GLOBAL EXISTENCE OF SOLUTIONS TO ONE-DIMENSIONAL VISCOUS QUANTUM HYDRODYNAMIC EQUATIONS

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Abstract. The existence of global-in-time weak solutions to the one-dimensional viscous quantum hydrodynamic equations is proved. The model consists of the conservation laws for the particle density and particle current density, including quantum corrections from the Bohm potential and viscous stabilizations arising from quantum Fokker-Planck collision terms in the Wigner equation. The model equations are coupled self-consistently to the Poisson equation for the electric potential and are supplemented with periodic boundary and initial conditions. When an additional diffusion term in the velocity is introduced in the momentum equation, the positivity of the particle density is proved. Without this regularization, only the nonnegativity of the density can be shown. The existence proof relies on the Faedo-Galerkin method together with a priori estimates from the energy functional.

1. Introduction

Diffusive corrections in quantum models are of great importance in open quantum systems modeling, for instance, an electron ensemble interacting with a background heat bath. Applications of such systems include quantum semiconductor structures in which diffusive effects may be relevant in some regimes. Caldeira and Leggett [3] and Diósi [8] have derived closed equations for a dissipative quantum-mechanical system related to quantum Brownian motion. Their approach was later improved by Castella et al. [4] and leads to a Wigner equation with Fokker-Planck-type collision operator. Since the Wigner equation is computationally very expensive, macroscopic models, which are numerically easier to solve, were derived. For instance, employing a moment method and a suitable closure condition to the Wigner equation, quantum hydrodynamic equations are obtained [7, 12]. Another derivation comes from the mixed-state Schrödinger system via the Madelung transform [13]. When a Fokker-Planck collision operator is included in the Wigner equation, viscosity terms appear in the macroscopic model (see [14, 17, 18] for a derivation), leading to *viscous quantum hydrodynamic equations*, which are the subject of this paper.

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The (scaled) viscous quantum hydrodynamic equations in one space dimension for the particle density ρ, the velocity u, and the electric potential V read as follows:

$$\rho_t + (\rho u)_x = \nu\rho_{xx}, \tag{1}$$

$$(\rho u)_t + (\rho u^2 + p(\rho))_x - \rho V_x - \frac{\delta^2}{2}\rho\Big(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\Big)_x = \nu(\rho u)_{xx} - \frac{\rho u}{\tau}, \tag{2}$$

$$\lambda^2 V_{xx} = \rho - C(x), \tag{3}$$

The pressure $p(\rho)$ is assumed to depend on the particle density; typical examples are $p(\rho) = p_0\rho^\alpha$ for some $p_0 > 0$ and $\alpha \geq 1$. The function $C(x)$ is the doping profile modeling charged background ions in, for instance, semiconductor crystals. The (constant and positive) physical parameters are the viscosity ν, the scaled Planck constant δ, the momentum relaxation time τ, and the Debye length λ. Equations (1)-(3) are considered on the one-dimensional torus $\mathbb{T}$ (with size one) and are complemented with the initial conditions

$$\rho(0,\cdot) = \rho_0, \quad (\rho u)(0,\cdot) = \rho_0 u_0 \quad \text{in } \mathbb{T}. \tag{4}$$

Equation (1) expresses mass conservation with the effective current density $J_0 = \rho u - \nu\rho_x$. The second equation is the balance equation for the particle current density or momentum ρu including the electric force term ρV_x, the relaxation-time term $-\rho u/\tau$, and the quantum correction with the Bohm potential $(\sqrt{\rho})_{xx}/\sqrt{\rho}$. The electric potential V is self-consistently given by the Poisson equation (3). In the absence of viscous and quantum effects, i.e. $\nu = \delta = 0$, the above equations represent the hydrodynamic semiconductor equations [2]. When no viscous effects are present, $\nu = 0$, we obtain the quantum hydrodynamic equations, studied in, e.g., [11, 16]. For more recent papers, we refer to [1, 15, 21, 22, 23].

The viscous quantum hydrodynamic model can be derived from the Wigner-Fokker-Planck equation by a moment method [14, 18]. The viscous regularizations arise from the Fokker-Planck collision operator. More precisely, the part of the scattering operator yielding the viscous terms is

$$Q(w) = \nu w_{xx},$$

where $w(x,k,t)$ is the Wigner function on the position-wave vector space (x,k), and $\nu > 0$ depends on the friction and the lattice temperature. Introducing the moments $\rho = \int_{\mathbb{R}^3} w dk$ and $\rho u = \int_{\mathbb{R}^3} wk dk$ gives

$$\int_{\mathbb{R}^3} Q(w)dk = \nu\rho_{xx}, \quad \int_{\mathbb{R}^3} Q(w)kdk = \nu(\rho u)_{xx},$$

which are the viscous terms in (1)-(2). Thus, they are *not* artificial regularizations but coming from the choice of the quantum collision operator in the Wigner equation.

There are only few mathematical results for the viscous quantum hydrodynamic model due to difficulties coming from the third-order derivatives in the quantum correction. The existence of classical solutions to the one-dimensional stationary model with physical boundary conditions was shown in [18]. The transient equations are considered in [5, 6, 9], and the local-in-time existence and exponential stability of solutions were proved. Global-in-time solutions in one space dimension are obtained if the initial energy is assumed to be sufficiently small [5]. In [18, 19], numerical solutions of the model and applications to resonant tunneling diodes were presented. However, no *global-in-time* existence result *without* smallness conditions seems to be available up to now for the transient system (1)-(3). In this paper, we prove such a result.

The main problem of the existence analysis lies in the strongly nonlinear third-order differential operator and the dispersive structure of the momentum equation. There are several attempts in the literature to deal with the quantum term. Integrating the stationary momentum equation leads to a second-order differential equation to which maximum principle arguments can be applied [11]. A fourth-order wave equation is obtained after differentiating the equation with respect to the spatial variable. This approach was employed in [20] to prove the existence of global solutions to the quantum hydrodynamic equations with $\nu = 0$, but only for initial data close to thermal equilibrium. The main idea of [5] was to introduce a bi-Laplacian regularization in the viscous model and to employ energy estimates to conclude local existence of solutions. Global existence of solutions to the inviscid model $\nu = 0$ with nonnegative particle density was achieved recently by a wave function polar decomposition method [1].

In this paper, we pursue a different strategy. We employ the Faedo-Galerkin method introduced by Feireisl in [10] for the analysis of the compressible Navier-Stokes equations. In particular, we introduce in the momentum equation (2) an additional regularization in the velocity, and we prove first the existence of solutions to the system

$$\rho_t + (\rho u)_x = \nu \rho_{xx}, \tag{5}$$

$$(\rho u)_t + (\rho u^2 + p(\rho))_x - \rho V_x - \frac{\delta^2}{2}\rho\Big(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\Big)_x = \nu(\rho u)_{xx} + \varepsilon u_{xx} - \frac{\rho u}{\tau}, \tag{6}$$

$$\lambda^2 V_{xx} = \rho - C(x), \quad t > 0,\ x \in \mathbb{T}, \tag{7}$$

with the initial conditions (4), where $\varepsilon > 0$. Let us mention that the diffusive velocity term can be derived from the *ad-hoc* quantum collision operator

$$Q(w) = \varepsilon \partial_x^2 \Big(\frac{w}{\int_{\mathbb{R}^3} w dk} \Big),$$

since

$$\int_{\mathbb{R}^3} Q(w) dk' = \varepsilon \partial_x^2 \int_{\mathbb{R}^3} \frac{w}{\int_{\mathbb{R}^3} w dk} dk' = 0,$$

$$\int_{\mathbb{R}^3} Q(w) k' dk' = \varepsilon \partial_x^2 \int_{\mathbb{R}^3} \frac{w k'}{\int_{\mathbb{R}^3} w dk} dk' = \varepsilon \partial_x^2 \frac{\rho u}{\rho} = \varepsilon u_{xx}.$$

The existence proof relies on the following ideas. First, for given u in a finite-dimensional Galerkin space, we solve (5). Since u is given and (5) is parabolic, a lower bound for the particle density can be concluded from the maximum principle. Classically, this bound depends on the L^∞ norm of u_x which is prohibitive to set up the fixed-point argument. We prove that the lower bound for ρ depends only on the L^2 norm of u_x (Lemma 3). In the second step we solve the Poisson equation and then the momentum equation in the Galerkin space, for given ρ, which yields the existence of local-in-time solutions via Banach's fixed-point theorem.

A priori bounds (and thus global-in-time existence) are obtained from the energy inequality. Let the enthalpy function h be defined by $h'(y) = p'(y)/y$ for $y > 0$ and $h(1) = 0$, and let H be a primitive of h. Furthermore, let the energy, consisting of the internal, kinetic, electric, and quantum energy, be given by

$$E(\rho,u)=\int_{\mathbb{T}}\Big(H(\rho)+\frac{1}{2}\rho u^2+\frac{\lambda^2}{2}V_x^2+\frac{\delta^2}{2}(\sqrt{\rho})_x^2\Big)dx.$$

Then we show that

$$\frac{dE}{dt}+\nu\int_{\mathbb{T}}(\rho u_x^2+\delta^2(\sqrt{\rho})_{xx}^2)dx+\varepsilon\int_{\mathbb{T}}u_x^2dx\le K,$$

where the constant $K>0$ depends only on $C(x)$, ν, and λ. This yields H^2 estimates for $\sqrt{\rho}$ and, for fixed $\varepsilon>0$, L^2 estimates for u_x, needed in the proof of the lower bound for ρ. We consider the one-dimensional equations since we need several times in the proof the embedding $H^1(\mathbb{T})\hookrightarrow L^\infty(\mathbb{T})$ which is valid in one space dimension only. Our first main result reads as follows.

Theorem 1. *Let $T>0$, $\varepsilon>0$, and $C\in L^2(\mathbb{T})$. Let the pressure function $p\in C^1([0,\infty))$ be monotone and let the primitive H of the enthalpy satisfy $H(y)\ge h_0(y-1)$ for all $y\ge 0$ and for some $h_0>0$. Furthermore, let the initial datum $(\rho_0,u_0)\in H^1(\mathbb{T})\times L^\infty(\mathbb{T})$ satisfy $\int_{\mathbb{T}}\rho_0(x)dx=\int_{\mathbb{T}}C(x)dx$, $\rho(x)\ge\eta_0>0$ for $x\in\mathbb{T}$ and for some $\eta_0>0$, and $E(\rho_0,u_0)<\infty$. Then there exists a constant $\eta>0$ and a weak solution (ρ,u,V) to (5)-(7) satisfying*

$$\begin{aligned}&\rho(t,x)\ge\eta>0\quad\text{for } t>0,\ x\in\mathbb{T},\quad V\in L^\infty(0,T;H^2(\mathbb{T})),\\&\rho_t\in L^2(0,T;L^2(\mathbb{T})),\quad(\rho u)_t\in L^2(0,T;H^{-2}(\mathbb{T})),\\&\rho\in L^\infty(0,T;H^1(\mathbb{T}))\cap L^2(0,T;H^2(\mathbb{T})),\\&u\in L^2(0,T;H^1(\mathbb{T}))\cap L^\infty(0,T;L^2(\mathbb{T})),\end{aligned}$$

where the lower bound $\eta>0$ depends on ε. The initial conditions (4) are satisfied in the sense of $H^{-2}(\mathbb{T})$.

The condition $H(y)\ge h_0(y-1)$ is satisfied, for instance, if the pressure is given by $p(\rho)=p_0\rho^\alpha$, where $p_0>0$ and $\alpha>0$. The regularity of ρ and u implies that $\rho u\in L^2(0,T;H^1(\mathbb{T}))$ and $\rho u^2\in L^2(0,T;W^{1,1}(\mathbb{T}))$.

Let $(\rho_\varepsilon,u_\varepsilon)$ be a solution to (5)-(7) in the sense of the above theorem. In the limit $\varepsilon\to 0$ we loose the lower bound for ρ_ε since it depends on ε. Furthermore, it is not clear how to pass to the limit in $\rho_\varepsilon u_\varepsilon^2$, since we have only weak convergence of $\sqrt{\rho_\varepsilon}u_\varepsilon$ in L^2. Moreover, we loose the control on u_ε and obtain results for the current density $J=\lim_{\varepsilon\to 0}\rho_\varepsilon u_\varepsilon$ only. In order to overcome these difficulties, we multiply the momentum equation by $\rho_\varepsilon^{3/2}$ and pass to the limit $\varepsilon\to 0$ in the resulting equation. This allows us to control the convective part since

$$\rho_\varepsilon^{3/2}(\rho_\varepsilon u_\varepsilon^2)_x=\big(\sqrt{\rho_\varepsilon}(\rho_\varepsilon u_\varepsilon)^2\big)_x-3(\sqrt{\rho_\varepsilon})_x(\rho_\varepsilon u_\varepsilon)^2.$$

Our second main result is summarized in the following theorem.

Theorem 2. *Let $T>0$ and let the assumptions of Theorem 1 hold. Then there exists a weak solution (ρ,J,V) to (1)-(3) with the regularity*

$$\begin{aligned}&\rho(t,x)\ge 0\quad\text{for } t>0,\ x\in\mathbb{T},\quad V\in L^\infty(0,T;H^2(\mathbb{T})),\\&\rho_t\in L^2(0,T;L^2(\mathbb{T})),\quad(\rho^{3/2}J)_t\in L^2(0,T;H^{-1}(\mathbb{T})),\\&\sqrt{\rho}\in L^\infty(0,T;H^1(\mathbb{T}))\cap L^2(0,T;H^2(\mathbb{T})),\quad J\in L^2(0,T;H^1(\mathbb{T})),\end{aligned}$$

satisfying $\rho_t + J_x = \nu\rho_{xx}$ *and* $\lambda^2 V_{xx} = \rho - C(x)$ *almost everywhere in* $(0,T)\times\mathbb{T}$ *and, for all* $\phi \in L^\infty(0,T;\, H^1(\mathbb{T}))$,

$$\begin{aligned}(8)\qquad &\int_0^T \langle(\rho^{3/2}J)_t, \phi\rangle_{H^{-1},H^1}dt - \frac{3}{2}\int_0^T\int_{\mathbb{T}}\sqrt{\rho}\rho_t J\phi dxdt \\ &- \int_0^T\int_{\mathbb{T}} J^2\big(3(\sqrt{\rho})_x\phi + \sqrt{\rho}\phi_x\big)dxdt + \int_0^T\int_{\mathbb{T}}\big((p(\rho))_x - \rho V_x\big)\rho^{3/2}\phi dxdt \\ &+ \frac{\delta^2}{2}\int_0^T\int_{\mathbb{T}}(\sqrt{\rho})_{xx}\big(5\rho^{3/2}(\sqrt{\rho})_x\phi + \rho^2\phi_x\big)dxdt \\ &= -\nu\int_0^T\int_{\mathbb{T}} J_x\rho\big(3(\sqrt{\rho})_x\phi + \sqrt{\rho}\phi_x\big)dxdt - \frac{1}{\tau}\int_0^T\int_{\mathbb{T}}\rho^{3/2}J\phi dxdt.\end{aligned}$$

The initial conditions are fulfilled in the following sense:

$$\rho(0,\cdot) = \rho_0 \text{ in } L^2(\mathbb{T}), \quad (\rho^{3/2}J)(0,\cdot) = \rho_0^{5/2}u_0 \text{ in } H^{-1}(\mathbb{T}).$$

Equation (8) is the weak formulation of

$$\begin{aligned}&(\rho^{3/2}J)_t - (\rho^{3/2})_t J + (\sqrt{\rho}J^2)_x - 3J^2(\sqrt{\rho})_x - \frac{\delta^2}{2}\big(\rho^2(\sqrt{\rho})_{xx}\big)_x + \frac{5\delta^2}{8}(\rho^2)_x(\sqrt{\rho})_{xx} \\ &\quad - \nu(\rho^{3/2}J_x)_x + \nu J_x(\rho^{3/2})_x + \rho^{3/2}\Big((p(\rho))_x - \rho V_x + \frac{J}{\tau}\Big) = 0,\end{aligned}$$

which is obtained from (2) after multiplication of $\rho^{3/2}$ and setting $J = \rho u$. If the limit density ρ is positive and smooth, we can divide the above equation by $\rho^{3/2}$ and recover the original formulation (2).

The paper is organized as follows. In the next section, we solve, for given velocity u, equation (5) for ρ, prove a lower bound for ρ only depending on the L^2 norm of u_x, and solve (6) locally in time. In section 3 we show the energy estimates and infer a global existence result for the nonlinear Faedo-Galerkin problem. Theorem 1 is proved in section 4, whereas section 5 is concerned with the proof of Theorem 2.

2. Linear Faedo-Galerkin approximation

In this section, we prove the existence of solutions to the linearized viscous quantum hydrodynamic equations with $\varepsilon > 0$. Let $T > 0$ and let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{T})$ which is also an orthogonal basis of $H^1(\mathbb{T})$. For instance, one may take the eigenfunctions of $-\partial_x^2$ with eigenvalues $\mu_n > 0$, given by

$$\begin{aligned} e_{2n}(x) &= \sqrt{2}\cos(2n\pi x), \quad \mu_{2n} = 8n^2\pi^2, \\ e_{2n+1}(x) &= \sqrt{2}\sin(2n\pi x), \quad \mu_{2n+1} = 8n^2\pi^2, \quad n \in \mathbb{N}_0.\end{aligned}$$

Introduce the finite-dimensional space $X_n = \text{span}(e_0,\ldots,e_n)$. We denote by $C^k(0, T; Z)$ the space of C^k functions on $[0,T]$ with values in the Banach space Z. Furthermore, let $(\rho_0, u_0) \in C^\infty(\mathbb{T})^2$ be some initial data satisfying $\rho_0(x) \ge \eta_0 > 0$ for $x \in \mathbb{T}$ and $\int_{\mathbb{T}}\rho_0 dx = \int_{\mathbb{T}} C(x)dx$. Finally, let $v \in C^0(0,T;X_n)$ be given. We notice that v can be written as

$$v(t,x) = \sum_{i=1}^n \lambda_i(t)e_i(x), \quad t\in[0,T],\ x\in\mathbb{T},$$

for some $\lambda_i(t)$, and we have

$$\|v\|_{C^0(0,T;X_n)} = \max_{t\in[0,T]} \sum_{i=1}^{n} |\lambda_i(t)|.$$

As a consequence, v can be bounded in $C^0(0,T;C^k(\mathbb{T}))$ for any $k \in \mathbb{N}$, and there exists a constant $K_k > 0$ depending on k such that

$$\|v\|_{C^0(0,T;C^k(\mathbb{T}))} \le K_k \|v\|_{C^0(0,T;L^2(\mathbb{T}))}.$$

Now, we define the approximate system. Let ρ be the classical solution to

$$\rho_t + (\rho v)_x = \nu \rho_{xx}, \quad x \in \mathbb{T},\ t > 0, \tag{9}$$

$$\rho(0,x) = \rho_0(x), \quad x \in \mathbb{T}. \tag{10}$$

The solution satisfies $\rho \in C^0(0,T;C^k(\mathbb{T}))$ for any $k \in \mathbb{N}$. Furthermore, it holds $\int_{\mathbb{T}} \rho dx = \int_{\mathbb{T}} \rho_0 dx = \int_{\mathbb{T}} C(x)dx$. We introduce the operator $S : C^0(0,T;X_n) \to C^0(0,T;C^3(\mathbb{T}))$ by $S(v) = \rho$. Since v is smooth, the maximum principle shows that $\rho = S(v)$ is bounded from above and below, i.e., for $\|v\|_{C^0(0,T;L^2(\mathbb{T}))} \le c$, there exist positive constants $K_0(c)$ and $K_1(c)$ depending on c such that

$$0 < K_0(c) \le (S(v))(t,x) \le K_1(c), \quad t \in [0,T],\ x \in \mathbb{T}. \tag{11}$$

Furthermore, since the equation for ρ is linear, there exists $K_2 > 0$ depending on k and n such that for all v_1, $v_2 \in C^0(0,T;X_n)$,

$$\|S(v_1) - S(v_2)\|_{C^0(0,T;C^k(\mathbb{T}))} \le K_2 \|v_1 - v_2\|_{C^0(0,T;L^2(\mathbb{T}))}. \tag{12}$$

We claim that the lower bound for $S(v)$ only depends on the $L^2(0,T;L^2(\mathbb{T}))$ norm of v_x,

$$\rho = S(v) \ge \eta = \eta(\|v_x\|_{L^2(0,T;L^2(\mathbb{T}))}) > 0 \quad \text{in } [0,T] \times \mathbb{T}. \tag{13}$$

This result is a consequence of the following lemma whose proof is presented at the end of this section.

Lemma 3. *Let $T > 0$ and $v \in L^2(0,T;H^1(\mathbb{T}))$. Let ρ be the solution to (9)-(10) with initial datum $\rho_0 \in L^\infty(\mathbb{T})$ satisfying $\rho_0(x) \ge \eta_0 > 0$ for $x \in \mathbb{T}$. Then there exists a constant $\eta > 0$ only depending on ν, ρ_0, and the $L^2(0,T;L^2(\mathbb{T}))$ norm of v_x such that*

$$\rho(t,x) \ge \eta > 0, \quad t \in [0,T],\ x \in \mathbb{T}.$$

Next, for given $\rho = S(v)$, we wish to solve the following linear problem on X_n for u_n:

$$\begin{aligned}(\rho u_n)_t + (\rho v u_n + p(\rho))_x - \rho(V[\rho])_x - \frac{\delta^2}{2}\rho\Big(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\Big)_x \\ = \nu(\rho u_n)_{xx} + \varepsilon(u_n)_{xx} - \frac{\rho u_n}{\tau},\end{aligned} \tag{14}$$

where $V[\rho] \in C^0(0,T;C^2(\mathbb{T}))$ is the unique solution to

$$\lambda^2 (V[\rho])_{xx} = \rho - C(x) \quad \text{in } \mathbb{T} \tag{15}$$

satisfying $\int_{\mathbb{T}} V dx = 0$. More explicitly, we are looking for a function $u_n \in C^0(0,T; X_n)$ verifying, for all test functions $\phi \in C^1(0,T;X_n)$ with $\phi(T,\cdot) = 0$,

$$\int_{\mathbb{T}} \rho u_n \phi_t dx + \int_{\mathbb{T}} \big(\rho v u_n + p(\rho)\big)\phi_x dx + \int_{\mathbb{T}} \rho (V[\rho])_x \phi dx - \frac{\delta^2}{2}\int_{\mathbb{T}} \frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}(\rho\phi)_x dx$$
$$- \int_{\mathbb{T}} \big(\nu \rho u_n + \varepsilon u_n\big)_x \phi_x dx - \frac{1}{\tau}\int_{\mathbb{T}} \rho u_n \phi dx = \int_{\mathbb{T}} \rho_0 u_0 \phi(0,\cdot) dx.$$

For given $\rho \in N_\eta = \{\rho \in L^1(\mathbb{T}) : \inf_{x\in\mathbb{T}} \rho \geq \eta > 0\}$, we introduce the following family of operators, following [10]:

$$M[\rho] : X_n \to X_n^*, \quad \langle M[\rho]u, w\rangle = \int_{\mathbb{T}} \rho u w dx, \quad u, w \in X_n.$$

These operators are symmetric and positive definite with the smallest eigenvalue

$$\inf_{\|w\|_{L^2(\mathbb{T})}=1} \langle M[\rho]w, w\rangle = \inf_{\|w\|_{L^2(\mathbb{T})}=1} \int_{\mathbb{T}} \rho w^2 dx \geq \inf_{x\in\mathbb{T}} \rho(x) > \eta,$$

employing the bound (13). Hence, as we are working in finite dimensions, the operators are invertible with

$$\|M^{-1}[\rho]\|_{\mathcal{L}(X_n^*, X_n)} \leq \eta^{-1},$$

where $\mathcal{L}(X_n^*, X_n)$ is the set of bounded linear mappings from X_n^* to X_n. Moreover, similar as in [10], it holds:

$$\|M^{-1}[\rho_1] - M^{-1}[\rho_2]\|_{\mathcal{L}(X_n^*, X_n)} \leq K(n,\eta)\|\rho_1 - \rho_2\|_{L^1(\mathbb{T})} \tag{16}$$

for all ρ_1, $\rho_2 \in N_\eta$. With these notations, we can rephrase problem (14) as an ordinary differential equation on the finite-dimensional space X_n:

$$\frac{d}{dt}\big(M[\rho(t)]u_n(t)\big) = N[v, u_n(t)], \quad t > 0, \quad M[\rho_0]u_n(0) = M[\rho_0]u_0, \tag{17}$$

where

$$\langle N[v,u_n], \phi\rangle = \int_{\mathbb{T}} \Big(- \big(\rho v u_n + p(\rho)\big)_x + \frac{\delta^2}{2}\Big(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\Big)_x + \rho(V[\rho])_x$$
$$+ \nu(\rho u_n)_{xx} + \varepsilon(u_n)_{xx} - \frac{1}{\tau}\rho u_n\Big)\phi dx, \quad \phi \in X_n.$$

Recall that $\rho = S(v) \in C^0(0,T;C^3(\mathbb{T}))$ is bounded from below, so the above integral is well defined. The operator $N[v,\cdot]$, defined for every $t \in [0,T]$ as an operator from X_n to X_n^*, is continuous in time. Then, standard theory of finite-dimensional systems of differential equations provides the existence of a unique C^1 solution of (17). In other words, there exists a unique solution $u_n \in C^1(0,T;X_n)$ to (14).

Proof of Lemma 3. We introduce the function

$$L(t,x) = \ln\frac{1}{\rho}\left(t, x + \int_0^t \int_{\mathbb{T}} v(s,y) dy ds\right),$$

which is a solution to

$$L_t - \nu L_{xx} = v_x - \tilde{v}L_x - \nu(L_x)^2,$$

where

$$\tilde{v} = v - \int_{\mathbb{T}} v dx.$$

Since

$$|\tilde{v}L_x| = \left|\sqrt{\frac{1}{2\nu}}\tilde{v}\sqrt{2\nu}L_x\right| \leq \frac{\tilde{v}^2}{4\nu} + \nu(L_x)^2,$$

we obtain

$$L_t - \nu L_{xx} \leq v_x + \frac{\tilde{v}^2}{4\nu}.$$

The idea is to show an upper bound for L which only depend on η_0, ν, and the L^2-norm of v_x and from which the lower bound for ρ follows. This is achieved by estimating the solution ψ to a certain parabolic problem and using the comparison principle to obtain $L \leq \psi$. We introduce the functions ψ_1, which is a solution to

$$\begin{aligned} &(\psi_1)_t - \nu(\psi_1)_{xx} = v_x, \quad x \in \mathbb{T},\ t > 0, \\ &\psi_1(0,x) = 0, \quad x \in \mathbb{T}, \end{aligned}$$

and ψ_2, which solves

$$\begin{aligned} &(\psi_2)_t - \nu(\psi_2)_{xx} = \frac{\tilde{v}^2}{4\nu}, \quad x \in \mathbb{T},\ t > 0, \\ &\psi_1(0,x) = L(0,x) = \ln\frac{1}{\rho_0(x)}, \quad x \in \mathbb{T}. \end{aligned}$$

First, notice that $\tilde{v}_x = v_x$ and $\int_{\mathbb{T}} \tilde{v}\,dx = 0$. Hence, by the Poincaré inequality in one space dimension,

$$\|\tilde{v}^2\|_{L^1(0,T;L^\infty(\mathbb{T}))} = \int_0^T \|\tilde{v}\|^2_{L^\infty(\mathbb{T})}dt \leq \int_0^T \|\tilde{v}_x\|^2_{L^2(\mathbb{T})}dt = \|v_x\|^2_{L^2(0,T;L^2(\mathbb{T}))}.$$

This shows that

$$\psi_2(t,x) \leq \frac{1}{4\nu}\|v_x\|^2_{L^2(L^2)} + \ln\frac{1}{\eta_0}, \quad t > 0, x \in \mathbb{T}. \tag{18}$$

Multiplying the equation for ψ_1 by $-(\psi_1)_{xx}$ and integrating over $\mathbb{T}$, we find that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}(\psi_1)_x^2 dx + \nu\int_{\mathbb{T}}(\psi_1)_{xx}^2 dx &= -\int_{\mathbb{T}} v_x(\psi_1)_{xx}\,dx \\ &\leq \frac{1}{4\nu}\int_{\mathbb{T}} v_x^2 dx + \nu\int_{\mathbb{T}}(\psi_1)_{xx}^2 dx, \end{aligned}$$

from which we conclude that

$$\|(\psi_1)_x\|^2_{L^\infty(0,T;L^2(\mathbb{T}))} \leq \frac{1}{2\nu}\|v_x\|^2_{L^2(0,T;L^2(\mathbb{T}))}.$$

Finally, for any $t > 0$, the integral of $\psi_1(t,\cdot)$ over $\mathbb{T}$ vanishes, and an application of the Poincaré inequality then gives

$$\|\psi_1\|_{L^\infty(0,T;L^\infty(\mathbb{T}))} \leq \frac{1}{2\nu}\|v_x\|^2_{L^2(0,T;L^2(\mathbb{T}))}. \tag{19}$$

Consider now the sum $\psi = \psi_1 + \psi_2$ which is a solution to

$$\begin{aligned} &\psi_t - \nu\psi_{xx} = v_x + \frac{\tilde{v}^2}{4\nu}, \quad x \in \mathbb{T},\ t > 0, \\ &\psi(0,x) = L(0,x) = \ln\frac{1}{\rho_0(x)}, \quad x \in \mathbb{T}. \end{aligned}$$

Then, by the comparison principle, $L \leq \psi$ in $[0,T] \times \mathbb{T}$, and, together with (19) and (18), we obtain for any $t > 0$ and $x \in \mathbb{T}$:

$$L(t,x) \leq \frac{1}{\nu} \|v_x\|^2_{L^2(0,T;L^2(\mathbb{T}))} + \ln \frac{1}{\eta_0}.$$

This leads to

$$\rho(t,x) \geq \eta_0 \exp\Big(-\frac{1}{\nu} \|v_x\|^2_{L^2(0,T;L^2(\mathbb{T}))} \Big),$$

for every $x \in \mathbb{T}$, $t > 0$. ☐

3. Solution of the nonlinear approximate problem

In this section, we show that there exists a solution to the system (9)-(10) and (14) on $\mathbb{T}$. More precisely, we prove the following result.

Proposition 4. *Let the assumptions of Theorem 1 hold and let the initial data be smooth with positive particle density. Then there exists a solution* $(\rho, u_n) \in C^0(0,T;C^3(\mathbb{T})) \times C^1(0,T;X_n)$ *to* (9)-(10) *and* (14), *with* $v = u_n$ *and* $\rho = \rho_n = S(u_n)$, *satisfying the following estimates:*

$$\rho_n(t,x) \geq \eta(\varepsilon) > 0, \quad t \in [0,T],\ x \in \mathbb{T}, \tag{20}$$

$$\|\sqrt{\rho_n}\|_{L^\infty(0,T;H^1(\mathbb{T}))} + \|\sqrt{\rho_n}\|_{L^2(0,T;H^2(\mathbb{T}))} \leq K, \tag{21}$$

$$\|\sqrt{\rho_n} u_n\|_{L^\infty(0,T;L^2(\mathbb{T}))} + \|\sqrt{\rho_n}(u_n)_x\|_{L^2(0,T;L^2(\mathbb{T}))} \leq K, \tag{22}$$

$$\varepsilon \|(u_n)_x\|_{L^2(0,T;L^2(\mathbb{T}))} \leq K, \tag{23}$$

$$\|V[\rho_n]\|_{L^\infty(0,T;H^1(\mathbb{T}))} \leq K, \tag{24}$$

where $\eta(\varepsilon) > 0$ *depends on* ε, *the initial data and the* $L^2(\mathbb{T})$ *norm of* $C(x)$, *and* $K > 0$ *only depends on* ν, λ, *the initial data, and* $C(x)$. *The potential* $V[\rho_n]$ *is defined by* (15) *with* $\rho = \rho_n$.

Proof. Integrating (17) over $(0,t)$, we can write the problem as the following nonlinear equation:

$$u_n(t) = M^{-1}[(S(u_n))(t)]\Big(M[\rho_0](u_0) + \int_0^t N[u_n, u_n(s)]ds \Big) \quad \text{in } X_n.$$

Taking into account (12) and (16), this equation can be solved with the fixed-point theorem of Banach, at least on a short time interval $[0,T']$, where $T' \leq T$, in the space $C^0(0,T';X_n)$. In fact, we obtain even $u_n \in C^1(0,T';X_n)$. We have to show that we can choose $T' = T$. It is sufficient to prove that u_n is bounded in X_n on the whole interval $[0,T']$. This is achieved by employing the energy estimate. We multiply (9) by $\phi = h(\rho_n) - V[\rho_n] - u_n^2/2 - (\delta^2/2)(\sqrt{\rho_n})_{xx}/\sqrt{\rho_n}$, use the test function u_n in (14), with $v = u_n$ and $\rho = \rho_n$, and add both equations. This leads to

$$
\begin{aligned}
0 &= \int_{\mathbb{T}} \Big((\rho_n)_t h(\rho_n) - (\rho_n)_t \frac{u_n^2}{2} + (\rho_n u_n)_t u_n \Big) dx \\
&\quad + \int_{\mathbb{T}} \big(- (\rho_n)_t V[\rho_n] - (\rho_n u_n)_x V[\rho_n] + \nu (\rho_n)_{xx} V[\rho_n] - \rho_n (V[\rho_n])_x u_n \big) dx \\
&\quad + \int_{\mathbb{T}} \big((\rho_n u_n)_x h(\rho_n) + (p(\rho_n))_x u_n \big) dx \\
&\quad - \frac{\delta^2}{2} \int_{\mathbb{T}} \Big((\rho_n u_n)_x \frac{(\sqrt{\rho_n})_{xx}}{\sqrt{\rho_n}} + \rho_n u_n \Big(\frac{(\sqrt{\rho_n})_{xx}}{\sqrt{\rho_n}} \Big)_x + (\rho_n)_t \frac{(\sqrt{\rho_n})_{xx}}{\sqrt{\rho_n}} \Big) dx \\
&\quad + \int_{\mathbb{T}} \Big(- \frac{1}{2} (\rho_n u_n)_x u_n^2 + (\rho_n u_n^2)_x u_n \Big) dx \\
&\quad + \nu \int_{\mathbb{T}} \Big(- (\rho_n)_{xx} h(\rho_n) + \frac{1}{2} (\rho_n)_{xx} u_n^2 + \frac{\delta^2}{2} (\rho_n)_{xx} \frac{(\sqrt{\rho_n})_{xx}}{\sqrt{\rho_n}} - (\rho_n u_n)_{xx} u_n \Big) dx \\
&\quad - \varepsilon \int_{\mathbb{T}} u_n (u_n)_{xx} dx + \frac{1}{\tau} \int_{\mathbb{T}} \rho_n u_n^2 dx \\
&= I_1 + \cdots + I_8.
\end{aligned}
$$

Notice that at this point, we need a pointwise solution to (9) such that this equation can be multiplied by ϕ. If (9) was solved in a Galerkin space only, we could not use ϕ as a test function since it is not admissible. On the other hand, u_n is an admissible test function for the Galerkin equation (14).

We estimate the above expression integral by integral. The first integral can be reformulated as

$$
I_1 = \partial_t \int_{\mathbb{T}} \Big(H(\rho) + \frac{1}{2} \rho_n u_n^2 \Big) dx,
$$

where we recall that H is a primitive of h. The second integral becomes, after integrating by parts and employing the Poisson equation,

$$
\begin{aligned}
I_2 &= \int_{\mathbb{T}} \big(- \lambda^2 (V[\rho_n])_{xxt} V[\rho_n] + \nu \rho_n (V[\rho_n])_{xx} \big) dx \\
&= \int_{\mathbb{T}} \Big(\frac{\lambda^2}{2} \partial_t (V[\rho_n])_x^2 + \nu \lambda^{-2} \rho_n (\rho_n - C(x)) \Big) dx.
\end{aligned}
$$

Integrating by parts in the first member of the third integral, we see that

$$
I_3 = \int_{\mathbb{T}} \big(\rho_n u_n h'(\rho_n) (\rho_n)_x + p'(\rho_n) (\rho_n)_x u_n \big) dx = 0,
$$

since, by definition, $p'(\rho_n) = \rho_n h'(\rho_n)$. Again by integrating by parts, the fourth integral simplifies to

$$
I_4 = -\delta^2 \int_{\mathbb{T}} (\sqrt{\rho_n})_t (\sqrt{\rho_n})_{xx} dx = \frac{\delta^2}{2} \partial_t \int_{\mathbb{T}} (\sqrt{\rho_n})_x^2 dx.
$$

The fifth integral vanishes since, in view of the periodic boundary conditions,

$$
I_5 = \frac{1}{2} \int_{\mathbb{T}} (\rho_n u_n^3)_x dx = 0.
$$

Integrating by parts in the sixth integral gives

$$\begin{aligned} I_6 &= \nu\int_{\mathbb{T}}\Big(h'(\rho_n)(\rho_n)_x^2 - (\rho_n)_x u_n(u_n)_x + \delta^2(\sqrt{\rho_n})_{xx}^2 + \frac{\delta^2}{3}\frac{((\sqrt{\rho_n})_x^3)_x}{\sqrt{\rho_n}} \\ &\quad + (\rho_n u_n)_x(u_n)_x\Big)dx \\ &= \nu\int_{\mathbb{T}}\Big((G(\rho_n))_x^2 + \rho_n(u_n)_x^2 + \delta^2(\sqrt{\rho_n})_{xx}^2 + \frac{16}{3}\delta^2(\sqrt[4]{\rho_n})_x^4\Big)dx, \end{aligned}$$

where $G'(y)=\sqrt{h'(y)}$, $y\ge 0$. Summarizing, we obtain

$$\begin{aligned} &\partial_t\int_{\mathbb{T}}\Big(H(\rho_n)+\frac{1}{2}\rho_n u_n^2 + \frac{\lambda^2}{2}(V[\rho_n])_x^2 + \frac{\delta^2}{2}(\sqrt{\rho_n})_x^2\Big)dx \\ &\quad + \nu\int_{\mathbb{T}}\Big((G(\rho_n))_x^2 + \rho_n(u_n)_x^2 + \delta^2(\sqrt{\rho_n})_{xx}^2 + \frac{16}{3}\delta^2(\sqrt[4]{\rho_n})_x^4\Big)dx \\ &\quad + \varepsilon\int_{\mathbb{T}}(u_n)_x^2dx + \frac{1}{\tau}\int_{\mathbb{T}}\rho_n u_n^2 dx \\ &= -\nu\lambda^{-2}\rho_n(\rho_n - C(x)) \le \frac{\nu}{4\lambda^2}\int_{\mathbb{T}}C(x)^2dx. \end{aligned}$$

From this estimate and the assumption $H(y)\ge h_0(y-1)$, the uniform bounds (21)-(24) follow. Using (13) and (23), we infer the lower bound (20). Then, the estimate (22) shows that (u_n) is bounded in $L^\infty(0,T;L^2(\mathbb{T}))$ with a bound which depends on ε. Together with the Lipschitz estimates (12) and (16), this allows us to apply the fixed-point theorem recursively until $T'=T$. ☐

We end this section by proving some estimates uniform in n and ε.

Lemma 5. *The following estimates holds:*

$$\|\partial_t\rho_n\|_{L^2(0,T;L^2(\mathbb{T}))} + \|\sqrt{\rho_n}\|_{L^6(0,T;W^{1,6}(\mathbb{T}))} \le K, \tag{25}$$

$$\|\rho_n\|_{L^\infty(0,T;H^1(\mathbb{T}))} + \|\rho_n\|_{L^2(0,T;H^2(\mathbb{T}))} \le K, \tag{26}$$

$$\|\partial_t(\rho_n u_n)\|_{L^2(0,T;H^{-2}(\mathbb{T}))} \le K, \tag{27}$$

$$\|\rho_n^\alpha\partial_t(\rho_n u_n)\|_{L^2(0,T;H^{-1}(\mathbb{T}))} \le K, \tag{28}$$

for all $\alpha\ge 1/2$, where $K>0$ is independent of n and ε.

Proof. By the Galiardo-Nirenberg inequality with $\theta=1/3$, we have

$$\begin{aligned} \|(\sqrt{\rho_n})_x\|_{L^6(0,T;L^6(\mathbb{T}))}^6 &\le K\int_0^T \|(\sqrt{\rho_n})_x\|_{H^1(\mathbb{T})}^{6\theta}\|(\sqrt{\rho_n})_x\|_{L^2(\mathbb{T})}^{6(1-\theta)}dt \\ &\le K\|\sqrt{\rho_n}\|_{L^\infty(0,T;H^1(\mathbb{T}))}^4\int_0^T\|\sqrt{\rho_n}\|_{H^2(\mathbb{T})}^2dt \le K, \end{aligned} \tag{29}$$

taking into account the bound (21). This shows that $\sqrt{\rho_n}$ is bounded in $L^6(0,T;W^{1,6}(\mathbb{T}))$. The function ρ_n solves (9)-(10), with $v=u_n$, written as

$$\partial_t\rho_n = -\sqrt{\rho_n}\sqrt{\rho_n}(u_n)_x - 2\sqrt{\rho_n}u_n(\sqrt{\rho_n})_x + 2\nu\sqrt{\rho_n}(\sqrt{\rho_n})_{xx} + 2\nu(\sqrt{\rho_n})_x^2.$$

In view of (21), (22), and (29), we infer that $\partial_t\rho_n\in L^2(0,T;L^2(\mathbb{T}))$. Furthermore, by (29), $(\rho_n)_{xx} = 2\sqrt{\rho_n}(\sqrt{\rho_n})_{xx} + 2(\sqrt{\rho_n})_x^2$ is bounded in $L^2(0,T;L^2(\mathbb{T}))$.

We claim that $\partial_t(\rho_n u_n)$ is bounded in $L^2(0,T;H^{-2}(\mathbb{T}))$. We have to verify that all terms in (14), with $\rho=\rho_n$ and $v=u_n$, except $\partial_t(\rho_n u_n)$ lie in this space. This is clear for the terms $(p(\rho_n))_x$, $\rho_n(V[\rho_n])_x$, $\varepsilon(u_n)_{xx}$, and $\rho_n u_n/\tau$.

Furthermore, $\rho_n u_n^2 = (\sqrt{\rho_n} u_n)^2$ is bounded in $L^\infty(0,T;L^1(\mathbb{T}))$, by (22), such that $(\rho_n u_n^2)_x$ is bounded in $L^\infty(0,T;W^{-1,1}(\mathbb{T})) \hookrightarrow L^\infty(0,T;H^{-2}(\mathbb{T}))$; $\nu(\rho_n u_n)_{xx} = \nu(2\sqrt{\rho_n}u_u(\sqrt{\rho_n})_x + \rho_n(u_n)_x)_x$ is bounded in $L^2(0,T;H^{-1}(\mathbb{T}))$; and

$$\rho_n\Big(\frac{(\sqrt{\rho_n})_{xx}}{\sqrt{\rho_n}}\Big)_x = \big(\sqrt{\rho_n}(\sqrt{\rho_n})_{xx}\big)_x - 2(\sqrt{\rho_n})_x(\sqrt{\rho_n})_{xx}$$

is bounded in $L^2(0,T;H^{-1})$. This shows the claim.

Next, let $\alpha \geq 1/2$. We want to show that $\rho_n^\alpha \partial_t(\rho_n u_n)$ is bounded in $L^2(0,T;H^{-1}(\mathbb{T}))$. The term

$$\rho_n^\alpha(\rho_n u_n^2)_x = 2\rho_n^{\alpha-1/2}(\sqrt{\rho_n})_x\rho_n u_n^2 + 2\rho_n^\alpha\sqrt{\rho_n}u_n\sqrt{\rho_n}(u_n)_x$$

is bounded in $L^2(0,T;L^1(\mathbb{T}))$ and hence also in $L^2(0,T;H^{-1}(\mathbb{T}))$. Notice that we have used here that $\alpha \geq 1/2$. If $\alpha \geq 0$ only, the bound depends on ε through the lower bound of ρ_n. Furthermore, $\rho_n^\alpha(p(\rho_n))_x$, $\rho^{\alpha+1}(V[\rho_n])_x$, and $\rho^{\alpha+1}u_n/\tau$ are bounded in $L^2(0,T;L^2(\mathbb{T}))$. The first term of

$$\rho_n^{\alpha+1}\Big(\frac{(\sqrt{\rho_n})_{xx}}{\sqrt{\rho_n}}\Big)_x = \big(\rho_n^{\alpha+1/2}(\sqrt{\rho_n})_{xx}\big)_x - 2(\alpha+1)\rho_n^\alpha(\sqrt{\rho_n})_x(\sqrt{\rho_n})_{xx}$$

is bounded in $L^2(0,T;H^{-1}(\mathbb{T}))$, the second term in $L^2(0,T;L^1(\mathbb{T}))$, so the sum is bounded in $L^2(0,T;H^{-1}(\mathbb{T}))$. Similarly, the sequences

$$\begin{aligned}
\varepsilon\rho_n^\alpha(u_n)_{xx} &= \big(\rho_n^\alpha\varepsilon(u_n)_x\big)_x - 2\alpha\rho_n^{\alpha-1/2}(\sqrt{\rho_n})_x\varepsilon(u_n)_x, \\
\rho_n^\alpha(\rho_n u_n)_{xx} &= \big(\rho_n^{\alpha+1/2}\sqrt{\rho_n}(u_n)_x\big)_x - 2(\alpha-1)\rho_n^\alpha(\sqrt{\rho_n})_x\sqrt{\rho_n}(u_n)_x \\
&\quad + \rho_n^{\alpha-1/2}(\rho_n)_{xx}\sqrt{\rho_n}u_n
\end{aligned}$$

are bounded in $L^2(0,T;H^{-1}(\mathbb{T}))$. ☐

4. PROOF OF THEOREM 1

In this section, we perform the limit $n \to \infty$, for fixed $\varepsilon > 0$, in the system (9)-(10), (14), and (15), with $\rho = \rho_n$ and $v = u_n$.

In view of (25), (26), and the compactness of the embeddings $H^1(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$ and $H^2(\mathbb{T}) \hookrightarrow H^1(\mathbb{T})$, the Aubin lemma provides the existence of a subsequence of (ρ_n) (not relabeled) such that, as $n \to \infty$,

$$\begin{aligned}
\rho_n &\to \rho \quad \text{strongly in } L^2(0,T;H^1(\mathbb{T})) \text{ and } L^\infty(0,T;L^\infty(\mathbb{T})), \\
\rho_n &\rightharpoonup \rho \quad \text{weakly in } L^2(0,T;H^2(\mathbb{T})), \\
\partial_t\rho_n &\rightharpoonup \partial_t\rho \quad \text{weakly in } L^2(0,T;L^2(\mathbb{T})).
\end{aligned}$$

Since (ρ_n) is bounded from below and above, $(\sqrt{\rho_n})_x$ converges weakly (up to a subsequence) to $(\sqrt{\rho})_x$ in $L^2(0,T;L^2(\mathbb{T}))$. Moreover, since $\varepsilon > 0$ is fixed, by (22) and (23), u_n converges weakly to a function u in $L^2(0,T;H^1(\mathbb{T}))$. These results show that

$$\partial_t\rho_n + (\rho_n u_n)_x - \nu(\rho_n)_{xx} \rightharpoonup \partial_t\rho + (\rho u)_x - \nu\rho_{xx} \quad \text{weakly in } L^1(0,T;L^2(\mathbb{T}))$$

and that

$$(p(\rho_n))_x - \frac{\delta^2}{2}\rho_n\Big(\frac{(\sqrt{\rho_n})_{xx}}{\sqrt{\rho_n}}\Big)_x - \nu(\rho_n u_n)_{xx} - \varepsilon(u_n)_{xx} + \frac{1}{\tau}\rho_n u_n$$

converges weakly in $L^1(0,T;H^{-1}(\mathbb{T}))$ to

$$(p(\rho))_x - \frac{\delta^2}{2}\rho\Big(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\Big)_x - \nu(\rho u)_{xx} - \varepsilon u_{xx} + \frac{1}{\tau}\rho u.$$

We also have $V[\rho_n] \to V[\rho]$ in $L^\infty(0,T;H^2(\mathbb{T}))$. In order to pass to the limit in the convection term, we observe first that $\rho_n u_n \rightharpoonup \rho u$ weakly* in $L^2(0,T;L^\infty(\mathbb{T}))$ since (ρ_n) converges strongly in $L^\infty(0,T;L^\infty(\mathbb{T}))$ and (u_n) converges weakly* in $L^2(0,T;L^\infty(\mathbb{T}))$. On the other hand, taking into account (27) and the bound for $(\rho_n u_n)$ in $L^2(0,T;H^1(\mathbb{T}))$, Aubin's lemma implies that $\rho_n u_n \to \rho u$ strongly in $L^2(0,T;L^\infty(\mathbb{T}))$. Thus,

$$\rho_n u_n^2 \rightharpoonup \rho u^2 \quad \text{weakly* in } L^1(0,T;L^\infty(\mathbb{T})).$$

Thus, passing to the limit $n \to \infty$ in (14), with $\rho = \rho_n$ and $v = u_n$, shows that $(\rho, u, V[\rho])$ is a solution to (5)-(7). This finishes the proof of Theorem 1.

5. Proof of Theorem 2

Let $(\rho_\varepsilon, u_\varepsilon, V[\rho_\varepsilon])$ be the solution to (5)-(7) constructed in the previous section. In this section, we will perform the limit $\varepsilon \to 0$.

By the ε-independent estimates (25) and (26), the Aubin lemma gives the existence of a subsequence (again not relabeled) such that

$$\begin{aligned} \rho_\varepsilon \to \rho \quad & \text{strongly in } L^2(0,T;H^1(\mathbb{T})) \text{ and in } L^\infty(0,T;L^\infty(\mathbb{T})), \\ \rho_\varepsilon \rightharpoonup \rho \quad & \text{weakly in } L^2(0,T;H^2(\mathbb{T})), \\ \partial_t\rho_\varepsilon \rightharpoonup \partial_t\rho \quad & \text{weakly in } L^2(0,T;L^2(\mathbb{T})). \end{aligned}$$

Furthermore, by (21), up to a subsequence,

$$\sqrt{\rho_\varepsilon} \rightharpoonup \sqrt{\rho} \quad \text{weakly* in } L^\infty(0,T;H^1(\mathbb{T})) \text{ and weakly in } L^2(0,T;H^2(\mathbb{T})).$$

By (22),

$$(\rho_\varepsilon u_\varepsilon)_x = 2(\sqrt{\rho_\varepsilon})_x\sqrt{\rho_\varepsilon}u_\varepsilon + \sqrt{\rho_\varepsilon}\sqrt{\rho_\varepsilon}(u_\varepsilon)_x$$

is bounded in $L^2(0,T;L^2(\mathbb{T}))$, and hence, $(\rho_\varepsilon u_\varepsilon)$ is bounded in $L^2(0,T;H^1(\mathbb{T}))$. Therefore,

$$\rho_\varepsilon u_\varepsilon \rightharpoonup J \quad \text{weakly in } L^2(0,T;H^1(\mathbb{T})).$$

Thus, by (25) and (26), letting $\varepsilon \to 0$ in the mass conservation equation (9) with $\rho = \rho_\varepsilon$ and $v = u_\varepsilon$ yields

$$\rho_t + J_x = \nu\rho_{xx} \quad \text{in } L^2(0,T;L^2(\mathbb{T})).$$

In order to let $\varepsilon \to 0$ in the momentum equation, we need to multiply (14) by $\rho_\varepsilon^{3/2}$. The reason is that we cannot control (u_ε) but only $(\rho_\varepsilon u_\varepsilon)$ which makes it difficult to pass to the limit in $(\rho_\varepsilon u_\varepsilon^2)_x$. The $L^2(0,T;H^1(\mathbb{T}))$ bound for $(\rho_\varepsilon u_\varepsilon)$ together with (27) implies that, by Aubin's lemma,

$$\rho_\varepsilon u_\varepsilon \to J \quad \text{strongly in } L^2(0,T;L^\infty(\mathbb{T})).$$

Thus, for any test function $\phi \in L^\infty(0,T;H^1(\mathbb{T}))$, as $\varepsilon \to 0$,

$$\begin{aligned} \int_{\mathbb{T}} \rho_\varepsilon^{3/2}(\rho_\varepsilon u_\varepsilon^2)_x\phi dx &= -\int_{\mathbb{T}} \big(3(\sqrt{\rho_\varepsilon})_x\phi + \sqrt{\rho_\varepsilon}\phi_x\big)(\rho_\varepsilon u_\varepsilon)^2 dx \\ &\to \int_{\mathbb{T}} \big(3(\sqrt{\rho})_x\phi + \sqrt{\rho}\phi_x\big)J^2 dx. \end{aligned}$$

By (28), $\rho_\varepsilon^{3/2}(\rho_\varepsilon u_\varepsilon)_t$ is bounded in $L^2(0,T;H^{-1}(\mathbb{T}))$. Hence, also

$$(\rho_\varepsilon^{5/2}u_\varepsilon)_t = \rho_\varepsilon^{3/2}(\rho_\varepsilon u_\varepsilon)_t + \frac{3}{2}\rho_\varepsilon(\rho_\varepsilon)_t(\sqrt{\rho_\varepsilon}u_\varepsilon)$$

is bounded in this space and we infer that

$$(\rho_\varepsilon^{5/2}u_\varepsilon)_t \rightharpoonup (\rho^{3/2}J)_t \quad \text{weakly in } L^2(0,T;H^{-1}(\mathbb{T})). \tag{30}$$

Using $\rho_\varepsilon^{3/2}\phi$ with $\phi \in L^\infty(0,T;H^1(\mathbb{T}))$ as a test function in the weak formulation of (6), it holds

$$\begin{aligned}
0 &= \int_0^T\int_{\mathbb{T}} \rho_\varepsilon^{3/2}(\rho_\varepsilon u_\varepsilon)_t\phi dxdt - \int_0^T\int_{\mathbb{T}}\big(\rho_\varepsilon u_\varepsilon^2 + p(\rho_\varepsilon)\big)(\rho_\varepsilon^{3/2}\phi)_x dxdt \\
&\quad - \int_0^T\int_{\mathbb{T}}\rho_\varepsilon^{5/2}(V[\rho_\varepsilon])_x\phi dxdt + \frac{\delta^2}{2}\int_0^T\int_{\mathbb{T}}\frac{(\sqrt{\rho_\varepsilon})_{xx}}{\sqrt{\rho_\varepsilon}}(\rho_\varepsilon^{5/2}\phi)_x dxdt \\
&\quad + \nu\int_0^T\int_{\mathbb{T}}(\rho_\varepsilon u_\varepsilon)_x(\rho_\varepsilon^{3/2}\phi)_x + \varepsilon\int_0^T\int_{\mathbb{T}}(u_\varepsilon)_x\phi_x dxdt \\
&\quad + \frac{1}{\tau}\int_0^T\int_{\mathbb{T}}\rho_\varepsilon^{5/2}u_\varepsilon\phi dxdt \\
&= K_1 + \cdots + K_7.
\end{aligned}$$

Employing (30), we have

$$\begin{aligned}
K_1 &= \int_0^T \langle(\rho_\varepsilon^{5/2}u_\varepsilon)_t, \phi\rangle_{H^{-1},H^1}dt - \frac{3}{2}\int_0^T\int_{\mathbb{T}}\rho_\varepsilon^{3/2}(\rho_\varepsilon)_t u_\varepsilon\phi dxdt \\
&\to \int_0^T \langle(\rho^{3/2}J)_t,\phi\rangle_{H^{-1},H^1}dt - \frac{3}{2}\int_0^T\int_{\mathbb{T}}\sqrt{\rho}\rho_t J\phi dxdt.
\end{aligned}$$

For the second integral, we obtain

$$\begin{aligned}
K_2 &= \int_0^T\int_{\mathbb{T}}\big((\rho_\varepsilon u_\varepsilon)^2 + \rho_\varepsilon p(\rho_\varepsilon)\big)\big(3(\sqrt{\rho_\varepsilon})_x\phi + \sqrt{\rho_\varepsilon}\phi_x\big)dxdt \\
&\to \int_0^T\int_{\mathbb{T}}\big(J^2 + \rho p(\rho)\big)\big(3(\sqrt{\rho})_x\phi + \sqrt{\rho}\phi_x\big)dxdt,
\end{aligned}$$

since $(\rho_\varepsilon u_\varepsilon)^2$ converges strongly in $L^1(0,T;L^\infty(\mathbb{T}))$ and $(\sqrt{\rho_\varepsilon})_x$ converges weakly* in $L^\infty(0,T;L^2(\mathbb{T}))$. Furthermore, $(V[\rho_\varepsilon])_x$ converges weakly* in $L^\infty(0,T;\,H^1(\mathbb{T}))$ to $V[\rho]$:

$$K_3 \to \int_0^T\int_{\mathbb{T}}\rho^{5/2}(V[\rho])_x\phi dxdt.$$

The fourth integral can be written as

$$K_4 = \frac{\delta^2}{2}\int_0^T\int_{\mathbb{T}}(\sqrt{\rho_\varepsilon})_{xx}\Big(\frac{5}{2}\rho_\varepsilon(\rho_\varepsilon)_x\phi + \rho_\varepsilon^2\phi_x\Big)dxdt.$$

Since ρ_ε converges strongly in $L^\infty(0,T;L^\infty(\mathbb{T}))$ and in $L^2(0,T;H^1(\mathbb{T}))$, it follows that

$$K_4 \to \frac{\delta^2}{2}\int_0^T\int_{\mathbb{T}}(\sqrt{\rho})_{xx}\big(5\rho^{3/2}(\sqrt{\rho})_x\phi + \rho^2\phi_x\big)dxdt.$$

The weak convergence of $\rho_\varepsilon u_\varepsilon$ in $L^2(0,T;H^1(\mathbb{T}))$ and the strong convergences of $\sqrt{\rho_\varepsilon}$ in $L^\infty(0,T;L^\infty(\mathbb{T}))$ and of $(\rho_\varepsilon)_x$ in $L^2(0,T;L^2(\mathbb{T}))$ imply that

$$\begin{aligned} K_5 &= \nu \int_0^T \int_{\mathbb{T}} (\rho_\varepsilon u_\varepsilon)_x \Big(\frac{3}{2}\sqrt{\rho_\varepsilon}(\rho_\varepsilon)_x \phi + \rho_\varepsilon^{3/2}\phi_x \Big) dxdt \\ &\to \nu \int_0^T \int_{\mathbb{T}} J_x \Big(\frac{3}{2}\sqrt{\rho}\rho_x \phi + \rho^{3/2}\phi_x \Big) dxdt = \nu \int_0^T \int_{\mathbb{T}} J_x (\rho^{3/2}\phi)_x dxdt. \end{aligned}$$

Finally, the estimate (23) shows that $K_6 \to 0$, and

$$K_7 \to \frac{1}{\tau} \int_0^T \rho^{3/2} J \phi dxdt.$$

This proves that (ρ, J, V) solves the system (1)-(3) for smooth initial data. A standard approximation procedure gives the result for initial data $(\rho_0, u_0) \in H^1(\mathbb{T}) \times L^\infty(\mathbb{T})$ with positive particle density and finite energy.

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