On global motions of a compressible barotropic and selfgravitating gas with density-dependent viscosities.

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SUMMARY

We consider the Cauchy problem for the equations of selfgravitating motions of a barotropic gas with density-dependent viscosities $\mu(\rho)$, and $\lambda(\rho)$ satisfying the Bresch-Desjardins condition, when the pressure $P(\rho)$ is not necessarily a monotone function of the density. We prove that this problem admits a global weak solution provided that the adiabatic exponent γ associated to $P(\rho)$ satisfies $\gamma > \frac{4}{3}$.

1 Introduction

We consider the Navier-Stokes-Poisson system for a compressible isentropic gas in $\mathbb{R}^3$ with density-dependent viscosities

$$\left\{\begin{array}{c} \partial_t\rho + \mathrm{div}(\rho\vec{v}) = 0, \\ \\ \partial_t(\rho\vec{v}) + \mathrm{div}(\rho\vec{v}\otimes\vec{v}) - \mathrm{div}\,(2\mu(\rho)D(\vec{v})) - \nabla\left(\lambda(\rho)\mathrm{div}\vec{v}\right) + \nabla P - \rho\nabla\Phi = \vec{0}, \\ \\ \Delta\Phi = -4\pi G\rho. \end{array}\right. \tag{1}$$

Here ρ is the density, $\vec{v}$ is the velocity, Φ is the Newtonian gravitational potential, P is the pressure, D is the stain tensor with $D(\vec{v}) = 1/2\,(\nabla\vec{v} +^t \nabla\vec{v})$. The two Lamé viscosity coefficients λ and μ depend on density and satisfy the stability requirements

$$\mu(\rho) > 0, \qquad 2\mu(\rho) + 3\lambda(\rho) \geqslant 0.$$

The pressure $P(\rho)$ is related to the density ρ by a general barotropic constitutive law (see [3] for motivations) satisfying

$$\left\{\begin{array}{c} P \in C^1(\mathbb{R}_+), \quad P(0) = 0, \\ \dfrac{1}{a}\, z^{\gamma-1} - b \leqslant P'(z) \leqslant a z^{\gamma-1} + b \quad \text{for all } z \geqslant 0, \end{array}\right. \tag{2}$$

for two constants $a > 0$ and $b \geqslant 0$.

It is known after [3] that, under condition (2) on P, the Cauchy problem associated to (1) admits a (renormalized) weak solution when the viscosity coefficients are positive constants, modulo mild conditions on the initial data (in particular vacuum is allowed) and provided $\gamma > \frac{3}{2}$ (in space dimension $N = 3$).

Recently Mellet and Vasseur [5] proved the existence of a solution for the barotropic Navier-Stokes system, when the viscosity coefficients are density dependent functions related by the Bresch-Desjardins relation [1] [2], for any "physical" adiabatic exponent $\gamma > 1$.

In the following, we extend the work of Mellet and Vasseur [5] to the Navier-Stokes-Poisson system, when the viscosity coefficients are density dependent functions also related by the Bresch-Desjardins relation (see below) and the pressure P satisfies (2). In other words, we prove that the Cauchy problem for the Navier-Stokes-Poisson system, in the framework of [5], admits a global weak solution, provided that $\gamma > \frac{4}{3}$.

2 Main result

Solving Poisson equation leads to an integro-differential system equivalent to (1)

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho \vec{v}) = 0, \\ \partial_t(\rho\vec{v}) + \mathrm{div}(\rho\vec{v}\otimes\vec{v}) - \mathrm{div}\,(2\mu(\rho)D(\vec{v})) - \nabla\left(\lambda(\rho)\mathrm{div}\vec{v}\right) + \nabla P - 4\pi G\rho\nabla\Phi = \vec{0}, \end{cases} \tag{3}$$

where

$$\Phi(x,t) := \int_{\mathbb{R}^3} \frac{\rho(y,t)}{|x-y|}\, dy, \tag{4}$$

for $t > 0$, with initial conditions

$$\rho|_{t=0} = \rho^0(x), \quad \rho\vec{v}|_{t=0} = \vec{m}^0(x), \qquad \text{on } \mathbb{R}^3. \tag{5}$$

Following Mellet and Vasseur [5], we suppose that μ and λ satisfy the Bresch-Desjardins relations [1]

$$\mu(\rho) = \mu_1\Psi(\rho), \quad \lambda(\rho) = 2\mu_1\left(\rho\Psi'(\rho) - \Psi(\rho)\right), \tag{6}$$

where Ψ is an increasing function of ρ, with the constraints introduced in [5]

$$\begin{cases} \mu'(\rho) \geqslant m, \quad \mu(0) \geqslant 0 \\ |\lambda'(\rho)| \leqslant \dfrac{1}{\nu}\, \mu'(\rho), \\ \nu\mu(\rho) \leqslant 2\mu(\rho) + 3\lambda(\rho) \leqslant \dfrac{1}{\nu}\, \mu(\rho), \end{cases} \tag{7}$$

for a suitable positive constant ν.

If $\gamma \geqslant 3$, we also require that

$$\liminf_{\rho\to\infty} \frac{\mu(\rho)}{\rho^{\frac{\gamma}{3}+\varepsilon}} > 0, \tag{8}$$

for some $\varepsilon > 0$.

We also define the auxiliary functions ϕ and Π

$$\rho\phi'(\rho) = \Psi'(\rho), \quad \Pi(\rho) := \rho\int_1^\rho \frac{P(z)}{z^2}\, dz.$$

The pair (ρ, v) is said to be a weak solution of (3) (5) on $\mathbb{R}^3 \times [0,T]$ if

$$\begin{cases} \rho \in L^\infty\left(0,T); L^1(\mathbb{R}^3)\cap L^\gamma(\mathbb{R}^3)\right), \\ \sqrt{\rho} \in L^\infty\left(0,T); H^1(\mathbb{R}^3)\right), \quad \sqrt{\rho}\,\vec{v} \in \left(L^\infty([0,T], L^2(\mathbb{R}^3))\right)^3, \\ \mu(\rho)\ D(\vec{v}) \in \left(L^2\left(0,T); W^{-1,1}_{loc}(\mathbb{R}^3)\right)\right)^9, \quad \lambda(\rho)\ \mathrm{div}\vec{v} \in \left(L^2\left(0,T); W^{-1,1}_{loc}(\mathbb{R}^3)\right)\right)^9, \end{cases} \tag{9}$$

with $\rho \geqslant 0$ and $(\rho, \sqrt{\rho}\ \vec{v})$ satisfying

$$\begin{cases} \partial_t\rho + \mathrm{div}(\sqrt{\rho}\sqrt{\rho}\vec{v}) = 0, \\ \rho(0,x) = \rho^0(x), \end{cases}$$

in the distributional sense.

Moreover for any test function $\vec{\eta} \in \left(C_c^2(\overline{\Omega_R} \times [0,T])\right)^3$ such that $\vec{\eta}(\cdot,T) = \vec{0}$, the following equality holds

$$\int_{\mathbb{R}^3} \vec{m}^0 \cdot \vec{\eta}(x,0)\; dx + \int_0^T \int_{\mathbb{R}^3} [\sqrt{\rho}(\sqrt{\rho}\vec{v}) \cdot \vec{\eta}_t + \sqrt{\rho}\vec{v} \otimes \sqrt{\rho}\vec{v} : \nabla\vec{\eta}]\; dx\; dt + \int_0^T \int_{\mathbb{R}^3} P(\rho)\; \mathrm{div}\; \vec{\eta}\; dx\; dt$$

$$+ \int_0^T \int_{\mathbb{R}^3} \frac{\mu(\rho)}{\sqrt{\rho}}\; \sqrt{\rho}v_j\; \partial_{ii}\eta_j\; dx\; dt + \int_0^T \int_{\mathbb{R}^3} 2\mu'(\rho)\sqrt{\rho}v_j\; \partial_i\sqrt{\rho}\; \partial_i\eta_j\; dx\; dt$$

$$+ \int_0^T \int_{\mathbb{R}^3} \frac{\mu(\rho)}{\sqrt{\rho}}\; \sqrt{\rho}v_i\; \partial_{ji}\eta_j\; dx\; dt + \int_0^T \int_{\mathbb{R}^3} 2\mu'(\rho)\sqrt{\rho}v_j\; \partial_i\sqrt{\rho}\; \partial_j\eta_j\; dx\; dt$$

$$+ \int_0^T \int_{\mathbb{R}^3} \frac{\lambda(\rho)}{\sqrt{\rho}}\; \sqrt{\rho}v_i\; \partial_{ij}\eta_j\; dx\; dt + \int_0^T \int_{\mathbb{R}^3} 2\lambda'(\rho)\sqrt{\rho}v_i\partial_i\sqrt{\rho}\; \partial_j\eta_j\; dx\; dt$$

$$= \int_0^T \int_{\mathbb{R}^3} 4\pi G\rho\nabla\Delta^{-1}\rho \cdot \vec{\eta}\; dx\; dt. \tag{10}$$

In this formula, the five last terms in the left-hand side are requested in order to provide a rigorous meaning to the viscous contribution in the equation of motion (see [5]), and the right hand side is a shortened notation for

$$\int_0^T \int_{\Omega} 4\pi G\rho\nabla \left(\int_{\mathbb{R}^3} \frac{\rho(y,t)}{|x-y|}\; dy \right) \cdot \vec{\eta}\; dx\; dt.$$

Then our main result is the following

Theorem 1 *Suppose that $\gamma > 4/3$ and that $\lambda(\rho)$ and $\mu(\rho)$ are two $C^2(\mathbb{R}_+)$ functions of ρ satisfying (6),(7) and (8).*

Assume that $(\rho_n, \vec{v}_n)_{n\in\mathbb{N}}$ is a sequence of weak solutions of (3) (5) satisfying energy-entropy inequalities (14), (18) and (25), with the following conditions on the initial data

$$\rho_n^0\big|_{t=0} = \rho_n^0(x), \quad \vec{m}_n^0\big|_{t=0} = m_n^0(x) = \rho_n^0(x)\vec{v}_n^0(x),$$

where ρ_n^0 and $\vec{v}_n^0$ satisfy

$$\rho_n^0 \geqslant 0, \quad \rho_n^0 \to \rho^0 \text{ in } L^1(\mathbb{R}^3), \quad \rho_n^0\vec{v}_n^0 \to \rho^0\; \vec{v}^0 \text{ in } L^1(\mathbb{R}^3), \tag{11}$$

and satisfy the bounds

$$\int_{\mathbb{R}^3} \frac{1}{2}\; \rho_n^0|\vec{v}_n^0|^2 + \Pi(\rho_n^0) - 4\pi G\Phi(\rho_n^0)\; dx < C, \quad \int_{\mathbb{R}^3} \frac{1}{\rho_n^0}\; |\nabla\mu(\rho_n^0)|^2\; dx < C, \tag{12}$$

and

$$\int_{\mathbb{R}^3} \frac{1}{2}\; \rho_n^0 \left(1 + |\vec{v}_n^0|^2\right)\; \log\left(1 + |\vec{v}_n^0|^2\right)\; dx < C. \tag{13}$$

Then, up to a subsequence, $(\rho_n, \sqrt{\rho_n}\vec{v}_n)$ converges strongly to a solution of the problem (3)(5) satisfying entropy inequalities (14), (18) and (25) in the following sense

$$\rho_n \to \rho \text{ in } C^0(0,T; L^{3/2}(\mathbb{R}^3)) \text{ strongly,}$$

$$\sqrt{\rho_n}\vec{v}_n \to \sqrt{\rho}\vec{v} \text{ in } L^2(0,T; L^2_{loc}(\mathbb{R}^3) \text{ strongly,}$$

$$\vec{m}_n = \rho_n\vec{v}_n \to \rho\vec{v} \text{ in } L^1(0,T; L^1_{loc}(\mathbb{R}^3) \text{ strongly,}$$

for any $T > 0$.

3 A priori estimates

Let us first recall the energy estimate

Lemma 1 *1. For any smooth solution of (3)*

$$\frac{d}{dt}\int_{\mathbb{R}^3}\left[\frac{1}{2}\ \rho|\vec{v}|^2+\Pi(\rho)-\frac{1}{2}\ G\rho\Phi(\rho)\right]\ dx+\int_{\mathbb{R}^3}2\mu(\rho)|D(\vec{v})|^2dx+\int_{\mathbb{R}^3}\lambda(\rho)(div\ \vec{v})^2dx=0, \tag{14}$$

where $\Pi(\rho)=\rho\int_1^\rho \frac{P(z)}{z^2}\ dz$.

2. If $\int_{\mathbb{R}^3}\left[\frac{1}{2}\ \rho^0|\vec{v}^0|^2+\Pi(\rho^0)-\frac{1}{2}\ G\rho^0\Phi(\rho^0)\right]\ dx<\infty$*, the following estimates hold*

$$\|\sqrt{\rho}\ \vec{v}\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\leqslant C, \|\rho\|_{L^\infty(0,T;L^\gamma(\mathbb{R}^3))}\leqslant C, \tag{15}$$

$$\|\sqrt{\mu(\rho)}\ D(\vec{v})\|_{L^2(0,T;L^2(\mathbb{R}^3))}\leqslant C, \tag{16}$$

$$\|\rho\|_{L^\infty(0,T;L^1(\mathbb{R}^3))}\leqslant C. \tag{17}$$

Proof:

1. Multiplying the momentum equation by $\vec{v}$ and integrating on $\mathbb{R}^3$ gives (14).
2. Integrating (2), we have first

$$P(z)\geqslant\frac{1}{a\gamma}\ z^\gamma-bz,$$

then

$$\Pi(\rho)\geqslant\frac{1}{a(\gamma-1)}\ (\rho^\gamma-\rho)-b\rho\log\rho.$$

Using Hardy-Littlewood-Sobolev inequality, we get also

$$\left|\int_{\mathbb{R}^3}\rho\Phi(\rho)\ dx\right|\leqslant C\|\rho(t,\cdot)\|^2_{L^{6/5}(\mathbb{R}^3)}\leqslant C\|\rho(t,\cdot)\|^{\frac{5\gamma-6}{3(\gamma-1)}}_{L^1(\mathbb{R}^3)}\ \|\rho(t,\cdot)\|^{\frac{\gamma}{3(\gamma-1)}}_{L^\gamma(\mathbb{R}^3)}.$$

Using the continuity equation we have the mass conservation, which gives (17).

Plugging all of these estimates into (14), we get

$$\begin{aligned}&\int_{\mathbb{R}^3}\left(\frac{1}{2}\ \rho|\vec{v}|^2+\frac{1}{a(\gamma-1)}\ \rho^\gamma\right)\ dx\\ &-\frac{1}{a(\gamma-1)}\int_{\mathbb{R}^3}\rho\ dx-b\int_{\mathbb{R}^3}\rho\log\rho\ dx-C\|\rho(t,\cdot)\|^{\frac{5\gamma-6}{3(\gamma-1)}}_{L^1(\mathbb{R}^3)}\ \|\rho(t,\cdot)\|^{\frac{\gamma}{3(\gamma-1)}}_{L^\gamma(\mathbb{R}^3)}\\ &+\frac{4}{3}\int_{\mathbb{R}^3}\mu(\rho)|D(\vec{v})|^2dx+\frac{1}{3}\int_{\mathbb{R}^3}(2\mu(\rho)+3\lambda(\rho))(\mathrm{div}\ \vec{v})^2dx\leqslant C.\end{aligned}$$

Using the previous estimates, all the negative contributions in the left-hand side can be absorbed in the pressure term, providing that $\gamma>4/3$ which implies (15) and (16) □

The following "gravitational version" of the Bresch-Desjardins lemma [1] holds

Lemma 2 *1. If λ and μ are two $C^2(\mathbb{R}_+)$ satisfying (6) and (7), the following identity holds for any smooth solution of (3)*

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^3}\left[\frac{1}{2}\ \rho|\vec{v}+\nabla\phi|^2+\Pi(\rho)-\frac{1}{2}\ G\rho\Phi(\rho)\right]\ dx+\int_{\mathbb{R}^3}\nabla\phi(\rho)\cdot\nabla P(\rho)\ dx\\ &+\int_{\mathbb{R}^3}\mu(\rho)\left|\nabla\vec{v}-^t\nabla\vec{v}\right|^2dx-G\int_{\mathbb{R}^3}\rho\nabla\phi\cdot\nabla\Phi\ dx=0,\end{aligned} \tag{18}$$

where $\phi':=2\frac{\mu'}{\rho}$.

2. If $\int_{\mathbb{R}^3}\left[\frac{1}{2}\ \left|\rho^0+\vec{v}^0\right|^2+\Pi(\rho^0)-\frac{1}{2}\ G\rho^0\Phi(\rho^0)\right]\ dx<\infty$*, and if* $\int_{\mathbb{R}^3}\rho^0|\nabla\phi(\rho^0)|^2dx<\infty$*, the following estimates hold*

$$\|\sqrt{\rho}\ \nabla\phi(\rho)\|_{L^\infty(0,T;L^2(\mathbb{R}^3))}\leqslant C, \tag{19}$$

$$\|\sqrt{\mu'(\rho)\rho^{\gamma-2}}\ \nabla\rho)\|_{L^2(0,T;L^2(\mathbb{R}^3))}\leqslant C, \tag{20}$$

$$\|\sqrt{\mu(\rho)}\ \vec{v}\|_{L^2(0,T;L^2(\mathbb{R}^3))}\leqslant C. \tag{21}$$

Proof:

1. Multiplying the continuity equation by $|\nabla\phi|^2$ (although this is formal, all the following manipulations may be made rigorous by standard regularization: see [1][2][5]), we have

$$\frac{d}{dt}\int_{\mathbb{R}^3}\frac{1}{2}\ \rho|\nabla\phi|^2\ dx=-\int_{\mathbb{R}^3}\rho\nabla\vec{v}:\nabla\phi\otimes\nabla\phi\ dx+\int_{\mathbb{R}^3}\rho^2\phi'\Delta\phi\ \mathrm{div}\vec{v}\ dx.$$
$$+\int_{\mathbb{R}^3}\rho|\nabla\phi|^2\ \mathrm{div}\vec{v}\ dx=0. \tag{22}$$

Now we observe that

$$\frac{d}{dt}\int_{\mathbb{R}^3}\rho\vec{v}\cdot\nabla\phi\ dx=\int_{\mathbb{R}^3}\nabla\phi\cdot\partial_t(\rho\vec{v})|\ dx+\int_{\mathbb{R}^3}\phi'(\mathrm{div}\rho\vec{v})^2\ dx. \tag{23}$$

Taking the scalar product of the momentum equation by $\nabla\phi$ and integrating by parts, we get

$$\int_{\mathbb{R}^3}\partial_t(\rho\vec{v})\cdot\nabla\phi\ dx$$
$$=-\int_{\mathbb{R}^3}(2\mu(\rho)+\lambda(\rho))\Delta\phi\mathrm{div}\vec{v}\ dx+2\int_{\mathbb{R}^3}\nabla\vec{v}:\nabla\phi\otimes\nabla\mu(\rho)\ dx-2\int_{\mathbb{R}^3}\nabla\phi\cdot\nabla\mu(\rho)\ \mathrm{div}\vec{v}\ dx$$
$$-\int_{\mathbb{R}^3}\nabla\phi\cdot\nabla P(\rho)\ dx-\int_{\mathbb{R}^3}\nabla\phi\ \mathrm{div}(\rho\vec{v}\otimes\vec{v})\ dx+G\int_{\mathbb{R}^3}\rho\nabla\phi\cdot\nabla\Phi\ dx. \tag{24}$$

Now from (22) and (23)

$$\frac{d}{dt}\int_{\mathbb{R}^3}\left[\frac{1}{2}\ \rho|\nabla\phi|^2+\rho\vec{v}\cdot\nabla\phi\right]\ dx+\int_{\mathbb{R}^3}\nabla\phi\cdot\nabla P(\rho)\ dx$$
$$=-\int_{\mathbb{R}^3}\nabla\phi\ \mathrm{div}(\rho\vec{v}\otimes\vec{v})\ dx+\int_{\mathbb{R}^3}\phi'(\mathrm{div}\rho\vec{v})^2\ dx+G\int_{\mathbb{R}^3}\rho\nabla\phi\cdot\nabla\Phi\ dx.$$

Simplifying the first two terms in the right-hand side as

$$-\int_{\mathbb{R}^3}\nabla\phi\ \mathrm{div}(\rho\vec{v}\otimes\vec{v})\ dx+\int_{\mathbb{R}^3}\phi'(\mathrm{div}\rho\vec{v})^2\ dx=\int_{\mathbb{R}^3}\lambda(\rho)(\mathrm{div}\vec{v})^2\ dx+\int_{\mathbb{R}^3}2\mu(\rho)\partial_jv_i\partial_iv_j\ dx,$$

we get

$$\frac{d}{dt}\int_{\mathbb{R}^3}\left[\frac{1}{2}\ \rho|\nabla\phi|^2+\rho\vec{v}\cdot\nabla\phi\right]\ dx+\int_{\mathbb{R}^3}\nabla\phi\cdot\nabla P(\rho)\ dx$$
$$-\int_{\mathbb{R}^3}\lambda(\rho)(\mathrm{div}\vec{v})^2\ dx-\int_{\mathbb{R}^3}2\mu(\rho)|D(\vec{v})|^2\ dx+\int_{\mathbb{R}^3}\frac{1}{2}\ \mu(\rho)(\partial_jv_i-\partial_iv_j)^2\ dx-G\int_{\mathbb{R}^3}\rho\nabla\phi\cdot\nabla\Phi\ dx=0.$$

Adding this equality to (14) gives (18).

2. In order to control all of the terms in (18) we first observe that, as in the energy equality, the first integral satisfies

$$\int_{\mathbb{R}^3}\left[\frac{1}{2}\ \rho|\vec{v}+\nabla\phi|^2+\Pi(\rho)-\frac{1}{2}\ G\rho\Phi(\rho)\right]\ dx\geqslant\int_{\mathbb{R}^3}\left[\frac{1}{2}\ \rho|\vec{v}+\nabla\phi|^2+\alpha\rho^\gamma\right]\ dx,$$

for a suitable positive α.

We have now after (2)

$$\int_{\mathbb{R}^3} \nabla\phi(\rho)\cdot\nabla P(\rho)\ dx = \int_{\mathbb{R}^3} \phi'(\rho)P'(\rho)|\nabla\rho|^2\ dx$$

$$\geqslant \frac{1}{a}\int_{\mathbb{R}^3} \phi'(\rho)\rho^{\gamma-1}|\nabla\rho|^2\ dx - b\int_{\mathbb{R}^3} \phi'(\rho)|\nabla\rho|^2\ dx.$$

Using (6) and (7), the last term is bounded as follows

$$b\int_{\mathbb{R}^3} \phi'(\rho)|\nabla\rho|^2\ dx = b\int_{\mathbb{R}^3} \rho|\nabla\phi|^2(2\mu'(\rho))^{-1}\ dx \leqslant \frac{b}{2m}\int_{\mathbb{R}^3} \rho|\nabla\phi|^2\ dx.$$

We have also, using Poisson equation

$$-G\int_{\mathbb{R}^3} \rho\nabla\phi\nabla\Phi\ dx = G\int_{\mathbb{R}^3} \rho\mu(\rho)\ dx.$$

Finally, integrating in t and plugging all of these bounds into (18), we get

$$\int_{\mathbb{R}^3}\left[\frac{1}{2}\ \rho|\vec{v}+\nabla\phi|^2+\alpha\rho^\gamma\right]\ dx + \int_0^T\int_{\mathbb{R}^3} \phi'(\rho)\rho^{\gamma-1}|\nabla\rho|^2\ dx\ dt$$

$$+\int_0^T\int_{\mathbb{R}^3}\mu(\rho)\left|\nabla\vec{v}-^t\nabla\vec{v}\right|^2 dx\ dt \leqslant C + C\int_0^T\int_{\mathbb{R}^3}\rho|\nabla\phi|^2\ dx\ dt.$$

Using Gronwall's lemma, we end with the inequality

$$\sup_{(0,T)}\int_{\mathbb{R}^3}\rho|\nabla\phi|^2\ dx + \int_0^T\int_{\mathbb{R}^3}\phi'(\rho)\rho^{\gamma-1})|\nabla\rho|^2\ dx\ dt + \int_0^T\int_{\mathbb{R}^3}\mu(\rho)\left|\nabla\vec{v}-^t\nabla\vec{v}\right|^2 dx\ dt \leqslant C,$$

which gives the estimates (20) and (20) and, together with (16), the last bound (21) □

Lemma 3 *If λ and μ are two $C^2(\mathbb{R}_+)$ satisfying (6) and (7), the following inequality holds for any smooth solution of (3)*

$$\frac{d}{dt}\int_{\mathbb{R}^3}\frac{1}{2}\ \rho\left(1+|\vec{v}|^2\right)\ \log\left(1+|\vec{v}|^2\right)\ dx + \frac{\nu}{2}\int_{\mathbb{R}^3}\mu(\rho)\ \log\left(1+|\vec{v}|^2\right)|D(\vec{v})|^2\ dx$$

$$\leqslant C\left(\int_{\mathbb{R}^3}\left(\frac{\rho^{-\frac{\delta}{2}}P^2}{\mu(\rho)}\right)^{\frac{2}{2-\delta}} dx\right)^{\frac{2-\delta}{2}}\times\left(\int_{\mathbb{R}^3}\rho\left[\log\left(1+|\vec{v}|^2\right)\right]^{\frac{2}{\delta}}\ dx\right)^{\frac{\delta}{2}}$$

$$+C\int_{\mathbb{R}^3}\mu(\rho)|\nabla\vec{v}|^2dx + C\left(\|\rho\|^{5/12}_{L^{3/2}(\mathbb{R}^3)}+\|\rho\|^{\frac{5(1-\epsilon)}{12(1+\epsilon}}_{L^{\frac{3(1-\epsilon)}{2-3\epsilon}}(\mathbb{R}^3)}\right)\ \|\sqrt{\rho}\ \vec{v}\|^{5/6}_{L^2(\mathbb{R}^3)}\ \|\rho\|_{L^2(\mathbb{R}^3)}, \qquad (25)$$

for any δ such that $0<\delta<2$.

Proof:

Multiplying the second equation (3) by $\log\left(1+|\vec{v}|^2\right)$ and using (7), we get

$$\frac{d}{dt}\int_{\mathbb{R}^3}\frac{1}{2}\ \rho\left(1+|\vec{v}|^2\right)\ \log\left(1+|\vec{v}|^2\right)\ dx + \int_{\mathbb{R}^3}\frac{1}{2}\ \rho\vec{v}\cdot\nabla\left[\left(1+|\vec{v}|^2\right)\ \log\left(1+|\vec{v}|^2\right)\right]\ dx$$

$$+\nu\int_{\mathbb{R}^3}\mu(\rho)\ \log\left(1+|\vec{v}|^2\right)|D(\vec{v})|^2\ dx \leqslant -\int_{\mathbb{R}^3}\left[1+\log\left(1+|\vec{v}|^2\right)\right]\vec{v}\cdot\nabla P\ dx$$

$$+C\int_{\mathbb{R}^3}\mu(\rho)|\nabla\vec{v}|^2dx + G\int_{\mathbb{R}^3}\rho\log\left(1+|\vec{v}|^2\right)\vec{v}\cdot\nabla\Phi\ dx =: I_1+I_2+I_3. \qquad (26)$$

First we note that, by multiplying the first equation (3) by $\frac{1}{2}\ \left(1+|\vec{v}|^2\right)\log\left(1+|\vec{v}|^2\right)$ and integrating by parts, the second contribution in the left-hand side of (26) rewrites

$$\int_{\mathbb{R}^3} \frac{1}{2}\ \rho\vec{v}\cdot\nabla\left[\left(1+|\vec{v}|^2\right)\ \log\left(1+|\vec{v}|^2\right)\right]\ dx = \int_{\mathbb{R}^3} \frac{1}{2}\ \left(1+|\vec{v}|^2\right)\log\left(1+|\vec{v}|^2\right)\partial_t\rho\ dx.$$

Let us estimate now the various terms I_j in the right-hand side of (26).

Using Cauchy-Schwarz

$$|I_1| \leqslant \left|\int_{\mathbb{R}^3} \frac{2v_iv_k}{1+|\vec{v}|^2}\ \partial_i v_k\ P\ dx\right| + \left|\int_{\mathbb{R}^3} \left[1+\log\left(1+|\vec{v}|^2\right)\right]\mathrm{div}\vec{v}\ P\ dx\right|$$

$$\leqslant 2\left(\int_{\mathbb{R}^3}\mu(\rho)|\nabla\vec{v}|^2\ dx\right)^{1/2}\left(\int_{\mathbb{R}^3}\frac{P^2}{\mu(\rho)}\ dx\right)^{1/2} + \left|\int_{\mathbb{R}^3}\left[1+\log\left(1+|\vec{v}|^2\right)\right]\mathrm{div}\vec{v}\ P\ dx\right|.$$

Bounding the last integral by using Cauchy-Schwarz and Young's inequalities, taking into account (2) and (7), we find

$$|I_1| \leqslant \int_{\mathbb{R}^3}\mu(\rho)|\nabla\vec{v}|^2dx + \frac{\nu}{2}\int_{\mathbb{R}^3}\mu(\rho)\left[1+\log\left(1+|\vec{v}|^2\right)\right]|D(\vec{v})|^2\ dx + C_\nu\int_{\mathbb{R}^3}\frac{P^2}{\mu(\rho)}\left[1+\log\left(1+|\vec{v}|^2\right)\right]\ dx,$$

where the last term, using Cauchy-Schwarz, is bounded by

$$\left(\int_{\mathbb{R}^3}\left(\frac{\rho^{-\frac{\delta}{2}}P^2}{\mu(\rho)}\right)^{\frac{2}{2-\delta}}dx\right)^{\frac{2-\delta}{2}} \times \left(\int_{\mathbb{R}^3}\rho\left[\log\left(1+|\vec{v}|^2\right)\right]^{\frac{2}{\delta}}\ dx\right)^{\frac{\delta}{2}}.$$

Observing that $\log\left(1+|\vec{v}|^2\right) \leqslant C_\epsilon\left(1+|\vec{v}|^\epsilon\right)$ for any $\epsilon\in(0,1)$, and using Hardy-Littlewood-Sobolev inequality, we have

$$|I_3| \leqslant C\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\rho(t,x)\left(1+|\vec{v}(t,x)|^\epsilon\right)|\vec{v}(t,x)|\rho(t,y)}{|x-y|^2}\ dy\ dx$$

$$\leqslant C\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\rho(t,x)|\vec{v}(t,x)|\rho(t,y)}{|x-y|^2}\ dy\ dx + C\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\rho(t,x)|\vec{v}(t,x)|^{1+\epsilon}\rho(t,y)}{|x-y|^2}\ dy\ dx$$

$$\leqslant C\|\rho\|^{5/12}_{L^{3/2}(\mathbb{R}^3)}\ \|\sqrt{\rho}\ \vec{v}\|^{5/6}_{L^2(\mathbb{R}^3)}\ \|\rho\|_{L^2(\mathbb{R}^3)} + C\|\rho\|^{\frac{5(1-\epsilon)}{12(1+\epsilon}}_{L^{\frac{3(1-\epsilon)}{2-3\epsilon}}(\mathbb{R}^3)}\ \|\sqrt{\rho}\ \vec{v}\|^{5/6}_{L^2(\mathbb{R}^3)}\ \|\rho\|_{L^2(\mathbb{R}^3)}.$$

Plugging the previous estmates into (26) ends the proof of Lemma 3 $\square$

4 Proof of Theorem 1

Suppose that the initial data $(\rho_n^0,\vec{v}_n^0)$ satisfy

$$\begin{cases} \rho_n^0 \geqslant 0 \text{ a.e. in } \mathbb{R}^3, \quad \rho_n^0 \text{ bounded in } L^1\cap L^\gamma(\mathbb{R}^3), \\ \rho_n^0|\vec{v}_n^0|^2 \text{ bounded in } L^1(\mathbb{R}^3), \\ \sqrt{\rho_n^0}\nabla\phi(\rho_n^0) \text{ bounded in } L^2(\mathbb{R}^3), \\ \rho_n^0|\vec{v}_n^0|^2\ \log\left(1+|\vec{v}_n^0|^2\right) \text{ bounded in } L^1(\mathbb{R}^3). \end{cases} \tag{27}$$

Using Lemmas 1 and 2, we get first

$$
\begin{cases}
\|\sqrt{\rho_n}\ \vec{v}_n\|_{L^\infty(0,T;L^2(\mathbb{R}^3)} \leqslant C, \\[2ex]
\|\rho_n\|_{L^\infty(0,T;L^1\cap L^\gamma(\mathbb{R}^3)} \leqslant C, \\[2ex]
\left\|\sqrt{\mu(\rho_n)}\ \nabla\vec{v}_n\right\|_{L^2(0,T;L^2(\mathbb{R}^3)} \leqslant C, \\[2ex]
\|\mu'(\rho_n)\ \nabla\sqrt{\rho_n}\|_{L^\infty(0,T;L^2(\mathbb{R}^3)} \leqslant C, \\[2ex]
\left\|\sqrt{\mu'(\rho_n)\rho_n^{\gamma/2}}\ \nabla\rho_n\right\|_{L^2(0,T;L^2(\mathbb{R}^3)} \leqslant C.
\end{cases}
\tag{28}
$$

Using (7) together with (28), we obtain finally

$$
\begin{cases}
\|\sqrt{\rho_n}\ \nabla\vec{v}_n\|_{L^2(0,T;L^2(\mathbb{R}^3)} \leqslant C, \\[2ex]
\|\nabla\rho_n\|_{L^\infty(0,T;L^2(\mathbb{R}^3)} \leqslant C, \\[2ex]
\left\|\nabla\rho_n^{\gamma/2}\right\|_{L^2(0,T;L^2(\mathbb{R}^3)} \leqslant C.
\end{cases}
\tag{29}
$$

The proof of Theorem 1 is then achieved, in the same way as in [5], by dividing it into several parts: showing first the convergence of the density and the pressure, then improving the convergence of $\rho_n\ |\vec{v}_n|^2$, proving the convergence of the momentum, showing the strong convergence of $\sqrt{\rho_n}\ \nabla\vec{v}_n$ and finally the convergence of diffusion terms.

We only sketch the proof hereafter, only stressing the specific roles played by gravitation and pressure, and we refer to [5] for details.

1) Convergence of $\sqrt{\rho_n}$.

Lemma 4 *If μ satisfies (7), up to an extracted subsequence*

$$\sqrt{\rho_n} \to \sqrt{\rho} \quad a.e.\ in\ (0,T)\times\mathbb{R}^3\ and\ strongly\ in\ L^2(0,T;L^2_{loc}(\mathbb{R}^3)).$$

Moreover $\rho_n \to \rho$ in $C^0(0,T;L^{3/2}(\mathbb{R}^3))$.

Proof: After the mass conservation and the second inequality (29), the sequence $\{\sqrt{\rho_n}\}$ is bounded in $L^\infty(0,T;H^1(\mathbb{R}^3))$ and, using the continuity equation, $\{\partial_t\sqrt{\rho_n}\}$ is bounded in $L^2(0,T;H^{-1}(\mathbb{R}^3))$. This implies, after Aubin-Lions lemma that $\{\sqrt{\rho_n}\}$ is bounded in $L^2(0,T;L^2_{loc}(\mathbb{R}^3))$. Next, using Sobolev embedding, $\{\sqrt{\rho_n}\}$ is also bounded in $L^2(0,T;L^6(\mathbb{R}^3))$ so, using the first bound (28), we see that the sequence $\{\sqrt{\rho_n}\vec{v}_n\}$ is bounded in $L^\infty(0,T;L^{3/2}(\mathbb{R}^3))$. Plugging into the continuity equation gives that$\{\partial_t\rho_n\}$ is bounded in $L^2(0,T;W^{-1,3/2}(\mathbb{R}^3))$. As $\{\nabla\rho_n\}$ is also bounded in $L^2(0,T;L^{3/2}(\mathbb{R}^3))$, we conclude that the sequence $\{\rho_n\}$ is compact in $C(0,T;L^{3/2}_{loc}(\mathbb{R}^3))$.

2) Convergence of the pressure and the gravitational force.

Lemma 5 *If $4/3 < \gamma$, the pressure $P(\rho_n)$ and the gravitational force $\rho_n\nabla\Phi(\rho_n)$ are bounded in $L^{5/3}((0,T)\times\mathbb{R}^3)$.*

In particular

$$P(\rho_n)\quad converges\ strongly\ to\ P(\rho)\quad in\ L^1_{loc}((0,T)\times\mathbb{R}^3),$$

and

$$\rho_n\nabla\Phi(\rho_n)\quad converges\ strongly\ to\ \rho\nabla\Phi(\rho)\quad in\ L^1_{loc}((0,T)\times\mathbb{R}^3).$$

Proof:

1. After (2)
$$|\nabla P(\rho_n)| \leqslant C\left(\nabla\rho_n^\gamma + \nabla\rho_n\right),$$
so the second inequality (28) and the third (29) give that $P(\rho_n)$ is bounded in $L^2(0,T;H^1(\mathbb{R}^3))$ which implies that $P(\rho_n)$ is also bounded in $L^2(0,T;L^6(\mathbb{R}^3))$. As it is also bounded in $L^\infty(0,T;L^1(\mathbb{R}^3))$ we get that, using Hölder inequality, it is also bounded in $L^{5/3}((0,T)\times\mathbb{R}^3)$, and, after Lemma 4, we conclude that $P(\rho_n)$ converges strongly to $P(\rho)$ in $L^1_{loc}((0,T)\times\mathbb{R}^3)$.

2. After Hölder's inequality, we have
$$\|\rho_n\nabla\Phi(\rho_n)\|_{5/3} \leqslant \|\rho_n\|_q\|\nabla\Phi(\rho_n)\|_r,$$
for $\frac{1}{q}+\frac{1}{r}=\frac{3}{5}$, and after Sobolev theorem applied to the weakly singular operator $\nabla\Phi$, we see that
$$\|\nabla\Phi(\rho_n)\|_r \leqslant C(q,r) \leqslant \|\rho_n\|_q,$$
provided that $\frac{1}{q}=\frac{1}{r}-\frac{1}{3}$. We get then the constraint $q=\frac{30}{14}$ which is allowed, as we know that ρ_n^γ is bounded in $L^{5/3}((0,T)\times\mathbb{R}^3)$. This implies that $\rho_n\nabla\Phi(\rho_n)$ is bounded in $L^{5/3}((0,T)\times\mathbb{R}^3)$, and after Lemma 4, we conclude that $\rho_n\nabla\Phi(\rho_n)$ converges strongly to $\rho\nabla\Phi(\rho)$ in $L^1_{loc}((0,T)\times\mathbb{R}^3)$.

3) Convergence of the momentum.

Lemma 6 *Up to an extracted subsequence*
$$\rho_n\vec{v}_n \to m \quad \text{a.e. in } (0,T)\times\mathbb{R}^3 \text{ and strongly in } L^2(0,T;L^p_{loc}(\mathbb{R}^3)),$$
for any $1\leqslant p<3/2$.

Proof:

1. We first need to bound $\sqrt{\rho_n}\,\vec{v}_n$.

   Using Lemma 2, we have
$$\frac{d}{dt}\int_{\mathbb{R}^3}\frac{1}{2}\,\rho_n\left(1+|\vec{v}_n|^2\right)\,\log\left(1+|\vec{v}_n|^2\right)\,dx+\frac{\nu}{2}\int_{\mathbb{R}^3}\mu(\rho_n)\,\log\left(1+|\vec{v}_n|^2\right)|D(\vec{v}_n)|^2\,dx$$
$$\leqslant C\left(\int_{\mathbb{R}^3}\left(\frac{\rho_n^{-\frac{\delta}{2}}P^2(\rho_n)}{\mu(\rho_n)}\right)^{\frac{2}{2-\delta}}dx\right)^{\frac{2-\delta}{2}}\times\left(\int_{\mathbb{R}^3}\rho_n\left[\log\left(1+|\vec{v}_n|^2\right)\right]^{\frac{2}{\delta}}\,dx\right)^{\frac{\delta}{2}}+C\int_{\mathbb{R}^3}\mu(\rho_n)|\nabla\vec{v}_n|^2dx$$
$$+C\left(\|\rho_n\|^{5/12}_{L^{3/2}(\mathbb{R}^3)}+\|\rho_n\|^{\frac{5(1-\epsilon)}{12(1+\epsilon}}_{L^{\frac{3(1-\epsilon)}{2-3\epsilon}}(\mathbb{R}^3)}\right)\|\sqrt{\rho_n}\,\vec{v}_n\|^{5/6}_{L^2(\mathbb{R}^3)}\,\|\rho_n\|_{L^2(\mathbb{R}^3)}, \tag{30}$$
for any δ such that $0<\delta<2$.

   After Lemma 2 (as $\gamma>4/3$) and the first estimate (28), the last contribution is bounded provided that ϵ is small enough; moreover, using the energy bounds together with (7), inequality (30) implies the following
$$\frac{d}{dt}\int_{\mathbb{R}^3}\frac{1}{2}\,\rho_n\left(1+|\vec{v}_n|^2\right)\,\log\left(1+|\vec{v}_n|^2\right)\,dx\leqslant C+C\left(\int_{\mathbb{R}^3}\left(\frac{\rho_n^{-\frac{\delta}{2}}P^2(\rho_n)}{\mu(\rho_n)}\right)^{\frac{2}{2-\delta}}dx\right)^{\frac{2-\delta}{2}}.$$

   Now, provided $2\gamma-1<\frac{5\gamma}{3}$ (i.e. $\gamma<3$), the right-hand side is in $L^1(0,T)$. One checks finally that, as in [5], the same conclusion holds when $\gamma\geqslant 3$, provided that the hypothesis (8) is valid, and we end with the required bound
$$\rho_n|\vec{v}_n|^2\,\log\left(1+|\vec{v}_n|^2\right)\text{ is bounded in } L^\infty(0,T;L^1(\mathbb{R}^3)). \tag{31}$$

2. Using the identity

$$\partial_i(\rho_n v_{nj}) = \sqrt{\rho_n}\sqrt{\rho_n}\ \partial_i(v_{nj}) + 2\sqrt{\rho_n}\ v_{nj}\ \partial_i(\sqrt{\rho_n}),$$

we see that the second contribution in the right-hand side is bounded in $L^\infty(0,T;L^1(\mathbb{R}^3))$ after (31). As the first one is bounded in $L^2(0,T;L^p(\mathbb{R}^3))$ for any $1 \leqslant p \leqslant 3/2$, we find that $\nabla(\rho_n\vec{v}_n)$ is bounded in $L^2(0,T;L^1(\mathbb{R}^3))$, then $\rho_n\vec{v}_n$ is bounded in $L^2(0,T;W^{1,1}(\mathbb{R}^3))$.

Now the proof is achieved if one shows that $\partial_t(\rho_n\vec{v}_n)$ is bounded in $L^2(0,T;W^{-2,4/3}(K))$, for any compact $K \subset \mathbb{R}^3$, which can be proved by using the technique of Mellet and Vasseur (see Lemma 4.4 of [5]). We omit the proof.

The last two points to check in order to complete the proof of Theorem 1 are the convergence of $\sqrt{\rho_n}\ \vec{v}_n$ the diffusion terms. As the proof of Mellet and Vasseur applies verbatim, we only quote their result for completeness (see Lemmas 4.6 and 4.7 in [5])

Lemma 7 *1. We have*

$$\sqrt{\rho_n}\ \vec{v}_n \to \frac{\vec{m}}{\sqrt{\rho}} \text{ strongly in } L^2((0,T)\times\mathbb{R}^3),$$

In particular: $\vec{m}(x,t) = 0$ a.e. on $\{\rho(x,t) = 0\}$ and there is a function $\vec{v}(x,t)$ such that $\vec{m}(x,t) = \rho(x,t)\vec{v}(x,t)$ and

$$\rho_n\ \vec{v}_n \to \rho\vec{v} \text{ strongly in } L^2(0,T;L^p_{loc})(\mathbb{R}^3)), \quad 1 \leqslant p < 3/2,$$

$$\sqrt{\rho_n}\ \vec{v}_n \to \sqrt{\rho}\ \vec{v} \text{ strongly in } L^2_{loc}((0,T)\times\mathbb{R}^3).$$

2. We have

$$\mu(\rho_n)D(\vec{v}_n) \to \mu(\rho)D(\vec{v}) \text{ in } \mathcal{D}'(\mathbb{R}^3),$$

and

$$\lambda(\rho_n)\,div(\vec{v}_n) \to \lambda(\rho)\,div(\vec{v}) \text{ in } \mathcal{D}'(\mathbb{R}^3).$$

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