \sqrt{n}\|_{L^2(0,T;L^2(\mathbb{R}))}^2 \\ &\leq C(T_0 + 1)^2 \leq C. \end{aligned}$$

We now introduce, to show (4.20), a new variable

$$w := n - \partial_x q.$$

Then, it follows from (1.7) that

$$(4.23) \quad \partial_t w + nw = n^2 + q\partial_x n.$$

Since $n > 0$, we have

$$(4.24) \quad \partial_t |w| \leq n^2 + |q\partial_x n|.$$

To estimate n^2 , we observe

$$n^2 = n(n - \hat{n} + \hat{n}) = n(m_1 + m_2 + \hat{n}) = \underbrace{nm_1}_{=:k_1} + \underbrace{n(m_2 + \hat{n})}_{=:k_2}.$$

Since $|m_2| = |\hat{n}(\frac{n}{\hat{n}} - 1)\mathbf{1}_{\{|\frac{n}{\hat{n}} - 1| < \frac{1}{2}\}}| \leq \frac{n-}{2} \leq C$, we have

$$\| |m_2| + \hat{n} \|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \leq C.$$

By (4.22) and (4.12), we have $n^2 = k_1 + k_2$ with

$$(4.25) \quad \|k_1\|_{L^1(0,T;L^1(\mathbb{R}))} + \|k_2\|_{L^1(0,T;L^\infty(\mathbb{R}))} \leq C(T_0 + 1)^3 \leq C.$$

To estimate $|q\partial_x n|$, we first observe that since $\partial_x n = 2\sqrt{n}\partial_x \sqrt{n}$ with

$$\|\partial_x \sqrt{n}\|_{L^2(0,T;L^2(\mathbb{R}))} \leq C(T_0 + 1) \leq C$$

by (4.16) and

$$\|\sqrt{n}\|_{L^2(0,T;L^\infty(\mathbb{R}))} \leq C(T_0 + 1) \leq C$$

by (4.22), we have

$$\|\partial_x n\|_{L^1(0,T;L^2(\mathbb{R}))} \leq C(T_0 + 1)^2 \leq C.$$

It implies

$$\|(q - \hat{q}) \cdot \partial_x n\|_{L^1(0,T;L^1(\mathbb{R}))} \leq C(T_0 + 1)^{5/2} \leq C.$$

Note $\sqrt{n} \leq \sqrt{|m_1|} + \sqrt{|m_2 + \hat{n}|}$ from $n = m_1 + m_2 + \hat{n}$ with

$$\|\sqrt{|m_1|}\|_{L^\infty(0,T;L^2(\mathbb{R}))} \leq \|m_1\|_{L^\infty(0,T;L^1(\mathbb{R}))}^{1/2} \leq C(T_0 + 1)^{1/2} \leq C$$

and $\|\sqrt{|m_2 + \hat{n}|}\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \leq C$.

Thus we get $|\partial_x n| = 2|\sqrt{n}| \cdot |\partial_x \sqrt{n}| \leq C \left(\underbrace{|\sqrt{|m_1|}| \cdot |\partial_x \sqrt{n}|}_{=:h_1} + \underbrace{(|\sqrt{|m_2 + \hat{n}|}| \cdot |\partial_x \sqrt{n}|)}_{=:h_2} \right)$ with

$$\|h_1\|_{L^1(0,T;L^1(\mathbb{R}))} \leq \sqrt{T_0} \cdot \|h_1\|_{L^2(0,T;L^1(\mathbb{R}))} \leq C(T_0 + 1)^2 \leq C$$

and

$$\|h_2\|_{L^2(0,T;L^2(\mathbb{R}))} \leq C(T_0 + 1) \leq C$$

We put $h_2 = \underbrace{h_2 \mathbf{1}_{\{|h_2| > 1\}}}_{=:h_{2,1}} + \underbrace{h_2 \mathbf{1}_{\{|h_2| \leq 1\}}}_{=:h_{2,2}}$, then we get

$$\|h_{2,1}\|_{L^1(0,T;L^1(\mathbb{R}))} \leq \|h_{2,1}\|_{L^2(0,T;L^2(\mathbb{R}))}^2 \leq C(T_0 + 1)^2 \leq C$$

and

$$\|h_{2,2}\|_{L^1(0,T;L^\infty(\mathbb{R}))} \leq T_0 \cdot \|h_{2,2}\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \leq C \cdot T_0 \leq C.$$

Thus we have $|\hat{q}\partial_x n| \leq C|\hat{q}| \cdot (h_1 + h_{2,1} + h_{2,2}) = \underbrace{C|\hat{q}| \cdot (h_1 + h_{2,1})}_{=:l_1} + \underbrace{C|\hat{q}| \cdot h_{2,2}}_{=:l_2}$ with

$$\|l_1\|_{L^1(0,T;L^1(\mathbb{R}))} \leq C(T_0 + 1)^2 \leq C$$

and

$$\|l_2\|_{L^1(0,T;L^\infty(\mathbb{R}))} \leq C \cdot T_0 \leq C$$

and

In sum, we have $|q\partial_x n| \leq \underbrace{|(q - \hat{q}) \cdot \partial_x n|}_{=:l_0} + l_1 + l_2$ with

$$(4.26) \quad \|l_0 + l_1\|_{L^1(0,T;L^1(\mathbb{R}))} + \|l_2\|_{L^1(0,T;L^\infty(\mathbb{R}))} \leq C(T_0 + 1)^{5/2} \leq C.$$

Therefore, it follows from (4.24), (4.25) and (4.26) that

$$\partial_t |w| \leq \underbrace{(k_1 + l_0 + l_1)}_{=:w_1} + \underbrace{(k_2 + l_2)}_{=:w_2}$$

with

$$\|w_1\|_{L^1(0,T;L^1(\mathbb{R}))} + \|w_2\|_{L^1(0,T;L^\infty(\mathbb{R}))} \leq C(T_0 + 1)^3 \leq C.$$

Moreover, since

$$w_0 = n_0 - \partial_x q_0 = \underbrace{-\partial_x(q_0 - \tilde{q})\mathbf{1}_{\{|\partial_x(q_0 - \tilde{q})| > 1\}}}_{=:j_1} + \underbrace{\left(n_0 - \partial_x(q_0 - \tilde{q})\mathbf{1}_{\{|\partial_x(q_0 - \tilde{q})| \leq 1\}} - \partial_x \tilde{q}\right)}_{=:j_2} \text{ with}$$

$$\|j_1\|_{L^1(\mathbb{R})} \leq \|\partial_x(q_0 - \tilde{q})\|_{L^2(\mathbb{R})}^2 \leq \|U_0 - \tilde{U}\|_{H^1(\mathbb{R})}^2 \leq C$$

and

$$\|j_2\|_{L^\infty(\mathbb{R})} \leq \|n_0\|_{L^\infty(\mathbb{R})} + C \leq \|n_0 - \tilde{n}\|_{L^\infty(\mathbb{R})} + C \leq \|U_0 - \tilde{U}\|_{H^1(\mathbb{R})} + C \leq C.$$

Therefore, we have $|w| \leq i_1 + i_2$ with

$$\|i_1\|_{L^\infty(0,T;L^1(\mathbb{R}))} + \|i_2\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \leq C(T_0 + 1)^3 \leq C.$$

Indeed, for $x \in \mathbb{R}$ and for $t \in [0, T]$, we have

$$|w(t, x)| = |w_0(x)| + \int_0^t (\partial_t |w|)(s, x) ds \leq \underbrace{|j_1(x)| + \int_0^t w_1(s, x) ds}_{=: \alpha_1(t, x)} + \underbrace{\left(|j_2(x)| + \int_0^t w_2(s, x) ds\right)}_{=: \alpha_2(t, x)}$$

with

$$\|\alpha_1\|_{L^\infty(0,T;L^1(\mathbb{R}))} \leq \|j_1\|_{L^1(\mathbb{R})} + \|w_1\|_{L^1(0,T;L^1(\mathbb{R}))} \leq C(T_0 + 1)^3 \leq C$$

and

$$\|\alpha_2\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \leq \|j_2\|_{L^\infty(\mathbb{R})} + \|w_2\|_{L^1(0,T;L^\infty(\mathbb{R}))} \leq C(T_0 + 1)^3 \leq C.$$

This implies

$$|\partial_x q| = |n - w| \leq |n - \hat{n}| + |\hat{n}| + |w| = \underbrace{|m_1| + \alpha_1}_{=:g_1} + \underbrace{\left(|m_2| + \hat{n} + \alpha_2\right)}_{=:g_2}$$

with

$$(4.27) \quad \|g_1\|_{L^\infty(0,T;L^1(\mathbb{R}))} + \|g_2\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \leq C(T_0 + 1)^3 \leq C.$$

Note from (4.11) that $q = q - \hat{q} + \hat{q} = \underbrace{(q - \hat{q})\mathbf{1}_{\{|q-\hat{q}|>1\}}}_{=:f_1} + \underbrace{(q - \hat{q})\mathbf{1}_{\{|q-\hat{q}|\leq 1\}} + \hat{q}}_{=:f_2}$ and

$$(4.28) \quad \|f_1\|_{L^\infty(0,T;L^1(\mathbb{R}))} + \|f_2\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \leq C(T_0 + 1) \leq C.$$

Therefore, using Lemma 4.1 below, together with (4.28) and (4.27), we have

$$\|q\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \leq C(T_0 + 1)^3 \leq C.$$

Lemma 4.1. *Let f be any function on \mathbb{R} such that $f = f_1 + f_2$ and $|f'| \leq g_1 + g_2$ with $f_1, g_1 \in L^1(\mathbb{R})$ and $f_2, g_2 \in L^\infty(\mathbb{R})$.*

Then, $f \in L^\infty(\mathbb{R})$ with

$$\|f\|_{L^\infty(\mathbb{R})} \leq 2\left(\|f_1\|_{L^1(\mathbb{R})} + \|f_2\|_{L^\infty(\mathbb{R})} + \|g_1\|_{L^1(\mathbb{R})} + \|g_2\|_{L^\infty(\mathbb{R})}\right).$$

Proof. Since

$$f(x) = f(y) + \int_y^x f'(z)dz,$$

for any $x, y \in \mathbb{R}$, we have, by taking $\frac{1}{2} \int_{x-1}^{x+1} dy$,

$$\begin{aligned} |f(x)| &\leq \frac{1}{2} \int_{x-1}^{x+1} (|f_1(y)| + |f_2(y)|)dy + \frac{1}{2} \int_{x-1}^{x+1} \int_{x-1}^{x+1} (|g_1(z)| + |g_2(z)|)dzdy \\ &\leq \frac{1}{2} \|f_1\|_{L^1(\mathbb{R})} + \|f_2\|_{L^\infty(\mathbb{R})} + \|g_1\|_{L^1(\mathbb{R})} + 2\|g_2\|_{L^\infty(\mathbb{R})}. \end{aligned}$$

for any $x \in \mathbb{R}$. □

4.5. Uniform bounds on $\|n\|_{L^\infty}$ and $\|1/n\|_{L^\infty}$. We now use Lemma 2.2 (De Giorgi type lemma) to get uniform bounds on $\|n\|_{L^\infty(0,T;L^\infty(\mathbb{R}))}$ and $\|1/n\|_{L^\infty(0,T;L^\infty(\mathbb{R}))}$. First, to control $\|n\|_{L^\infty(0,T;L^\infty(\mathbb{R}))}$, we set

$$(4.29) \quad m = n, \quad m_1 = n - \hat{n}, \quad m_2 = \hat{n}, \quad p_1 = -q, \quad p_2 = -(q - \hat{q}), \quad \text{and} \quad p_3 = -\hat{q}.$$

Since

$$\partial_t n - \partial_{xx} n - q\partial_x n - n\partial_x q = 0,$$

the above quantities in (4.29) satisfy the assumption of Lemma 2.2. More precisely, we use (4.20) and (4.11) to estimate

$$\begin{aligned} (4.30) \quad &\|m\|_{t=0} \|_{L^\infty(\mathbb{R})} + \| |p_1| + |p_2| + |p_3| + |m_2| \|_{L^\infty((0,T)\times\mathbb{R})} + \| |p_2| + |\partial_x p_3| \|_{L^2((0,T)\times\mathbb{R})} \\ &= \|n_0\|_{L^\infty(\mathbb{R})} + \| |q| + |q - \hat{q}| + |\hat{q}| + |\hat{n}| \|_{L^\infty((0,T)\times\mathbb{R})} + \| |q - \hat{q}| + |\partial_x \hat{q}| \|_{L^2((0,T)\times\mathbb{R})} \\ &\leq \|n_0\|_{L^\infty(\mathbb{R})} + 2\|q\|_{L^\infty((0,T)\times\mathbb{R})} + 2\| |\hat{q}| + |\hat{n}| \|_{L^\infty((0,T)\times\mathbb{R})} + \sqrt{T} \| |q - \hat{q}| + |\partial_x \hat{q}| \|_{L^\infty(0,T;L^2(\mathbb{R}))} \\ &\leq C(T_0 + 1) \leq C. \end{aligned}$$

Since the above constant C does not depend on T , by Lemma 2.2, we obtain

$$(4.31) \quad \|n\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \leq C_{T_0} \leq C.$$

Similarly, we can obtain

$$(4.32) \quad \left\| \frac{1}{n} \right\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \leq C_{T_0} \leq C.$$

Indeed, in order to apply Lemma 2.2, let

(4.33)

$$m = 1/n, \quad m_1 = \frac{1}{n} - \frac{1}{\hat{n}} = \frac{\hat{n} - n}{n\hat{n}}, \quad m_2 = \frac{1}{\hat{n}}, \quad p_1 = -q, \quad p_2 = (q - \hat{q}), \quad \text{and} \quad p_3 = \hat{q}.$$

Notice that it follows from (1.7) and (3.4) that

$$\partial_t \left(\frac{1}{n} \right) - \partial_{xx} \left(\frac{1}{n} \right) + (-q) \partial_x \left(\frac{1}{n} \right) + \left(\frac{1}{n} \right) \partial_x q = -\frac{2(\partial_x n)^2}{n^3} \leq 0, \quad \text{for a.e. } t \in [0, T],$$

where $\frac{(\partial_x n)^2}{n^3} \in L_t^2 L_x^2$ by the interpolation $\partial_x n \in L_t^\infty L_x^2 \cap L_t^2 L_x^\infty \subset L_t^4 L_x^4$. Thus, (3.4) implies that the quantities of (4.33) satisfy (2.5) and (2.6) on $[0, \tilde{T}]$.

Furthermore, the quantities of (4.33) satisfy (2.7) as follows:

$$\begin{aligned} & \|m|_{t=0}\|_{L^\infty(\mathbb{R})} + \| |p_1| + |p_2| + |p_3| + |m_2| \|_{L^\infty((0,T) \times \mathbb{R})} + \| |p_2| + |\partial_x p_3| \|_{L^2((0,T) \times \mathbb{R})} \\ &= \|1/n_0\|_{L^\infty(\mathbb{R})} + \| |q| + |q - \hat{q}| + |\hat{q}| + |1/\hat{n}| \|_{L^\infty((0,T) \times \mathbb{R})} + \| |q - \hat{q}| + |\partial_x \hat{q}| \|_{L^2((0,T) \times \mathbb{R})} \\ &\leq C(T_0 + 1) \leq C. \end{aligned}$$

Thus (4.32) follows from Lemma 2.2.

4.6. Uniform bound on $\|n - \hat{n}\|_{L^2}$. We first recall from (4.14), (4.15), (4.31) that $m_1 = (n - \hat{n}) - m_2$, and

$$\|m_1\|_{L^\infty(0,T;L^1(\mathbb{R}))} \leq C(T_0 + 1) \leq C,$$

and

$$\|(n - \hat{n}) - m_2\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \leq \|n\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} + C \leq C.$$

Since

$$\int_{\mathbb{R}} |m_1(t)|^2 dx = \int_{\mathbb{R}} |m_1(t)| \cdot |(n - \hat{n}) - m_2| dx \leq C_{T_0}(T_0 + 1) \leq C$$

for any $t \in [0, T]$, we get

$$\|m_1\|_{L^\infty(0,T;L^2(\mathbb{R}))}^2 \leq C_{T_0}(T_0 + 1) \leq C.$$

Thus we have

$$\|n - \hat{n}\|_{L^\infty(0,T;L^2(\mathbb{R}))} \leq \|m_1\|_{L^\infty(0,T;L^2(\mathbb{R}))} + \|m_2\|_{L^\infty(0,T;L^2(\mathbb{R}))} \leq C_{T_0}(T_0 + 1)^{1/2} \leq C$$

by (4.15).

4.7. Uniform bounds on $\|\partial_x n\|_{L^2}$, $\|\partial_x q\|_{L^2}$ and $\|\partial_{xx} n\|_{L^2}$. From the system (1.7), we do the energy method to obtain

$$\begin{aligned} & \frac{d}{dt} \left(\int_{\mathbb{R}} \frac{|\partial_x n|^2}{2} dx + \int_{\mathbb{R}} \frac{|\partial_x q|^2}{2} dx \right) + \int_{\mathbb{R}} |\partial_{xx} n|^2 dx \\ &= \int_{\mathbb{R}} \left((\partial_x n)(\partial_{xx} n)q + 2(\partial_x n)^2 \partial_x q - \partial_x [n(\partial_x n)](\partial_x q) + (\partial_x q)(\partial_{xx} n) \right) dx \end{aligned}$$

By integration by parts and using the dissipation term, we get

$$\begin{aligned}
(4.34) \quad & \frac{d}{dt} \left(\int_{\mathbb{R}} \frac{|\partial_x n|^2}{2} dx + \int_{\mathbb{R}} \frac{|\partial_x q|^2}{2} dx \right) + \frac{1}{2} \int_{\mathbb{R}} |\partial_{xx} n|^2 dx \\
& \leq C \int_{\mathbb{R}} \left(|\partial_x n|^2 |q|^2 + |\partial_x q|^2 |n|^2 + |\partial_x q|^2 \right) dx \\
& \leq C_{T_0} \int_{\mathbb{R}} \left(|\partial_x n|^2 + |\partial_x q|^2 \right) dx
\end{aligned}$$

where we used (4.20) and (4.31) in the last inequality. Then by Grönwall's inequality, we get

$$\|\partial_x n\|_{L^\infty(0,T;L^2(\mathbb{R}))} + \|\partial_x q\|_{L^\infty(0,T;L^2(\mathbb{R}))} \leq C_{T_0} \leq C.$$

In addition, by (4.34), we obtain

$$\|\partial_{xx} n\|_{L^2(0,T;L^2(\mathbb{R}))} \leq C_{T_0} \leq C.$$

4.8. Conclusion. Since

$$\|\partial_x \hat{n}\|_{L^\infty(0,T;L^2(\mathbb{R}))} + \|\partial_x \hat{q}\|_{L^\infty(0,T;L^2(\mathbb{R}))} \leq C,$$

we get

$$\|\partial_x(U - \hat{U})\|_{L^\infty(0,T;L^2(\mathbb{R}))} \leq C_{T_0} \leq C.$$

Hence we conclude

$$\sup_{t \in [0,T]} \|U(t) - \hat{U}(t)\|_{H^1(\mathbb{R})} \leq C_{T_0} \leq C.$$

Note that the above constant C does not depend on any choice of T satisfying $T < T_0$, which completes the proof.

APPENDIX A. PROOF OF LEMMA 2.2

For any constant $M > 2R$, we consider a sequence $(c_k)_{k \geq 0}$ defined by

$$c_k := M(1 - 2^{-k-1}), \quad k \geq 0.$$

Note that $M > c_{k+1} \geq c_k \geq c_0 = M/2 > R$ for all k , and $\lim_{k \rightarrow \infty} c_k = M$. Let

$$m_k := (m - c_k)_+,$$

and

$$E_k := \sup_{[0,T]} \int_{\mathbb{R}} m_k^2 dx + \int_0^T \int_{\mathbb{R}} |\partial_x m_k|^2 dx dt.$$

Note that E_k is well defined since

$$m - R = |m| - R \leq |m| - |m_2| \leq |m - m_2| = |m_1| \in L^\infty(0, T; L^2(\mathbb{R}))$$

implies

$$0 \leq m_k = (m - c_k)_+ \leq (m - R)_+ \leq |m_1| \in L^\infty(0, T; L^2(\mathbb{R}))$$

and

$$|\partial_x m_k| = |\partial_x m \mathbf{1}_{\{m > c_k\}}| \leq |\partial_x m| \in L^2(0, T; L^2(\mathbb{R})).$$

Observe that E_k is non-increasing in k since $0 \leq m_{k+1} \leq m_k$ and $|\partial_x m_{k+1}| \leq |\partial_x m_k|$ due to $\{m > c_{k+1}\} \subset \{m > c_k\}$. We also see

$$\int_{\mathbb{R}} (m - R)_+|_{t=0} dx = 0 \quad \text{and} \quad \int_{\mathbb{R}} m_k|_{t=0} dx = 0 \quad \text{for any } k$$

due to $R \geq \|m|_{t=0}\|_{L^\infty(\mathbb{R})}$.

Our goal is to show that there exists $M = M(R, T) > 0$ such that

$$(A.1) \quad \lim_{k \rightarrow \infty} E_k = 0.$$

Once we prove it, then we obtain

$$\sup_{[0, T]} \int_{\mathbb{R}} (m - M)_+^2 dx = 0 \quad \text{due to} \quad 0 \leq (m - M)_+ \leq m_k, \quad \text{for any } k,$$

which gives the desired result. Therefore, it remains to prove (A.1) in the following steps.

Step1) Since for any constant c ,

$$\partial_t(m - c) - \partial_{xx}(m - c) + p_1 \partial_x(m - c) + m \partial_x(p_2 + p_3) \leq 0,$$

$\bar{m} := (m - R)_+$ satisfies

$$\frac{d}{dt} \int_{\mathbb{R}} \bar{m}^2 dx + \int_{\mathbb{R}} |\partial_x \bar{m}|^2 dx \leq - \int_{\mathbb{R}} \bar{m} (\partial_x \bar{m}) p_1 dx - \int_{\mathbb{R}} \bar{m} m \partial_x (p_2 + p_3) dx.$$

Then, using the integration by parts and

$$\bar{m} \mathbf{1}_{\bar{m} > 0} = (m - R) \mathbf{1}_{\bar{m} > 0},$$

we have

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} \bar{m}^2 dx + \int_{\mathbb{R}} |\partial_x \bar{m}|^2 dx \\ & \leq \int_{\mathbb{R}} \left(|\bar{m}| |\partial_x \bar{m}| (|p_1| + 2|p_2| + 2|p_3|) + R|p_2| |\partial_x \bar{m}| + \bar{m} R |\partial_x p_3| \right) dx, \end{aligned}$$

which yields

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} \bar{m}^2 dx + \frac{1}{2} \int_{\mathbb{R}} |\partial_x \bar{m}|^2 dx \\ & \leq C (\| |p_1| + |p_2| + |p_3| \|_{L^\infty((0, T) \times \mathbb{R})}^2 + 1) \|\bar{m}\|_{L^2(\mathbb{R})}^2 + CR^2 (\|p_2\|_{L^2(\mathbb{R})}^2 + \|\partial_x p_3\|_{L^2(\mathbb{R})}^2) \\ & \leq C(R^2 + 1) \|\bar{m}\|_{L^2(\mathbb{R})}^2 + CR^2 (\|p_2\|_{L^2(\mathbb{R})}^2 + \|\partial_x p_3\|_{L^2(\mathbb{R})}^2). \end{aligned}$$

Therefore, by the Grönwall's inequality with (2.7) and the fact $\bar{m}|_{t=0} = 0$, there exists a positive constant $C_* = C_*(R, T_0)$ such that

$$\sup_{[0, T]} \int_{\mathbb{R}} \bar{m}^2 dx + \int_0^T \int_{\mathbb{R}} |\partial_x \bar{m}|^2 dx dt \leq C_*.$$

This together with $c_0 \geq R$ implies

$$(A.2) \quad E_0 \leq C_*.$$

Step2) Since for each $k \geq 1$,

$$\frac{d}{dt} \int_{\mathbb{R}} |m_k|^2 dx + \int_{\mathbb{R}} |\partial_x m_k|^2 dx \leq - \int_{\mathbb{R}} m_k (\partial_x m_k) p_1 dx - \int_{\mathbb{R}} m_k m \partial_x (p_2 + p_3) dx,$$

it follows from the integration by parts with $m_k \mathbf{1}_{m_k > 0} = (m - c_k) \mathbf{1}_{m_k > 0}$ that

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} |m_k|^2 dx + \int_{\mathbb{R}} |\partial_x m_k|^2 dx \\ & \leq \int_{\mathbb{R}} |\partial_x m_k| \left(|m_k| |p_1| + (2|m_k| + c_k) (|p_2| + |p_3|) \right) \mathbf{1}_{m_k > 0} dx, \end{aligned}$$

which yields

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} |m_k|^2 dx + \frac{1}{2} \int_{\mathbb{R}} |\partial_x m_k|^2 dx \\ & \leq C (\|p_1\|_{L^\infty((0,T) \times \mathbb{R})}^2 + \|p_2\|_{L^\infty((0,T) \times \mathbb{R})}^2 + \|p_3\|_{L^\infty((0,T) \times \mathbb{R})}^2) \int_{\mathbb{R}} (|m_k|^2 + c_k^2) \mathbf{1}_{m_k > 0} dx. \end{aligned}$$

Thus, using (2.7) and $m_k|_{t=0} = 0$

$$E_k \leq CR^2 \int_0^T \int_{\mathbb{R}} (|m_k|^2 + c_k^2) \mathbf{1}_{m_k > 0} dx dt.$$

Note that since $m_{k-1} \geq c_k - c_{k-1} = M2^{-k-1}$ when $m_k > 0$, we have

$$\mathbf{1}_{m_k > 0} \leq M^{-1} 2^{k+1} m_{k-1} \leq (M^{-1} 2^{k+1} m_{k-1})^\beta, \quad \forall \beta > 1.$$

This together with $m_k^2 \mathbf{1}_{m_k > 0} \leq m_{k-1}^2 \mathbf{1}_{m_k > 0}$ and $c_k \leq M$ implies

$$E_k \leq \frac{CR^2 16^k}{M^2} \int_0^T \int_{\mathbb{R}} |m_{k-1}|^4 dx dt, \quad k \geq 1.$$

But, using $\|m_k\|_{L^\infty(\mathbb{R})} \leq C \|m_k\|_{H^1(\mathbb{R})}$ by Sobolev embedding, and

$$\|m_k\|_{L^4((0,T) \times \mathbb{R})}^4 \leq \|m_k\|_{L^\infty(0,T;L^2(\mathbb{R}))}^2 \|m_k\|_{L^2(0,T;L^\infty(\mathbb{R}))}^2,$$

we find that

$$\frac{C}{1+T} \|m_k\|_{L^4((0,T) \times \mathbb{R})}^4 \leq (E_k)^2, \quad k \geq 0.$$

Therefore, there exists a positive constant $C_1 = C_1(R, T_0)$ such that

$$E_{k+1} \leq \frac{C_1 16^k}{M^2} (E_k)^2, \quad \forall k \geq 0.$$

In particular, putting $C_2 = 16(C_1 + 1) > 1$,

$$E_{k+1} \leq \frac{(C_2)^k}{M^2} (E_k)^2, \quad \forall k \geq 0.$$

Set $F_k := E_k/M^2$. Then,

$$0 \leq F_{k+1} \leq (C_2)^k (F_k)^2, \quad \forall k \geq 0.$$

Moreover, since it follows from (A.2) that

$$F_0 = \frac{E_0}{M^2} \leq \frac{C_*}{M^2} \rightarrow 0 \quad \text{as } M \rightarrow \infty,$$

using Lemma A.1 below, there exists a constant $M > 0$ with $M > 2R$ such that

$$\lim_{k \rightarrow \infty} F_k = 0, \quad \text{so we get} \quad \lim_{k \rightarrow \infty} E_k = 0.$$

Hence we completes the proof of Lemma 2.2.

The following lemma can be proved in a standard way (or see the proof of [32, Lemma 1])

Lemma A.1. [32, Lemma 1] *For $C > 1$ and $\beta > 1$, there exists a constant $C_0 = C_0(C, \beta)$ such that for every sequence $\{W_k\}_{k=0}^\infty$ verifying $0 < W_0 < C_0$ and for every $k \geq 0$:*

$$0 \leq W_{k+1} \leq C^k W_k^\beta,$$

we have

$$\lim_{k \rightarrow \infty} W_k = 0.$$

APPENDIX B. PROOF OF PROPOSITION 3.1

Step 1 (Iteration Scheme) We first set

$$(n^0(t, x), q^0(t, x)) = (n_0(x), q_0(x)).$$

Then, for $k \geq 1$, and given (n^{k-1}, q^{k-1}) , we iteratively define (n^k, q^k) as a solution of the following linear system:

$$\begin{aligned} \partial_t n^k &= \partial_{xx} n^k + \partial_x (n^{k-1} q^{k-1}), \\ \partial_t q^k &= \partial_x n^k, \\ (n^k, q^k)|_{t=0} &= (n_0, q_0). \end{aligned} \tag{B.1}$$

By the general theory of the heat equation together with (3.1), for each $k \geq 1$, if $\partial_x (n^{k-1} q^{k-1}) \in L^\infty(0, T; L^2(\mathbb{R}))$ for some $\tilde{T} > 0$, then (B.1) has a unique solution (n^k, q^k) such that

$$(n^k - \tilde{n}, q^k - \tilde{q}) \in \left(C^0(0, \tilde{T}; H^1(\mathbb{R})) \cap L^2(0, \tilde{T}; H^2(\mathbb{R})) \right) \times C^0(0, \tilde{T}; H^1(\mathbb{R})).$$

Step 2 (Uniform bound) For convenience, we set

$$\begin{aligned} N^k(t, x) &:= n^k(t, x) - \hat{n}(t, x), \quad \hat{n}(t, x) := \tilde{n}(x - \sigma t), \\ Q^k(t, x) &:= q^k(t, x) - \hat{q}(t, x), \quad \hat{q}(t, x) := \tilde{q}(x - \sigma t). \end{aligned} \tag{B.2}$$

In this step, we will prove the following: for any $M_0 > 0$, and any initial datum (n_0, q_0) satisfying (3.1), there exists $T > 0$ such that

$$\begin{aligned} \sup_{t \in [0, T]} \|(N^k(t), Q^k(t))\|_{H^1(\mathbb{R})} &\leq 2M_0, \quad \forall k \geq 0, \\ \|\partial_x N^k\|_{L^2(0, T; H^1(\mathbb{R}))} &\leq 2M_0, \quad \forall k \geq 1. \end{aligned} \tag{B.3}$$

As the initial step, we first show (B.3) when $k = 0$. Since $\tilde{n}', \tilde{q}' \in H^1(\mathbb{R})$, and

$$\begin{aligned} \|(N^0(t), Q^0(t))\|_{H^1(\mathbb{R})} &\leq \|N^0(t)\|_{H^1(\mathbb{R})} + \|Q^0(t)\|_{H^1(\mathbb{R})} \\ &\leq \|n_0 - \tilde{n}\|_{H^1(\mathbb{R})} + \|\tilde{n} - \tilde{n}(\cdot - \sigma t)\|_{H^1(\mathbb{R})} + \|q_0 - \tilde{q}\|_{H^1(\mathbb{R})} + \|\tilde{q} - \tilde{q}(\cdot - \sigma t)\|_{H^1(\mathbb{R})}, \end{aligned}$$

we use (3.1) to have

$$\|(N^k(t), Q^k(t))\|_{H^1(\mathbb{R})} \leq M_0 + Ct,$$

where the constant C depends on $\|\tilde{n}'\|_{H^1(\mathbb{R})}$, $\|\tilde{q}'\|_{H^1(\mathbb{R})}$ and σ . Thus, taking $T > 0$ small enough such that $CT \leq M_0$, we obtain (B.3) when $k = 0$. Now, as the inductive step, for any $k \geq 1$, we assume

$$(B.4) \quad \sup_{t \in [0, T]} \|(N^{k-1}(t), Q^{k-1}(t))\|_{H^1(\mathbb{R})} \leq 2M_0.$$

Since (\hat{n}, \hat{q}) is a solution to (1.7), we use (B.1) to find that

$$(B.5) \quad \begin{aligned} \partial_t N^k &= \partial_{xx} N^k + \partial_x (n^{k-1} Q^{k-1} + \hat{q} N^{k-1}), \\ \partial_t Q^k &= \partial_x N^k, \\ (N^k, Q^k)|_{t=0} &= (n_0 - \hat{n}, q_0 - \hat{q}). \end{aligned}$$

Since it follows from (B.5) that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \frac{|N^k|^2}{2} + \int_{\mathbb{R}} |\partial_x N^k|^2 &= - \int_{\mathbb{R}} (n^{k-1} Q^{k-1} - \hat{q} N^{k-1}) \partial_x N^k, \\ \frac{d}{dt} \int_{\mathbb{R}} \frac{|Q^k|^2}{2} &= \int_{\mathbb{R}} Q^k \partial_x N^k, \end{aligned}$$

we use Young's inequality to have

$$\frac{d}{dt} \int_{\mathbb{R}} (|N^k|^2 + |Q^k|^2) + \int_{\mathbb{R}} |\partial_x N^k|^2 \leq 2 \int_{\mathbb{R}} |n^{k-1} Q^{k-1} - \hat{q} N^{k-1}|^2 + 2 \int_{\mathbb{R}} |Q^k|^2.$$

Since $n^{k-1} \in L^\infty([0, T] \times \mathbb{R})$ by (B.4) together with Sobolev embedding and the boundedness of \hat{n} , we use (B.4) again to have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} (|N^k|^2 + |Q^k|^2) + \int_{\mathbb{R}} |\partial_x N^k|^2 &\leq 4(\|n^{k-1}\|_\infty^2 + \|\hat{q}\|_\infty^2) \int_{\mathbb{R}} (|Q^{k-1}|^2 + |N^{k-1}|^2) + 2 \int_{\mathbb{R}} |Q^k|^2 \\ &\leq C(M_0) + 2 \int_{\mathbb{R}} |Q^k|^2, \end{aligned}$$

which implies that for some $C = C(M_0)$,

$$\begin{aligned} \int_{\mathbb{R}} (|N^k(t)|^2 + |Q^k(t)|^2) + \int_0^t \int_{\mathbb{R}} |\partial_x N^k|^2 &\leq e^{Ct} \int_{\mathbb{R}} (|n_0 - \tilde{n}|^2 + |q_0 - \tilde{q}|^2) + Cte^{Ct} \\ &\leq e^{Ct} M_0^2 + Cte^{Ct}. \end{aligned}$$

Thus, taking T small again (if needed) such that $\sqrt{e^{CT}(M_0^2 + CT)} \leq 2M_0$, we have

$$\begin{aligned} \sup_{t \in [0, T]} \|(N^k(t), Q^k(t))\|_{L^2(\mathbb{R})} &\leq 2M_0, \\ \|\partial_x N^k\|_{L^2(0, T; L^2(\mathbb{R}))} &\leq 2M_0. \end{aligned}$$

Next, to estimate the higher norm, we use (B.5) to get

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \frac{|\partial_x N^k|^2}{2} + \int_{\mathbb{R}} |\partial_{xx} N^k|^2 &= - \int_{\mathbb{R}} \partial_x (n^{k-1} Q^{k-1} - \hat{q} N^{k-1}) \partial_{xx} N^k, \\ \frac{d}{dt} \int_{\mathbb{R}} \frac{|\partial_x Q^k|^2}{2} &= \int_{\mathbb{R}} \partial_x Q^k \partial_{xx} N^k, \end{aligned}$$

which gives

$$\frac{d}{dt} \int_{\mathbb{R}} (|\partial_x N^k|^2 + |\partial_x Q^k|^2) + \int_{\mathbb{R}} |\partial_{xx} N^k|^2 \leq 2 \int_{\mathbb{R}} |\partial_x (n^{k-1} Q^{k-1} - \hat{q} N^{k-1})|^2 + 2 \int_{\mathbb{R}} |\partial_x Q^k|^2.$$

Likewise, since (B.4) implies

$$\begin{aligned} \int_{\mathbb{R}} |\partial_x (n^{k-1} Q^{k-1} - \hat{q} N^{k-1})|^2 &\leq \|\partial_x n^{k-1}\|_{L^2(\mathbb{R})} \|Q^{k-1}\|_{L^\infty(\mathbb{R})} + \|n^{k-1}\|_{L^\infty(\mathbb{R})} \|\partial_x Q^{k-1}\|_{L^2(\mathbb{R})} \\ &\quad + \|\partial_x \hat{q}\|_{L^2(\mathbb{R})} \|N^{k-1}\|_{L^\infty(\mathbb{R})} + \|\hat{q}\|_{L^\infty(\mathbb{R})} \|\partial_x N^{k-1}\|_{L^2(\mathbb{R})} \\ &\leq C(M_0), \end{aligned}$$

we have that (for T smaller than above if needed)

$$\begin{aligned} \sup_{t \in [0, T]} \|(\partial_x N^k(t), \partial_x Q^k(t))\|_{L^2(\mathbb{R})} &\leq 2M_0, \\ \|\partial_{xx} N^k\|_{L^2(0, T; L^2(\mathbb{R}))} &\leq 2M_0. \end{aligned}$$

Therefore, we have (B.3).

Step 3 (Uniform bound for $1/n$) Since it follows from (B.3) and from Sobolev embedding that for all $k \geq 1$,

$$\begin{aligned} \partial_x (n^{k-1} q^{k-1}) &= (\partial_x n^{k-1}) q^{k-1} + n^{k-1} \partial_x q^{k-1} \\ &= (\partial_x n^{k-1})(q^{k-1} - \hat{q}) + (\partial_x n^{k-1}) \hat{q} + (n^{k-1} - \hat{n}) \partial_x q^{k-1} + \hat{n} \partial_x q^{k-1} \in L^\infty(0, T; L^2(\mathbb{R})), \end{aligned}$$

and $\|\partial_x (n^{k-1} q^{k-1})\|_{L^\infty(0, T; L^2(\mathbb{R}))} \leq C(M_0)$, we use Duhamel's principle to represent

$$n^k(t, x) = \int_{\mathbb{R}} \Phi(t, x - y) n_0(y) dy + \int_0^t \int_{\mathbb{R}} \Phi(t - s, x - y) [\partial_x (n^{k-1} q^{k-1})](s, y) dy ds$$

where $\Phi(t, x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x|^2}{4t}}$ is the heat kernel in 1D.

Thus, by Young's inequality, we have the following estimate: for all $x \in \mathbb{R}$, $t \in [0, T]$,

$$\begin{aligned} n^k(t, x) &\geq \inf_{\mathbb{R}} n_0 - \int_0^t \|\Phi(t - s, \cdot)\|_{L^2(\mathbb{R})} \cdot \|[\partial_x (n^{k-1} q^{k-1})](s, \cdot)\|_{L^2(\mathbb{R})} ds \\ &\geq r_0 - \|[\partial_x (n^{k-1} q^{k-1})]\|_{L^\infty(0, T; L^2(\mathbb{R}))} \int_0^T \|\Phi(s, \cdot)\|_{L^2(\mathbb{R})} ds \\ &\geq r_0 - C \cdot \|[\partial_x (n^{k-1} q^{k-1})]\|_{L^\infty(0, T; L^2(\mathbb{R}))} \cdot T^{3/4} \geq r_0 - C(M_0) T^{3/4}. \end{aligned}$$

Therefore, taking T small again (if needed) such that $r_0 - C(M_0) T^{3/4} \geq r_0/2$, we have

$$(B.6) \quad \inf_{t \in [0, T]} \inf_{x \in \mathbb{R}} n^k(x, t) \geq \frac{r_0}{2}.$$

Step 4 (Convergence) We will first prove that the sequence $\{(N^k, Q^k)\}_{k \geq 1}$ is Cauchy in \mathcal{S} , where

$$\mathcal{S} := (L^\infty(0, T; L^2(\mathbb{R})) \cap L^2(0, T; H^1(\mathbb{R}))) \times L^\infty(0, T; L^2(\mathbb{R})).$$

For convenience, we set

$$\bar{N}^k := N^{k+1} - N^k, \quad \bar{Q}^k := Q^{k+1} - Q^k.$$

Then, using $\bar{N}^{k-1} = n^k - n^{k-1}$ and $\bar{Q}^{k-1} = q^k - q^{k-1}$, it follows from (B.5) that

$$(B.7) \quad \begin{aligned} \partial_t \bar{N}^k &= \partial_{xx} \bar{N}^k + \partial_x (n^k \bar{Q}^{k-1} + q^{k-1} \bar{N}^{k-1}), \\ \partial_t \bar{Q}^k &= \partial_x \bar{N}^k, \\ (\bar{N}^k, \bar{Q}^k)|_{t=0} &= (0, 0). \end{aligned}$$

Thus, using the same estimate as in Step 2, we find that for all $t \leq T$,

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}} (|\bar{N}^k|^2 + |\bar{Q}^k|^2) + \int_{\mathbb{R}} |\partial_x \bar{N}^k|^2 \\ & \leq 4(\|n^k\|_{L^\infty([0,T] \times \mathbb{R})}^2 + \|q^{k-1}\|_{L^\infty([0,T] \times \mathbb{R})}^2) \int_{\mathbb{R}} (|\bar{Q}^{k-1}|^2 + |\bar{N}^{k-1}|^2) + 2 \int_{\mathbb{R}} |\bar{Q}^k|^2. \end{aligned}$$

Using the uniform-in- k bound (B.3) with Sobolev embedding, we have

$$\frac{d}{dt} \int_{\mathbb{R}} (|\bar{N}^k|^2 + |\bar{Q}^k|^2) + \int_{\mathbb{R}} |\partial_x \bar{N}^k|^2 \leq C(M_0) \int_{\mathbb{R}} (|\bar{Q}^{k-1}|^2 + |\bar{N}^{k-1}|^2) + 2 \int_{\mathbb{R}} |\bar{Q}^k|^2.$$

Integrating it over $[0, T]$, we have

$$\int_{\mathbb{R}} (|\bar{N}^k(t)|^2 + |\bar{Q}^k(t)|^2) + \int_0^t \int_{\mathbb{R}} |\partial_x \bar{N}^k|^2 \leq C \int_0^t \left(\int_{\mathbb{R}} (|\bar{Q}^{k-1}|^2 + |\bar{N}^{k-1}|^2) + 2 \int_{\mathbb{R}} |\bar{Q}^k|^2 \right).$$

This implies

$$\int_{\mathbb{R}} (|\bar{N}^k(t)|^2 + |\bar{Q}^k(t)|^2) + \int_0^t \int_{\mathbb{R}} |\partial_x \bar{N}^k|^2 \leq \frac{Ct^k}{k!}, \quad \forall t \leq T.$$

Therefore, the sequence $\{(N^k, Q^k)\}_{k \geq 1}$ is Cauchy in \mathcal{S} , which implies that there exists a limit (N^∞, Q^∞) such that

$$(B.8) \quad (N^k, Q^k) \rightarrow (N^\infty, Q^\infty) \quad \text{in } \mathcal{S}.$$

Furthermore, using (B.8) and (B.3), we have

$$(B.9) \quad \sup_{t \in [0, T]} \|(N^\infty(t), Q^\infty(t))\|_{H^1(\mathbb{R})} \leq 2M_0, \quad \|\partial_x N^\infty\|_{L^2(0, T; H^1(\mathbb{R}))} \leq 2M_0.$$

Step 5 (Existence) Let $n := N^\infty + \hat{n}$ and $q := Q^\infty + \hat{q}$. Then, by (B.2) and (B.8), we obtain that

$$(B.10) \quad (n^k - \hat{n}, q^k - \hat{q}) \rightarrow (n - \hat{n}, q - \hat{q}) \quad \text{in } \mathcal{S},$$

and

$$(B.11) \quad \|(n - \hat{n}, q - \hat{q})\|_{L^\infty(0, T; H^1(\mathbb{R}))} \leq 2M_0, \quad \partial_x n \in L^2(0, T; H^1(\mathbb{R})).$$

This implies $n^k - \hat{n} \rightarrow n - \hat{n}$ in $L^\infty(0, T; H^{3/4}(\mathbb{R}))$, and thus $n^k \rightarrow n$ in $L^\infty(0, T; L^\infty(\mathbb{R}))$, which together with (B.6) yields

$$\inf_{t \in [0, T]} \inf_{x \in \mathbb{R}} n(x, t) \geq \frac{r_0}{2}.$$

Moreover, (B.10) and (B.11) together with (B.1) imply that (n, q) solves (1.7) with $(n, q)|_{t=0} = (n_0, q_0)$ in the sense of distributions. In particular, (1.7) and (B.11) yield that $\partial_t(n - \tilde{n}) \in L^2(0, T; H^2(\mathbb{R}))$, which together with Aubin-Lions lemma implies $n - \tilde{n} \in C([0, T]; H^1(\mathbb{R}))$, and thus $q - \tilde{q} \in C([0, T]; H^1(\mathbb{R}))$.

Step 6 (Uniqueness) Let (n_1, q_1) and (n_2, q_2) be solutions to (1.7) with the initial datum (n_0, q_0) , and satisfy (B.11). Then, set $\bar{n} := n_1 - n_2$, $\bar{q} := q_1 - q_2$. Then, it follows from (1.7) that

$$\begin{aligned}\partial_t \bar{n} &= \partial_{xx} \bar{n} + \partial_x (n_1 \bar{q} + q_2 \bar{n}), \\ \partial_t \bar{q} &= \partial_x \bar{n}, \\ (\bar{n}, \bar{q})|_{t=0} &= (0, 0).\end{aligned}$$

which has the same structure as in (B.7). Thus, using the same estimates as above, we have

$$\int_{\mathbb{R}} (|\bar{n}(t)|^2 + |\bar{q}(t)|^2) \leq C \int_0^T \int_{\mathbb{R}} (|\bar{n}(t)|^2 + |\bar{q}(t)|^2), \quad \forall t \leq T,$$

which implies that $n_1 = n_2$ and $q_1 = q_2$ on $[0, T] \times \mathbb{R}$.

REFERENCES

- [1] Jose A Carrillo, Jingyu Li, and Zhian Wang. Boundary spike-layer solutions of the singular keller-segel system: existence and stability. *preprint, arXiv:1908.07485*.
- [2] Myeongju Chae and Kyudong Choi. Nonlinear stability of planar traveling waves in a chemotaxis model of tumor angiogenesis with chemical diffusion. *J. Differential Equations, to appear*.
- [3] Myeongju Chae, Kyudong Choi, Kyungkeun Kang, and Jihoon Lee. Stability of planar traveling waves in a Keller-Segel equation on an infinite strip domain. *J. Differential Equations*, 265(1):237–279, 2018.
- [4] Kyudong Choi, Moon-Jin Kang, Young-Sam Kwon, and Alexis Vasseur. Contraction for large perturbations of traveling waves in a hyperbolic-parabolic system arising from a chemotaxis model. *preprint, arXiv:1904.12169*.
- [5] L. Corrias, B. Perthame, and H. Zaag. A chemotaxis model motivated by angiogenesis. *C. R. Math. Acad. Sci. Paris*, 336(2):141–146, 2003.
- [6] C. M. Dafermos. The second law of thermodynamics and stability. *Arch. Rational Mech. Anal.*, 70(2):167–179, 1979.
- [7] Ennio De Giorgi. Sulla differenziabilità e l’analiticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3)*, 3:25–43, 1957.
- [8] R. J. DiPerna. Uniqueness of solutions to hyperbolic conservation laws. *Indiana Univ. Math. J.*, 28(1):137–188, 1979.
- [9] Marco A. Fontelos, Avner Friedman, and Bei Hu. Mathematical analysis of a model for the initiation of angiogenesis. *SIAM J. Math. Anal.*, 33(6):1330–1355, 2002.
- [10] Avner Friedman and J. Ignacio Tello. Stability of solutions of chemotaxis equations in reinforced random walks. *J. Math. Anal. Appl.*, 272(1):138–163, 2002.
- [11] Jonathan Goodman. Remarks on the stability of viscous shock waves. In *Viscous profiles and numerical methods for shock waves (Raleigh, NC, 1990)*, pages 66–72. SIAM, Philadelphia, PA, 1991.
- [12] Jun Guo, Jixiong Xiao, Huijiang Zhao, and Changjiang Zhu. Global solutions to a hyperbolic-parabolic coupled system with large initial data. *Acta Math. Sci. Ser. B (Engl. Ed.)*, 29(3):629–641, 2009.
- [13] Dirk Horstmann. From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. *I. Jahresber. Deutsch. Math.-Verein.*, 105(3):103–165, 2003.
- [14] Hai-Yang Jin, Jingyu Li, and Zhi-An Wang. Asymptotic stability of traveling waves of a chemotaxis model with singular sensitivity. *J. Differential Equations*, 255(2):193–219, 2013.
- [15] Moon-Jin Kang. L^2 -type contraction for shocks of scalar viscous conservation laws with strictly convex flux. *preprint, arXiv:1901.02969*.
- [16] Moon-jin Kang and Alexis Vasseur. Contraction property for large perturbations of shocks of the barotropic navier-stokes system,. *J. Eur. Math. Soc. (JEMS), to appear*.

- [17] Moon-Jin Kang and Alexis Vasseur. Uniqueness and stability of entropy shocks to the isentropic Euler system in a class of inviscid limits from a large family of Navier-Stokes systems. *preprint, arXiv:1902.01792*.
- [18] Moon-Jin Kang and Alexis Vasseur. L^2 -contraction for shock waves of scalar viscous conservation laws. *Annales de l'Institut Henri Poincaré (C) : Analyse non linéaire*, 34(1):139156, 2017.
- [19] Shuichi Kawashima and Akitaka Matsumura. Stability of shock profiles in viscoelasticity with non-convex constitutive relations. *Comm. Pure Appl. Math.*, 47(12):1547–1569, 1994.
- [20] E. F. Keller and L. A. Segel. Traveling bands of chemotactic bacteria: A theoretical analysis. *J. theor. Biol.*, 30:235–248, 1971.
- [21] Howard A. Levine, Brian D. Sleeman, and Marit Nilsen-Hamilton. A mathematical model for the roles of pericytes and macrophages in the initiation of angiogenesis. I. The role of protease inhibitors in preventing angiogenesis. *Math. Biosci.*, 168(1):77–115, 2000.
- [22] Dong Li, Tong Li, and Kun Zhao. On a hyperbolic-parabolic system modeling chemotaxis. *Math. Models Methods Appl. Sci.*, 21(8):1631–1650, 2011.
- [23] Dong Li, Ronghua Pan, and Kun Zhao. Quantitative decay of a one-dimensional hybrid chemotaxis model with large data. *Nonlinearity*, 28(7):2181–2210, 2015.
- [24] Jingyu Li, Tong Li, and Zhi-An Wang. Stability of traveling waves of the Keller-Segel system with logarithmic sensitivity. *Math. Models Methods Appl. Sci.*, 24(14):2819–2849, 2014.
- [25] Tong Li and Zhi-An Wang. Nonlinear stability of traveling waves to a hyperbolic-parabolic system modeling chemotaxis. *SIAM J. Appl. Math.*, 70(5):1522–1541, 2010.
- [26] Tong Li and Zhi-An Wang. Steadily propagating waves of a chemotaxis model. *Mathematical Biosciences*, 240(2):161–168, 2012.
- [27] Vincent R. Martinez, Zhian Wang, and Kun Zhao. Asymptotic and viscous stability of large-amplitude solutions of a hyperbolic system arising from biology. *Indiana Univ. Math. J.*, 67(4):1383–1424, 2018.
- [28] Benoît Perthame. PDE models for chemotactic movements: parabolic, hyperbolic and kinetic. *Appl. Math.*, 49(6):539–564, 2004.
- [29] Gerald Rosen. Steady-state distribution of bacteria chemotactic toward oxygen. *Bull. Math. Biol.*, 40(5):671–674, 1978.
- [30] Jonathan A. Sherratt. Traveling wave solutions of a mathematical model for tumor encapsulation. *SIAM J. Appl. Math.*, 60(2):392–407, 2000.
- [31] Alexis Vasseur. Recent results on hydrodynamic limits. In *Handbook of differential equations: evolutionary equations. Vol. IV*, Handb. Differ. Equ., pages 323–376. Elsevier/North-Holland, Amsterdam, 2008.
- [32] Alexis F. Vasseur. A new proof of partial regularity of solutions to Navier-Stokes equations. *NoDEA Nonlinear Differential Equations Appl.*, 14(5-6):753–785, 2007.
- [33] Zhi-An Wang. Mathematics of traveling waves in chemotaxis—review paper. *Discrete Contin. Dyn. Syst. Ser. B*, 18(3):601–641, 2013.
- [34] Zhi-An Wang and Thomas Hillen. Shock formation in a chemotaxis model. *Math. Models Appl. Sci.*, 31(1):45–70, 2008.

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