SIAM J. NUMER. ANAL.
Vol. 36, No. 2, pp. 465–474

KINETIC SEMIDISCRETIZATION OF SCALAR CONSERVATION LAWS AND CONVERGENCE BY USING AVERAGING LEMMAS*

ALEXIS VASSEUR†

Abstract. We consider a time discrete kinetic scheme (known as "transport collapse method") for the inviscid Burgers equation $\partial_t u + \partial_x \frac{u^2}{2} = 0$. We prove the convergence of the scheme by using averaging lemmas without bounded variation estimate. Then, the extension of this result to the kinetic model of Brenier and Corrias is discussed.

Key words. scalar conservation laws, kinetic schemes, averaging lemmas

AMS subject classifications. 65M12, 35L65

PII. S0036142996313610

1. Introduction. Lions, Perthame, and Tadmor have shown in [10] that multidimensional scalar conservation laws can be reformulated as a kinetic equation, using an additional kinetic variable. Then the unknown becomes a "density-like" function $f(t, x, v)$. For the inviscid Burgers equation, the kinetic formulation is described in the following way:

$$\partial_t f + v\partial_x f = \partial_v \mu \quad \text{in } \mathbb{R}_t^+ \times \mathbb{R}_x \times \mathbb{R}_v^+ \tag{1.1}$$

with the constraint

$$f(t, x, v) = \mathbf{1}_{\{0 \le v \le \int f dv\}} \text{ a.e.,}$$

where $\mu(t, x, v)$ is a nonnegative bounded measure. Notice that μ is not given a priori; it can be seen as a multiplier related to the constraint on the shape of f. It is similar to the pression term in the incompressible Navier–Stokes equation, which is a multiplier related to the constraint that the flow is divergence free. Finally we recover the solution u of the inviscid Burgers equation denoting $u(t, x) = \int f(t, x, v)dv$ and integrating (1.1) with respect to v. Moreover, since μ is nonnegative, u is the classical entropy solution.

A generalization of this kinetic model was introduced by Brenier and Corrias in [4] to define a rigorous concept of entropy multivalued solutions with at most K branches for the inviscid Burgers equation. It is described in the following way:

$$\partial_t f + v\partial_x f = (-1)^{K-1}\partial_v^K \mu \quad \text{in } \mathbb{R}_t^+ \times \mathbb{R}_x \times \mathbb{R}_v^+ \tag{1.2}$$

with the constraint

$$f(t, x, v) = \mathcal{M}f(t, x, v) \text{ a.e.,}$$

where $\mu(t, x, v)$ is a nonnegative bounded measure and $\mathcal{M}f(t, x, v)$ is the so-called "Maxwellian" function related to f for almost every (t, x), defined by

$$\begin{aligned} \mathcal{M}f(t, x, v) &= \mathbf{1}_{\{\cup_{i=1}^p [a_i, b_i]\}}(v) &\quad \text{if } K = 2p, \\ \mathcal{M}f(t, x, v) &= \mathbf{1}_{\{[0,b_1]\cup_{i=2}^{p+1} [a_i, b_i]\}}(v) &\quad \text{if } K = 2p + 1, \end{aligned}$$

*Received by the editors December 11, 1996; accepted for publication (in revised form) March 12, 1998; published electronically February 11, 1999.
http://www.siam.org/journals/sinum/36-2/31361.html

†École Normale Supérieure, DMI, 45 rue d'Ulm, 75230 Paris Cedex 05, France and Laboratoire d'Analyse Numérique, Tour 55-65, 5ème étage, Université Pierre et Marie Curie 4, Place Jussieu, 75252 Paris Cedex 05, France (vasseur@dmi.ens.fr).

where $a_i(t,x)$ and $b_i(t,x)$ verify $a_i \le b_i \le a_{i+1}$ and are defined such that for every $0 \le k \le K-1$

$$\int_0^\infty v^k \mathcal{M}f(t,x,v)dv = \int_0^\infty v^k f(t,x,v)dv.$$

As above, μ is not given a priori and is a multiplier related to the constraint that f is equal to its Maxwellian. Notice that the definition of the Maxwellian above is equivalent to the following one which will be more convenient:

$$\left\{\begin{array}{l} \text{for all } k \text{ integer, } 0 \le k \le K-1, \\ \displaystyle\int_0^\infty v^k \mathcal{M}f(t,x,v)dv = \int_0^\infty v^k f(t,x,v)dv, \\ \text{for all } v \;\; \mathcal{M}f(t,x,v) \in \{0;1\}, \\ TV_v(\mathcal{M}f(t,x,\cdot)) \le K, \end{array}\right. \tag{1.3}$$

where TV_v denotes the total variation with respect to the v variable. Moreover (as shown in [4]), $v \mapsto \mathcal{M}f(t,x,v)$ minimizes $\int_0^\infty v^K g(v)dv$ among all functions g valued in $[0,1]$ such that $\int_0^\infty v^k g(v)dv = \int_0^\infty v^k f(t,x,v)dv$ for $k = 0,\dots,K-1$. In the cases $K=1$ and $K=2$, this system corresponds exactly to the kinetic formulation, respectively, of the inviscid Burgers equation written above and of the isentropic gas dynamic equations with $\gamma = 3$ due to Lions, Perthame, and Tadmor [11].

A natural time discrete scheme for this model is defined by

$$\begin{array}{l} \text{for all } t \in]n\Delta t; (n+1)\Delta t], \\ f_{\Delta t}(t,x,v) = \mathcal{M}f_{\Delta t}(n\Delta t; x-(t-n\Delta t)v; v), \end{array} \tag{1.4}$$

where $\Delta t > 0$ denotes the time step. It means that for $t \in]n\Delta t; (n+1)\Delta t]$, $f_{\Delta t}$ evolves by free streaming and at each time $n\Delta t$ the associated Maxwellian is substituted for the values of $f_{\Delta t}$. It is reminiscent of Chorin's projection method in incompressible fluid mechanics [5]. The problem of convergence of this scheme for two branches or more is open. Notice that in both [10] and [4] the existence of solutions is obtained by using the so-called Bhatnagar, Gross, and Krook (BGK) approximation

$$\partial_t f^\epsilon + v\partial_x f^\epsilon = \frac{1}{\epsilon}\left(\mathcal{M}f^\epsilon - f^\epsilon\right)$$

and letting ϵ go to zero. The time discrete scheme corresponds to

$$\partial_t f_{\Delta t} + v\partial_x f_{\Delta t} = \sum_{n\ge 0} \delta(t-n\Delta t)\left(\mathcal{M}f_{\Delta t} - f_{\Delta t}\right).$$

We see that the right-hand side is more singular than in the BGK approximation, which may induce a time oscillatory behavior of the approximate solutions. Related problems have been investigated by Desvillettes and Mischler for the time discretization of the Boltzmann equation [6]. However, for $K=1$, we easily see that this scheme is nothing but the "transport-collapse" method [2, 3] for the inviscid Burgers equation:

$$\partial_t u + \partial_x \frac{u^2}{2} = 0$$

(see also [8]). Indeed, in this case, $f_{\Delta t}(t,x,v) = \mathbf{1}_{[0,u_{\Delta t}(t,x)]}(v)$, where $u_{\Delta t}(t,x) = \int_0^\infty f_{\Delta t}(t,x,v)dv$. Thus, according to (1.4),

$$\text{for all } t \in [n\Delta t; (n+1)\Delta t[,$$

$$u_{\Delta t}(t,x) = \int_0^\infty \mathbf{1}_{\{0\le v\le u_{\Delta t}(n\Delta t, x-(t-n\Delta t)v)\}}(v)dv.$$

It is known in this case that

$$\text{for all } n \;\; TV_x\left(u_{\Delta t}((n+1)\Delta t,\cdot)\right) \le TV_x\left(u_{\Delta t}(n\Delta t,\cdot)\right)$$

(where TV denotes the total variation). Then the convergence of $u_{\Delta t}$ to u can be proved by using classical compactness tools. But the scheme is no longer total variation diminishing for $K \ge 2$ (see section 4).

The goal of this paper is to provide a new proof of convergence of the scheme for $K = 1$ by using one of the most powerful tools in kinetic theory, the so-called averaging lemmas [1, 9, 7]. It is based on the study of evolution for $t \in [n\Delta t, (n+1)\Delta t]$ of the entropy gap between the solution of the free transport equation and its Maxwellian. This proof does not need bounded variation (BV) estimates and shows the convergence even in the case when the initial datas are Δt-dependent and converge weakly $*$ in L^∞ as Δt goes to 0, which is impossible by using only BV estimates and L^1 contraction arguments. However, the method of proof cannot be easily extended to the general model of [4] and we explain why.

2. Convergence of the scheme. The first part of the proof is valid for arbitrary K. Therefore the general notations are kept in this section. Let $L > 0$ and $f^0(x,v) \in L^1_{(x,v)}$ Maxwellian, supported in v in $[0,L]$. We consider $f_{\Delta t}$ solution of the scheme defined by (1.4) and the initial data

$$f_{\Delta t}(0,x,v) = f^0(x,v).$$

We introduce

$$Df_{\Delta t}(t) = \int_x \int_v v^K \left(f_{\Delta t}(t,x,v) - \mathcal{M}f_{\Delta t}(t,x,v)\right) dvdx$$

which is the entropy gap between $f_{\Delta t}$ and its Maxwellian at time t. For all $t \ge 0$, we have $Df_{\Delta t}(t) \ge 0$. We can see it directly by using the second definition of $\mathcal{M}f(v)$. Moreover, $Df_{\Delta t}(t) = 0$ if and only if $f_{\Delta t}(t,x,v) = \mathcal{M}f_{\Delta t}(t,x,v)$ for almost every (x,v). So $Df_{\Delta t}(t)$ characterizes the gap between functions $f_{\Delta t}(t,\cdot,\cdot)$ and $\mathcal{M}f_{\Delta t}(t,\cdot,\cdot)$. For the general case we have the following easy result.

THEOREM 2.1. *The family $f_{\Delta t}$ has the following properties:*

(i) *For all $t \ge 0$, $f_{\Delta t}(t,x,v)$ and $\mathcal{M}f_{\Delta t}(t,x,v)$ lie in $L^1_{(x,v)}$ and are supported in v in $[0,L]$.*

(ii) *There exists a nonnegative measure $\mu_{\Delta t}(t,x,v)$ supported in v in $[0,L]$ so that*

$$\partial_t f_{\Delta t} + v\partial_x f_{\Delta t} = (-1)^{K-1}\partial_v^K \mu_{\Delta t}. \tag{2.1}$$

(iii) *We have the following estimate:*

$$\begin{aligned}\sum_{n=0}^{\infty} Df_{\Delta t}(n\Delta t) &= K! \int_0^\infty \int_{x,v} d\mu_{\Delta t}(t,x,v) \\ &\le L^K \int_{x,v} f^0(x,v)dxdv =: C(f^0).\end{aligned} \tag{2.2}$$

(iv) *There exist a bounded nonnegative measure* $\mu(t,x,v)$, *and* $f(t,x,v) \in L^\infty_t([0,+\infty[; L^1_{(x,v)}(\mathbb{R}^2))$ *with* $0 \le f(t,x,v) \le 1$ *so that, up to extraction,*

$$f_{\Delta t} \overset{L^\infty}{} \underset{\rightharpoonup}{weak*} f, \qquad \mu_{\Delta t} \overset{measure}{\rightharpoonup} \mu, \tag{2.3}$$

with

$$\partial_t f + v\partial_x f = (-1)^{K-1}\partial_v^K \mu.$$

Remarks.

(i) The results of this theorem are quite simple and do not depend on the number of branches K. However, they do not prove convergence of the scheme unless we can show that $f(t,x,v) = \mathcal{M}f(t,x,v)$ for almost every (t,x,v). This will be achieved only in the case $K=1$.

(ii) Property (2.2) implies that for all fixed $T>0$,

$$\lim_{\Delta t\to 0}\left(\sum_{0\le n\le \frac{T}{\Delta t}} Df_{\Delta t}(n\Delta t)\Delta t\right) = 0.$$

If we could write

$$\int_0^T Df_{\Delta t}(t)dt \sim \sum_{0\le n\le \frac{T}{\Delta t}} Df_{\Delta t}(n\Delta t)\Delta t,$$

passing to the limit would yield $\int_0^T Df(t)dt = 0$ and then, according to the properties of $Df(t)$, we would conclude that $f(t,x,v) = \mathcal{M}f(t,x,v)$ for almost every (t,x,v). Theorem 2.2 will provide such a result but only for the case $K=1$.

Proof of Theorem 2.1. From [4] we know that if $g(v)$ is supported in $[0,L]$ so is $\mathcal{M}g(v)$ and that $\int g(v)dv = \int \mathcal{M}g(v)dv$. Thus $\mathcal{M}f_{\Delta t}$ and $f_{\Delta t}$ are supported in $[0,L]$ in v and belong to $L^1_{(x,v)}$ since $f^0 \in L^1_{(x,v)}$.

(i) For every C^∞ function $\theta(v)$ on $[0,L]$ with $\frac{\partial^K\theta}{\partial v^K}(v) \ge 0$, we have

$$\begin{aligned} &\partial_t \int_v \theta(v) f_{\Delta t}(t,x,v)dv + \partial_x \int_v v\theta(v) f_{\Delta t}(t,x,v)dv \\ &= \sum_n \delta_{n\Delta t}(t)\int_v \theta(v)\left[f_{\Delta t}(n\Delta t,x,v) - \mathcal{M}f_{\Delta t}(n\Delta t,x,v)\right]dv. \end{aligned} \tag{2.4}$$

But according to the properties of $\mathcal{M}f_{\Delta t}$, the second term is nonpositive. This ensures the existence of a nonnegative measure supported in v in $[0,L]$ (as in [10] or [4]) satisfying (2.1).

(ii) If we integrate equation (2.4) on $[0,T]\times\mathbb{R}_x$ with $\theta(v) = v^K$, we get

$$\begin{aligned} &-\int v^K f_{\Delta t}(T,x,v)dxdv + \int v^K f^0(x,v)dxdv \\ &= K!\int_0^T\int_{x,v} d\mu_{\Delta t}(t,x,v) \end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \left(\sum_n \delta_{n\Delta t}(t) \int_{v,x} v^K \left[f_{\Delta t}(n\Delta t, x, v) - \mathcal{M} f_{\Delta t}(n\Delta t, x, v) \right] dvdx \right) dt \\
&= \sum_{0 \le n \le \frac{T}{\Delta t}} \int_x \int_v v^K \left(f_{\Delta t}(n\Delta t, x, v) - \mathcal{M} f_{\Delta t}(n\Delta t, x, v) \right) dvdx \\
&= \sum_{0 \le n \le \frac{T}{\Delta t}} Df_{\Delta t}(n\Delta t).
\end{aligned}
$$

The last step is obtained by the definition of $Df_{\Delta t}$ itself. Thus (2.2) is proved.

(iii) By definition, $f_{\Delta t}$ is uniformly bounded in L^∞. Moreover, according to (2.2), $\mu_{\Delta t}$ is uniformly bounded, too. Therefore there exist $f(t, x, v)$ and a nonnegative measure $\mu(t, x, v)$ so that, up to extraction, $f_{\Delta t}$ converges weakly $*$ in L^∞ to f and $\mu_{\Delta t}$ converges weakly to μ. Moreover, for all Δt, $0 \le f_{\Delta t}(t, x, v) \le 1$ so $0 \le f(t, x, v) \le 1$ and, for all Δt, $f_{\Delta t}$ and $\mu_{\Delta t}$ are supported in v in $[0, L]$, so $f(t, x, v)$ and $\mu(t, x, v)$ are supported in v in $[0, L]$, too. Finally, after passing to the limit in the sense of distributions in (2.1), we find

$$
\partial_t f + v \partial_x f = (-1)^{K-1} \partial_v^K \mu,
$$

which ends the proof of Theorem 2.1. $\square$

From now on we restrict our study to the case $K = 1$. The main observation which allows us to conclude, and which replaces the property of L^1-contraction and BV-boundedness used in [2, 3, 8], is the following one.

THEOREM 2.2. *Let $f^0 \in L^1(x, v)$, Maxwellian, supported in v in $[0, L]$. Let $f(t, x, v) = f^0(x - tv, v)$ be the solution of free transport equation $\partial_t f + v \partial_x f = 0$ with initial data f^0. The function $tDf(t)$ is nondecreasing with respect to t.*

This result shows that, up to a normalization factor, the gap between the solution of the free transport equation and its Maxwellian is time nondecreasing. We will give the proof in section 3, and it will be shown that it no longer works for $K = 2$ in section 4. Let us now show the convergence of the discrete scheme for $K = 1$ using this result.

So, for all $t \in]n\Delta t, (n+1)\Delta t]$,

$$
\frac{t - n\Delta t}{\Delta t} Df_{\Delta t}(t) \le Df_{\Delta t}((n+1)\Delta t).
$$

We introduce $h_{\Delta t}$ function of t defined on $]n\Delta t, (n+1)\Delta t]$ by

$$
h_{\Delta t}(t) = \frac{t - n\Delta t}{\Delta t}.
$$

After integrating in t we find $0 \le \int_0^T h_{\Delta t}(t) Df_{\Delta t}(t) dt \le C(f_0)\Delta t$ which shows that $\lim_{\Delta t \to 0} (\int_0^T h_{\Delta t}(t) Df_{\Delta t}(t) dt) = 0$. We notice that $h_{\Delta t} \overset{L^p \text{ weak}}{\rightharpoonup} \frac{1}{2}$ for $p \in [1, +\infty[$. So it is now sufficient to show the following theorem to conclude.

THEOREM 2.3. *For all $p \in [1, +\infty[$, up to extraction,*

$$
Df_{\Delta t} \overset{L^p \; strong}{\longrightarrow} Df.
$$

Indeed, it implies that $\lim_{\Delta t \to 0} (\int_0^T h_{\Delta t}(t) Df_{\Delta t}(t) dt) = \frac{1}{2} \int_0^T Df(t) dt$. It immediately follows that $\frac{1}{2} \int_0^T Df(t) dt = 0$. So according to the properties of Df, for almost every (t,x,v),

$$f(t,x,v) = \mathcal{M}f(t,x,v). \qquad \square$$

For the proof of Theorem 2.3 we use an averaging lemma from [7] or [4].

PROPOSITION 2.4. *Let $f_{\Delta t} \in L^\infty([0,T]\times\mathbb{R}_x\times\mathbb{R}_v^+)$ satisfying equation* (2.1) *for some nonnegative measure $\mu_{\Delta t}$ on $[0,T]\times\mathbb{R}_x\times\mathbb{R}_v^+$, bounded uniformly in Δt. If $f_{\Delta t}$ is bounded uniformly in Δt, then $\int_v f_{\Delta t}(t,x,v)\psi(v)dv$ belongs to a compact set of $L^p_{loc}([0,T]\times\mathbb{R}_x)$, $1<p<\infty$, for any $\psi \in L^{p'}(\mathbb{R}_v^+)$ with compact support.*

Remark. Notice that averaging lemmas give more than compactness. In fact, the function $\int_v f_{\Delta t}(t,x,v)\psi(v)dv$ is bounded in the Besov space $B_2^{s,p}(\mathbb{R}^+\times\mathbb{R})$ for a $p<2$ and an $s<1$. Unfortunately this amount of time regularity is not sufficient to prove the convergence of the scheme without the help of Theorem 2.2.

Proof of Theorem 2.3. Let us show it in the general case with K branches. Using Proposition 2.4, up to extraction, we have

$$\int_v v^K f_{\Delta t}(t,x,v)dv \overset{L^p_{t,x}\text{strong}}{\longrightarrow} \int_v v^K f(t,x,v)dv.$$

In order to prove Theorem 2.3, it is sufficient to prove

$$\int_v v^K \mathcal{M}f_{\Delta t}(t,x,v)dv \overset{L^p_{t,x}\text{strong}}{\longrightarrow} \int_v v^K \mathcal{M}f(t,x,v)dv.$$

Using the averaging lemma with $\psi(v)=v^k$ and $0\le k\le K-1$, and using (1.3), up to extraction, we find

$$\int_v v^k \mathcal{M}f_{\Delta t}(t,x,v)dv \overset{\text{ae }(t,x)}{\longrightarrow} \int_v v^k \mathcal{M}f(t,x,v)dv.$$

Let us assume for a moment the following lemma.

LEMMA 2.5. *Let $\mathcal{M}g_\epsilon(v)$ be a sequence of Maxwellian functions in v only, supported in $[0,L]$. We suppose that $\int_0^\infty v^k\mathcal{M}g_\epsilon(v)dv$ converges for all integers $0\le k<K$ to $\int_0^\infty v^k g(v)dv$. Then the entire sequence $\mathcal{M}g_\epsilon$ converges in L^1 strong to $\mathcal{M}g$.*

Thanks to this lemma, for almost every (t,x), $\mathcal{M}f_{\Delta t}(t,x,.)$ converges strongly in L^1_v to $\mathcal{M}f(t,x,.)$ (without further extraction). So,

$$\int_v v^K \mathcal{M}f_{\Delta t}(t,x,v)dv \overset{\text{ae }(t,x)}{\longrightarrow} \int_v v^K \mathcal{M}f(t,x,v)dv.$$

Finally, using Lebesgue's convergence theorem, we have

$$\int_v v^K \mathcal{M}f_{\Delta t}(t,x,v)dv \overset{L^p\text{strong}}{\longrightarrow} \int_v v^K \mathcal{M}f(t,x,v)dv,$$

which concludes the proof of the theorem. $\square$

Proof of Lemma 2.5. The definition of the Maxwellian gives $TV(g_\epsilon)\le K$. Thus there exists a subsequence ϵ_j such that $\mathcal{M}g_{\epsilon_j}$ converges in L^1 strong to a function h. So there exists a subsequence ϵ_{j_k} such that, for almost every v, $\mathcal{M}g_{\epsilon_{j_k}}(v)$ converges to $h(v)$, which shows that $h(v)$ is valued in $\{0,1\}$ and that $TV(h)\le K$. Thus h is a Maxwellian function, and so $h=\mathcal{M}g$. We have proved that $\mathcal{M}g_{\epsilon_j}$ converges to $\mathcal{M}g$ in L^1_v strong. Therefore, according to the uniqueness of the limit, the entire sequence $\mathcal{M}g_\epsilon$ converges to $\mathcal{M}g$ in L^1_v. $\square$

3. Estimate of the entropy gap between the solution of the free transport equation and its Maxwellian in the case $K = 1$. This section is devoted to the proof of Theorem 2.2. In this section K is fixed to one. The difficulties in extending this result to the case $K = 2$ will be shown in section 4.

Let $f^0(x, v) \in L^1(\mathbb{R}_x \times \mathbb{R}_v)$ be Maxwellian such that $f^0(x, v) = \mathbf{1}_{[0,m^0(x)]}(v)$ with $\mathrm{Supp}_v\left(f^0(x, v)\right) \subseteq [0, L]$. Let $f(t, x, v) = f_0(x - tv, v)$ be the solution of the free transport equation with f_0 as initial data and $m(t, x) = \int_0^L f(t, x, v)dv$. Let

$$\begin{aligned} l(t, x, v) &= \mathcal{L}\{v' \leq v \;/\; f(t, x, v') = 0\} \\ &= v - \int_0^v f(t, x, v')dv', \end{aligned}$$

where $\mathcal{L}$ is Lebesgue's measure on $[0, L]$. A new expression for $Df(t)$ depending on $l(t, x, v)$ will be shown. For this, the following lemma is needed.

LEMMA 3.1. *For all (t, x, v^*), $\mathcal{M}f(t, x, v^*) = 1$ if and only if there exists $v \geq 0$ such that $\int_0^v f(t, x, v')dv' = v^*$.*

Proof. Let f^* be the function defined in the following way: if there exists $v \geq 0$ such that $\int_0^v f(t, x, v')dv' = v^*$, then $f^*(t, x, v^*) = 1$ else $f^*(t, x, v^*) = 0$.

If $f^*(t, x, v^*) = 1$, then there exists $v \geq 0$ such that $\int_0^v f(t, x, v')dv' = v^*$. But

$$\int_0^v f(t, x, v')dv' \leq \int_0^L f(t, x, v')dv' = m(t, x),$$

so $\mathrm{Supp}_v f^*(t, x, \cdot) \subseteq [0, m(t, x)]$. Finally the function $F(t, x, v) = \int_0^v f(t, x, v')dv'$ is continuous in v, and we have $F(t, x, 0) = 0$ and $F(t, x, L) = m(t, x)$. So according to the mean value theorem, for all $v^* \leq m(t, x)$ there exists v such that $v^* = \int_0^v f(t, x, v')dv'$ and $f^*(t, x, v^*) = 1$. Therefore,

$$f^*(t, x, v^*) = \mathbf{1}_{[0,m(t,x)]}(v^*) = \mathcal{M}f(t, x, v^*). \qquad \square$$

This next proposition follows.

PROPOSITION 3.2. *The function Df can be written as*

$$\begin{aligned} Df(t) &= \int_{x,v} l(t, x, v)f(t, x, v)dvdx \\ &= \int_{x,v} \bar{l}(t, x, v)f_0(x, v)dvdx, \end{aligned}$$

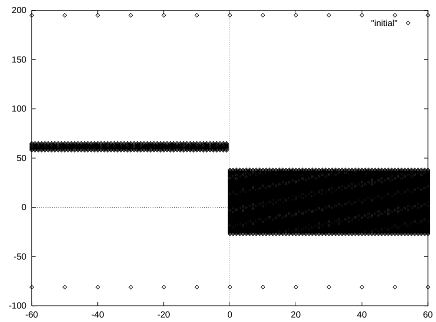
where $\bar{l}(t, x, v) = l(t, x + tv, v)$.

Proof.

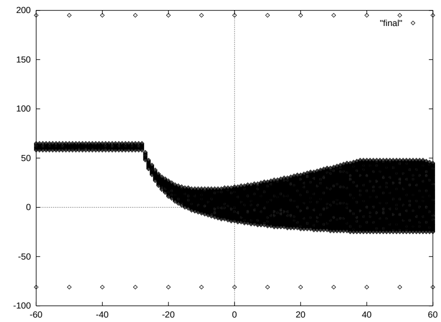
$$\begin{aligned} \int_0^L v^*\mathcal{M}f(t, x, v^*)dv^* &= \int_0^{m(t,x)} v^*\mathcal{M}f(t, x, v^*)dv^* \\ &= \int_0^L \left[\int_0^v f(t, x, v')dv'\right]\mathcal{M}f\left(t, x, \int_0^v f(t, x, v')dv'\right) f(t, x, v)dv. \end{aligned}$$

We have used the variable change $v^* = \int_0^v f(t, x, v')dv'$. Moreover, according to Lemma 3.1,

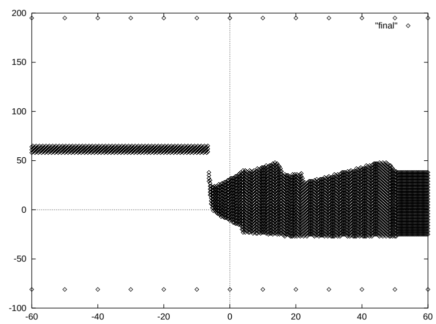
$$\mathcal{M}f\left(t, x, \int_0^v f(t, x, v')dv'\right) = 1;$$



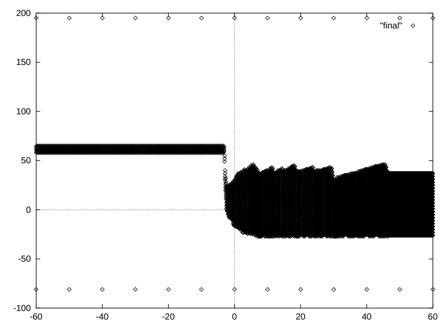
Exact solution



Approximation $\Delta t = 1$



Approximation $\Delta t = 0.3$



Approximation $\Delta t = 0.14$

FIG. 1.

therefore,

$$\begin{aligned}\int_0^L vf(t,x,v)dv - \int_0^L v^*\mathcal{M}f(t,x,v^*)dv^* &= \int_0^L \left[v - \int_0^v f(t,x,v')dv'\right] f(t,x,v)dv \\ &= \int_0^L l(t,x,v)f(t,x,v)dv.\end{aligned}$$

We conclude by integrating in x. □

PROPOSITION 3.3. *For all* (x,v), $t\,\bar{l}(t,x,v)$ *is a nondecreasing function with respect to* t.

Notice that this property is no longer true for $K = 2$, as shown in the next section. Finally, Theorem 2.2 is a straightforward consequence of Propositions 3.2 and 3.3.

Proof of Proposition 3.3. According to the definitions of $l(t,x,v)$ and $\bar{l}(t,x,v)$,

$$\begin{aligned}t\,\bar{l}(t,x,v) &= t\mathcal{L}\left\{v'/m_0\left(x + t(v - v')\right) - v \le v' - v \le 0\right\} \\ &= \mathcal{L}\left\{u/v - m_0(x+u) \ge \frac{u}{t} \ge 0\right\}.\end{aligned}$$

If $v - m_0(x+u) \ge \frac{u}{t}$, then for all $t' \ge t$, $v - m_0(x+u) \ge \frac{u}{t'}$. Therefore, for (x,v) fixed, $t\,\bar{l}(t,x,v)$ is nondecreasing with respect to t. □

4. Numerical examples for $K = 2$. This section deals with numerical examples for the case with two branches which illustrate the differences from the case

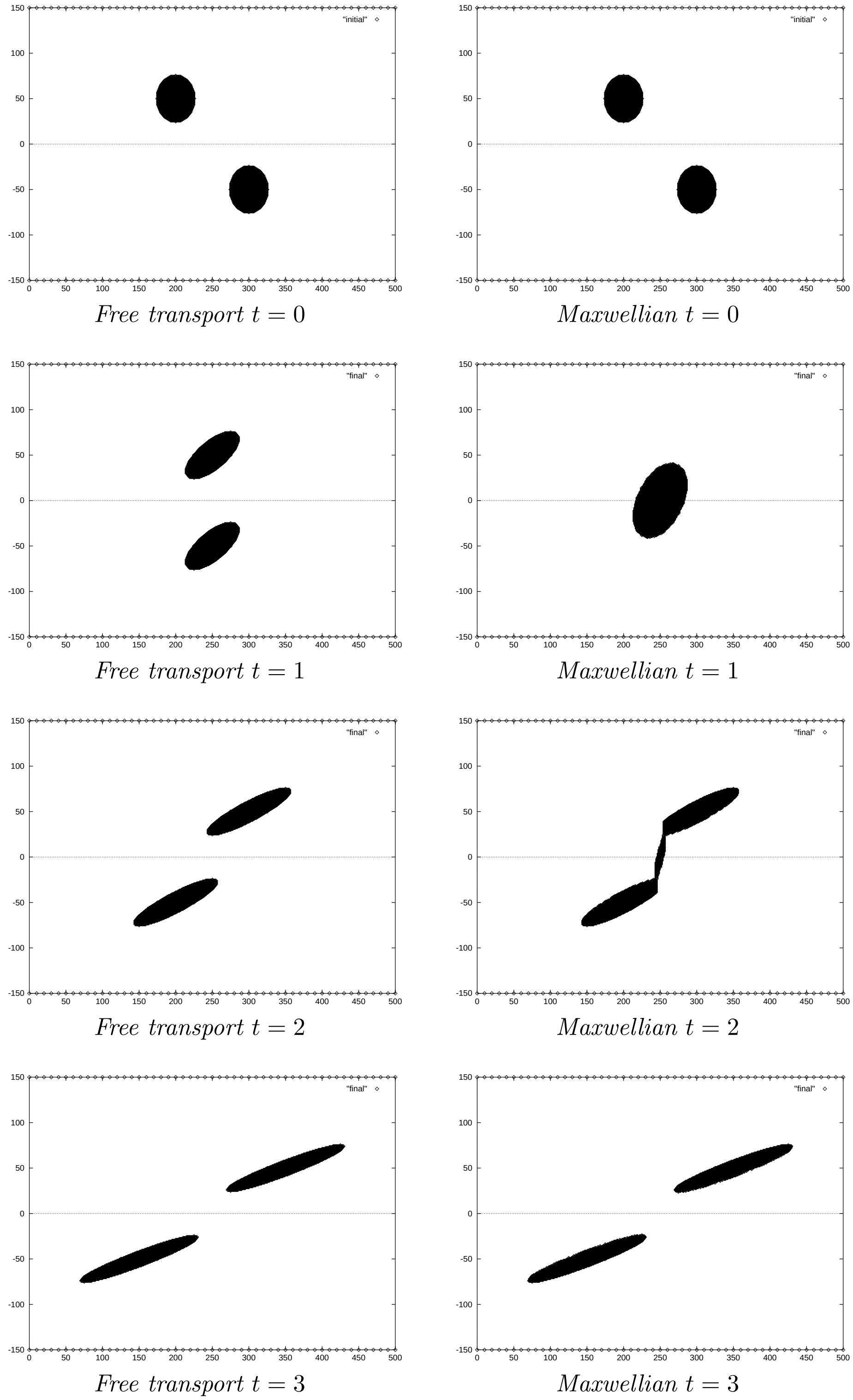

"initial"
"initial"
Free transport $t = 0$
Maxwellian $t = 0$
"final"
"final"
Free transport $t = 1$
Maxwellian $t = 1$
"final"
"final"
Free transport $t = 2$
Maxwellian $t = 2$
"final"
"final"
Free transport $t = 3$
Maxwellian $t = 3$

FIG. 2.

with only one branch. We recall here that this case corresponds to the isentropic gas dynamic equations with $p = \frac{\rho^3}{12}$, namely,

$$\partial_t \rho + \partial_x(\rho v) = 0,$$
$$\partial_t(\rho v) + \partial_x \left(\rho v^2 + \frac{\rho^3}{12} \right) = 0.$$

A particle method with a very large number of particles is used here in order to have a good approximation of the time discretization scheme which has no space grid.

First we consider a problem of Riemann's (Figure 1). The exact solution of (1.2) is here stationary. The momentum ρv is everywhere $P = 400$ and the ratio of left and right densities is $\frac{\rho_L}{\rho_R} = 0.1$. We show the computations at time $t = 1$ and for several values of Δt. At each computation the number of particles is $N = \frac{2000}{\Delta t}$.

Notice that there are more and more oscillations when Δt decreases. Therefore, we cannot expect a simple bound for $TV_x\,(m_{\Delta t}(\cdot, n\Delta t))$. (However, the amplitude of oscillations seems to decrease with the number of iterations.)

In the second example (Figure 2), we consider a function f^0 which is Maxwellian. On the left graphs, we show its transform by the free transport equation, and on the right graphs, we show the associated Maxwellian. The initial function is composed of two circular patches. Their radii are $R = 25$, and their centers are $(200, 50)$ and $(300, -50)$. The computations are given here with 4000 particles.

We can see that for $\Delta t \geq 3$, we have $Df_{\Delta t}(\Delta t) = 0$ (because the solution of the free transport equation is Maxwellian for $t \geq 3$). On the other hand, both $Df_{\Delta t}(1)$ and $Df_{\Delta t}(2)$ are positive. Therefore, we are not able to control $Df_{\Delta t}(t)$ for $t \in [0, \Delta t]$ by $Df_{\Delta t}(\Delta t)$.

Both examples show obstacles to the generalization of the proof of convergence of the time discrete kinetic scheme. The hopes to find a proof lie in a sharp study of oscillation decay.

REFERENCES

[1] V. I. Agoshkov, *Spaces of functions with differential-difference characteristics and the smoothness of solutions of the transport equation*, Sov. Math. Dokl., 29 (1984), pp. 662–666.

[2] Y. Brenier, *Résolution d'équations d'évolution quasilinéaires en dimension N d'espace à l'aide d'équations linéaires en dimension $N+1$*, J. Differential Equations, 50 (1983), pp. 375–390.

[3] Y. Brenier, *Averaged multivalued solutions for scalar conservation laws*, SIAM J. Numer. Anal., 21 (1984), pp. 1013–1037.

[4] Y. Brenier and L. Corrias, *A kinetic formulation for multi-branch entropy solution of scalar conservation laws*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 15 (1998), pp. 169–190.

[5] A. Chorin, *Numerical solutions of the Navier-Stokes equations*, Math. Comp., 22 (1968), pp. 745–762.

[6] L. Desvillettes and S. Mischler, *About the splitting algorithm for Boltzmann and B.G.K. equations*, Math. Models Methods Appl. Sci., 6 (1996), pp. 1079–1101.

[7] R. J. DiPerna, P. L. Lions, and Y. Meyer, *L^p regularity of velocity averages*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 8 (1991), pp. 271–287.

[8] Y. Giga and T. Miyakawa, *A kinetic construction of global solutions of first order quasilinear equations*, Duke Math. J., 50 (1983), pp. 505–515.

[9] F. Golse, B. Perthame, and R. Sentis, *Un résultat de compacité pour les équations de transport*, C. R. Acad. Sci. Paris Sér. I Math., 301 (1985), pp. 341–344.

[10] P. L. Lions, B. Perthame, and E. Tadmor, *A kinetic formulation of multidimensional scalar conservation laws and related equations*, J. Amer. Math. Soc., 7 (1994), pp. 169–191.

[11] P. L. Lions, B. Perthame, and E. Tadmor, *Kinetic formulation of the isentropic gas dynamics and p-systems*, Comm. Math. Phys., 163 (1994), pp. 415–431.