New perspectives in fluid dynamics: Mathematical analysis of a model proposed by Howard Brenner

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1 Introduction

1.1 Field equations

In a series of papers [3], [4], [5], Howard Brenner proposed a daring new approach to continuum fluid mechanics based on the concept of two different velocities: the mass-based (Eulerian) *mass velocity* $\mathbf{v}_m$ derived from the classical notion of mass transport, and the fluid-based (Lagrangian) *volume velocity* $\mathbf{v}$ associated to the motion of individual particles (molecules). According to overwhelming majority of standard works and research studies on continuum fluid mechanics, these two velocities are implicitly assumed to be one and the same entity (see [3]). This point of view, remaining unchallenged from the time of Euler, led to the nowadays classical mathematical theory of fluid mechanics based on the *Navier-Stokes-Fourier* system of partial differential equations. Brenner argues that, in general, $\mathbf{v} \neq \mathbf{v}_m$, this inequality being significant for compressible fluids with high density gradients. He provides a number of purely theoretical as well as experimental arguments in support of his theory involving: (i) Öttinger's generic theory [19] of non-equilibrium irreversible processes; (ii) Klimontovich's molecularly based theory [14] of rarefied gases; (iii) existing thermophoretic and diffusiophoretic experimental data (see [3]).

At the level of mathematical modeling, Brenner's modification of the standard Navier-Stokes-Fourier system is significantly simpler than the alternatives provided by the theories of extended thermodynamics. In the absence of external body forces and heat sources, a mathematical description of a single-component fluid is based on the following trio of balance laws:

Brenner-Navier-Stokes-Fourier (BNSF) System:

Mass Conservation (continuity equation):

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{v}_m) = 0; \tag{1.1}$$

Balance of Linear Momentum:

$$\partial_t(\varrho \mathbf{v}) + \mathrm{div}_x(\varrho \mathbf{v} \otimes \mathbf{v}_m) + \nabla_x p = \mathrm{div}_x \mathbb{S}; \tag{1.2}$$

Total Energy Conservation:

$$\partial_t \left(\varrho(\frac{1}{2}|\mathbf{v}|^2 + e) \right) + \mathrm{div}_x \left(\varrho(\frac{1}{2}|\mathbf{v}|^2 + e)\mathbf{v}_m \right) \tag{1.3}$$
$$+\mathrm{div}_x(p\mathbf{v}) + \mathrm{div}_x \mathbf{q} = \mathrm{div}_x(\mathbb{S}\mathbf{v}),$$

where ϱ is the *mass density*, p is the *pressure*, e the specific *internal energy*, $\mathbb{S}$ the *viscous stress* tensor, and $\mathbf{q}$ stands for the internal energy flux. In addition, we assume that the fluid occupies a bounded (regular) domain $\Omega \subset R^3$ so that all quantities depend on the time $t \in [0,T]$, and the spatial position $x \in \Omega$.

1.2 Constitutive relations

A constitutive equation relating $\mathbf{v}_m$ to $\mathbf{v}$ is a corner stone of Brenner's approach. After a thorough discussion (see [3] - [5]), Brenner proposes a universal constitutive equation in the form:

$(\mathbf{v} - \mathbf{v}_m)$ - Constitutive Relation:

$$\mathbf{v} - \mathbf{v}_m = K \nabla_x \log(\varrho), \tag{1.4}$$

where $K \geq 0$ is a purely *phenomenological* coefficient. A specific relation of K to other thermodynamic quantities is open to discussion. Note that Brenner's original hypothesis $K = \frac{\kappa}{c_p \varrho}$, with κ the heat conductivity coefficient and c_p the specific heat at constant volume, has been tested and subsequently modified by several authors (see Greenshields and Reese [12]). In the incompressible regime, namely when $\varrho = \mathrm{const}$, the two velocities coincide converting (1.1 - 1.3) to the conventional Navier-Stokes-Fourier system describing the motion of an "incompressible" fluid.

The specific form of the remaining constitutive relations is determined, to a certain extent, by Second law of thermodynamics. In accordance with the fundamental principles of statistical physics (see Gallavotti [11]), the pressure $p = p(\varrho, \vartheta)$ as well as the internal energy $e = e(\varrho, \vartheta)$ are numerical functions of the density ϱ and the *absolute temperature* ϑ interrelated through

GIBBS' EQUATION:

$$\vartheta Ds(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta) D\left(\frac{1}{\varrho}\right), \tag{1.5}$$

where the symbol $s = s(\varrho, \vartheta)$ denotes the *specific entropy*.

Equation (1.2) multiplied on $\mathbf{v}$ provides, by help of (1.1), the balance of kinetic energy

$$\partial_t \left(\frac{1}{2}\varrho|\mathbf{v}|^2\right) + \mathrm{div}_x \left(\frac{1}{2}\varrho|\mathbf{v}|^2\mathbf{v}_m\right) + \mathrm{div}_x(p\mathbf{v}) = \mathrm{div}_x(\mathbb{S}\mathbf{v}) + p\mathrm{div}_x\mathbf{v} - \mathbb{S} : \nabla_x\mathbf{v}, \tag{1.6}$$

which can be subtracted from (1.3) in order to obtain the balance of internal energy in the form

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e\mathbf{v}_m) + \nabla_x\mathbf{q} = \mathbb{S} : \nabla_x\mathbf{v} - p\mathrm{div}_x\mathbf{v}. \tag{1.7}$$

For the sake of simplicity, we shall assume that

$$p(\varrho, \vartheta) = \underbrace{p_e(\varrho)}_{\text{elastic pressure}} + \underbrace{\vartheta p_t(\varrho)}_{\text{thermal pressure}}. \tag{1.8}$$

Albeit rather restrictive, formula (1.8) still includes the physically relevant case of *perfect gas*, where $p_e = 0$, $p_t = \mathrm{R}\varrho$. The reader may consult the book by Bridgeman [8] concerning general state equations in the form (1.8).

In accordance with Gibbs' equation (1.5), the internal energy splits into two parts:

$$e(\varrho, \vartheta) = \underbrace{e_e(\varrho)}_{\text{elastic energy}} + \underbrace{e_t(\vartheta)}_{\text{thermal energy}}, \text{ where } e_e(\varrho) = \int_1^\varrho \frac{p_e(z)}{z^2}\, \mathrm{d}z. \tag{1.9}$$

Simplifying again we take

$$e_t(\vartheta) = c_v\vartheta, \tag{1.10}$$

where $c_v > 0$ is the specific heat at constant volume. Accordingly, we have

$$s(\varrho, \vartheta) = c_v \log(\vartheta) - \int_1^\varrho \frac{p_t(z)}{z^2}\, \mathrm{d}z. \tag{1.11}$$

Thus, after a simple manipulation, we deduce from (1.7) the thermal energy balance

$$c_v \left(\partial_t(\varrho\vartheta) + \mathrm{div}_x(\varrho\vartheta\mathbf{v}_m)\right) + \mathrm{div}_x \left(\mathbf{q} + Kp_e(\varrho)\nabla_x \log(\varrho)\right) \tag{1.12}$$

$$= \mathbb{S} : \nabla_x \mathbf{v} + K \frac{p_e'(\varrho)}{\varrho} |\nabla_x \varrho|^2 - \vartheta p_t(\varrho) \mathrm{div}_x \mathbf{v}.$$

Moreover, we suppose the *heat flux* obeys Fourier's law, specifically,

FOURIER'S LAW:

$$\mathbf{q} + K p_e(\varrho) \nabla_x \log(\varrho) = -\kappa \nabla_x \vartheta, \tag{1.13}$$

where κ is the heat conductivity coefficient.

Finally, dividing (1.12) by ϑ yields the *entropy balance*

$$\partial_t (\varrho s) + \mathrm{div}_x (\varrho s \mathbf{v}_m) - \mathrm{div}_x \left(\frac{\kappa}{\vartheta} \nabla_x \vartheta - K p_t(\varrho) \nabla_x \log(\varrho) \right) \tag{1.14}$$
$$= \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{v} + K \frac{p_e'(\varrho)}{\varrho} |\nabla_x \varrho|^2 + \frac{\kappa}{\vartheta} |\nabla_x \vartheta|^2 + K \frac{p_t'(\varrho) \vartheta}{\varrho} |\nabla_x \varrho|^2 \right).$$

By virtue of Second law of thermodynamics, the quantity on the right-hand side of (1.14) representing the *entropy production rate* must be non-negative for any admissible physical process. Accordingly, and in sharp contrast to the standard theory, it is the velocity $\mathbf{v}$ rather than $\mathbf{v}_m$ that must appear in the rheological law for the viscous stress. For a linearly viscous (Newtonian) fluid such a stipulation yields:

NEWTON'S RHEOLOGICAL LAW:

$$\mathbb{S} = \mu \left(\nabla_x \mathbf{v} + \nabla_x^T \mathbf{v} - \frac{2}{3} \mathrm{div}_x \mathbf{v} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{v} \, \mathbb{I}, \tag{1.15}$$

where $\mu \geq 0$ and $\eta \geq 0$ stand for the shear and bulk viscosity coefficients, respectively. By the same token, the quantities K, κ, p_e', p_t' must be non-negative.

1.3 Boundary conditions

Another inovative aspect of Brenner's theory is the claim that it is the volume velocity $\mathbf{v}$ rather than $\mathbf{v}_m$ that should be considered in the otherwise well-accepted *no-slip* boundary condition for viscous fluids

$$\mathbf{v}|_{\partial\Omega} = 0 \tag{1.16}$$

(cf. Brenner [5]). On the other hand, the standard *impermeability* condition hypothesis keeps its usual mass-based form

$$\mathbf{v}_m \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{1.17}$$

where $\mathbf{n}$ stands for the outer normal vector, or, equivalently,

$$\nabla_x \varrho \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{1.18}$$

As we will see below, such a stipulation is in perfect agreement with the variational formulation of the problem in the spirit of the modern theory of partial differential equations.

Finally, we focus in this study on energetically closed systems, in particular,

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0 \tag{1.19}$$

yielding, in view of (1.13), (1.18),

$$\nabla_x \vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{1.20}$$

1.4 Mathematics of Brenner's model

Besides the necessity of experimental evidence, an ultimate criterion of validity of any mathematical model is its solvability in the framework of physically relevant data. The main goal of the present study is to show that Brenner's model provides a very attractive alternative to both the classical Navier-Stokes-Fourier system and the mathematically almost untractable problems provided by extended thermodynamics. In particular, we establish *a priori* estimates for solutions of BNSF system strong enough in order to ensure the property of *weak sequential stability.* That means any sequence of solutions bounded by *a priori* estimates possesses a subsequence converging weakly to a (weak) solution of the same problem. This is a remarkable property that allows us to develop a rigorous *existence theory* for the evolutionary problem (1.1 - 1.4), without any restriction imposed on the size of the data and the length of the time interval. Such a theory can be viewed as a counterpart of the seminal work of Leray [16] devoted to the classical incompressible Navier-Stokes system.

It is worth-noting here that, despite the concerted effort of generations of mathematicians, the weak solutions identified by Leray [16], Hopf [13], and Ladyzhenskaya [15] provide the only available framework, where global-in-time existence for the (standard) incompressible Navier-Stokes model can be rigorously verified for any choice of (large) data. A comparable theory for the compressible isentropic fluids was developed by P.-L.Lions [17] and later extended to a specific class of solutions for the full Navier-Stokes-Fourier system (see [9], [10]), where the energy equation (1.3) is replaced by an entropy or thermal energy inequality supplemented with an integrated total energy balance.

Rigorous (large data) existence results for the classical Navier-Stokes-Fourier system, meaning system (1.1 - 1.3) with $\mathbf{v} = \mathbf{v}_m$ are in short supply. Quite recently, Bresch and Desjardins [6], [7] discovered an interesting integral identity yielding *a priori* bounds on the density gradient and the property of weak sequential stability for system (1.1 - 1.3) provided the viscosity coefficients μ and η depend on the density ϱ in a specific way and the pressure p is given through formula (1.8), where the elastic component p_e is assumed to be singular for the density ϱ approaching zero.

The idea that a density dependent *bulk* viscosity coefficient η may actually provide better *a priori* estimates goes back to the remarkable study by Vaigant and Kazhikhov [20]. To the best of our knowledge, this is the only result where the authors establish global existence of *smooth* solutions although conditioned by the 2-D periodic geometry of the physical space and rather unrealistic hypotheses concerning the viscosity coefficients. A suitable functional relation satisfied by the viscosity coefficients $\mu = \mu(\varrho)$, $\eta = \eta(\varrho)$ was also used in [18] to prove global existence for the isentropic Navier-Stokes system with a general pressure law.

The main stumbling blocks encountered in the mathematical theory of compressible fluids based on the classical Navier-Stokes-Fourier system that have been identified in the previously cited studies can be summarized as follows:

- absence of *uniform bounds* on the density ϱ, in particular, the hypothetical possibility of concentration phenomena in the pressure term;
- a possibility of appearance of *vacuum regions*, meaning sets of positive measure on which $\varrho = 0$, even in the situation when strict positivity is imposed on the initial density distribution;
- low regularity of the velocity field allowing for development of uncontrollable *oscillations* experienced by the transported quantities, in particular, the density.

Quite remarkably, Brenner's modifications of the Navier-Stokes-Fourier system offer a new inside and at least a partial remedy to each of the issues listed above. The principal features of this new approach based on (BNSF) system read as follows:

- The new model provides a relatively simple and rather transparent modification of the classical system replacing the Eulerian mass velocity $\mathbf{v}_m$ by its volume counterpart $\mathbf{v}$ in the viscous stress tensor and the specific momentum, where $\mathbf{v}_m$ and $\mathbf{v}$ are interrelated through formula (1.4). The two velocities coincide in the "incompressible" regime when $\varrho = \text{const}$.
- The model conveniently unifies the principles of statistical mechanics with thermodynamics of continuum models of large multiparticle systems, in particular, it is consistent with First and Second laws of thermodynamics.
- The associated mathematical theory developed below allows for a rather general class of state equations, in particular and unlike all comparable results for the classical Navier-Stokes-Fourier system, the perfect gas state equation expressed through Boyle-Marriot's law can be handled.
- Weak solutions of (BNSF) system do not contain vacuum zones for positive times, the time-space Lebesgue measure of the set where ϱ vanishes is zero.
- Possible oscillations of the density as well as other fields are effectively damped by diffusion, the weak stability property is preserved even under non-isotropic perturbations of the transport terms.

Outline of the paper

Mathematical theory of (BNSF) system developed in this study is based on the standard framework of Sobolev spaces. Similarly to any *non-linear* problem, the class of function spaces is determined by means of the available *a priori* estimates established in Section 2. A remarkable new feature of the present setting is that the continuity equation (1.1) can be rewritten in terms of the volume velocity as

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{v}) = \mathrm{div}_x(K \nabla_x \varrho). \tag{1.21}$$

Equation (1.21) is *parabolic* yielding strong *a priori* estimates on ϱ provided $\mathbf{v}$ enjoys certain smoothness. On the other hand, the standard energy and gradient estimates for the velocity $\mathbf{v}$ and the temperature ϑ are obtained on the basis of entropy equation (1.14).

Section 3 develops refined estimates on $\mathbf{v}$ and ϑ necessary in order to establish equi-integrability of the fluxes appearing in the total energy balance (1.3). We point out that the bounds obtained for the volume velocity $\mathbf{v}$ are actually *better* than those for the velocity field considered in the standard incompressible Navier-Stokes system because we control the pressure in a rather strong L^2-norm.

Section 4 discusses the issue of *weak sequential stability.* It is shown that any sequence of (regular) solutions bounded by *a priori* bounds established in the previous part converges weakly to a distributional solution of the same problem. The most delicate task here is to control possible *concentrations* rather than *oscillations* of the weakly converging fields.

Finally, Section 5 proposes an approximation scheme analogous to that developed in [9] in order to establish a rigorous existence result for the corresponding initial-boundary value problem without any essential restrictions imposed on the size of the initial data and the length of the time interval.

2 A priori estimates

A priori estimates are natural bounds imposed on a family of solutions to a system of partial differential equations by the data, boundary conditions, and other parameters as the case may be. When deriving *a priori* estimates it is customary to assume that all quantities appearing in the equations are as smooth as necessary unless such a stipulation violated some obvious physical principles. In order to fix ideas, and in addition to the hypotheses discussed in Section 1, we suppose throughout the whole text the following technical assumptions:

HYPOTHESES:

(A 2.1) The phenomenological coefficient K introduced in (1.4) is a positive constant, say, $K \equiv 1$.
(A 2.2) The pressure p obeys the classical Boyle-Marriot law:

$$p(\varrho, \vartheta) = \mathrm{R}\varrho\vartheta,$$

with a positive constant R.
(A 2.3) The specific internal energy e satisfies

$$e = c_v \vartheta,$$

with a positive constant c_v.
(A 2.4) The viscosity coefficients are constant satisfying

$$\mu > 0, \ \eta = 0.$$

(A 2.5) The heat conductivity coefficient κ depends on the temperature, specifically,

$$\kappa(\vartheta) = \kappa_0(1 + \vartheta^3),$$

with a positive constant κ_0.

While hypotheses (A2.2 - A2.4) have obvious physical interpretation, hypotheses (A2.1), (A2.5) are of technical nature facilitating analysis of the problem, in particular at the level of *a priori* estimates. Note however that an active discussion is going on concerning the appropriate value of the phenomenological coefficient K (see [12]), while (A 2.5) is physically relevant as radiation heat conductivity at least for large values of ϑ (see [21]).

2.1 A priori estimates based on mass and energy conservation

Equation (1.1), or, equivalently, (1.21), expresses the physical principle of mass conservation. To begin, the standard maximum principle applies to the parabolic equation (1.21) yielding

$$\varrho(t, x) \geq 0 \text{ for all } t \in (0, T), \ x \in \Omega \tag{2.1}$$

provided

$$\varrho(0, \cdot) \equiv \varrho_0 \geq 0 \text{ in } \Omega. \tag{2.2}$$

Moreover, integrating (1.21) over Ω gives rise to

$$\int_\Omega \varrho(t, \cdot) \, \mathrm{d}x = \int_\Omega \varrho_0 \, \mathrm{d}x \text{ for all } t \in [0, T]$$

provided ϱ, $\mathbf{v}$ satisfy the boundary conditions (1.16), (1.18). As both ϱ_0 and ϱ are non-negative, we conclude that

$$\sup_{t\in(0,T)} \|\varrho(t,\cdot)\|_{L^1(\Omega)} = \int_\Omega \varrho_0 \ \mathrm{d}x \equiv M_0 \tag{2.3}$$

Similarly, the total energy balance equation (1.3) integrated over Ω yields

$$\int_\Omega \varrho \left(\frac{1}{2}|\mathbf{v}|^2 + c_v\vartheta \right)(t,\cdot) \ \mathrm{d}x = \int_\Omega \varrho_0 \left(\frac{1}{2}|\mathbf{v}_0|^2 + c_v\vartheta_0 \right) \ \mathrm{d}x \equiv E_0, \tag{2.4}$$

where $\mathbf{v}_0$ and ϑ_0 denote the initial distribution of the volume velocity and the temperature, respectively. Thus

$$\sup_{t\in(0,T)} \|\sqrt{\varrho}\mathbf{v}\|_{L^2(\Omega;R^3)} + \sup_{t\in(0,T)} \|\varrho\vartheta\|_{L^1(\Omega)} \le c(E_0). \tag{2.5}$$

Note that we have tacitly anticipated that the absolute temperature ϑ is a non-negative quantity. This stipulation is justified in the next section.

2.2 A priori estimates stemming from Second law of thermodynamics

Second law of thermodynamics is expressed through the entropy balance (1.14). In accordance with the boundary conditions (1.17), (1.18), the normal component of the entropy flux vanishes on the boundary; whence

$$\int_\Omega \varrho s(\tau,\cdot) \ \mathrm{d}x - \int_\Omega \varrho_0 s(\varrho_0,\vartheta_0) \ \mathrm{d}x \tag{2.6}$$

$$= \int_0^T \int_\Omega \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{v} + \kappa_0 \frac{1+\vartheta^3}{\vartheta} |\nabla_x\vartheta|^2 + K\frac{\vartheta}{\varrho}|\nabla_x\varrho|^2 \right) \ \mathrm{d}x \ \mathrm{d}t$$

Gibbs' equation (1.5) combined with hypotheses (A 2.2), (A 2.3) yields

$$s(\varrho,\vartheta) = c_v \log(\vartheta) - \mathrm{R}\log(\varrho),$$

in particular, since Ω is assumed to be bounded, the uniform bounds already established in (2.3), (2.5) imply that

$$\int_\Omega \varrho[\log(\vartheta)]^+ \ \mathrm{d}x - \int_\Omega \varrho[\log(\varrho)]^- \ \mathrm{d}x \le c(M_0,E_0) \text{ uniformly in } (0,T),$$

where we have denoted $[z]^+ = \max\{z,0\}$, $[z]^- = \min\{z,0\}$.

Consequently, we easily deduce from (2.6) that

$$\sup_{t\in(0,T)} \|\varrho\log(\varrho)\|_{L^1(\Omega)} + \sup_{t\in(0,T)} \|\varrho\log(\vartheta)\|_{L^1(\Omega)} \le c(M_0,E_0,S_0), \tag{2.7}$$

and

$$\int_0^T \int_\Omega \frac{1}{\vartheta}\mathbb{S} : \nabla_x \mathbf{v} \ \mathrm{d}x \ \mathrm{d}t \le c(M_0,E_0,S_0), \tag{2.8}$$

$$\|\nabla_x \log(\vartheta)\|_{L^2(0,T;L^2(\Omega;R^3)} + \|\nabla_x \vartheta^{3/2}\|_{L^2(0,T;L^2(\Omega;R^3))} \leq c(M_0, E_0, S_0), \quad (2.9)$$

$$\|\nabla_x \sqrt{\varrho}\|_{L^2(0,T;L^2(\Omega;R^3))} \leq c(M_0, E_0, S_0), \quad (2.10)$$

where we have denoted

$$S_0 = \int_\Omega \varrho_0 s(\varrho_0, \vartheta_0) \ \mathrm{d}x.$$

At this stage we need the following version of *Poincare's inequality*:

Lemma 2.1 *Let $\Omega \subset R^N$ be a bounded Lipschitz domain. Let $B \subset \Omega$ be a measurable set such that $|B| \geq m > 0$.*

Then we have

$$\|v\|_{W^{1,2}(\Omega;R^N)} \leq c(m,\beta) \left(\|\nabla_x v\|_{L^2(\Omega;R^N)} + \left(\int_B |v|^\beta \ \mathrm{d}x \right)^{1/\beta} \right)$$

for any $v \in W^{1,2}(\Omega)$, where the constant $c = c(m,\beta)$ depends solely on m and the parameter $\beta > 0$.

Our goal is to apply Lemma 2.1 first to $v = \vartheta^{3/2}$ and then $v = \log(\vartheta)$. To this end, we first show that there exist $\delta > 0$, $m > 0$ independent of $t \in (0,T)$ such that

$$|\{x \in \Omega \mid \varrho(t,x) > \delta\}| > m. \quad (2.11)$$

In order to see this, use (2.7) to deduce there exists $\alpha > 0$ such that

$$\int_{\{\varrho(t,\cdot) \geq \alpha\}} \varrho(t,\cdot) \ \mathrm{d}x \leq \frac{M_0}{3}$$

for any $t \in (0,T)$, where M_0 is the total mass of the fluid defined in (2.3). We fix $\delta = M_0/(3|\Omega|)$.

On the other hand, in accordance with (2.3),

$$M_0 = \int_\Omega \varrho(t,\cdot) \ \mathrm{d}x = \int_{\{\varrho(t,\cdot)\leq\delta\}} \varrho(t,\cdot) \ \mathrm{d}x + \int_{\{\delta<\varrho(t,\cdot)<\alpha\}} \varrho(t,\cdot) \ \mathrm{d}x + \int_{\{\varrho(t,\cdot)\geq\alpha\}} \varrho(t,\cdot) \ \mathrm{d}x$$

$$\leq \delta|\Omega| + \alpha|\{x \in \Omega \mid \varrho(t,x) > \delta\}| + \frac{M_0}{3};$$

whence we can take $m = M_0/3\alpha$. Note that the value of m, δ, derived on the basis of (2.3), (2.7), depend only on M_0, E_0, S_0.

Consequently, combining the uniform bounds established in (2.5), (2.9) with the conclusion of Lemma 2.1 we obtain

$$\|\vartheta^{3/2}\|_{L^2(0,T;W^{1,2}(\Omega;R^3))} \leq c(M_0, E_0, S_0). \quad (2.12)$$

Similarly, estimates (2.7), (2.9) yield

$$\|\log(\vartheta)\|_{L^2(0,T;W^{1,2}(\Omega;R^3))} \leq c(M_0, E_0, S_0), \quad (2.13)$$

and, finally, (2.3), (2.10) give rise to

$$\|\sqrt{\varrho}\|_{L^2(0,T;W^{1,2}(\Omega;R^3))} \leq c(M_0, E_0, S_0). \quad (2.14)$$

2.3 A priori bounds based on maximal regularity

A priori bounds established so far depend solely on the integral means M_0, E_0, S_0 representing the total amount of mass, energy, and entropy at the initial instant $t = 0$. In order to get more information, better summability of the initial data is necessary.

Equation (1.21) can be written in the form

$$\partial_t \varrho - \Delta \varrho = -\mathrm{div}_x(\varrho \mathbf{v}), \ \nabla_x \varrho \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \varrho(0, \cdot) = \varrho_0 \tag{2.15}$$

that can be viewed as a non-homogeneous linear parabolic equation, where, by virtue of (2.5), (2.14) combined with the standard imbedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$,

$$\|\varrho \mathbf{v}\|_{L^2(0,T;L^{3/2}(\Omega;R^3))} \leq c(M_0, E_0, S_0).$$

Now, we evoke the *maximal regularity estimates* applicable to the parabolic problem (2.15) (see Amann [1], [2]):

MAXIMAL REGULARITY ESTIMATES:

Proposition 2.1 *Let $\Omega \subset R^3$ be a regular bounded domain. Assume that*

$$f \in L^p(0, T; [W^{1,q'}(\Omega)]^*), \ 1 < p, q < \infty$$

is a given function.
Then problem

$$\partial_t r - \Delta r = f, \ \nabla_x r \cdot \mathbf{n}|_{\partial\Omega} = 0, \ r(0, \cdot) = 0$$

admits a unique weak solution r in the class

$$r \in L^p(0, T; W^{1,q}(\Omega)), \ \partial_t r \in L^p(0, T; [W^{1,q'}(\Omega)]^*),$$

$$r \in C\left([0, T]; \{[W^{1,q'}(\Omega)]^*; W^{1,q}(\Omega)\}_{1/p',p}\right),$$

where the symbol $\{;\}_{1/p',p}$ stands for the real interpolation space. Moreover,

$$\sup_{t \in [0,T]} \|r(t, \cdot)\|_{\{[W^{1,q'}(\Omega)]^*; W^{1,q}(\Omega)\}_{1/p',p}}$$

$$+\|\partial_t r\|_{L^p(0,T;[W^{1,q'}(\Omega)]^*)} + \|r\|_{L^p(0,T;W^{1,q}(\Omega))}$$

$$\leq \|f\|_{L^p(0,T;[W^{1,q'}(\Omega)]^*)}.$$

Thus, writing $\varrho = \varrho_1 + \varrho_2$, where ϱ_1 solves the homogeneous problem

$$\partial_t \varrho_1 - \Delta \varrho_1 = 0, \ \nabla_x \varrho_1 \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \varrho_1(0, \cdot) = \varrho_0,$$

while

$$\partial_t \varrho_2 - \Delta \varrho_2 = -\mathrm{div}_x(\varrho \mathbf{u}),\ \nabla_x \varrho_2 \cdot \mathbf{n}|_{\partial\Omega} = 0,\ \varrho_2(0,\cdot) = 0,$$

we obtain

$$\|\varrho_2\|_{L^2(0,T;W^{1,3/2}(\Omega))} \leq c\|\varrho \mathbf{v}\|_{L^2(0,T;L^{3/2}(\Omega))} \leq c(M_0, E_0, S_0), \tag{2.16}$$

and

$$\sup_{t\in(0,T)} \|\varrho_1(t,\cdot)\|_{L^3(\Omega)} \leq c\|\varrho_0\|_{L^3(\Omega)}.$$

On the other hand, as $W^{1,3/2}(\Omega) \hookrightarrow L^3(\Omega)$, we deduce, exactly as above,

$$\|\varrho \mathbf{v}\|_{L^4(0,T;L^{3/2}(\Omega;R^3))} \leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}).$$

Consequently, a simple iteration of the previous argument (bootstrap) yields, finally,

$$\|\varrho\|_{L^p(0,T;L^3(\Omega))} \leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}) \text{ for any } 1 \leq p < \infty. \tag{2.17}$$

At this stage, we are ready to exploit the thermal energy balance (1.12) in order to obtain uniform bounds on the volume velocity gradient. Indeed integrating (1.12) over Ω we easily deduce a uniform bound

$$\int_0^T \int_\Omega \mathbb{S} : \nabla_x \mathbf{v}\ \mathrm{d}x\ \mathrm{d}t \leq c$$

provided we are able to control the term

$$\int_0^T \int_\Omega \varrho\vartheta \mathrm{div}_x \mathbf{v}\ \mathrm{d}x\ \mathrm{d}t.$$

To this end, we evoke (2.12), (2.17), which, together with the imbedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, yield

$$\|\varrho\vartheta\|_{L^q(0,T;L^{9/4}(\Omega))} \leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, q) \text{ for any } 1 \leq q < 3. \tag{2.18}$$

On the other hand, the linear form associated to the viscous stress $\mathbb{S}$ satisfies a variant of Korn's inequality

$$\int_\Omega \mathbb{S} : \nabla_x \mathbf{v}\ \mathrm{d}x \geq c\|\nabla_x \mathbf{v}\|^2_{L^2(\Omega;R^{3\times 3})} \tag{2.19}$$

that can be easily verified by means of by parts integration since the volume velocity $\mathbf{v}$ vanishes on $\partial\Omega$.

Consequently, using (2.18), (2.19), and the standard Poincare inequality, we infer that

$$\|\mathbf{v}\|_{L^2(0,T;W^{1,2}(\Omega;R^3))} \leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}). \tag{2.20}$$

2.4 Positivity of the density

A remarkable feature of Brenner's model is the possibility to eliminate the regions with vanishing density. To this end, we multiply equation (2.15) on $1/\varrho$ to obtain

$$\partial_t \log(\varrho) - \Delta \log(\varrho) = |\nabla_x \log(\varrho)|^2 - \mathrm{div}_x \mathbf{v} - \mathbf{v} \cdot \nabla_x \log(\varrho). \tag{2.21}$$

It follows from (2.3) that

$$\sup_{t \in (0,T)} \|[\log(\varrho)(t,\cdot)]^+\|_{L^p(\Omega)} \leq c(M_0) \text{ for all } 1 \leq p < \infty; \tag{2.22}$$

whence integrating (2.21) over Ω yields

$$\sup_{t \in (0,T)} \|\log(\varrho)(t,\cdot)\|_{L^1(\Omega)} \leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\log(\varrho_0)\|_{L^1(\Omega)}), \tag{2.23}$$

$$\|\log(\varrho)\|_{L^2(0,T;W^{1,2}(\Omega))} \leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\log(\varrho_0)\|_{L^1(\Omega)}), \tag{2.24}$$

where we have used the uniform bound on the velocity field established in (2.20). In particular, we have

$$|\{x \in \Omega \mid \varrho(t,x) = 0\}| = 0 \text{ for any } t \in (0,T), \tag{2.25}$$

meaning, the vacuum zones, if any, have zero Lebesgue measure.

The lower bound on $\log(\varrho)$ can be improved by means of the classical comparison argument. Specifically, we deduce from (2.21) that

$$\log(\varrho) \geq V,$$

where V is a solution to the problem

$$\partial_t V - \Delta V = -\mathrm{div}_x \mathbf{v} - \frac{1}{2}|\mathbf{v}|^2, \ \nabla_x V \cdot \mathbf{n}|_{\partial\Omega} = 0, \ V(0,\cdot) = \log(\varrho_0), \tag{2.26}$$

with $\mathbf{v}$ fixed. Seeing that, by virtue of (2.20), the right-hand side $-\mathrm{div}_x \mathbf{v} - (1/2)|\mathbf{v}|^2$ is bounded in the space

$$L^2((0,T) \times \Omega) \oplus L^1(0,T; L^3(\Omega)),$$

we conclude, using the standard parabolic theory, that

$$\sup_{t \in (0,T)} \|V(t,\cdot)\|_{L^3(\Omega)} \leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\log(\varrho_0)\|_{L^3(\Omega)}),$$

which, together with (2.22), yields

$$\sup_{t \in (0,T)} \|\log(\varrho)(t,\cdot)\|_{L^3(\Omega)} \leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\log(\varrho_0)\|_{L^3(\Omega)}). \tag{2.27}$$

3 Refined velocity and temperature estimates

The *a priori* bounds derived in this section are quite non-standard and to a certain extent even better than those that can be obtained for the weak solutions of the classical incompressible Navier-Stokes system. This is due to the fact that we are able to control the pressure by means of estimate (2.18).

3.1 Refined estimates of the volume velocity

We start rewriting the momentum equation (1.2) in the form

$$\varrho\left(\partial_t \mathbf{v} + \mathbf{v}_m \cdot \nabla_x \mathbf{v}\right) + \mathrm{R}\nabla_x(\varrho\vartheta) = \mu\Delta\mathbf{v} + \frac{1}{3}\mu\nabla_x \mathrm{div}_x \mathbf{v}. \tag{3.1}$$

Following [18], the main idea is to multiply (3.1) on $|\mathbf{v}|^{2\alpha}\mathbf{v}$, where $\alpha > 0$ is a positive parameter to be fixed below. Integrating the resulting expression, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2(\alpha+1)}\int_\Omega \varrho|\mathbf{v}|^{2(\alpha+1)}\ \mathrm{d}x + \mu\int_\Omega \left(|\mathbf{v}|^{2\alpha}|\nabla_x\mathbf{v}|^2 + \frac{1}{3}|\mathbf{v}|^{2\alpha}|\mathrm{div}_x\mathbf{v}|^2\right)\ \mathrm{d}x \tag{3.2}$$

$$= \mathrm{R}\int_\Omega \left(|\mathbf{v}|^{2\alpha}\varrho\vartheta\mathrm{div}_x\mathbf{v} + \varrho\vartheta\nabla_x|\mathbf{v}|^{2\alpha}\cdot\mathbf{v}\right)\ \mathrm{d}x$$

$$-\mu\int_\Omega\left([(\nabla_x\mathbf{v})\mathbf{v}]\cdot\nabla_x|\mathbf{v}|^{2\alpha} + \frac{1}{3}\mathrm{div}_x\mathbf{v}\nabla_x|\mathbf{v}|^{2\alpha}\cdot\mathbf{v}\right)\ \mathrm{d}x.$$

It is easy to check that the second integral on the right-hand side of (3.2) is controlled by its counterpart on the left-hand side as soon as $\alpha > 0$ is small enough. By the same token, using estimates (2.18), (2.20) we get

$$\left|\int_0^T\int_\Omega |\mathbf{v}|^{2\alpha}\varrho\vartheta\mathrm{div}_x\mathbf{v}\ \mathrm{d}x\ \mathrm{d}t\right| \le c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)})$$

provided $\alpha > 0$ is sufficiently small.

Finally,

$$\left|\int_0^\tau\int_\Omega \varrho\vartheta\nabla_x|\mathbf{v}|^{2\alpha}\cdot\mathbf{v}\ \mathrm{d}x\ \mathrm{d}t\right|$$

$$\le \alpha\int_0^\tau\int_\Omega \varrho^2\vartheta^2|\mathbf{v}|^{2\alpha}\ \mathrm{d}x\ \mathrm{d}t + \alpha\int_0^T\int_\Omega |\mathbf{v}|^{2\alpha}|\nabla_x\mathbf{v}|^2\ \mathrm{d}x\ \mathrm{d}t \text{ for any } \tau\in(0,T).$$

Consequently, relation (3.2) gives rise to the following bounds:

$$\sup_{t\in(0,T)}\int_\Omega \varrho|\mathbf{v}|^{2(1+\alpha)}(t,\cdot)\ \mathrm{d}x \le c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}), \tag{3.3}$$

and

$$\||\mathbf{v}|^\alpha\mathbf{v}\|_{L^2(0,T;W^{1,2}(\Omega;R^3))} \le c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}) \tag{3.4}$$

for a certain $\alpha > 0$.

Since $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ and $W^{1,3/2}(\Omega) \hookrightarrow L^3(\Omega)$, estimate (3.4) combined with (2.17) imply

$$\|\varrho\mathbf{v}\|_{L^2(0,T;L^2(\Omega;R^3))} \leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}), \tag{3.5}$$

which can be used in (2.15) in order to obtain

$$\sup_{t\in(0,T)} \|\varrho(t,\cdot)\|_{L^2(\Omega)} \leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}), \tag{3.6}$$

together with

$$\|\varrho\|_{L^2(0,T;W^{1,2}(\Omega))} \leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}). \tag{3.7}$$

As a matter of fact, a slightly better estimate may be obtained, namely

$$\|\varrho\|_{L^{(2+\alpha)}(0,T;W^{1,(2+\alpha)}(\Omega))} \leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}) \tag{3.8}$$

for a certain $\alpha > 0$. Here, similarly to (3.4), the symbol $\alpha > 0$ denotes a generic positive parameter that may be different in different formulas.

Combining the previous estimates, in particular (3.4), (3.6), (3.7), we observe that

$$\mathrm{div}_x(\varrho\mathbf{v}) \text{ belongs to the space } L^{(1+\alpha)}((0,T)\times\Omega;R^3) \text{ for a certain } \alpha > 0;$$

whence equation (2.15), together with the standard L^p-theory for linear parabolic problems, yield

$$\begin{aligned} &\|\partial_t\varrho\|_{L^{(1+\alpha)}((0,T)\times\Omega)} + \|\varrho\|_{L^{(1+\alpha)}(0,T;W^{2,(1+\alpha)}(\Omega))} \\ &\leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}) \end{aligned} \tag{3.9}$$

for a certain $\alpha > 0$.

On the point of conclusion, we note that the *a priori* bounds established in this section, notably (3.4), imply equi-integrability of the "viscous flux" of the total energy that is necessary in order to handle (1.3).

3.2 Temperature estimates

A short inspection of the total energy balance (1.3) reveals immediately one of the main technical problems involved in Brenner's model, namely a possibility of concentrations in the heat flux

$$\mathbf{q} = -\kappa_0(1+\vartheta^3)\nabla_x\vartheta.$$

Note that, for the time being, we have shown only (2.12), which is clearly insufficient not even to control the L^1-norm of $\mathbf{q}$.

To begin, we exploit again the thermal energy balance. Multiplying (1.12) on $H'(\vartheta)$, where H is a suitable function specified below, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho H(\vartheta)\ \mathrm{d}x + \kappa_0 \int_\Omega H''(\vartheta)(1+\vartheta^3)|\nabla_x \vartheta|^2\ \mathrm{d}x$$

$$= \int_\Omega H'(\vartheta)\mathbb{S} : \nabla_x \mathbf{v}\ \mathrm{d}x - \mathrm{R}\int_\Omega H'(\vartheta)\varrho\vartheta \mathrm{div}_x \mathbf{v}\ \mathrm{d}x.$$

By virtue of estimates (2.18), (2.20), the right-hand side is integrable in t as soon as the derivative H' is bounded. Thus the choice $H(\vartheta) = (1+\vartheta)^{1-\omega}$, with $\omega > 0$, leads to a uniform bound

$$\int_0^T \int_\Omega \frac{\vartheta^3}{(1+\vartheta)^{1+\omega}} |\nabla_x \vartheta|^2\ \mathrm{d}x\ \mathrm{d}t \tag{3.10}$$

$$\leq c(\omega, M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)})$$

for any $\omega > 0$. Here, the best estimates would be obtained in the limit case $\omega \to 0$ unfortunately not attainable.

Writing $\vartheta^3 \nabla_x \vartheta \approx \nabla_x \vartheta^4$ we need uniform bounds on ϑ^4 that would be "slightly better" than in L^1, more precisely, we need *equi-integrability* of ϑ^4 in the Lebesgue space $L^1((0,T)\times\Omega)$. To this end, we claim first that such a bound follows immediately from (2.5), (3.10) at least on the region where ϱ is bounded below away from zero. Indeed by virtue of the standard imbedding relation $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ estimate (3.10) yields

$$\|\vartheta\|_{L^{(4-\omega)}(0,T;L^{(12-\omega)}(\Omega))} \tag{3.11}$$

$$\leq c(\omega, M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)})$$

for any $\omega > 0$. Consequently, by means of (2.5) and a simple interpolation argument, we deduce that for any $\varepsilon > 0$ there exists $\alpha = \alpha(\varepsilon) > 0$ such that

$$\int_{\{\varrho > \varepsilon\}} |\vartheta|^{4+\alpha}\ \mathrm{d}x\ \mathrm{d}t \leq c(\varepsilon, M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}). \tag{3.12}$$

Boundedness of ϑ on the vacuum set, though the latter is of zero measure, is a more delicate task. The first step is to obtain L^4- integrability of ϑ on the whole set $(0,T)\times\Omega$. To this end, we multiply the thermal energy equation (1.12) by

$$\varphi = \Delta_N^{-1}\left[\varrho - \frac{1}{|\Omega|}\int_\Omega \varrho\ \mathrm{d}x\right],$$

where Δ_N stands for the Laplacean defined on the space of functions of zero mean and supplemented with the homogeneous Neumann boundary conditions. Observe that, by virtue of estimates (2.17), (3.6) combined with the standard elliptic regularity for Δ_N,

$$\left\| \Delta_N^{-1}\left[\varrho - \frac{1}{|\Omega|}\int_\Omega \varrho\ \mathrm{d}x\right]\right\|_{L^\infty((0,T)\times\Omega)} \tag{3.13}$$

$$\leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}),$$

and

$$\left\|\nabla_x \Delta_N^{-1}\left[\varrho - \frac{1}{|\Omega|}\int_\Omega \varrho \ {\rm d}x\right]\right\|_{L^p(0,T;L^p(\Omega))} \tag{3.14}$$

$$\leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}) \text{ for any } 1 < p < \infty.$$

Thus multiplying the thermal energy equation on φ yields

$$-\kappa_0 \int_0^T \int_\Omega \vartheta^4 \left(\varrho - \frac{1}{|\Omega|}\int_\Omega \varrho \ {\rm d}x\right) \ {\rm d}x \ {\rm d}t = \sum_{i=1}^3 I_i, \tag{3.15}$$

where we have set

$$I_1 = \int_0^T \int_\Omega (\mathbb{S} : \nabla_x \mathbf{v} - {\rm R}\varrho\vartheta {\rm div}_x \mathbf{v})\, \Delta_N^{-1}\left[\varrho - \frac{1}{|\Omega|}\int_\Omega \varrho \ {\rm d}x\right] \ {\rm d}x \ {\rm d}t,$$

$$I_2 = c_v \int_0^T \int_\Omega \varrho\vartheta\mathbf{v}_m \cdot \nabla_x \Delta_N^{-1}\left[\varrho - \frac{1}{|\Omega|}\int_\Omega \varrho \ {\rm d}x\right] \ {\rm d}x \ {\rm d}t,$$

and

$$I_3 = -c_v \int_0^T \int_\Omega \partial_t(\varrho\vartheta)\Delta_N^{-1}\left[\varrho - \frac{1}{|\Omega|}\int_\Omega \varrho \ {\rm d}x\right] \ {\rm d}x \ {\rm d}t.$$

In accordance with the uniform bounds (2.18), (2.20), and (3.13), the integral I_1 is bounded in terms of the norm of the initial data.

Similarly, writing

$$\varrho\vartheta\mathbf{v}_m = \varrho\vartheta\mathbf{v} - \vartheta\nabla_x\varrho,$$

we can use estimates (2.12), (2.20), (3.6), and (3.9) in order to conclude that

$$\|\varrho\vartheta\mathbf{v}_m\|_{L^q(0,T;L^q(\Omega;R^3))} \tag{3.16}$$

$$\leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}) \text{ for a certain } q > 1.$$

Thus, in view of (3.14), the integral I_2 is controlled by the data.

Next, we use equation (2.15) in order to write I_3 in the form

$$I_3 = c_v \left[\int_\Omega \varrho\vartheta\Delta_N^{-1}\left[\varrho - \frac{1}{|\Omega|}\int_\Omega \varrho \ {\rm d}x\right] \ {\rm d}x\right]_{t=0}^{t=T}$$

$$+ \int_0^T \int_\Omega \varrho\vartheta\Delta_N^{-1}\left[{\rm div}_x(\varrho\mathbf{v}) + \Delta\varrho\right] \ {\rm d}x \ {\rm d}t,$$

where, by virtue of (2.5), (3.13), the first term on the right-hand side is bounded. Moreover, by virtue of the standard elliptic regularity estimates combined with (2.20), (3.6), we have

$$\|\Delta_N^{-1}[{\rm div}_x(\varrho\mathbf{v})]\|_{L^2(0,T;L^3(\Omega))}$$

$$\leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)});$$

whence, evoking (2.18), we control

$$\int_0^T \int_\Omega \varrho\vartheta\Delta_N^{-1}[\mathrm{div}_x(\varrho\mathbf{v})] \ \mathrm{d}x \ \mathrm{d}t.$$

Finally, seeing that

$$\Delta_N^{-1}[\Delta\varrho] = \varrho - \frac{1}{|\Omega|}\int_\Omega \varrho \ \mathrm{d}x,$$

we can use (3.6) in order to conclude that I_3 is bounded.

Now, returning to (3.15) we write

$$-\int_0^T \int_\Omega \vartheta^4 \left(\varrho - \frac{1}{|\Omega|}\int_\Omega \varrho \ \mathrm{d}x\right) \ \mathrm{d}x \ \mathrm{d}t \geq \frac{M_0}{2|\Omega|}\int_{\{\varrho < M_0/(2|\Omega|)\}} \vartheta^4 \ \mathrm{d}x \ \mathrm{d}t$$

$$-\int_{\{\varrho \geq M_0/(2|\Omega|)\}} \vartheta^4 \left(\varrho - \frac{1}{|\Omega|}\int_\Omega \varrho \ \mathrm{d}x\right) \mathrm{d}x \ \mathrm{d}t,$$

where the last integral is bounded. Indeed interpolating (2.5), (3.11) on the set $\{\varrho \geq M_0/(2|\Omega|)\}$ yields

$$\|1_{\{\varrho > M_0/(2|\Omega|)}\vartheta\|_{L^{(4+\alpha)}(0,T;L^{(8+\alpha)}(\Omega))} \tag{3.17}$$

$$\leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}) \text{ for a certain } \alpha > 0;$$

which, combined with (3.6) yields the desired conclusion.

Thus we infer from (3.15) that

$$\int_{\{\varrho < M_0/(2|\Omega|)\}} \vartheta^4 \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}),$$

which yields, together with (3.12),

$$\|\vartheta^4\|_{L^1((0,T)\times\Omega)} \leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\varrho_0|\mathbf{v}_0|^{2(1+\alpha)}\|_{L^1(\Omega)}). \tag{3.18}$$

Our ultimate goal is to improve the estimates on ϑ^4 in the area where the density ϱ is small. To this end, we use "test" functions in the form

$$\varphi = \chi\Delta^{-1}[\chi\log(\varrho)],$$

where $\chi \in C_c^\infty(\Omega)$, $\chi \geq 0$, and the symbol Δ denotes the standard Laplace operator defined via its Fourier symbol $-|\xi|^2$ on the whole space R^3.

To begin, we claim that, on the basis of the refined velocity estimates obtained in (3.4), we are allowed to use the comparison argument exactly as in Section 2.4, in order to strengthen (2.27) to

$$\sup_{t\in(0,T)} \|\log(\varrho)(t,\cdot)\|_{L^{(3+\alpha)}(\Omega)} \leq c(M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\log(\varrho_0)\|_{L^3(\Omega)}) \tag{3.19}$$

for a certain $\alpha > 0$. In particular, by virtue of the standard elliptic theory,

$$\|\Delta^{-1}[\chi \log(\varrho)]\|_{L^\infty((0,T)\times\Omega)} + \|\nabla_x \Delta^{-1}[\chi \log(\varrho)]\|_{L^\infty((0,T)\times\Omega;R^3)} \tag{3.20}$$
$$\leq c(\chi, M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\log(\varrho_0)\|_{L^3(\Omega)}).$$

Similarly to the preceding step, we use φ as "test" functions in the thermal energy equation (2.12). Using the same arguments as above combined with the bounds established in (3.20), we deduce

$$\frac{\kappa_0}{4}\int_0^T \int_\Omega \Delta\vartheta^4 \chi \Delta^{-1}[\chi \log(\varrho)] \ \mathrm{d}x \ \mathrm{d}t = c_v \int_0^T \int_\Omega \partial_t(\varrho\vartheta)\chi\Delta^{-1}[\chi \log(\varrho)] \ \mathrm{d}x \ \mathrm{d}t, \tag{3.21}$$

where, furthermore,

$$\int_0^T \int_\Omega \Delta\vartheta^4 \chi \Delta^{-1}[\chi \log(\varrho)] \ \mathrm{d}x \ \mathrm{d}t = \int_0^T \int_\Omega \chi^2 \vartheta^4 \log(\varrho) \ \mathrm{d}x \ \mathrm{d}t \tag{3.22}$$
$$+ \int_0^T \int_\Omega \Big(2\vartheta^4 \nabla_x \chi \cdot \nabla_x \Delta^{-1}[\chi \log(\varrho)] + \vartheta^4 \Delta\chi \Delta^{-1}[\chi \log(\varrho)] \Big) \ \mathrm{d}x \ \mathrm{d}t.$$

In order to handle the right-hand side of (3.21), we use equation (2.21) to obtain

$$\int_0^T \int_\Omega \partial_t(\varrho\vartheta)\chi\Delta^{-1}[\chi \log(\varrho)] \ \mathrm{d}x \ \mathrm{d}t = \left[\int_\Omega \varrho\vartheta\chi\Delta^{-1}[\chi\log(\varrho)] \ \mathrm{d}x \right]_{t=0}^{t=T} \tag{3.23}$$
$$- \int_0^T \int_\Omega \varrho\vartheta\chi\Delta^{-1} \left[\chi\Delta\log(\varrho) + \chi|\nabla_x \log(\varrho)|^2 - \chi \mathrm{div}_x \mathbf{v} - \chi\mathbf{v}\cdot\nabla_x \log(\varrho) \right] \ \mathrm{d}x \ \mathrm{d}t$$
$$\geq - \int_0^T \int_\Omega \varrho\vartheta\chi\Delta^{-1} \left[\chi\Delta\log(\varrho) - \chi\mathrm{div}_x\mathbf{v} - \frac{1}{2}|\mathbf{v}|^2 \right] \ \mathrm{d}x \ \mathrm{d}t,$$

where, similarly to Section 2.4, we have used positivity of the operator $-\Delta$.

At this stage, we can use the uniform estimates (2.5), (2.18), (2.20) and (3.13) in order to observe that the last integral on the right-hand side of (3.23) is bounded. Consequently, relations (3.22), (3.23) allow us to conclude that

$$\int_0^T \int_{\mathcal{K}} \vartheta^4 |\log(\varrho)| \mathrm{d}x \ \mathrm{d}t \leq c(\mathcal{K}, M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\log(\varrho_0)\|_{L^3(\Omega)}) \tag{3.24}$$

for any *compact* $\mathcal{K} \subset \Omega$ (keep in mind that $\log(\varrho)$ is negative on the "vacuum" set where ϱ approaches zero).

Estimate (3.24), together with (3.12), (3.18), imply *equi-integrability* of ϑ^4 at least on any compact subset of Ω. In order to extend this property up to the boundary, we simply use

$$\varphi = \Delta_N^{-1}[\omega]$$

as a "test" function in (1.12), where

$$\omega = \omega(x), \ \omega(x) \geq -\underline{\omega} \text{ for all } x \in \Omega, \ \omega \in L^4(\Omega), \ \lim_{x\to\partial\Omega} \omega(x) = \infty,$$

and

$$\int_\Omega \omega \, \mathrm{d}x = 0.$$

Thus we have shown the following result:

EQUI-INTEGRABILITY OF THE TEMPERATURE FLUX:

For any $\varepsilon > 0$, there exists $\delta > 0$,

$$\delta = \delta(\varepsilon, M_0, E_0, S_0, \|\varrho_0\|_{L^3(\Omega)}, \|\log(\varrho_0)\|_{L^3(\Omega)}),$$

such that

$$\int_B \vartheta^4 \, \mathrm{d}x \, \mathrm{d}t < \varepsilon \tag{3.25}$$

for any measurable set $B \subset (0,T) \times \Omega$ such that $|B| < \delta$.

4 Weak sequential stability

The problem of weak sequential stability can be formulated in the following way:

Assume that $\{\varrho_n, \mathbf{v}_n, \vartheta_n\}_{n=1}^\infty$ is a sequence of, say, regular solutions to problem (1.1 - 1.4), supplemented with the constitutive equations (1.13), (1.15), and the boundary conditions (1.16), (1.20). In addition, suppose that

$$\varrho_n(0,\cdot) = \varrho_{0,n}, \ \mathbf{v}_n(0,\cdot) = \mathbf{v}_{0,n}, \ \vartheta_n(0,\cdot) = \vartheta_{0,n},$$

where the initial data $\varrho_{0,n}$, $\mathbf{v}_{0,n}$, $\vartheta_{0,n}$ satisfy:

$$\left. \begin{array}{c} 0 < \underline{\varrho} \le \varrho_{0,n}(x) \le \overline{\varrho} \text{ for all } x \in \Omega, \\[2ex] 0 < \underline{\vartheta} \le \vartheta_{0,n}(x) \le \overline{\vartheta} \text{ for all } x \in \Omega, \\[2ex] \|\mathbf{v}_{0,n}\|_{L^\infty(\Omega;R^3)} \le U_0 \end{array} \right\} \tag{4.1}$$

uniformly for $n = 1, 2, \dots$

Thus, if hypotheses (A 2.1 - A 2.5) are satisfied, the sequence $\{\varrho_n, \mathbf{v}_n, \vartheta_n\}_{n=1}^\infty$ admits the uniform bounds established in the preceding section. In particular, passing to subsequences if necessary, we may assume that

$$\left\{ \begin{array}{c} \varrho_n \to \varrho \text{ weakly in } L^1((0,T) \times \Omega), \\[2ex] \mathbf{v}_n \to \mathbf{v} \text{ weakly in } L^1((0,T) \times \Omega; R^3), \\[2ex] \vartheta_n \to \vartheta \text{ weakly in } L^1((0,T) \times \Omega). \end{array} \right\}$$

The problem of *weak sequential stability* consists in showing that the limit quantities ϱ, $\mathbf{v}$, and ϑ represent a weak solution to the same system. To this end, two fundamental properties have to be verified: **(i)** pointwise a.a. convergence of all field variables, **(ii)** equi-integrability of all fluxes and production terms in the field equations (1.1 - 1.4).

4.1 Pointwise convergence

With the relatively strong *a priori* estimates at hand, the pointwise a.a. convergence of the field variables can be resolved easily. To begin, the uniform bound established in (3.9) is sufficient to conclude that

$$\left.\begin{array}{c} \varrho_n \to \varrho, \\ \\ \nabla_x \varrho_n \to \nabla_x \varrho \end{array}\right\} \text{ a.a. in } (0,T) \times \Omega. \tag{4.2}$$

Next, using (2.5), (3.6), we deduce from the momentum equation (1.2) that

$$\varrho_n \mathbf{v}_n \to \varrho \mathbf{v} \text{ in } C_{\text{weak}}([0,T]; L^{4/3}(\Omega; R^3)).$$

On the other hand, by virtue of (2.20),

$$\mathbf{v}_n \to \mathbf{v} \text{ weakly in } L^2(0,T; W^{1,2}(\Omega; R^3));$$

whence

$$\int_0^T \int_\Omega \varrho_n |\mathbf{v}_n|^2 \ \mathrm{d}x \ \mathrm{d}t \to \int_0^T \int_\Omega \varrho |\mathbf{v}|^2 \ \mathrm{d}x \ \mathrm{d}t. \tag{4.3}$$

Now it is easy to check that (4.3), together with the uniform bounds derived in the previous section, imply pointwise (a.a.) convergence of $\mathbf{v}_n$ on the set $\{x \mid \varrho(x) > 0\}$. But since the latter is of full measure, we conclude that

$$\mathbf{v}_n \to \mathbf{v} \text{ a.a. in } (0,T) \times \Omega. \tag{4.4}$$

Exactly the same argument can be used to show

$$\vartheta_n \to \vartheta \text{ a.a. in } (0,T) \times \Omega. \tag{4.5}$$

To conclude, let us remark that, by virtue of (4.2), (4.4), and (2.24),

$$\mathbf{v}_{mn} \to \mathbf{v}_m \text{ a.a. in } (0,T) \times \Omega. \tag{4.6}$$

4.2 Equi-integrability of the fluxes and production rates

We concentrate only on the most difficult terms appearing in (1.2), (1.3), namely,

$$\{\varrho_n \mathbf{v}_n \otimes \mathbf{v}_{mn}\}_{n=1}^\infty, \ \{\varrho_n |\mathbf{v}_n|^2 \mathbf{v}_{mn}\}_{n=1}^\infty, \ \{\varrho_n \vartheta_n \mathbf{v}_{mn}\}_{n=1}^\infty, \ \{\mathbf{q}\}_{n=1}^\infty, \ \{\mathbb{S}_n \mathbf{v}_n\}_{n=1}^\infty.$$

We recall that the term $\varrho\vartheta\mathbf{v}_m$ has already been handled in (3.16), while equi-integrability of $\{\mathbb{S}_n\mathbf{v}_n\}_{n=1}^{\infty}$ follows directly from the refined velocity estimates (3.4). Moreover, writing

$$\int_0^T \int_\Omega \mathbf{q}_n \cdot \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t = \frac{\kappa_0}{4} \int_0^T \int_\Omega \vartheta_n^4 \Delta\varphi \ \mathrm{d}x \ \mathrm{d}t$$

for any test function φ satisfying $\nabla_x \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$, we observe that convergence of the integral on the left-hand side follows from (3.25).

Furthermore,

$$\varrho_n \mathbf{v}_n \otimes \mathbf{v}_{mn} = \varrho_n \mathbf{v}_n \otimes \mathbf{v}_n - \mathbf{v}_n \otimes \nabla_x \varrho_n;$$

whence equi-integrability of the sequence $\{\varrho_n\mathbf{v}_n \otimes \mathbf{v}_{mn}\}_{n=1}^{\infty}$ follows directly from (3.6), (3.7), and the refined velocity estimates (3.4).

Finally,

$$\varrho_n |\mathbf{v}_n|^2 \mathbf{v}_{mn} = \varrho_n |\mathbf{v}_n|^2 \mathbf{v}_n - |\mathbf{v}_n|^2 \nabla_x \varrho_n,$$

Seeing that

$$\varrho_n |\mathbf{v}_n|^2 \mathbf{v}_n = \sqrt{\varrho_n}\sqrt{\varrho_n}\mathbf{v}_n |\mathbf{v}_n|^2,$$

we can deduce equi-integrability of this expression from (2.5), (2.17), and (3.4). Furthermore,

$$\int_0^T \int_\Omega |\mathbf{v}_n|^2 \nabla_x \varrho_n \cdot \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t$$

$$= - \int_0^T \int_\Omega \varrho_n |\mathbf{v}_n|^2 \Delta\varphi \ \mathrm{d}x \ \mathrm{d}t - 2\int_0^T \int_\Omega \varrho_n [\nabla_x \mathbf{v}_n \mathbf{v}_n] \cdot \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t$$

for any $\varphi \in C_c^\infty((0,T)\times\Omega)$, where the last integral can be controlled by means of (2.17) and the refined estimates on the volume velocity established in (3.4).

5 Global existence

The question of *existence* of solutions is an ultimate criterion of validity of any mathematical model. Fortunately, Brenner's model exhibits strong similarity to the *approximate system* of equations introduced in [9] in order to show existence of weak solutions to the standard Navier-Stokes-Fourier system. Taking advantage of this remarkable coincidence, we propose the following family of approximate problems:

Approximate System:

$$\partial_t \varrho - \Delta \varrho = -\mathrm{div}_x(\varrho \mathbf{v}), \tag{5.1}$$

$$\nabla_x \varrho \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{5.2}$$

$$\varrho(0,\cdot) = \varrho_0; \tag{5.3}$$

$$\partial_t(\varrho \mathbf{v}) + \mathrm{div}_x(\varrho \mathbf{v} \otimes \mathbf{v}) + \mathrm{R}\nabla_x(\varrho\vartheta) - \mathrm{div}_x(\mathbf{v} \otimes \nabla_x \varrho) = \mathrm{div}_x \mathbb{S} - \varepsilon |\mathbf{v}|^{\Gamma-2}\mathbf{v}, \tag{5.4}$$

$$\mathbf{v}|_{\partial\Omega} = 0, \tag{5.5}$$

$$(\varrho \mathbf{v})(0,\cdot) = \varrho_0 \mathbf{v}_0; \tag{5.6}$$

$$c_v\left(\partial_t(\varrho\vartheta) + \mathrm{div}_x(\varrho\vartheta\mathbf{v})\right) - \kappa_0 \mathrm{div}_x\left((1+\vartheta^3)\nabla_x\vartheta\right) - c_v \mathrm{div}_x(\vartheta \nabla_x \varrho) \tag{5.7}$$

$$= \mathbb{S} : \nabla_x \mathbf{v} - \mathrm{R}\varrho\vartheta \mathrm{div}_x \mathbf{v} + \varepsilon |\mathbf{v}|^{\Gamma},$$

$$\nabla_x \vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{5.8}$$

$$(\varrho\vartheta)(0,\cdot) = \varrho_0 \vartheta_0. \tag{5.9}$$

Here $\varepsilon > 0$ is a small parameter and $\Gamma > 0$ a fixed (large) number, the value of which is to be chosen below. A general strategy developed in [9, Chapter 7] applies almost literally to the above system. Specifically, the approximate momentum equation (5.4) can be solved via the Faedo-Galerkin approximation scheme while equations (5.1), (5.7) are solved directly by means of the standard theory of parabolic problems. Even more specifically, replacing (5.4) by a system of ordinary differential equations resulting from the Galerkin projections on a finite number of modes, we fix $\mathbf{v}$ satisfying the initial condition (5.6), solve (5.1 - 5.3) with this $\mathbf{v}$ obtaining ϱ, then solve (5.7 - 5.9) with given ϱ, ϑ, and go back to solve (5.4 - 5.6) closing the circle via the Schauder fixed point argument.

This procedure, carried out and discussed in detail in [9, Chapter 7], yields the existence of solutions to the approximate system (5.1 - 5.9) provided we are able to show that our scheme is compatible with the *a priori* estimates obtained in Section 2. In order to see this, we note first that the total energy balance

$$\partial_t \left(\varrho \left(\frac{1}{2}|\mathbf{v}|^2 + c_v\vartheta \right) \right) + \mathrm{div}_x \left(\varrho \left(\frac{1}{2}|\mathbf{v}|^2 + c_v\vartheta \right) \mathbf{v} \right)$$

$$-\mathrm{div}_x \left(\left(\frac{1}{2}|\mathbf{v}|^2 + c_v\vartheta \right) \nabla_x \varrho \right) - \kappa_0 \mathrm{div}_x \left((1+\vartheta^3)\nabla_x\vartheta \right) + \mathrm{div}_x(p\mathbf{v}) = \mathrm{div}_x(\mathbb{S}\mathbf{v})$$

deduced on the basis of (5.4), (5.7) does not contain any $\varepsilon-$dependent terms.

5.1 Regularity of the approximate velocities

Unfortunately, the refined velocity estimates obtained in Section 3.1 are not compatible with the Faedo-Galerkin approximations as they are based on multiplying the momentum equation on a *nonlinear* function of $\mathbf{v}$. In order to substitute for these estimates at the first level of the approximation procedure, the ε-dependent quentities have been added in (5.4), (5.7). In particular, integrating the thermal energy balance equation (5.7), we deduce that the approximate volume velocities are bounded in the Lebesgue space $L^\Gamma((0,T)\times\Omega;R^3)$. As ϱ solves the parabolic equation (5.1), better summability of $\mathbf{v}$ gives rise to higher regularity of ϱ. Specifically, we have the following result.

Lemma 5.1 *Let $\Omega\subset R^3$ be a bounded regular domain. Let $\mathbf{v}$ be a given velocity field satisfying*

$$\|\mathbf{v}\|_{L^\Gamma((0,T)\times\Omega;R^3)} + \|\mathbf{v}\|_{L^2(0,T;W^{1,2}(\Omega;R^3))} \leq \overline{v} \tag{5.10}$$

Assume that $\varrho\geq 0$ is a weak solution of problem (5.1 - 5.3) belonging to the class $L^1(0,T;L^3(\Omega))$, and such that $\varrho_0\in C(\overline{\Omega})$,

$$\frac{1}{r}\leq \inf_{x\in\Omega}\varrho_0(x)\leq \sup_{x\in\Omega}\varrho_0(x)+\|\varrho\|_{L^1(0,T;L^3(\Omega))}\leq r. \tag{5.11}$$

Finally, suppose that

$$\operatorname{ess}\sup_{t\in(0,T)}\|(\sqrt{\varrho}\mathbf{v})(t,\cdot)\|_{L^2(\Omega;R^3)}\leq\overline{m}. \tag{5.12}$$

Then, for any $\Gamma>0$ large enough, the density ϱ belongs to the spaces $C([0,T]\times\overline{\Omega})$ and is strictly positive in $[0,T]\times\overline{\Omega}$. In addition,

$$\partial_t\varrho,\ \Delta\varrho\in L^2((0,T)\times\Omega) \tag{5.13}$$

provided $\varrho_0\in W^{1,2}(\Omega)$. The norm of ϱ in the aforementioned spaces depends only on $\overline{v}$, r, and $\overline{m}$.

Proof:

(i) Assume first that $\varrho(0,\cdot)=0$. Following the line of arguments used in Section 2.3, specifically the maximal regularity estimates established in Proposition 2.1, we deduce from (5.11), (5.12) that

$$\operatorname{ess}\sup_{t\in(0,T)}\|\varrho\|_{L^p(\Omega)}+\|\varrho\|_{L^q(0,T;L^3(\Omega))}\leq c(p,q,r,\overline{m}) \text{ for any } 1\leq p<3,\ q<\infty. \tag{5.14}$$

(ii) Estimate (5.14) combined with hypothesis (5.10) can be used iteratively to improve integrability of ϱ. To begin, employing again Proposition 2.1 we get

$$\varrho \in L^{p(\Gamma)}(0,T;W^{1,q(\Gamma)}(\Omega)),$$

where $p(\Gamma) \nearrow \infty$, $q(\Gamma) \nearrow 3$ provided $\Gamma \to \infty$. Since $W^{1,3}(\Omega) \hookrightarrow L^q(\Omega)$ for any finite q, we infer that

$$\varrho \in L^{p(\Gamma)}((0,T)\times\Omega), \text{ with } p(\Gamma) \nearrow \infty \text{ for } \Gamma \to \infty.$$

Thus, another application of Proposition 2.1 yields the desired conclusion

$$\varrho \in C([0,T]\times\overline{\Omega}).$$

The same can be shown, of course, in the case $\varrho_0 \neq 0 \in C(\overline{\Omega})$.

(iii) Strict positivity of ϱ can be shown via a comparison argument exactly as in Section 2.4. Specifically, we have

$$\log(\varrho) \geq V,$$

where V solves problem (2.26). Accordingly, for $\Gamma > 0$ large enough, V is bounded from below as required.

(iv) It remains to show that ϱ belongs to the "optimal" regularity class (5.13). Here again, it is enough to handle the case $\varrho_0 = 0$. Since we already know that ϱ is bounded, we deduce from Proposition 2.1 that

$$\varrho \in L^{p(\Gamma)}(0,T;W^{1,p(\Gamma)}(\Omega)), \ p(\Gamma) \nearrow \infty \text{ as } \Gamma \to \infty.$$

Thus the desired conclusion follows from the standard parabolic as

$$\partial_t \varrho - \Delta\varrho = -\nabla_x\varrho\cdot\mathbf{v} - \varrho\mathrm{div}_x\mathbf{v} \in L^2((0,T)\times\Omega).$$

q.e.d.

5.2 Refined velocity estimates revisited

As already pointed out, the refined velocity estimates based on *non-linear* multipliers $|\mathbf{v}|^{2\alpha}\mathbf{v}$ are not compatible with the Faedo-Galerkin approximations applied to problem (5.4 - 5.6). Fortunately, as we have observed in Lemma 5.1, even better regularity of ϱ is obtained as a consequence of the presence of the extra $\varepsilon-$terms in (5.4), (5.7). In view of the general arguments discussed in Sections 2, 4, we may therefore expect the approximate system (5.1 - 5.9) to be solvable in the regularity class induced by the *a priori* estimates obtained in Section 2, where, in addition, ϱ enjoys the same regularity as in the conclusion of Lemma 5.1. Thus, our ultimate goal is to carry out the limit $\varepsilon \to 0$.

At this stage, the $\varepsilon-$dependent bounds on the volume velocity field $\mathbf{v}$ have to be replaced by those obtained in Section 3.1. In other words, we have to show that the quantities $|\mathbf{v}|^{2\alpha}\mathbf{v}$ can be used as test functions in the weak formulation of (5.4) which reads

$$\left|\int_\Omega \varrho\mathbf{v}\cdot\varphi \ \mathrm{d}x\right|_{t=0}^{t=\tau} - \int_0^\tau\int_\Omega \varrho\mathbf{v}\cdot\partial_t\varphi \ \mathrm{d}x \ \mathrm{d}t - \int_0^\tau\int_\Omega \varrho\mathbf{v}\otimes(\mathbf{v}-\nabla_x\varrho):\nabla_x\varphi \ \mathrm{d}x \ \mathrm{d}t \tag{5.15}$$

$$-\mathrm{R}\int_0^\tau\int_\Omega \varrho\vartheta\mathrm{div}_x\varphi \ \mathrm{d}x \ \mathrm{d}t = -\int_0^\tau\int_\Omega \mathbb{S}:\nabla_x\varphi \ \mathrm{d}x \ \mathrm{d}t - \varepsilon\int_0^\tau\int_\Omega |\mathbf{v}|^{\Gamma-2}\mathbf{v}\cdot\varphi \ \mathrm{d}x \ \mathrm{d}t$$

for any $\tau\in[0,T]$ and any test function $\varphi\in C_c^\infty(R\times\Omega;R^3)$. Obviously, the principal difficulty stems from the lack of information on the *time derivative* of $\varphi\approx|\mathbf{v}|^\alpha\mathbf{v}$.

Extending ϱ, $\mathbf{v}$ to be ϱ_0, $\mathbf{v}_0$ for $t<0$ and $\varrho(\tau,\cdot)$, $\mathbf{v}(\tau,\cdot)$ for $\tau\in[T,\infty)$ we can use the quantities

$$\varphi(t,x)=\eta_\delta(\tau-t)\phi(x)$$

as test functions in (5.15), where $\{\eta\}_\delta$ is a suitable family of regularizing kernels with respect to the time variable. Denoting $[v]_\delta=\eta_\delta*v$ we deduce

$$\partial_t[\varrho\mathbf{v}]_\delta=[\mathrm{div}_x\mathbb{S}]_\delta-[\mathrm{div}_x(\varrho\mathbf{v}\otimes\mathbf{v}_m)]_\delta-\mathrm{R}[\nabla_x(\varrho\vartheta)]_\delta-\varepsilon[|\mathbf{v}|^{\Gamma-2}\mathbf{v}]_\delta \tag{5.16}$$

for $t\in R$ provided all quantities in the brackets on the right-hand side have been extended to be zero outside the interval $[0,T]$. Since $\mathbf{v}$ and ϱ belong to the regularity class specified in Lemma 5.1, we can identify the mapping $t\mapsto[\mathrm{div}_x\mathbb{S}]_\delta$ with a smooth function of time ranging in the dual $W^{-1,2}(\Omega)$, while the remaining terms on the right-hand side of (5.16) belong to $C^\infty(R;L^q(\Omega;R^3))$ for a certain $q>1$.

Now, consider the commutator

$$\omega_\delta=\partial_t[\varrho\mathbf{v}]_\delta-\partial_t(\varrho[\mathbf{v}]_\delta) \text{ on the time interval } [0,T].$$

Since $\partial_t\varrho$ belongs to the Lebesgue space $L^2((0,T)\times\Omega)$ and $\mathbf{v}\in L^\Gamma((0,T)\times\Omega;R^3)$, the classical regularity estimates of Friedrichs (see [9, Lemma 4.3]) yield

$$\omega_\delta\to0 \text{ as } \delta\to0 \text{ in } L^p((0,T)\times\Omega), \text{ where } \frac{1}{p}=\frac{1}{2}+\frac{1}{\Gamma}.$$

Thus we are allowed to replace $\partial_t[\varrho\mathbf{v}]_\delta$ by $\partial_t(\varrho[\mathbf{v}]_\delta)$ in (5.16) and multiply the resulting expression by $T_k(|[\mathbf{v}]_\delta|^{2\alpha})[\mathbf{v}]_\delta$, where T_k are the cut-off functions

$$T_k(z)=\min\{k,z\},$$

to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho H_k(|[\mathbf{v}]_\delta|) \ \mathrm{d}x+\int_\Omega T_k(|[\mathbf{v}]_\delta|^{2\alpha})\mathrm{div}_x[(\varrho\mathbf{v}_m\otimes\mathbf{v})]_\delta\cdot[\mathbf{v}]_\delta \ \mathrm{d}x \tag{5.17}$$

$$-\mathrm{R}\int_\Omega [\varrho\vartheta]_\delta T_k(|[\mathbf{v}]_\delta|^{2\alpha})\mathrm{div}_x([\mathbf{v}]_\delta) \ \mathrm{d}x-\mathrm{R}\int_\Omega [\varrho\vartheta]_\delta\nabla_x T_k(|[\mathbf{v}]_\delta|^{2\alpha})\cdot[\mathbf{v}]_\delta \ \mathrm{d}x$$

$$
+\mu \int_{\Omega} T_k(|[\mathbf{v}]_\delta|^{2\alpha}) |\nabla_x [\mathbf{v}]_\delta|^2 \ \mathrm{d}x + \frac{\mu}{3} \int_{\Omega} T_k(|[\mathbf{v}]_\delta|^{2\alpha}) |\mathrm{div}_x [\mathbf{v}]_\delta|^2 \ \mathrm{d}x
$$

$$
+\varepsilon \int_{\Omega} T_k(|[\mathbf{v}]_\delta|^{2\alpha}) [|\mathbf{v}|^{\Gamma-2}\mathbf{v}]_\delta \cdot [\mathbf{v}]_\delta \ \mathrm{d}x = \int_{\Omega} T_k(|[\mathbf{v}]_\delta|^{2\alpha}) \omega_\delta \cdot [\mathbf{v}]_\delta \ \mathrm{d}x +
$$

$$
-\mu \int_{\Omega} \left([(\nabla_x [\mathbf{v}]_\delta)[\mathbf{v}]_\delta] \cdot \nabla_x T_k(|[\mathbf{v}]_\delta|^{2\alpha}) + \frac{1}{3} \mathrm{div}_x [\mathbf{v}]_\delta \nabla_x T_k(|[\mathbf{v}]_\delta|^{2\alpha}) \cdot [\mathbf{v}]_\delta \right) \ \mathrm{d}x,
$$

where

$$
H_k'(z) = \begin{cases} z^{2\alpha+1} \text{ for } 0 \leq z \leq k^{1/2\alpha}, \\ \\ kz \text{ if } z \geq k^{1/2\alpha}. \end{cases}
$$

Thus letting first $\delta \to 0$ and then $k \to \infty$ we deduce the same estimates as in Section 3.1 that are independent of ε.

5.3 Global existence - conclusion

In accordance with the previous discussion, the existence of global-in-time solutions for the initial-boundary value problem associated to system (1.1 - 1.4) can be established in two steps:

- Solutions of the approximate system (5.1 - 5.9) are obtained by the method described in detail in [9].
- The approximate solutions enjoy the regularity properties established in Lemma 5.1. In particular, the approximate velocity field $\mathbf{v}$ belongs to the regularity class identified in Section 5.2, where the bounds are independent of ε.
- We let $\varepsilon \to 0$ to recover a weak solution of the original system. The limit is carried over by the same arguments as in Section 4.

On the point of conclusion, let us state our main existence result.

Global-in-Time Existence for Large Data:

Theorem 5.1 *Let* $\Omega \subset R^3$ *be a bounded domain of class* $C^{2+\nu}$, $\nu > 0$. *Assume that the thermodynamic functions satisfy hypotheses (A 2.1 - A 2.5). Finally, let the initial data are chosen so that*

$$\varrho_0,\ \vartheta_0 \in L^\infty(\Omega),\ \mathbf{v}_0 \in L^\infty(\Omega; R^3),$$

$$\operatorname{ess\,inf}_{\Omega} \varrho_0 > 0,\ \operatorname{ess\,inf}_{\Omega} \vartheta_0 > 0.$$

Then the initial-boundary value problem associated to (1.1 - 1.5), (1.13), (1.15 - 1.20) possesses at least one weak solution ϱ, $\mathbf{v}$, ϑ *on* $(0,T) \times \Omega$. *Moreover,*

$$\varrho(t,x) > 0,\ \vartheta(t,x) > 0 \textit{ for a.a. } (t,x) \in (0,T) \times \Omega.$$

Remark: *In accordance with our considerations in Section 4, we write*

$$\mathbf{q} = -\kappa_0 \nabla_x \vartheta - \frac{\kappa_0}{4} \nabla_x \vartheta^4,$$

and similarly.

$$\varrho |\mathbf{v}|^2 \mathbf{v}_m = \varrho |\mathbf{v}|^2 \mathbf{v} - \operatorname{div}_x (\varrho |\mathbf{v}|^2) + 2\varrho (\nabla_x \mathbf{v}\ \mathbf{v})$$

in the weak formulation of the total energy balance (1.3).

Under the hypotheses (A 2.1 - A 2.5), Brenner's model or (BNSF) system represents an interesting alternative to the classical approach. The velocity field $\mathbf{v}$ enjoys more regularity than the weak solutions to the incompressible Navier-Stokes system constructed by Leray. In addition, both ϱ and ϑ are *positive* although with a possible exception of a set of zero Lebesgue measure. To the best of our knowledge, such a result for the *standard* Navier-Stokes-Fourier system lies beyond the scope of the available existence theory. Of course, the model is open to discussion regarding, in particular, the relevant value of the phenomenological coefficient K set constant in the present study.

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