



Classical and quantum transport in random media

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Abstract

We study in this article the transport of particles in time-dependent random media, in the so-called weak coupling limit. We show the convergence of a Liouville equation to a Fokker–Planck equation. We also obtain the semi-classical limit of Schrödinger equations. This limit is described by a linear Boltzmann equation. In both cases, the ratio between a typical time scale and the scale of the media determines whether the limit diffusion and the collision process are elastic or not.

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Résumé

Nous étudions dans cet article le transport de particules dans un milieu aléatoire dépendant du temps et dans la limite de faibles couplages. Nous montrons d’une part la convergence d’une équation de Liouville vers une équation de Fokker–Planck. D’autre part nous obtenons que la limite semi-classique d’équations de Schrödinger est décrite par une équation de Boltzmann linéaire. Dans les deux cas le rapport entre un temps de décorrélation et une longueur caractéristique des inhomogénéités du milieu détermine si le processus de diffusion limite et l’opérateur de collision sont élastiques ou non.

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1. Introduction

In this paper we investigate the asymptotic behavior of particles dynamics in a random media. The random media is modeled by a random potential $V_t^h(x)$ which is time dependent. The small parameter h represents the correlation length and $V_t^h(x)$, $V_s^h(y)$

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are independent random variables as soon as $|t - s| \geq h\tau$ where τ is an other parameter. The amplitude of the potential is of order \sqrt{h} . We will prove that, starting from a classical Liouville equation, we obtain in the limit a Fokker–Planck equation. A parallel with quantum transport is done. We will show that the limit of a quantum dynamic is described by a linear Boltzmann equation. This kind of problems belongs to the class of rigorous derivations of an irreversible dynamic from a reversible one. It has motivated a lot of works in mathematical physics and mathematics. There are two kind of asymptotics corresponding to different physical situations.

The strong coupling corresponds to a situation where particles collide with scatterers very rarely. But one scattering changes the velocity of particles with order one.

On the contrary, in the weak coupling situation, the particles are not deviated much by one scattering but they collide with scatterers very often.

The scaling we study in this work correspond to the so-called weak coupling limit.

1.1. Existing results

In the context of strong coupling, one of the key reference of the field is the work of Lanford [14] who gave the first rigorous derivation of the Boltzmann equation for large classical systems, for short time. Obtaining this result globally in time after the possible breakdown of the smoothness of the solution of the Boltzmann equation is still an outstanding open problem. The case of noninteracting particles in a random media is also a difficult problem. G. Galavotti gave in [8] a rigorous derivation of the linear Boltzmann equation for such a Lorentz gas. The center of diffusion of the random potential is assumed to follow a Poisson law. It allows to perform a change of variable in the integrals with respect to the random variables which leads to an almost explicit computation of the solution. Following the same method his result has been generalized in various contexts by Spohn [19], Boldrighini et al. [2], Desvilletes and Pulvirenti [4]. Let us point out, that the hypothesis that the potential is random, is absolutely essential to the derivation. Indeed, Bourgain et al. [3] have recently proved that it was not possible to find a diffusion process when considering scattering centers localized on a periodic lattice.

It seems there is no mathematical result for quantum transport in the strong coupling regime.

Concerning the weak coupling limit for classical transport, the first result seems to be due to Kesten and Papanicolaou [13]. In their approach, particles are accelerated by a small, time independent, random field. The correlations in the random field die out rapidly with position. At the leading order, particles follow a free flight. Therefore the correlations in the field die out rapidly along a path. They prove that a rescaled velocity converges to a diffusion Markov process. Their result holds in dimension larger than 3. In the more difficult case of the dimension 2, the same result has been established by Dürr et al. [6]. Finally, Desvilletes and Ricci in [5], studied the case of strong force field with the same asymptotic as in this paper. They followed the Galavotti methods, and proved that the limit process is governed by a Vlasov–Fokker–Planck equation.

For the weak coupling limit of quantum transport, the first derivation of the linear Boltzmann equation for very short time is due to Spohn [18]. An extension of this result is due to Ho et al. [11] but the complete analysis of the problem has only recently been

performed by Erdős and Yau [7]. Their method is based on an iteration of the Duhamel formula. The numbers of iterations they used, depends on the small parameter of the problem and tends to infinity. Each term of the series is analyzed and very sharply estimated by using graph theory. By this extremely technical procedure they obtain a precise description of the asymptotic of wave functions. The final step consists in computing the corresponding Wigner transform whose limits allow to determine the limits of observables. Let us conclude this short presentation of the existing results by mentioning the paper of Ryzhik et al. [17] which gives elegant formal derivations of transport equations in the weak coupling limit for various waves equations. We have also learnt recently that Bal, Papanicolaou and Ryzhik have obtained a result similar to us in the quantum case where the parameter τ is kept fixed, [1].

In all the rigorous derivation we have described above, the limit is performed directly on the solution and not on the equation. The final equation is only recognized afterward. Also, one of the main difficulty of the above approaches is due to the correlation in time of particles paths if they collide with the same scattering center more than one time. The key points of the proofs are to show that this event occurs very rarely. It requires a deep understanding of the structure of particles paths. The drawback is that this program can be followed only for particular potential.

In our approach the stochasticity in time of the potential automatically implies the nonself correlation of particles paths. On one hand, the major mathematical difficulty disappear making the problem easier and perhaps less interesting from the strict mathematical point of view. On the other hand, it allows to consider more general potential, higher order correlation terms and a more general class of equations. It has also to be pointed out that this assumption is close to the concept of mixing property of Kesten and Papanicolaou [12,13]. In their approach the diffusion regime is obtained as a perturbation of a classical transport around a given velocity v . For small times the paths are almost straight-lines $x + vt$. Since the correlation in the force field is assumed to decrease fast in position, it also decreases fast with time along a path $x + vt$. In our approach, we use the more direct assumption that the force fields are not correlated after a given laps of time.

1.2. A new strategy of proof

Let us describe our method of proof in an abstract framework. Let $t \in \mathbb{R}$ be the time variable and h a small parameter. Let $u_h(t)$ be a time dependent random variable with value in some functional space. The function $u_h(t)$ represents some physical quantity. As in Galavotti or Erdős–Yau results, we average over randomness. That means that we look for many measurements of the variable $u_h(t)$ and that we want to describe the evolution of the mean value over randomness of these measurements.

Let us denote by $\langle X \rangle$ the expectation of the random variable X . Then the problem is to find the limit of the expectation value $\langle u_h(t) \rangle$. A more precise and difficult result would be to obtain the almost surely limit of $u_h(t)$.

We assume that $u_h(t)$ is the solution of an evolution equation:

$$\frac{d}{dt}u_h(t) + Au_h(t) = \theta_t^h(u_h(t)). \quad (1)$$

The linear operator A is deterministic and generates a group S_t , $t \in \mathbb{R}$, in some functional space. The linear operators θ_t^h are stochastic, time dependent and satisfy:

- (i) the expectation operators vanish $\langle \theta_t^h \rangle = 0$,
- (ii) the operators θ_t^h and θ_s^h are independent as soon as $|t - s| \geq h\tau$ where the parameter τ is fixed or depends on h in such a way $\tau \rightarrow \infty$, $h\tau \rightarrow 0$ when $h \rightarrow 0$,
- (iii) for every deterministic linear operator B , there is a family of deterministic operators $R^h(\sigma; B)$, $\sigma \in [0, \infty)$, such that

$$\forall s, t \in \mathbb{R}, \quad \langle \theta_t^h B \theta_s^h \rangle = R^h(|t - s|; B).$$

Assumption (i) means that the random operator has a vanishing mean value. Assumption (ii) can be seen as a Markov property. The independence property is somehow drastic in a physical point of view. It can certainly be replaced by an assumption that the correlations in the random operators θ_t^h decrease very fast with respect to time. Assumption (iii) is a stationarity property of the random operators. It expresses some time independence of the distribution of θ_t^h and that there is no arrow of time built in the random operators.

Taking the expectation of (1) we obtain:

$$\frac{d}{dt} \langle u_h(t) \rangle + A \langle u_h(t) \rangle = \langle \theta_t^h (u_h(t)) \rangle. \quad (2)$$

The problem is to find the limit of $\langle \theta_t^h (u_h(t)) \rangle$. For that purpose we use a 2 times iterated Duhamel formula

$$\begin{aligned} u_h(t) &= S_{h\tau} u_h(t - h\tau) + \int_0^{h\tau} S_\sigma \theta_{t-\sigma}^h S_{2h\tau-\sigma} u_h(t - 2h\tau) d\sigma \\ &+ \int_0^{h\tau} \int_0^{2h\tau-\sigma} S_\sigma \theta_{t-\sigma}^h S_s \theta_{t-\sigma-s}^h u_h(t - \sigma - s) ds d\sigma. \end{aligned}$$

We obtain

$$\begin{aligned} \langle \theta_t^h u_h(t) \rangle &= \langle \theta_t^h S_{h\tau} u_h(t - h\tau) \rangle + \int_0^{h\tau} \langle \theta_t^h S_\sigma \theta_{t-\sigma}^h S_{2h\tau-\sigma} u_h(t - 2h\tau) \rangle d\sigma + r_t^h \\ \text{with } r_t^h &= \int_0^{h\tau} \int_0^{2h\tau-\sigma} \langle \theta_t^h S_\sigma \theta_{t-\sigma}^h S_s \theta_{t-\sigma-s}^h u_h(t - \sigma - s) \rangle ds d\sigma. \end{aligned} \quad (3)$$

The origin of irreversibility. The crucial point is to assume that at some time t_0 the function $u_h(t_0)$ is independent of the operators θ_t^h , $t \in \mathbb{R}$. Without loss of generality we can choose this time as the origin, $t_0 = 0$. It creates 2 arrows of time starting from 0. In particular, in view of the assumption (ii), θ_t^h and $u_h(s)$ are independent as soon as $t \geq s + h\tau$ and $s \geq 0$, respectively $t \leq s - h\tau$ and $s \leq 0$.

Combining this fact with (i), Eq. (3) becomes for $t \geq 2h\tau$

$$\begin{aligned} \langle \theta_t^h(u_h(t)) \rangle &= \langle \theta_t^h \rangle \langle S_{h\tau} u_h(t - h\tau) \rangle + \int_0^{h\tau} \langle \theta_t^h S_\sigma \theta_{t-\sigma}^h S_{-\sigma} \rangle \langle S_{2h\tau} u_h(t - 2h\tau) \rangle d\sigma + r_t^h, \\ \langle \theta_t^h(u_h(t)) \rangle &= \int_0^{h\tau} \langle \theta_t^h S_\sigma \theta_{t-\sigma}^h S_{-\sigma} \rangle \langle u_h(t) \rangle d\sigma + r_t^h + e_t^h \\ \text{with } e_t^h &= \int_0^{h\tau} \langle \theta_t^h S_\sigma \theta_{t-\sigma}^h S_{-\sigma} \rangle \langle S_{2h\tau} u_h(t - 2h\tau) - u_h(t) \rangle d\sigma. \end{aligned} \tag{4}$$

Using (iii) and plugging this expression in (2) we obtain for $t \geq 2h\tau$,

$$\frac{d}{dt} \langle u_h(t) \rangle + A \langle u_h(t) \rangle = Q^h \langle u_h(t) \rangle + r_t^h + e_t^h, \quad \text{with } Q^h = \int_0^{h\tau} R^h(\sigma; S_\sigma) S_{-\sigma} d\sigma.$$

It remains only analysis problems:

- (a) prove that the remainders r_t^h and e_t^h vanish for a convenient topology,
- (b) determine the limit of $Q^h(\langle u_h(t) \rangle)$.

At least formally, we obtain $\langle u_h(t) \rangle \rightarrow u(t)$ where $u(t)$ is a solution of

$$\frac{d}{dt} u(t) + Au(t) = Q(u(t)), \quad \text{for } t > 0.$$

Let us remark that for negative time, the same analysis leads to

$$\langle \theta_t^h(u_h(t)) \rangle = \int_0^{h\tau} \langle \theta_t^h \rangle S_{-h\tau} \langle u_h(t + h\tau) \rangle - \int_0^{h\tau} \langle \theta_t^h S_\sigma \theta_{t+\sigma}^h S_{-\sigma} \rangle \langle u_h(t) \rangle d\sigma + r_t^h + e_t^h$$

and the limit equation becomes

$$\frac{d}{dt} u(t) + Au(t) = -Q(u(t)), \quad \text{for } t < 0.$$

It makes appear clearly the role of the initial data.

1.3. Obtained results

As a first step, we consider in this paper Liouville and Schrödinger equations, but acoustic equations, Maxwell equations, and so on . . . can also be analyzed in this very general program. This is allowed by the new technique of proof we use. We pass to the limit in the equation and not in the solution.

For classical systems, in the asymptotic regime, we obtain Fokker–Planck equations. For quantum systems we obtain linear Boltzmann equations. In both cases when τ is kept fixed the collision operators are diffusive: they do not leave the energy of particles invariant. On the converse when $\tau \rightarrow \infty$ with $h\tau \rightarrow 0$ we recover the results of Desvillettes and Ricci [5] (classical transport) or Erdős and Yau [7] (quantum transport). The diffusion (classical transport) or collision operators (quantum transport) are elastic.

1.4. Outline of the paper and notations

The paper is organized as follows. In the next section we give a brief description of the potentials we consider. In particular we give explicit examples. Section 3 is devoted to the analysis of classical transport. The particles are accelerated by a random force field. We prove that the distribution of particles satisfies in the limit a linear Fokker–Planck equation. In the last section we study quantum transport. We consider infinitely many Schrödinger equations with a random potential. The limit of the corresponding Wigner transform is proved to be a distribution function which satisfies a linear Boltzmann equation.

In all this paper we denote $\langle X \rangle$ the expectation value of the integrable random variable X . Spaces L^p , $W^{s,p}$, $H^s = W^{s,2}$ are the usual Sobolev spaces. The space C^p is the set of p -times continuously differentiable functions and C_c^p is the set of functions of C^p with compact support. The sphere of \mathbb{R}^D is denoted by \mathbf{S}_{D-1} .

2. The random potential

We study in this paper the dynamics of classical or quantum particles in a random potential. A typical situation which is studied in this work can be described as follows. At the microscopic level, this potential $U^\varepsilon = U_t^\varepsilon(x)$ is assumed to be a real function of time space variables $(t, x) \in \mathbb{R}^{1+D}$ where D is the space dimension. It can be the sum of microscopic potentials due to scatterers. In the microscopic variables, the range of interaction of the microscopic potential is of order 1. In these units, the distance between two scatterers is of order $\varepsilon^{-1/D}$ where ε is a small parameter. By this way there is one scatterer per volume of order $1/\varepsilon$. Up to order ε , the potential U^ε is stationary.

The difference with classical approaches is that the intensity of microscopic potentials and the places of scatterers depends on time. The expectation value of $U_t^\varepsilon(x)$ vanishes and two values of the potential are independent as soon as they are taken at a lapse of time large enough. If this lapse of time is chosen as a microscopic unit, these crucial assumptions read:

- $\langle U_t^\varepsilon(x) \rangle = 0$ for all $t \in \mathbb{R}$, $x \in \mathbb{R}^D$,
- $U_t^\varepsilon(x)$ and $U_s^\varepsilon(y)$ are independent random variables for $|t - s| \geq 1$,

- $\langle U_t^\varepsilon(x)U_s^\varepsilon(y) \rangle = R_{t-s}(x-y) + \varepsilon g_{t,s}^\varepsilon(x,y)$ for parameters $\varepsilon \in (0, 1]$ and some real functions $R = R_t(x)$ and $g^\varepsilon = g_{t,s}^\varepsilon(x,y)$ with g^ε bounded, uniformly with respect to ε , for convenient topologies.

The last assumption allows, up to order ε , to obtain the Property (iii) of Section 1.2. Some examples of random potential satisfying these assumptions can be given as follows.

Example 1. Let $(T_k^n, X_k^n) \in \mathbb{R}^{1+D}$ with $(k, n) \in \mathbb{Z} \times \mathbb{Z}^D$ be independent random variables equidistributed in $\{k/2 + [-1/4, 1/4]\} \times \{n + [-1/2, 1/2]^D\}$. Let $v \in C_c^\infty(\mathbb{R}^{1+D})$ compactly supported in $[-1/4, 1/4] \times \mathbb{R}^D$ and such that $\int_{\mathbb{R}^{1+D}} v(t, x) dt dx = 0$. We define:

$$U_t^\varepsilon(x) = \sum_{(k,n) \in \mathbb{Z} \times \mathbb{Z}^D} \frac{1}{\sqrt{\varepsilon}} v\left(t - T_k^n, x - \frac{X_k^n}{\varepsilon^{1/D}}\right).$$

Actually, the sum defining $U_t^\varepsilon(x)$ is finite. We have $U^\varepsilon \in C^\infty(\mathbb{R}^{1+D})$.

For a fixed $t \in \mathbb{R}$ the potential U_t^ε depends only on $(T_{k_0}^n, X_{k_0}^n)$ with $k_0 \in (t - 1/2, t + 1/2)$. If $|s - t| \geq 1$, U_s^ε depends only on $(T_{k_1}^n, X_{k_1}^n)$ with $k_0 \neq k_1$. Therefore $U_t^\varepsilon(x)$ and $U_s^\varepsilon(y)$ are independent random variable for any $(x, y) \in \mathbb{R}^{2D}$.

We have:

$$\langle U_t^\varepsilon(x) \rangle = \sum_{(k,n) \in \mathbb{Z}^{1+D}} \frac{1}{\sqrt{\varepsilon}} \left\langle v\left(t - T_k^n, x - \frac{X_k^n}{\varepsilon^{1/D}}\right) \right\rangle = \sqrt{\varepsilon} \int_{\mathbb{R}^{1+D}} v(t-s, x-y) ds dy = 0;$$

$$\begin{aligned} \langle U_t^\varepsilon(x)U_s^\varepsilon(y) \rangle &= \sum_{(k,n) \in \mathbb{Z}^{1+D}} \sum_{(l,m) \in \mathbb{Z}^{1+D}} \frac{1}{\varepsilon} \left\langle v\left(t - T_k^n, x - \frac{X_k^n}{\varepsilon^{1/D}}\right) v\left(s - T_l^m, y - \frac{X_l^m}{\varepsilon^{1/D}}\right) \right\rangle, \\ &= \langle U_t^\varepsilon(x) \rangle \langle U_s^\varepsilon(y) \rangle \\ &\quad + \sum_{(k,n) \in \mathbb{Z}^{1+D}} \frac{1}{\varepsilon} \left\langle v\left(t - T_k^n, x - \frac{X_k^n}{\varepsilon^{1/D}}\right) v\left(s - T_k^n, y - \frac{X_k^n}{\varepsilon^{1/D}}\right) \right\rangle \\ &\quad - \sum_{(k,n) \in \mathbb{Z}^{1+D}} \frac{1}{\varepsilon} \left\langle v\left(t - T_k^n, x - \frac{X_k^n}{\varepsilon^{1/D}}\right) \right\rangle \left\langle v\left(s - T_k^n, y - \frac{X_k^n}{\varepsilon^{1/D}}\right) \right\rangle \end{aligned}$$

$$\begin{aligned} \langle U_t^\varepsilon(x)U_s^\varepsilon(y) \rangle &= \int_{\mathbb{R}^{1+D}} v(t-\sigma, x-z)v(s-\sigma, y-z) d\sigma dz \\ &\quad - \varepsilon \sum_{(k,n) \in \mathbb{Z}^{1+D}} \left\langle \frac{1}{\varepsilon} v\left(t - T_k^n, x - \frac{X_k^n}{\varepsilon^{1/D}}\right) \right\rangle \left\langle \frac{1}{\varepsilon} v\left(s - T_k^n, y - \frac{X_k^n}{\varepsilon^{1/D}}\right) \right\rangle. \end{aligned}$$

Then the assumptions are satisfied with

$$R_t(y) = \int_{\mathbb{R}^{D+1}} v(t + \sigma, y + z)v(\sigma, z) d\sigma dz, \tag{5}$$

$$g_{t,s}(x, y) = - \sum_{(k,n) \in \mathbb{Z}^{1+D}} \int_{[-1/4, 1/4]^2} \int_{[-1/2, 1/2]^{2D}} v\left(t + \frac{k}{2} + \sigma, x + n + z\right) \\ \times v\left(s + \frac{k}{2} + \sigma', y + n + z'\right) d\sigma d\sigma' dz dz'. \quad (6)$$

In this case g does not depend on the parameter ε .

Example 2. Let $(T_k^n, X_k^n) \in \mathbb{R}^{1+D}$ with $(k, n) \in \mathbb{Z} \times \mathbb{Z}^D$ as in the previous example. We introduce other independent random variables ω_k^n such that $\langle \omega_k^n \rangle = 0$, $\langle (\omega_k^n)^2 \rangle = 1$. Let $v \in C_c^\infty(\mathbb{R}^{1+D})$ be the profile of a potential whose support lies in $[-1/4, 1/4] \times \mathbb{R}^D$. We define $U_t^\varepsilon(x)$ by:

$$U_t^\varepsilon(x) = \sum_{(k,n) \in \mathbb{Z}^{1+D}} \omega_k^n v(t - T_k^n, x - X_k^n).$$

As in the previous example $U_t^\varepsilon(x)$ and $U_s^\varepsilon(y)$ are independent as soon as $|t - s| \geq 1$. We also have $\langle U_t^\varepsilon(x) \rangle = 0$ and

$$\begin{aligned} \langle U_t^\varepsilon(x) U_s^\varepsilon(y) \rangle &= \sum_{(k,n) \in \mathbb{Z}^{1+D}} \sum_{(l,m) \in \mathbb{Z}^{1+D}} \langle \omega_k^n \omega_l^m \rangle \langle v(t - T_k^n, x - X_k^n) v(s - T_l^m, y - X_l^m) \rangle \\ &= \sum_{(k,n) \in \mathbb{Z}^{1+D}} \langle v(t - T_k^n, x - X_k^n) v(s - T_k^n, y - X_k^n) \rangle \\ &= \int_{\mathbb{R}^{1+D}} v(t - \sigma, x - z) v(s - \sigma, y - z) d\sigma dz. \end{aligned}$$

Then the assumptions are satisfied with (5) and $g = 0$. In this example U does not depend on a parameter ε .

The force field which is the gradient of the potential is denoted $F^\varepsilon = \nabla_x U^\varepsilon$. Let $h \in (0, 1]$ and $\tau \geq 1$ be two real parameters. At the macroscopic level, we introduce a time scale of order $1/h\tau$ and a space scale of order $1/h$ in the microscopic units. The potential in convenient energy units is assumed to be of order \sqrt{h} . The parameters ε , τ and h are assumed to be linked by a relation $\varepsilon = \varepsilon(h)$, $\tau = \tau(h)$. Let us define:

$$V_t^h(x) = U_{t/h\tau}^\varepsilon\left(\frac{x}{h}\right), \quad E_t^h(x) = F_{t/h\tau}^\varepsilon\left(\frac{x}{h}\right).$$

The classical dynamics of particles is governed by Newton's law,

$$\frac{d}{dt} X = \mathcal{E}, \quad \frac{d}{dt} \mathcal{E} = -\frac{1}{\sqrt{h}} E_t^h(X).$$

The quantum dynamics is modeled by the Schrödinger equation,

$$i\hbar \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2} \Delta_x \psi + \sqrt{\hbar} V_t^h(x) \psi.$$

Remark that these equations correspond to the same potential because

$$\nabla_x (\sqrt{\hbar} V_t^h(x)) = \frac{1}{\sqrt{\hbar}} E_t^h(x).$$

In the next sections we consider these equations at a statistical level and we prove that in the limit $\hbar \rightarrow 0$, the distribution function satisfies a Fokker–Planck equation in the first case and a Boltzmann equation in the second case.

3. Classical transport

In this section we use the notation:

$$\begin{aligned} \nabla_{x,y}^{m,p} u &:= \left(\frac{\partial^{|\alpha|+|\beta|} u}{\partial x_1^{\alpha_1} \dots \partial x_D^{\alpha_D} \partial y_1^{\beta_1} \dots \partial y_D^{\beta_D}} \right)_{\alpha, \beta \in \mathbb{N}^D, |\alpha|=m, |\beta|=p} \\ |\nabla_{x,y}^{m,p} u| &:= \max \left\{ \left| \frac{\partial^{|\alpha|+|\beta|} u}{\partial x_1^{\alpha_1} \dots \partial x_D^{\alpha_D} \partial y_1^{\beta_1} \dots \partial y_D^{\beta_D}} \right|; \text{ for } |\alpha| = m, |\beta| = p \right\} \end{aligned}$$

for integers m, p , variables $x, y \in \mathbb{R}^D$ and u a vector valued function of these variables.

We are concerned with the following Liouville equation:

$$\frac{\partial}{\partial t} f_h + \xi \cdot \nabla_x f_h = \theta_t^h f_h, \tag{7}$$

$$f_h(0, x, \xi) = f^0(x, \xi), \tag{8}$$

where θ_t^h is the linear operator from $L^p(\mathbb{R}^{2D})$ to $W^{-1,p}(\mathbb{R}^{2D})$ defined by

$$\theta_t^h = -\frac{1}{\sqrt{\hbar}} E_t^h(x) \cdot \nabla_\xi.$$

As it is well known, the solution of (7), (8) is obtained by:

$$\begin{aligned} f^h(t, X(t, x, \xi), \mathcal{E}(t, x, \xi)) &= f^0(x, \xi), \\ \partial_t X &= \mathcal{E}, \quad \partial_t \mathcal{E} = -\frac{1}{\sqrt{\hbar}} E_t^h(X), \\ X(0, x, \xi) &= x, \quad \mathcal{E}(0, x, \xi) = \xi. \end{aligned} \tag{9}$$

Remembering that $E_t^h = \nabla_x U_{t/h\tau}^\varepsilon(x/h)$, we state the precise assumptions on the potential U^ε :

- (A1) $U^\varepsilon \in L^\infty(\mathbb{R}^+; W^{3,\infty}(\mathbb{R}^D))$ and $N(\varepsilon) := \langle (\|U^\varepsilon\|_{L^\infty(\mathbb{R}^+; W^{3,\infty}(\mathbb{R}^D))})^3 \rangle < \infty$,
 (A2) $\langle U_t^\varepsilon(x) \rangle = 0$, for all $t \in \mathbb{R}$, $x \in \mathbb{R}^D$,
 (A3) $U_t^\varepsilon(x)$, $U_s^\varepsilon(y)$ are independent random variables for $|t - s| \geq 1$,
 (A4) $\langle U_t^\varepsilon(x) U_s^\varepsilon(y) \rangle = R_{t-s}(x - y) + \varepsilon g_{t,s}^\varepsilon(x, y)$, for some real functions R and g^ε which satisfy

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} R \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^D)) \cap L^\infty(\mathbb{R}^+ \times \mathbb{R}^D), \quad \text{for } \alpha \in \mathbb{N}^D, |\alpha| \leq 3,$$

and for some positive constant C , independent on ε ,

$$\| |\nabla_{x,y}^{1,1} g^\varepsilon| + |\nabla_{x,y}^{1,2} g^\varepsilon| + |\nabla_{x,y}^{2,1} g^\varepsilon| \|_{L^\infty((\mathbb{R}^+)^2 \times \mathbb{R}^{2D})} \leq C.$$

In the asymptotic regime corresponding to $\tau \rightarrow \infty$ we need the following supplementary technical assumption:

$$(A') \quad \sup_{v \in \mathbf{S}_{D-1}} \int_0^\infty \sup_{t \in \mathbb{R}^+} (|\nabla_x R_t(rv)| + |\nabla_x^2 R_t(rv)|) dr < +\infty,$$

$$\sup_{v \in \mathbf{S}_{D-1}} \int_0^\infty \sup_{t \in \mathbb{R}^+} (|\nabla_x^3 R_t(rv)|) r dr < +\infty,$$

and for some constant $C > 0$ independent on ε

$$\sup_{v \in \mathbf{S}_{D-1}} \int_0^\infty \sup_{s,t,x \in \mathbb{R}^{D+2}} (|\nabla_{x,y}^{1,1} g_{s,t}^\varepsilon(x, x - rv)|) dr$$

$$+ \sup_{v \in \mathbf{S}_{D-1}} \int_0^\infty \sup_{s,t,x \in \mathbb{R}^{D+2}} (|\nabla_{x,y}^{1,2} g_{s,t}^\varepsilon(x, x - rv)|) r dr \leq C.$$

Property (A1) implies some regularity on U^ε when ε is fixed, but notice that we may have $N(\varepsilon) \rightarrow +\infty$ (actually, in Example 1 given in the previous section, it does). The three last properties have been yet announced in the introduction and in the beginning of Section 2. The technical assumption needed on R and g^ε for the classical transport problem are detailed in Property (A4) and (A').

Definition 3.1. The correlation matrix H of the force field and the corresponding remainder term j^ε are defined by

$$H_t(x) = -\nabla_x^2 R_t(x), \quad j_{t,s}^\varepsilon(x, y) = \nabla_{x,y}^{1,1} g_{s,t+s}^\varepsilon(x, x - y). \quad (10)$$

Remark 3.1. Using the above definitions, and by remarking that

$$\nabla_x^2 R_t(x - y) = -\nabla_{x,y}^{1,1} (R_t(x - y))$$

we obtain that

$$\langle E_t^h(x) \otimes E_s^h(y) \rangle = H_{(t-s)/(h\tau)}\left(\frac{x-y}{h}\right) + \varepsilon J_{s/(h\tau), (t-s)/(h\tau)}^{\varepsilon}\left(\frac{x}{h}, \frac{x-y}{h}\right).$$

Note also that $E^h \in L^\infty(\mathbb{R}^+; W^{2,\infty}(\mathbb{R}^D))$, $E_t^h(x)$, $E_s^h(y)$ are independent random variables for $|t-s| \geq h\tau$ and that $\langle E_t^h(x) \rangle = 0$.

The limit behavior of the solution of Liouville equations (7) is governed by a diffusion process. The corresponding diffusion matrix is given below.

Proposition 3.1. *Assume (A) holds. Let $\xi \in \mathbb{R}^D$, the diffusion matrix defined by*

$$A_\tau(\xi) = \int_0^\tau H_{s/\tau}(s\xi) ds, \quad \text{for } \tau \in (0, +\infty), \quad A_\infty(\xi) = \int_0^\infty H_0(s\xi) ds,$$

is symmetric and nonnegative. It lies in $(L^\infty(\mathbb{R}^D))^{D \times D}$ for $\tau < \infty$ and if (A') holds, $A_\infty(\xi)$ is bounded by $C/|\xi|$ for some positive constant C . Moreover for all $\xi \in \mathbb{R}^D \setminus \{0\}$ we have $A_\infty(\xi) \cdot \xi = 0$.

This section is devoted to the proof of the following theorem.

Theorem 3.1. *Let $E_t^h = \nabla_x U_{t/h\tau}^\varepsilon(x/h)$ where the random potential U^ε satisfies assumptions (A) and is independent on the random initial data f^0 . We assume that for a positive constant C_0 we have $\|f^0\|_{L^\infty \cap L^1(\mathbb{R}^{2D})} \leq C_0$ and that*

$$\varepsilon(1 + \tau^2) + h(\tau^2 + N(\varepsilon)^2 \tau^{10}) \rightarrow 0. \tag{11}$$

Then, up to a subsequence, $\langle f_h \rangle$ converges for $1 < p < \infty$ and for any time $T > 0$ in $C^0([0, T]; L^p(\mathbb{R}^{2D})\text{-weak})$ to a function $f \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^{2D}) \cap L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^{2D}))$ with $f(t=0) = \langle f_0 \rangle$. Moreover:

- if τ is fixed then f is solution of

$$\frac{\partial}{\partial t} f + \xi \cdot \nabla_x f - \operatorname{div}_\xi (A_\tau(\xi) \nabla_\xi f) = 0, \quad \text{for } t > 0, (x, \xi) \in \mathbb{R}^{2D}, \tag{12}$$

in the sense of distribution.

- if $\tau \rightarrow \infty$ and (A') holds then f is solution of

$$\begin{aligned} \frac{\partial}{\partial t} f + \xi \cdot \nabla_x f - \operatorname{div}_\xi (A_\infty(\xi) \nabla_\xi f) &= 0, \\ \text{for } t > 0, x \in \mathbb{R}^D, \xi \in \mathbb{R}^D \setminus \{0\}, \end{aligned} \tag{13}$$

in the sense of distribution. In (13), $|\xi|$ is only a parameter. If $D \geq 3$ then f is solution of (13) on the whole space $(0, \infty) \times \mathbb{R}^{2D}$.

Remark 3.2. In view of Proposition 3.1, we have that $A_\infty(\xi) \cdot \nabla_\xi \phi = 0$ for every function ϕ which depends only on the modulus $|\xi|$. Therefore (13) is equivalent to the following statement. For every $\phi \in C_c^\infty((0, \infty))$ the function f_ϕ defined by $f_\phi(t, x, \xi) = f(t, x, \xi)\phi(|\xi|^2)$ is a solution of (13) on the whole space $(0, \infty) \times \mathbb{R}^{2D}$ with initial data $\langle f_0(x, \xi) \rangle \phi(|\xi|^2)$.

From (9), by Liouville theorem, $dX d\mathcal{E} = dx d\xi$ which implies

$$\|f_h(t, \cdot, \cdot)\|_{L^p(\mathbb{R}^{2D})} = \|f^0\|_{L^p(\mathbb{R}^{2D})} \quad (14)$$

for every p , $1 \leq p \leq +\infty$. The aim is to find the evolution of the limit of $\langle f_h \rangle$ when $h \rightarrow 0$. Before proving the theorem we show the nonnegativity of the diffusion matrix.

Proof of Proposition 3.1. Assumption (A') leads to the bound in $C/|\xi|$ for $A_\infty(\xi)$ and (A4) leads to the bound in $(L^\infty(\mathbb{R}^D))^{D \times D}$ for A_τ . The matrix $H_t(x)$ is symmetric as limit of tensor products. Therefore $A_\tau(\xi)$ is also symmetric. The matrix $H_t(x)$ is also even with respect to time and position. It yields $A_\tau(\xi) = \frac{1}{2} \int_{-\tau}^{\tau} H_{s/\tau}(s\xi) ds$. We prove now that $A_\infty(\xi) \cdot \xi = 0$. In view of (10) we have:

$$A_\infty(\xi) \cdot \xi = -\frac{1}{2} \int_{-\infty}^{+\infty} \nabla_x^2 R_0(s\xi) \cdot \xi ds = -\frac{1}{2} \int_{-\infty}^{+\infty} \frac{d}{ds} (\nabla_x R_0(s\xi)) ds.$$

For $\xi \neq 0$, the function $s \mapsto \nabla_x R_0(s\xi)$ belongs to $W^{1,1}(0, \infty)$ because of (A'). Therefore $\nabla_x R_0(s\xi)$ vanishes at $s = \pm\infty$.

It remains to show that A_τ and A_∞ are nonnegative. We use the following lemma.

Lemma 3.1. For every $F \in L^1(\mathbb{R})$,

$$\int_{-\infty}^{+\infty} F(s) ds = \lim_{R \rightarrow +\infty} \frac{1}{2R} \int_{-R}^R \int_{-R}^R F(s-t) ds dt.$$

Let us postpone the proof of the lemma for a while. We find that

$$A_\infty(\xi) = \lim_{R \rightarrow +\infty} \frac{1}{4R} \int_{-R}^R \int_{-R}^R H_0((s-t)\xi) ds dt.$$

Using property (A') and Remark 3.1, with $x = s\xi$, $y = t\xi$,

$$\frac{1}{4R} \int_{-R}^R \int_{-R}^R H_0((s-t)\xi) \, ds \, dt = \lim_{\varepsilon \rightarrow 0} \frac{1}{4R} \left\langle \int_{-R}^R \nabla_x U_0^\varepsilon(s\xi) \, ds \otimes \int_{-R}^R \nabla_x U_0^\varepsilon(t\xi) \, dt \right\rangle,$$

which is a nonnegative matrix. Therefore $A_\infty(\xi)$ is nonnegative too. In the same way we find, using (A4), that

$$A_\tau(\xi) = \frac{1}{2} \int_{-\infty}^{+\infty} H_{s/\tau}(s\xi) \, ds = \lim_{R \rightarrow +\infty} \frac{1}{4R} \int_{-R}^R \int_{-R}^R H_{(s-t)/\tau}((s-t)\xi) \, ds \, dt.$$

We conclude in the same way that A_τ is nonnegative. \square

Proof of Lemma 3.1. We have

$$\int_{-\infty}^{+\infty} F(s) \, ds = \frac{1}{2R} \int_{-R}^R \int_{-\infty}^{+\infty} F(s) \, ds \, dt = \frac{1}{2R} \int_{-R}^R \int_{-\infty}^{+\infty} F(s-t) \, ds \, dt.$$

Therefore

$$\begin{aligned} & \left| \int_{-\infty}^{+\infty} F(s) \, ds - \frac{1}{2R} \int_{-R}^R \int_{-R}^R F(s-t) \, ds \, dt \right| \\ & \leq \frac{1}{2R} \int_{-R}^R \int_{|s| \geq R} |F(s-t)| \, ds \, dt \leq \frac{1}{2R} \int_{-R}^R \int_{|s+t| \geq R} |F(s)| \, ds \, dt \\ & \leq \frac{1}{2R} \int_{|t| \leq (1-\alpha)R} \int_{|s| \geq \alpha R} |F(s)| \, ds \, dt + \frac{1}{2R} \int_{R > |t| > (1-\alpha)R} \int_{\mathbb{R}} |F(s)| \, ds \, dt \\ & \leq \int_{|s| \geq \alpha R} |F(s)| \, ds + \alpha \|F\|_{L^1}, \end{aligned}$$

for every $0 < \alpha \leq 1$. When α is fixed the first term converges to 0 when R goes to $+\infty$. Therefore

$$\limsup_{R \rightarrow +\infty} \left| \int_{-\infty}^{+\infty} F(s) \, ds - \frac{1}{2R} \int_{-R}^R \int_{-R}^R F(s-t) \, ds \, dt \right| \leq \alpha \|F\|_{L^1}$$

for every $\alpha > 0$, so finally it is null. \square

Proof of Theorem 3.1. We define S_t the linear operator from $L^p(\mathbb{R}^{2D})$ to $L^p(\mathbb{R}^{2D})$ associated to the free streaming:

$$(S_t \eta)(x, \xi) = \eta(x - t\xi, \xi).$$

The quantity $\langle f_h \rangle$ verifies

$$\partial_t \langle f_h \rangle + \xi \cdot \nabla_x \langle f_h \rangle = \langle \theta_t^h f_h \rangle. \quad (15)$$

We have to study the limit when h goes to 0 of $\langle \theta_t^h f_h \rangle$. Remember that $\theta_t^h = -(1/\sqrt{h})E_t^h(x) \cdot \nabla_\xi$ with $E_t^h = \nabla_x U_{t/h\tau}^\varepsilon(x/h)$. Hence the assumptions (i) and (ii) of the introduction are straightforward consequences of hypothesis (A2), (A3). We can use its strategy of proof based on Duhamel formulae to find:

$$\langle \theta_t^h f_h \rangle = \int_0^{h\tau} \langle \theta_t^h S_\sigma \theta_{t-\sigma}^h S_{-\sigma} \rangle \langle f_h(t) \rangle d\sigma + r_t^h + e_t^h, \quad (16)$$

where:

$$r_t^h = \int_0^{h\tau} \int_0^{2h\tau-\sigma} \langle \theta_t^h S_\sigma \theta_{t-\sigma}^h S_s \theta_{t-\sigma-s}^h f_h(t-\sigma-s) \rangle ds d\sigma,$$

$$e_t^h = \int_0^{h\tau} \langle \theta_t^h S_\sigma \theta_{t-\sigma}^h S_{-\sigma} \rangle \langle S_{2h\tau} f_h(t-2h\tau) - f_h(t) \rangle d\sigma.$$

All these equalities have to be understood in the sense of distribution. We define the operator L_t^h from $W^{3,1}(\mathbb{R}^{2D})$ to $W^{1,1}(\mathbb{R}^{2D})$ by:

$$L_t^h \eta = \int_0^{h\tau} \langle S_\sigma \theta_{t-\sigma}^h S_{-\sigma} \theta_t^h \rangle \eta d\sigma. \quad (17)$$

We have the following lemma:

Lemma 3.2. Let $\eta \in W^{3,1}(\mathbb{R}^{2D})$ a test function, for all $t \in [2h\tau, +\infty)$ we have:

$$\int_{\mathbb{R}^{2D}} \langle \theta_t^h f_h(t) \rangle \eta dx d\xi = \int_{\mathbb{R}^{2D}} \langle f_h(t) \rangle L_t^h \eta dx d\xi + \int_{\mathbb{R}^{2D}} \langle S_{2h\tau} f_h(t-2h\tau) - f_h(t) \rangle L_t^h \eta dx d\xi$$

$$+ \rho_1^h(t) \quad (18)$$

where the remainder $\rho_1^h(t)$ is bounded by

$$\rho_1^h(t) \leq C\sqrt{h}\tau^5 N(\varepsilon)\|\eta\|_{W^{3,1}(\mathbb{R}^{2D})}\|f_0\|_{L^\infty} \tag{19}$$

for a positive constant C independent on h and t .

Proof. Multiplying (16) by the test function η and integrating with respect to x and ξ gives:

$$\begin{aligned} \int_{\mathbb{R}^{2D}} \langle \theta_t^h f_h(t) \rangle \eta \, dx \, d\xi &= \int_{\mathbb{R}^{2D}} \int_0^{h\tau} \langle f_h(t) \rangle \langle S_\sigma \theta_{t-\sigma} S_{-\sigma} \theta_t^h \eta \rangle \, d\sigma \, dx \, d\xi \\ &+ \int_{\mathbb{R}^{2D}} \int_0^{h\tau} \langle S_{2h\tau} f_h(t-2h\tau) - f_h(t) \rangle \langle S_\sigma \theta_{t-\sigma} S_{-\sigma} \theta_t^h \eta \rangle \, d\sigma \, dx \, d\xi \\ &+ \rho_1^h(t), \end{aligned} \tag{20}$$

with

$$\begin{aligned} \rho_1^h(t) &= \int_{\mathbb{R}^{2D}} r_t^h \eta \, dx \, d\xi \\ &= \int_{\mathbb{R}^{2D}} \int_0^{h\tau} \int_0^{2h\tau-\sigma} \langle f_h(t-\sigma-s) \rangle \langle \theta_{t-\sigma-s}^h S_{-s} \theta_{t-\sigma}^h S_{-\sigma} \theta_t^h \eta \rangle \, ds \, d\sigma \, dx \, d\xi. \end{aligned}$$

The definition of L_t^h gives (18). Using (14) we find that

$$|\rho_1^h| \leq 2h^2\tau^2\|f_0\|_{L^\infty} \left\langle \sup_{t,s,\sigma} \|\theta_{t-s-\sigma}^h S_{-\sigma} \theta_{t-s}^h S_{-s} \theta_t^h \eta\|_{L^1(\mathbb{R}^{2D})} \right\rangle. \tag{21}$$

Then, the estimate (19) is a consequence of the next lemma. It ends the proof of Lemma 3.2. \square

Lemma 3.3. For every $\eta \in W^{3,1}(\mathbb{R}^{2D})$ we have the following estimate:

$$\left\langle \sup_{s,\sigma \in [0,h\tau]} \|\theta_{t-s-\sigma}^h S_{-\sigma} \theta_{t-s}^h S_{-s} \theta_t^h \eta\|_{L^1(\mathbb{R}^{2D})} \right\rangle \leq C \frac{\tau^3}{h^{3/2}} N(\varepsilon)\|\eta\|_{W^{3,1}(\mathbb{R}^{2D})}.$$

Proof. We introduce the operator $T_{t,s}^h = \theta_{t-s}^h S_{-s}$ and we want to estimate

$$\sup_{s,\sigma \in [0,h\tau]} \|T_{t-s,\sigma}^h T_{t,s}^h \theta_t^h \eta\|_{L^1(\mathbb{R}^{2D})}.$$

Notice that

$$T_{t,s}^h \Psi(x, \xi) = -\frac{1}{\sqrt{h}} E_{t-s}^h(x) \cdot \nabla_{\xi} (\Psi(x + s\xi, \xi))$$

and

$$\nabla_{\xi} (\Psi(x + s\xi, \xi)) = \nabla_{\xi} \Psi(x + s\xi, \xi) + s \nabla_x \Psi(x + s\xi, \xi). \tag{22}$$

Therefore, since $|s| \leq h\tau$

$$\|T_{t,s}^h \Psi\|_{L^1(\mathbb{R}^{2D})} \leq \tau/\sqrt{h} \|E^h\|_{L^\infty} \|\Psi\|_{1,1,h}, \tag{23}$$

where we denote

$$\|\Psi\|_{\alpha,p,h} = \sum_{|k_1|+|k_2|=\alpha} \|\partial_{\xi}^{k_1} (h \partial_x)^{k_2} \Psi\|_{L^p(\mathbb{R}^{2D})}.$$

Using again (22) we find:

$$\|T_{t,s}^h \Psi\|_{1,1,h} \leq \tau/\sqrt{h} (\|h \nabla_x E^h\|_{L^\infty} \|\Psi\|_{1,1,h} + \tau \|E\|_{L^\infty} \|\Psi\|_{2,1,h}). \tag{24}$$

We need to estimate $\|\theta_t^h \eta\|_{1,1,h}$ and $\|\theta_t^h \eta\|_{2,1,h}$,

$$\left\{ \begin{array}{l} \|\theta_t^h \eta\|_{1,1,h} \leq \frac{1}{\sqrt{h}} (\|h \nabla_x E^h\|_{L^\infty} \|\nabla_{\xi} \eta\|_{L^1} + \|E^h\|_{L^\infty} \|\eta\|_{W^{2,1}(\mathbb{R}^{2D})}), \\ \|\theta_t^h \eta\|_{2,1,h} \leq \frac{1}{\sqrt{h}} (\|E^h\|_{L^\infty} \|\eta\|_{W^{3,1}(\mathbb{R}^{2D})} + \|h \nabla_x E^h\|_{L^\infty} \|\eta\|_{W^{2,1}(\mathbb{R}^{2D})} \\ \qquad \qquad \qquad + \|(h \nabla_x)^2 E^h\|_{L^\infty} \|\eta\|_{W^{1,1}(\mathbb{R}^{2D})}). \end{array} \right. \tag{25}$$

Therefore, thanks to (23)–(25),

$$\begin{aligned} & \sup_{s,\sigma \in [0,h\tau]} \|T_{t-s,\sigma}^h T_{t,s}^h \theta_t^h \eta\|_{L^1(\mathbb{R}^{2D})} \\ & \leq \tau/\sqrt{h} \sup_{s \in [0,h\tau]} \|E^h\|_{L^\infty} \|T_{t,s}^h \theta_t^h \eta\|_{1,1,h} \\ & \leq \tau^2/h \|E^h\|_{L^\infty} (\|h \nabla_x E^h\|_{L^\infty} \|\theta_t^h \eta\|_{1,1,h} + \tau \|E^h\|_{L^\infty} \|\theta_t^h \eta\|_{2,1,h}) \\ & \leq C \tau^3/h^{3/2} \|E^h\|_{L^\infty} (\|E^h\|_{L^\infty} + \|h \nabla_x E^h\|_{L^\infty}) \\ & \quad \times (\|E^h\|_{L^\infty} + \|h \nabla_x E^h\|_{L^\infty} + \|(h \nabla_x)^2 E^h\|_{L^\infty}) \|\eta\|_{W^{3,1}(\mathbb{R}^{2D})}. \end{aligned}$$

Therefore the expectation value of $\sup_{s,\sigma \in [0,h\tau]} \|T_{t-s,\sigma}^h T_{t,s}^h \theta_t^h \eta\|_{L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^{2D}))}$ is estimated by $C \tau^3/h^{3/2} N(\varepsilon) \|\eta\|_{W^{3,1}(\mathbb{R}^{2D})}$ which ends the proof. \square

It remains to compute the limit of $L_t^h \eta$.

Proposition 3.2. *Let $\eta \in C_c^\infty(\mathbb{R}^{2D})$ be a test function. If assumptions (A) hold, then for a fixed τ we have:*

$$\begin{aligned} L_t^h \eta &\rightarrow \operatorname{div}_\xi (A_\tau(\xi) \cdot \nabla_\xi \eta) \quad \text{as } (1 + \tau^2)\varepsilon + h\tau^2 N(\varepsilon)^2 \rightarrow 0, \\ &\text{in } C^0(\mathbb{R}_t^+; L^1(\mathbb{R}^{2D})). \end{aligned} \tag{26}$$

If we assume moreover (A') and $\|\nabla_\xi \eta|\xi|^{-2}\|_{L^1} + \|\nabla_\xi^2 \eta|\xi|^{-1}\|_{L^1} + \|\nabla_{x,\xi}^{1,1} \eta|\xi|^{-2}\|_{L^1} < +\infty$ then

$$\begin{aligned} L_t^h \eta &\rightarrow \operatorname{div}_\xi (A_\infty(\xi) \cdot \nabla_\xi \eta) \quad \text{as } \varepsilon + h(\tau + N(\varepsilon)^2 \tau^{10}) \rightarrow 0, \\ \tau &\rightarrow \infty, \text{ in } C^0(\mathbb{R}_t^+; L^1(\mathbb{R}^{2D})). \end{aligned} \tag{27}$$

We postpone the proof of this proposition at the end of this section.

Since $\|\langle f_h \rangle\|_{L^\infty} \leq \|f_0\|_{L^\infty}$, up to a subsequence there exists $f \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^{2D})$ such that $\langle f_h \rangle$ converges to f in $L^\infty(\mathbb{R}^+ \times \mathbb{R}^{2D})$ -weak*. Notice that $h\tau$ goes to zero, therefore $\langle S_{2h\tau} f_h(t - 2h\tau) - f_h(t) \rangle$ converges to 0 in $L^\infty(\mathbb{R}^+ \times \mathbb{R}^{2D})$ -weak* as well. Using Lemma 3.2, the definition of Proposition 3.1 and the above Proposition 3.2, we obtain in the two asymptotic regimes

$$\mathbf{1}_{\{t \geq 2h\tau\}} \int_{\mathbb{R}^{2D}} \langle \theta_t^h f_h \rangle \eta \, dx \, d\xi \rightarrow \int_{\mathbb{R}^{2D}} f \operatorname{div}_\xi (A_\tau(\xi) \nabla_\xi \eta) \, dx \, d\xi, \quad \text{in } L^\infty(0, \infty)\text{-weak}^*, \tag{28}$$

for every test functions $\eta \in C_c^\infty(\mathbb{R}^{2D})$ which have to satisfy, in the case $\tau = \infty$,

$$\|\nabla_\xi \eta|\xi|^{-2}\|_{L^1} + \|\nabla_\xi^2 \eta|\xi|^{-1}\|_{L^1} + \|\nabla_{x,\xi}^{1,1} \eta|\xi|^{-1}\|_{L^1} < +\infty.$$

In particular, for $D \geq 3$, the above condition is always satisfied. For $D \leq 2$, it holds for $\eta \in C_c^\infty(\mathbb{R}^D \times \mathbb{R}^D \setminus \{0\})$. We deduce from (15) that f verifies (12) for the first asymptotic regime. In the case $\tau \rightarrow \infty$, f satisfies (13) for $D \geq 3$ in the sense of distribution on $(0, \infty) \times \mathbb{R}^{2D}$. For $D \leq 2$, the same conclusion holds only on $(0, \infty) \times \mathbb{R}^D \times \mathbb{R}^D \setminus \{0\}$.

Continuity in time and the initial condition. The above convergence result (28) and (15) implies that

$$\frac{d}{dt} \int_{\mathbb{R}^{2D}} \langle f_h \rangle \eta \, dx \, d\xi = \int_{\mathbb{R}^{2D}} -\langle f_h \rangle \xi \cdot \nabla_x \eta + \langle \theta_t^h f_h \rangle \eta \, dx \, d\xi \leq C(\eta), \quad \forall t \geq 2h\tau,$$

for a positive constant $C(\eta)$ independent on t and on h and for every test function in $C_c^\infty(\mathbb{R}^D \times \mathbb{R}^D \setminus \{0\})$.

So if we denote $\tilde{f}_h(t) = \langle f_h \rangle(t + 2h\tau)$, $\int_{\mathbb{R}^{2D}} \tilde{f}_h \eta \, dx \, d\xi$ is compact in $C^0([0, T])$. Moreover, we have:

$$\begin{aligned} \left| \int_{\mathbb{R}^{2D}} \tilde{f}_h(t)\eta - \langle f_h \rangle(t)\eta \, dx \, d\xi \right| &\leq C_0 \int_0^{2h\tau} \int_{\mathbb{R}^{2D}} \frac{1}{\sqrt{h}} |\langle E_t^h(x) \rangle \cdot \nabla_\xi \eta| \, dx \, d\xi \, d\sigma \\ &\quad + C_0 \int_0^{2h\tau} \int_{\mathbb{R}^{2D}} |\xi| |\nabla_x \eta| \, dx \, d\xi \, d\sigma \\ &\leq C(\eta)\tau N(\varepsilon)\sqrt{h}. \end{aligned}$$

Therefore, since $h\tau^2 N(\varepsilon)^2$ converges to 0, $\int_{\mathbb{R}^{2D}} \langle f_h \rangle(t)\eta \, dx \, d\xi$ is also compact in $C^0([0, T])$ and converges uniformly toward $\int_{\mathbb{R}^{2D}} f(t)\eta \, dx \, d\xi$. In particular $f(t=0) = \langle f^0 \rangle$. Finally the bound in $L^\infty(\mathbb{R}^+; L^p(\mathbb{R}^{2D}))$ of $\langle f_h \rangle$ and the density of $C_c^\infty(\mathbb{R}^D \times \mathbb{R}^D \setminus \{0\})$ in $L^q(\mathbb{R}^{2D})$ ($1/p + 1/q = 1$) imply via an $\varepsilon/3$ argument that $\langle f_h \rangle \rightarrow f$ in $C^0([0, T]; L^p(\mathbb{R}^{2D}))$ -weak for any $T > 0$ and for $1 < p < \infty$. The proof of Theorem 3.1 is complete. \square

Proof of Proposition 3.2. Performing the change of variable $\sigma = hs$ in (17) we find:

$$L_t^h \eta = \int_0^\tau h \langle S_{hs} \theta_{t-hs}^h S_{-hs} \theta_t^h \eta \rangle \, ds.$$

We have:

$$\langle h S_{hs} \theta_{t-hs}^h S_{-hs} \theta_t^h \eta \rangle = (\operatorname{div}_\xi + hs \operatorname{div}_x) (\langle E_{t-hs}^h(x - hs\xi) \otimes E_t^h(\xi) \rangle \nabla_\xi \eta(x, \xi)).$$

Using Definition 3.1 and Remark 3.1, leads to

$$L_t^h \eta = \operatorname{div}_\xi \left(\int_0^\tau H_{s/\tau}(s\xi) \, ds \cdot \nabla_\xi \eta(x, \xi) \right) + \rho_2^h(t, x, \xi),$$

where

$$\begin{aligned} \rho_2^h(t, x, \xi) &= \int_0^\tau sh H_{s/\tau}(s\xi) \, ds : \nabla_{x,\xi}^{1,1} \eta(x, \xi) \\ &\quad + \varepsilon \int_0^\tau (\operatorname{div}_\xi + sh \operatorname{div}_x) (j_{s/\tau, t/\tau h}^\varepsilon(x/h, s\xi) \nabla_\xi \eta(x, \xi)) \, ds. \end{aligned}$$

Let us split the proof into two parts. First we consider the simplest case when τ is fixed, then the case $\tau \rightarrow +\infty$.

The case τ is fixed. Assumption (A4) and the fact that s remains bounded by τ imply

$$|\rho_2^h(t, x, \xi)| \leq Ch\tau^2 |\nabla_{x,\xi}^{1,1} \eta|(x, \xi) + C\varepsilon(1 + \tau^2) (|\nabla_\xi \eta|(x, \xi) + |\nabla_\xi^2 \eta|(x, \xi) + |\nabla_{x,\xi}^{1,1} \eta|(x, \xi)).$$

Hence ρ_2^h converges to 0 in $C^0(\mathbb{R}_t^+; L^1(\mathbb{R}^{2D}))$.

The case τ tends to ∞ . Now we consider the case $\tau \rightarrow +\infty$. In order to estimate the remainder term ρ_2^h let us state the following lemma:

Lemma 3.4. *Let $\alpha \in \mathbb{N}$, $g \in L^\infty(\mathbb{R}^D)$ and $\phi \in C_c^\infty(\mathbb{R}^{D+1})$. Then*

$$\int_{\mathbb{R}^D} \int_0^\infty s^\alpha |g(s\xi)\phi(s, \xi)| \, ds \, d\xi \leq \left\| \sup_s |\phi(s/|\xi|, \xi)| \cdot |\xi|^{-(1+\alpha)} \right\|_{L^1(\mathbb{R}^D)} \sup_{v \in \mathbf{S}_{D-1}} \int_0^\infty s^\alpha |g(sv)| \, ds$$

as soon as the right-hand side is well defined.

Proof. We use the change of variable $s|\xi| \rightarrow s$ to find

$$\begin{aligned} & \int_{\mathbb{R}^D} \int_0^\infty s^\alpha |g(s\xi)\phi(s, \xi)| \, ds \, d\xi \\ &= \int_{\mathbb{R}^D} \int_0^{+\infty} s^\alpha |\xi|^{-(\alpha+1)} |g(s\xi/|\xi|)\phi(s/|\xi|, \xi)| \, ds \, d\xi \\ &\leq \left\| \sup_s |\phi(s/|\xi|, \xi)| \cdot |\xi|^{-(\alpha+1)} \right\|_{L^1} \sup_{v \in \mathbf{S}_{D-1}} \left(\int_0^\infty s^\alpha |g(sv)| \, ds \right). \quad \square \end{aligned}$$

We go back to the proof of the proposition. We have:

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^{2D}} |\rho_2^h(t, x, \xi)| \, dx \, d\xi \\ &\leq h \int_0^\infty \int_{\mathbb{R}^D} \sup_\sigma |s H_{\sigma/\tau}(s\xi)| \left(\int_{\mathbb{R}^D} |\nabla_{x,\xi}^{1,1} \eta(x, \xi)| \, dx \right) \, d\xi \, ds \\ &\quad + \varepsilon \int_0^\infty \int_{\mathbb{R}^D} \sup_{x,\sigma,t} |s \nabla j_{\sigma,t}^\varepsilon(x/h, s\xi)| \left(\int_{\mathbb{R}^D} |\nabla_\xi \eta(x, \xi)| \, dx \right) \, d\xi \, ds \end{aligned}$$

$$\begin{aligned}
 & + \varepsilon \int_0^\infty \int_{\mathbb{R}^D} \left(\sup_{x, \sigma, t} |j_{\sigma, t}^\varepsilon(x/h, s\xi)| \int_{\mathbb{R}^D} (|\nabla_\xi^2 \eta|(x, \xi) + h\tau |\nabla_{\xi, x}^{1,1} \eta|(x, \xi)) dx \right) d\xi ds \\
 & \leq h \sup_v \int_0^\infty r \sup_s |H_s(rv)| dr \|\nabla_{x, \xi}^{1,1} \eta |\xi|^{-2}\|_{L^1} \\
 & \quad + \varepsilon \sup_v \int_0^\infty \sup_{s, t, x} |j_{s, t}^\varepsilon(x, rv)| dr (\|\nabla_\xi^2 \eta |\xi|^{-1}\|_{L^1} + h\tau \|\nabla_{x, \xi}^{1,1} \eta |\xi|^{-1}\|_{L^1}) \\
 & \quad + \varepsilon \sup_v \int_0^\infty r \sup_{s, t, x} |\operatorname{div}_y j_{s, t}^\varepsilon(x, rv)| dr \|\nabla_\xi \eta |\xi|^{-2}\|_{L^1}. \tag{29}
 \end{aligned}$$

In the last inequality we use Lemma 3.4 with successively

$$\begin{aligned}
 g(\xi) &= \sup_s |H_s(\xi)| \quad \text{and} \quad \phi(\xi) = \int |\nabla_{x, \xi}^{1,1} \eta|(x, \xi) dx, \\
 g(\xi) &= \sup_{x, s, t} |j_{s, t}^\varepsilon(x, \xi)| \quad \text{and} \quad \phi(\xi) = \int |\nabla_\xi^2 \eta|(x, \xi) dx, \\
 g(\xi) &= \sup_{x, s, t} |\nabla_y j_{s, t}^\varepsilon(x, \xi)| \quad \text{and} \quad \phi(\xi) = \int |\nabla_\xi \eta|(x, \xi) dx.
 \end{aligned}$$

Therefore, thanks to (A'), $\|\rho_2^h(t)\|_{L^1(\mathbb{R}^+ \times \mathbb{R}^{2D})}$ tends to 0 when h tends to 0 uniformly in time.

It remains to estimate the L^1 norm of

$$|\operatorname{div}_\xi ([H_{s/\tau}(s\xi) - H_0(s\xi)] \nabla_\xi \eta(x, \xi))|.$$

Notice that, up to a subsequence, it converges to 0 when $h \rightarrow 0$, $\tau \rightarrow \infty$, almost everywhere (since $H, \nabla_y H \in C(\mathbb{R}^+, L^1(\mathbb{R}^D))$). Moreover it is uniformly bounded by the function

$$2 \left(\sup_\sigma s |\operatorname{div}_y H_\sigma(s\xi)| |\nabla_\xi \eta| + \sup_\sigma |H_\sigma(s\xi)| |\nabla_\xi^2 \eta| \right)$$

which does not depend on h and which lies in $L^1_{s, x, \xi}$ thanks to Lemma 3.4 and provided that

$$\|\nabla_\xi \eta \cdot |\xi|^{-2}\|_{L^1} + \|\nabla_\xi^2 \eta \cdot |\xi|^{-1}\|_{L^1} < \infty.$$

So, thanks to Lebesgue Theorem, the L^1 norm of this term converges to 0. This does not depend on the subsequence by the uniqueness of the limit.

It ends the proof of Proposition 3.2. \square

4. Quantum transport

In Section 2, we have assumed that for parameters $\varepsilon \in (0, 1]$, there exist two real functions $R = R_t(x)$ and $g^\varepsilon = g_{t,s}^\varepsilon(x, y)$, such that the self correlation of the random potential, U^ε , at two different points in the microscopic variables is of the form:

$$\langle U_t^\varepsilon(x) U_s^\varepsilon(y) \rangle = R_{t-s}(x - y) + \varepsilon g_{t,s}^\varepsilon(x, y).$$

In order to give precise assumption on the potential and on the functions R and g^ε we need to introduce the following Fourier transforms.

Definition 4.2. The power spectrum of time–space fluctuations, \mathcal{R} , is defined by

$$\mathcal{R}(\omega, p) = \frac{1}{(2\pi)^{D+1}} \int_{\mathbb{R}^{D+1}} R_\sigma(y) e^{-iy \cdot p} e^{-i\sigma \omega} d\sigma dy.$$

The power spectrum of space fluctuations \mathcal{Q} is given by:

$$\mathcal{Q}_\sigma(p) = \frac{1}{(2\pi)^D} \int_{\mathbb{R}^D} R_\sigma(y) e^{-ip \cdot y} dy. \tag{30}$$

We define the remainder power spectrum by

$$G_{\sigma,s}^\varepsilon(z, p) = \frac{1}{(2\pi)^D} \int_{\mathbb{R}^D} g_{\sigma,s}^\varepsilon(z, z + y) e^{-ip \cdot (y+z)} dy. \tag{31}$$

Let us remark that \mathcal{R} is the Fourier transform of \mathcal{Q}_σ with respect to σ . We are now ready to give the assumptions on the random potential we will use in this section:

- (B1) $U_t^\varepsilon(x) \in L^\infty(\mathbb{R}^{D+1})$ and $M(\varepsilon) := \langle (\|U^\varepsilon\|_{L^\infty(\mathbb{R}^{D+1})})^3 \rangle < \infty$,
- (B2) $U_t^\varepsilon(x), U_s^\varepsilon(y)$ are independent random variables for $|t - s| \geq 1$,
- (B3) $\langle U_t^\varepsilon(x) \rangle = 0$, for all $t \in \mathbb{R}, x \in \mathbb{R}^D$,
- (B4) $\langle U_t^\varepsilon(x) U_s^\varepsilon(y) \rangle = R_{t-s}(x - y) + \varepsilon g_{t,s}^\varepsilon(x, y)$, with $R_\sigma(y) \in L^\infty(\mathbb{R}_\sigma; L^1(\mathbb{R}_y^D))$,
- (B5) The function $(1 + |p|)\mathcal{Q}_\sigma(p)$ belongs to $L^\infty(\mathbb{R}_\sigma; L^1(\mathbb{R}_p^D))$,
- (B6) We have

$$C(G) := \sup_{\varepsilon \in (0, 1]} \int_{\mathbb{R}^D} \sup_{(\sigma, s, z) \in \mathbb{R}^{2+D}} |G_{\sigma,s}^\varepsilon(z, p)| dp < \infty.$$

In dimension $D \geq 2$, we will also need the supplementary assumption

$$(B') \quad C(\mathcal{R}) := \int_{\mathbb{R}} \sup_{p \in \mathbb{R}^D} (1 + |\omega|^\gamma) |\mathcal{R}(\omega, p)| \, d\omega < \infty$$

$$\text{for some } \gamma > \frac{1}{4} \text{ if } D = 2, \gamma > \frac{D-2}{2} \text{ for } D \geq 3.$$

Remark 4.3. We may have $M(\varepsilon) \rightarrow \infty$ as $\varepsilon \rightarrow 0$. In Example 1 of Section 2, we have $M(\varepsilon) = \varepsilon^{-3/2}$.

Properties (B2) to (B4) have been introduced in the introduction and in the beginning of Section 2 and properties (B5), (B6) and (B') are the technical requirements needed for the quantum random problem. Compared with assumptions (A) of the Section on classical transport, less regularity on the potential is assumed.

The function $R_\sigma(y)$ is even with respect to σ and y and in view of (B2) we have $\text{supp}(R_\sigma(y)) \subset \{|\sigma| \leq 1\}$ and $\text{supp}(g_{t,s}^\varepsilon(x, y)) \subset \{|t-s| \leq 1\}$.

We also remark that R_t is, uniformly with respect to t , bounded in the Wiener algebra of real functions integrable on \mathbb{R}^D such that their Fourier transform is also integrable. Therefore $R \in L^\infty(\mathbb{R}^{1+D})$. In view of (B6), we also have that $\|g^\varepsilon\|_{L^\infty(\mathbb{R}^{2+2D})} \leq C$ for a positive constant $C > 0$ independent on ε .

The random potential will give rise to collision operators which are defined by their differential cross section given in the following definitions.

Definition 4.3. The diffusive differential cross section q_τ and diffusive total cross section Λ_τ are defined for any $\tau > 0$ by

$$q_\tau(\xi, \xi') = 2\pi\tau \mathcal{R}(\tau(|\xi|^2 - |\xi'|^2), \xi - \xi'), \quad \xi, \xi' \in \mathbb{R}^D,$$

$$\Lambda_\tau(\xi) = \int_{\mathbb{R}^D} q_\tau(\xi, \xi') \, d\xi', \quad \xi \in \mathbb{R}^D.$$

The elastic differential cross section k and elastic total cross section Σ are defined by

$$k(\xi, \nu) = \pi |\xi|^{D-2} Q_0(\xi - |\xi|\nu), \quad \xi \in \mathbb{R}^D, \nu \in \mathbf{S}_{D-1}, \quad \Sigma(\xi) = \int_{\mathbf{S}_{D-1}} k(\xi, \nu) \, d\nu,$$

where \mathbf{S}_{D-1} denotes the sphere of \mathbb{R}^D and $d\nu$ the corresponding surface measure.

The cross sections have to be nonnegative. This is proved below.

Lemma 4.5. Under the assumptions (B) the power spectrum fluctuations $Q_\sigma(p)$ and $G_{\sigma,s}^\varepsilon(z, p)$ (respectively $\mathcal{R}(\omega, p)$) depend continuously on p (respectively on p and ω) and vanish at infinity.

The function $\mathcal{R}(\omega, p)$ is real and even with respect to ω and p , $Q_\sigma(p)$ is real and even with respect to σ and p and Q_σ vanishes for $|\sigma| \geq 1$. The function $G_{\sigma,s}^\varepsilon(z, p)$ vanishes

for $|\sigma - s| \geq 1$. Moreover the functions $\mathcal{Q}_0(p)$ and $\mathcal{R}(\omega, p)$ and then the cross sections of Definition 4.3 are nonnegative.

The smoothness and vanishing properties are consequences of Remark 4.3. The function \mathcal{Q} and \mathcal{R} are real and even because R is also real and even. It remains to check that \mathcal{Q}_0 and \mathcal{R} are nonnegative. This classical result in probability theory is a consequence of the Bochner criterion. We give a proof for the sake to be self-contained. For any real valued test function η which belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^{1+D})$ we compute:

$$\begin{aligned} \left\langle \left(\int_{\mathbb{R}^{1+D}} \eta(t, x) U_t^\varepsilon(x) dt dx \right)^2 \right\rangle &= \int_{\mathbb{R}^{2+2D}} \eta(t, x) \eta(s, y) \langle U_t^\varepsilon(x) U_s^\varepsilon(y) \rangle dt dx ds dy \\ &= \int_{\mathbb{R}^{2+2D}} \eta(t, x) \eta(s, y) R_{t-s}(x - y) dt dx ds dy + O(\varepsilon) \\ &= \int_{\mathbb{R}^{2+2D}} \eta(t, x) \eta(t + s, x + y) R_s(y) dt dx ds dy + O(\varepsilon). \end{aligned}$$

We have:

$$\int_{\mathbb{R}^{1+D}} \eta(t, x) \eta(t + s, x + y) dt dx = \frac{1}{(2\pi)^{1+D}} \int_{\mathbb{R}^{1+D}} |\hat{\eta}(\omega, \xi)|^2 e^{is\omega} e^{iy \cdot \xi} d\omega d\xi,$$

where $\hat{\eta}$ is the Fourier transform of η . Because R is even we obtain:

$$\left\langle \left(\int_{\mathbb{R}^{1+D}} \eta(t, x) U_t^\varepsilon(x) dt dx \right)^2 \right\rangle = \frac{1}{2\pi} \int_{\mathbb{R}^{1+D}} |\hat{\eta}(\omega, \xi)|^2 \mathcal{R}(\omega, \xi) d\omega d\xi + O(\varepsilon).$$

The limit $\varepsilon \rightarrow 0$ yields

$$\forall \eta \in \mathcal{S}(\mathbb{R}^{1+D}), \quad \int_{\mathbb{R}^{1+D}} |\hat{\eta}(\omega, \xi)|^2 \mathcal{R}(\omega, \xi) d\omega d\xi \geq 0.$$

But any positive function of $\mathcal{S}(\mathbb{R}^{1+D})$ is of the form $|\hat{\eta}|^2$, therefore the above inequality shows that \mathcal{R} is nonnegative. We obtain the same result for \mathcal{Q}_0 by noticing that $\mathcal{Q}_0(\xi) = \int_{\mathbb{R}} \mathcal{R}(\omega, \xi) d\omega$ because \mathcal{R} is the Fourier transform of \mathcal{Q}_σ .

4.1. Schrödinger and Wigner equations

In this section we are concerned with the asymptotic behavior when $h \rightarrow 0$ of the solutions of the following Schrödinger equations

$$i\hbar \frac{\partial}{\partial t} \psi_{n,h} = -\frac{\hbar^2}{2} \Delta_x \psi_{n,h} + \sqrt{\hbar} V_t^h(x) \psi_{n,h}, \quad t \in \mathbb{R}, x \in \mathbb{R}^D, n = 1, 2, \dots, \quad (32)$$

$$\psi_{n,h}(0, x) = \psi_{n,h}^I(x), \quad x \in \mathbb{R}^D, n = 1, 2, \dots \quad (33)$$

The potential V^h is obtained from the potential satisfying the assumptions (B) by rescaling the time and the position. As in the previous section, we assume that the parameters h, ε and τ are linked by some relations $\varepsilon = \varepsilon(h)$ and $\tau = \tau(h)$. Then V^h is defined by:

$$V_t^h(x) = U_{t/h\tau}^\varepsilon \left(\frac{x}{h} \right). \quad (34)$$

The parameter h is the rescaled Planck constant. It is the usual small parameter in the context of semi-classical limit. The parameter $h\tau$ is the lapse of time in the macroscopic variables for which the potential taken at two different times is not correlated.

These equations model the quantum transport of particles in the random potential $\sqrt{\hbar} V^h$. We use the mixed state approach. The initial data is assumed to form an orthonormal system of $L^2(\mathbb{R}^D)$. It classically results that for all time $t \in \mathbb{R}$, the system $(\psi_{n,h}(t))_{n=1,2,\dots}$ is also orthonormal

$$\int_{\mathbb{R}^D} |\psi_{n,h}(t, x)|^2 dx = 1, \quad \int_{\mathbb{R}^D} \psi_{n,h}(t, x) \overline{\psi_{m,h}(t, x)} dx = 0, \quad (35)$$

$$t \in \mathbb{R}, n \neq m, n, m = 1, 2, \dots$$

To each index n corresponds an occupation probability $\lambda_{n,h}$ that the state of the particle is described by the wave function $\psi_{n,h}$. We assume that

$$\lambda_{n,h} \geq 0, \quad \sum_{n=1}^\infty \lambda_{n,h} = 1, \quad \sum_{n=1}^\infty (\lambda_{n,h})^2 \leq C_0 h^D \quad (36)$$

for some constant $C_0 > 0$, independent on h . A typical example is given by $\lambda_{n,h} = 1/N_h$ for $n \leq N_h$ and $\lambda_{n,h} = 0$ for $n > N_h$ with $N_h = O(1/h^D)$.

As in [9,10,15,16] we introduce the Wigner function which is associated to the mixed state:

$$w_h(t, x, \xi) = \sum_{n=1}^\infty \lambda_{n,h} \frac{1}{(2\pi)^D} \int_{\mathbb{R}^D} \psi_{n,h} \left(t, x + \frac{h}{2} y \right) \overline{\psi_{n,h} \left(t, x - \frac{h}{2} y \right)} e^{-iy \cdot \xi} dy. \quad (37)$$

$$w_h^I(x, \xi) = \sum_{n=1}^\infty \lambda_{n,h} \frac{1}{(2\pi)^D} \int_{\mathbb{R}^D} \psi_{n,h}^I \left(x + \frac{h}{2} y \right) \overline{\psi_{n,h}^I \left(x - \frac{h}{2} y \right)} e^{-iy \cdot \xi} dy. \quad (38)$$

We refer to [9,10,15,16] for properties of Wigner functions. We only emphasize that the weak limit of w_h allows to determine the limit of observables of quantum mechanics. We have:

Proposition 4.3. *Assume that the functions $\psi_{n,h}$ solve (32), (33) with initial data which form an orthonormal system. Then the Wigner functions w_h ($1 \geq h > 0$), defined by (37) with occupation probabilities satisfying (36) are real functions. They lie in a bounded set of $L^\infty(\mathbb{R}; L^2(\mathbb{R}^{2D}))$,*

$$\forall h \in (0, 1], \forall t \in \mathbb{R}, \quad \|w_h(t)\|_{L^2(\mathbb{R}^{2D})} \leq \sqrt{C_0}. \tag{39}$$

The cluster points of $(w_h)_{1 \geq h > 0}$ when $h \rightarrow 0$ in $L^\infty(\mathbb{R}; L^2(\mathbb{R}^{2D}))$ -weak are nonnegative functions f . They satisfy $\int_{\mathbb{R}^{2D}} f(t, x, \xi) dx d\xi \leq 1$ for all time $t \in \mathbb{R}$. The density defined by*

$$n_h(t, x) = \sum_{n=1}^{\infty} \lambda_{n,h} |\psi_{n,h}(t, x)|^2 \tag{40}$$

is bounded in $L^\infty(\mathbb{R}; L^1(\mathbb{R}^D))$. Moreover if for some subsequence $h_k \rightarrow 0$, $n_{h_k}(t, x) \rightarrow n$ in $L^\infty(\mathbb{R}; C_0^0(\mathbb{R}^D)')$ -weak and $w_{h_k} \rightarrow f$ in $L^\infty(\mathbb{R}; L^2(\mathbb{R}^{2D}))$ -weak*, then*

$$\int_{\mathbb{R}^D} f(t, x, \xi) d\xi \leq n(t, x), \quad \text{a.e. for } t \in \mathbb{R}, x \in \mathbb{R}^D. \tag{41}$$

In order to give the evolution equation which is satisfied by the Wigner function, we need to introduce the following pseudo-differential operator:

$$\theta_t^h := \frac{i}{\sqrt{h}} \left(U_{t/h\tau}^\varepsilon \left(\frac{x}{h} + \frac{D_\xi}{2} \right) - U_{t/h\tau}^\varepsilon \left(\frac{x}{h} - \frac{D_\xi}{2} \right) \right). \tag{42}$$

This operator is bounded in $L^2(\mathbb{R}^D)$ and its norm in $\mathcal{L}(L^2(\mathbb{R}^D))$ denoted by $\|\theta_t^h\|$ is bounded by:

$$\|\theta_t^h\| \leq \frac{2}{\sqrt{h}} \|U^\varepsilon\|_{L^\infty(\mathbb{R}^{D+1})}. \tag{43}$$

It is explicitly given by

$$\theta_t^h(\eta) := \frac{i}{\sqrt{h}(2\pi)^D} \int_{\mathbb{R}^D} \left(U_{t/h\tau}^\varepsilon \left(\frac{x}{h} + \frac{y}{2} \right) - U_{t/h\tau}^\varepsilon \left(\frac{x}{h} - \frac{y}{2} \right) \right) \mathcal{F}_{v \rightarrow y}(\eta(x, v)) e^{iy \cdot \xi} dy,$$

where $\mathcal{F}_{v \rightarrow y}$ is the Fourier transform between the dual variables v and y

$$\mathcal{F}_{v \rightarrow y}(\eta(v)) := \int_{\mathbb{R}^D} \eta(v) e^{-iv \cdot y} dv.$$

If W_t^h denotes the Fourier transform of $U_{t/h\tau}^\varepsilon$ with respect to the space variable (it is a tempered distribution) we obtain:

$$\theta_t^h(\eta) = \frac{i}{\sqrt{h}(2\pi)^D} \int_{\mathbb{R}^D} W_t^h(p) \left(\eta\left(x, \xi + \frac{p}{2}\right) - \eta\left(x, \xi - \frac{p}{2}\right) \right) e^{i\frac{p}{h} \cdot x} dp. \quad (44)$$

In the above formula the integral has to be understood as a duality between a distribution and a function. We are now ready to give the Wigner equation. We have (see [10,15,16]):

Proposition 4.4. *With the same assumption as in Proposition 4.3, the Wigner functions w_h solve the following Wigner equation:*

$$\frac{\partial}{\partial t} w_h + \xi \cdot \nabla_x w_h = \theta_t^h(w_h), \quad t \in \mathbb{R}, x \in \mathbb{R}^D, \xi \in \mathbb{R}^D, \quad (45)$$

$$w_h(0, x, \xi) = w_h^I(x, \xi), \quad x \in \mathbb{R}^D, \xi \in \mathbb{R}^D. \quad (46)$$

The operator θ_t^h is defined by (42). For all time $t \in \mathbb{R}$, it is a bounded skew operator on $L^2(\mathbb{R}^{2D})$.

4.2. Semi-classical limit in random media

The aim of this Section is to determine the asymptotic behavior of the expectation value $\langle w_h \rangle$ when $h \rightarrow 0$ together with $\varepsilon \rightarrow 0$. There are two different results depending on the parameter τ . This is precisely stated in the following theorem which is the main result of this section.

Theorem 4.2. *Assume that the random potential satisfies (B). Assume that the functions $\psi_{n,h}$ solve (32), (33) with initial data which form an orthonormal system. Assume that the occupation probabilities satisfy (36) and that the initial data $\psi_{n,h}^I$ and the random potential are independent random variables.*

When the parameters satisfy

$$\varepsilon\tau + h\tau^2(1 + \tau^2 M(\varepsilon)^2) \rightarrow 0$$

up to subsequences we have:

$$\langle w_h \rangle \rightarrow f \quad \text{in } C^0([0, T]; L^2(\mathbb{R}^{2D})\text{-weak}) \text{ for any } T > 0.$$

The function f is nonnegative and belongs to $L^\infty((0, \infty); L^1(\mathbb{R}^{2D}))$. We have $f(t=0) = f^I$ where f^I is the weak limit of $\langle w_h^I \rangle$ defined by (38). Depending on the parameter τ we obtain the following asymptotic behavior:

- *When τ is fixed, f is the solution of the linear Boltzmann equation with diffusive collision terms,*

$$\frac{\partial}{\partial t} f(t, x, \xi) + \xi \cdot \nabla_x f(t, x, \xi) = \int_{\mathbb{R}^D} q_\tau(\xi', \xi) f(t, x, \xi') d\xi' - \Lambda_\tau(\xi) f(t, x, \xi),$$

$$t > 0, (x, \xi) \in \mathbb{R}^{2D}. \tag{47}$$

- For $D \geq 2$, we assume moreover that the random potential satisfies (B'). Then when $\tau \rightarrow \infty$, f is the solution of the linear Boltzmann equation with elastic collision terms,

$$\frac{\partial}{\partial t} f(t, x, \xi) + \xi \cdot \nabla_x f(t, x, \xi) = \int_{\mathbb{S}^{D-1}} k(\xi, v) f(t, x, |\xi|v) dv - \Sigma(\xi) f(t, x, \xi),$$

$$t > 0, (x, \xi) \in \mathbb{R}^{2D}. \tag{48}$$

The cross sections q_τ , Λ_τ , k and Σ are given by Definition 4.3. In both cases, if moreover the initial data $\langle w_h^I \rangle$ converges in $L^2(\mathbb{R}^{2D})$ weak, the whole sequence converges. If the limit f^I satisfies $\int_{\mathbb{R}^{2D}} f^I(x, \xi) dx d\xi = 1$ then the concentration satisfies

$$n_h(t, x) = \sum_{n \geq 1} \lambda_{n,h} |\psi_{n,h}(t, x)|^2 \rightarrow \int_{\mathbb{R}^D} f(t, x, \xi) d\xi \quad \text{in } C^0([0, T]; L^1(\mathbb{R}^D)\text{-weak}).$$

Remark 4.4. The condition $\int_{\mathbb{R}^{2D}} f^I(x, \xi) dx d\xi = 1$ is satisfied if the initial data are h -oscillatory and compact at infinity, cf. [9,10]. This condition is fulfilled if for instance there is a constant $C > 0$ such that

$$\forall h \in (0, 1], \quad \sum_{n \geq 1} \lambda_{n,h} \int_{\mathbb{R}^D} h^2 |\nabla \psi_{n,h}^I(x)|^2 + |x| |\psi_{n,h}^I(x)|^2 dx \leq C.$$

The remainder of this section is devoted to the proof of this theorem. One of the main ingredient in the determination of the asymptotic behavior of w_h is the Duhamel formula. Therefore, as in the previous section, we introduce the unitary group on $L^2(\mathbb{R}^D)$, $(S_t)_{t \in \mathbb{R}}$ generated by the infinitesimal generator $\xi \cdot \nabla_x$,

$$\forall \eta \in L^2(\mathbb{R}^D), \quad S_t(\eta)(x, \xi) = \eta(x - t\xi, \xi), \quad x \in \mathbb{R}^D, \xi \in \mathbb{R}^D. \tag{49}$$

If w_h is a solution of (45), (46) we obtain:

$$w_h(t) = S_s w_h(t - s) + \int_0^s S_\sigma \theta_{t-\sigma}^h(w_h(t - \sigma)) d\sigma. \tag{50}$$

In particular, w_h can be obtained as the fixed point of the map

$$w \mapsto S_t w_h^I + \int_0^t S_\sigma \theta_{t-\sigma}^h(w(t - \sigma)) d\sigma.$$

If the initial data is assumed to be independent on the random potential, this formula shows that $w_h(t)$ depends only on V_s^h for $s \in [0, t]$ if $t \geq 0$ ($s \in [t, 0]$ if $t \leq 0$). In view of the assumption (B2), it follows:

Lemma 4.6. *Assume that for all $x, \xi, y \in \mathbb{R}^D$, for all $s \in \mathbb{R}$, $w_I(x, \xi)$ and $V_s^h(y)$ are independent variable then for all $t \geq 0$, $w_h(t, x, \xi)$ and for all $s \geq t + h\tau$ and $V_s^h(y)$ are independent random variables.*

We also have:

Lemma 4.7. *Assume that $V_t^h(y)$ and $\eta(x, \xi)$ are independent random variable for all $y, x, \xi \in \mathbb{R}^D$ then $\langle \theta_t^h(\eta) \rangle = 0$.*

This lemma is a direct consequence of the definition (42) and of (B3). We are now ready to use the strategy of proof described in Section 1.2. Thanks to Lemmas 4.6 and 4.7, we have for $t \geq 2h\tau$,

$$\frac{\partial}{\partial t} \langle w_h \rangle + \xi \cdot \nabla_x \langle w_h \rangle = \langle \theta_t^h(w_h) \rangle, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^D, \quad \xi \in \mathbb{R}^D,$$

with

$$\langle \theta_t^h w_h(t) \rangle = \int_0^{h\tau} \langle \theta_t^h S_\sigma \theta_{t-\sigma}^h S_{-\sigma} \rangle \langle w_h(t) \rangle d\sigma + r_t^h + e_t^h.$$

The remainder is given as in (4) by

$$\begin{aligned} r_t^h &= \left\langle \int_0^{h\tau} \theta_t^h S_\sigma \theta_{t-\sigma}^h \int_0^{2h\tau-\sigma} S_s \theta_{t-\sigma-s}^h w_h(t-\sigma-s) ds d\sigma \right\rangle, \\ e_t^h &= \int_0^{h\tau} \langle \theta_t^h S_\sigma \theta_{t-\sigma}^h S_{-\sigma} \rangle \langle S_{2h\tau} w_h(t-2h\tau) - w_h(t) \rangle d\sigma. \end{aligned} \quad (51)$$

Lemma 4.8. *Let w_h be the Wigner functions of Propositions 4.3 and 4.4 then we have:*

$$\frac{\partial}{\partial t} \langle w_h \rangle + \xi \cdot \nabla_x \langle w_h \rangle = \langle \theta_t^h(w_h) \rangle, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^D, \quad \xi \in \mathbb{R}^D. \quad (52)$$

Assume that for all $x, \xi, y \in \mathbb{R}^D$, for all $s \in \mathbb{R}$, $w_I(x, \xi)$ and $V_s^h(y)$ are independent variables, then the expectation value of $\theta_t^h(w_h)$ is given for $t \geq 2h\tau$ by

$$\langle \theta_t^h w_h(t) \rangle = (L_t^h)^* \langle w_h(t) \rangle + e_t^h + r_t^h \quad (53)$$

with $\|r_t^h\|_{L^2(\mathbb{R}^{2D})} + \|e_t^h\|_{L^2(\mathbb{R}^{2D})} \leq 32\sqrt{C_0} \tau^2 \sqrt{h} M(\varepsilon)$. The operator on $L^2(\mathbb{R}^{2D})$, $(L_t^h)^*$ is defined by

$$\forall \eta \in L^2(\mathbb{R}^{2D}), \quad (L_t^h)^*(\eta) = \int_0^{h\tau} \langle \theta_t^h S_\sigma \theta_{t-\sigma}^h S_{-\sigma} \rangle \eta \, d\sigma. \tag{54}$$

Proof. We estimate the L^2 norm of the first term of the remainder (51) by using (39), (43) and (B1)

$$\|r_t^h\|_{L^2(\mathbb{R}^{2D})} \leq \frac{8}{h\sqrt{h}} (\|U^\varepsilon\|_{L^\infty(\mathbb{R}^{D+1})})^3 \sqrt{C_0} \int_0^{h\tau} \int_0^{h\tau} ds \, d\sigma \leq 8\sqrt{C_0} \tau^2 \sqrt{h} M(\varepsilon).$$

To bound the second term we remark that

$$\begin{aligned} \|w_h(t) - S_{t-s} w_h(s)\|_{L^2(\mathbb{R}^{2D})} &= \left\| \int_s^t S_{t-\sigma} \theta_\sigma^h w_h(\sigma) \, d\sigma \right\|_{L^2(\mathbb{R}^{2D})} \\ &\leq 2\sqrt{C_0} \|U^\varepsilon\|_{L^\infty(\mathbb{R}^{D+1})} \frac{|t-s|}{\sqrt{h}}. \end{aligned} \tag{55}$$

Therefore we obtain:

$$\begin{aligned} \|(w_h(t) - S_{2h\tau} w_h(t - 2h\tau))\|_{L^2(\mathbb{R}^{2D})} &\leq 2\sqrt{C_0} \sqrt{h} \tau M(\varepsilon)^{1/3}, \\ e_t^h = (L_t^h)^*(S_{2h\tau} w_h(t - 2h\tau) - w_h(t)) &\leq 16\sqrt{C_0} \tau^2 \sqrt{h} M(\varepsilon). \end{aligned}$$

It gives the desired bound on e_t^h .

It remains to compute the limit of the leading term in (53). Let $\eta \in C_c^\infty(\mathbb{R}^{2D})$, for $t \geq 2h\tau$, we have:

$$\int_{\mathbb{R}^{2D}} \eta (L_t^h)^*(w_h(t)) \, dx \, d\xi = \int_{\mathbb{R}^{2D}} \langle w_h(t) \rangle L_t^h(\eta) \, dx \, d\xi, \tag{56}$$

where the operator L_t^h is the adjoint of $(L_t^h)^*$. Using that θ_t^h is a skew operator and that the adjoint of S_t is S_{-t} we have:

$$\forall \eta \in L^2(\mathbb{R}^{2D}), \quad L_t^h(\eta) = \int_0^{h\tau} \langle S_\sigma \theta_{t-\sigma}^h S_{-\sigma} \theta_t^h \rangle \eta \, d\sigma. \quad \square \tag{57}$$

Therefore the first step in the proof of Theorem 4.2 is to determine the asymptotic behavior of $L_t^h(\eta)$. This study allows to pass to the limit in (52). It will be done in the

second step of the proof. In the third step we precise the convergence of $\langle w_h \rangle$ and in particular we prove a uniform convergence in time which allow to pass to the limit for the initial data. The fourth step is concerned with the convergence of the concentration n_h . It uses classical arguments of semi classical asymptotic via Wigner measures.

Proof. *First step: computation of the operator L_t^h .* From (57) we obtain:

$$\begin{aligned}
 L_t^h(\eta) &= \frac{-1}{h(2\pi)^{2D}} \sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \varepsilon_1 \varepsilon_2 \int_0^{h\tau} \int_{\mathbb{R}^{2D}} \langle W_{t-\sigma}^h(q) W_t^h(p) \rangle \\
 &\quad \times \eta \left(x + \sigma \varepsilon_2 \frac{q}{2}, \xi + \varepsilon_1 \frac{p}{2} + \varepsilon_2 \frac{q}{2} \right) \\
 &\quad \times \exp \left(i \left(\frac{x}{h} \cdot (p+q) - \frac{\sigma}{h} \xi \cdot q + \varepsilon_2 \frac{\sigma}{h} \frac{q \cdot p}{2} \right) \right) dp dq d\sigma. \tag{58}
 \end{aligned}$$

In the above expression we used formula (44) and the integrals with respect to p, q have to be understood as a duality between distributions and test functions. In view of (B4) we have:

$$\langle W_{t-\sigma}^h(q) W_t^h(p) \rangle = \mathcal{F}_{(y,z) \rightarrow (p,q)} (R_{\sigma/h\tau}(y-z) + \varepsilon g_{t/h\tau, t/h\tau-\sigma/h\tau}^\varepsilon(y, z)). \tag{59}$$

Using Definition 4.2, Eq. (59) becomes

$$\begin{aligned}
 \langle W_{t-\sigma}^h(q) W_t^h(p) \rangle &= (2\pi)^{2D} \mathcal{Q}_{\sigma/h\tau}(p) \delta(p+q) \\
 &\quad + \varepsilon (2\pi)^D \mathcal{F}_{z \rightarrow p} (G_{t/h\tau, t/h\tau-\sigma/h\tau}^\varepsilon(z, q)). \tag{60}
 \end{aligned}$$

Putting (60) in (58) and using the change of variable $\sigma \rightarrow h\sigma$ we get

$$\begin{aligned}
 L_t^h(\eta) &= - \sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \varepsilon_1 \varepsilon_2 \int_0^\tau \int_{\mathbb{R}^D} \mathcal{Q}_{\sigma/\tau}(p) \eta \left(x - h\sigma \varepsilon_2 \frac{p}{2}, \xi + (\varepsilon_1 - \varepsilon_2) \frac{p}{2} \right) \\
 &\quad \times \exp \left(i \left(\sigma \xi \cdot p - \varepsilon_2 \sigma \frac{|p|^2}{2} \right) \right) dp d\sigma + \varepsilon T_t^h(\eta). \tag{61}
 \end{aligned}$$

The remainder $T_t^h(\eta)$ is given by

$$\begin{aligned}
 T_t^h(\eta) &= \frac{-1}{(2\pi)^D} \sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \varepsilon_1 \varepsilon_2 \int_0^\tau \int_{\mathbb{R}^{2D}} \mathcal{F}_{z \rightarrow p} (G(z, q)) \eta \left(x + h\sigma \varepsilon_2 \frac{q}{2}, \xi + \varepsilon_1 \frac{p}{2} + \varepsilon_2 \frac{q}{2} \right) \\
 &\quad \times \exp \left(i \left(\frac{x}{h} \cdot (p+q) - \sigma \xi \cdot q + \varepsilon_2 \sigma \frac{q \cdot p}{2} \right) \right) dp dq d\sigma, \tag{62}
 \end{aligned}$$

where for the sake of legibility, we skipped the dependence with respect to time and ε of G . We have:

Lemma 4.9. *The operator L_t^h reads*

$$L_t^h = Q_+^h - Q_-^h + \varepsilon T_t^h. \tag{63}$$

The gain term Q_+^h and the loss term Q_-^h are time independent operators defined by

$$Q_+^h(\eta) = \sum_{\varepsilon_1 = \pm 1} \int_0^\tau \int_{\mathbb{R}^D} \mathcal{Q}_{\sigma/\tau}(p) \eta \left(x + h\sigma\varepsilon_1 \frac{p}{2}, \xi + \varepsilon_1 p \right) e^{i\sigma p \cdot (\xi + \varepsilon_1 p/2)} dp d\sigma,$$

$$Q_-^h(\eta) = \sum_{\varepsilon_1 = \pm 1} \int_0^\tau \int_{\mathbb{R}^D} \mathcal{Q}_{\sigma/\tau}(p) \eta \left(x - h\sigma\varepsilon_1 \frac{p}{2}, \xi \right) e^{i\sigma p \cdot (\xi - \varepsilon_1 p/2)} dp d\sigma.$$

The remainder term T_t^h is estimated by

$$\|T_t^h(\eta)\|_{L^2(\mathbb{R}^{2D})} \leq \frac{4}{(2\pi)^D} \tau C(G) \|\eta\|_{L^2(\mathbb{R}^{2D})}. \tag{64}$$

Proof. In (61) we split the sum of the leading terms between the gain term, $\varepsilon_1\varepsilon_2 = -1$, and the loss term, $\varepsilon_1\varepsilon_2 = +1$. By this way we obtain the decomposition (63). It remains to estimate T_t^h .

Let $\mu \in C_c^\infty(\mathbb{R}^{2D})$ we define for $\varepsilon_1 = \pm 1$,

$$\Gamma_{\varepsilon_1}(\mu)(x, \xi, q) = \int_{\mathbb{R}^{2D}} \mu \left(x, \xi + \varepsilon_1 \frac{p}{2} \right) G(z, q) e^{-i(z-x/h) \cdot p} dz dp.$$

Putting $\xi' = \xi + \varepsilon_1 \frac{p}{2}$ we obtain:

$$\begin{aligned} \Gamma_{\varepsilon_1}(\mu)(x, \xi, q) &= \int_{\mathbb{R}^{2D}} \mu(x, \xi') G(z, q) e^{-i2\varepsilon_1(z-x/h) \cdot (\xi' - \xi)} dz 2^D d\xi' \\ &= \int_{\mathbb{R}^{2D}} \mu(x, \xi') G\left(\frac{x}{h} + \frac{\varepsilon_1}{2}z, q\right) e^{-iz \cdot (\xi' - \xi)} dz d\xi' \\ &= G\left(\frac{x}{h} + \frac{\varepsilon_1}{2}D_\xi, q\right) \mu. \end{aligned}$$

Therefore we have $\|\Gamma_{\varepsilon_1}(\mu)\|_{L^1(\mathbb{R}_q^D; L^2(\mathbb{R}_{x,\xi}^{2D}))} \leq C(G) \|\mu\|_{L^2(\mathbb{R}^{2D})}$ where $C(G)$ is the constant appearing in assumption (B6). The remainder $T_t^h(\eta)$ given by (62) can be expressed with the help of Γ_{ε_1} . We have:

$$\begin{aligned}
 T_t^h(\eta) &= \frac{-1}{(2\pi)^D} \sum_{\varepsilon_1, \varepsilon_2 = \pm 1} \varepsilon_1 \varepsilon_2 \int_0^\tau \int_{\mathbb{R}^D} \Gamma_{\varepsilon_1}(\eta) \left(x + h\sigma \varepsilon_2 \frac{q}{2}, \xi + \varepsilon_2 \frac{q}{2}, q \right) \\
 &\quad \times \exp\left(i \left(\frac{x}{h} \cdot q - \sigma \xi \cdot q \right) \right) dq d\sigma, \\
 \|T_t^h(\eta)\|_{L^2(\mathbb{R}^{2D})} &\leq \frac{2}{(2\pi)^D} \sum_{\varepsilon_1 = \pm 1} \int_0^\tau \|\Gamma_{\varepsilon_1}(\eta)\|_{L^1(\mathbb{R}_q^D; L^2(\mathbb{R}_{x, \xi}^{2D}))} d\sigma, \\
 &\leq \frac{4}{(2\pi)^D} \tau C(G) \|\eta\|_{L^2(\mathbb{R}^{2D})}.
 \end{aligned}$$

It ends the proof of the lemma. \square

Now we compute the gain term Q_+^h . We have using the change of variable $\xi' = \xi + \varepsilon_1 p$

$$Q_+^h(\eta) = \sum_{\varepsilon_1 = \pm 1} \int_0^\tau \int_{\mathbb{R}^D} \mathcal{Q}_{\sigma/\tau}(\varepsilon_1(\xi' - \xi)) \eta \left(x + h \frac{\sigma}{2} (\xi' - \xi), \xi' \right) e^{i\varepsilon_1 \sigma (|\xi'|^2 - |\xi|^2)} d\xi' d\sigma.$$

We use the relation $\mathcal{Q}_{\sigma/\tau}(-p) = \mathcal{Q}_{\sigma/\tau}(p) \in \mathbb{R}$ which is stated in Lemma 4.5. Let us also remark that $\mathcal{Q}_{\sigma/\tau}$ vanishes for $\sigma \geq \tau$ and is even with respect to σ . We obtain:

$$Q_+^h(\eta) = \int_{\mathbb{R}^D} \int_{\mathbb{R}} \mathcal{Q}_{\sigma/\tau}(\xi - \xi') e^{i\sigma (|\xi'|^2 - |\xi|^2)} \eta \left(x + h \frac{|\sigma|}{2} (\xi' - \xi), \xi' \right) d\sigma d\xi'.$$

Using Definition 4.3 we remark that

$$q_\tau(\xi, \xi') = \int_{\mathbb{R}} \mathcal{Q}_{\sigma/\tau}(\xi - \xi') e^{i\sigma (|\xi'|^2 - |\xi|^2)} d\sigma. \tag{65}$$

Therefore for any $\eta \in C_c^\infty(\mathbb{R}^{2D})$ we have:

$$\begin{aligned}
 Q_+^h(\eta) &= \int_{\mathbb{R}^D} q_\tau(\xi, \xi') \eta(x, \xi') d\xi' + h\tau^2 r_+^h, \\
 \text{with } r_+^h &= \int_{\mathbb{R}^D} \int_{\mathbb{R}} \mathcal{Q}_\sigma(\xi - \xi') e^{i\sigma (|\xi|^2 - |\xi'|^2)} \int_0^1 \nabla_x \eta(x + sh\tau\mu, \xi') \cdot \mu ds d\xi' d\sigma \\
 \mu &= \frac{|\sigma|}{2} (\xi' - \xi). \tag{66}
 \end{aligned}$$

Since Q_σ and η have compact support with respect to σ and ξ' , respectively, we have $|\mu| \leq C|\xi' - \xi|$. Using the change of variable $p = \xi - \xi'$, we obtain:

$$|r_+^h| \leq C \int_{\mathbb{R}^D} \int_{\mathbb{R}} |Q_\sigma(p)| |p| \int_0^1 |\nabla_x \eta(x + sh\tau\mu, \xi - p)| ds dp d\sigma.$$

It follows that $\|r_+^h\|_{L^2(\mathbb{R}^{2D})} \leq C(\eta) \|pQ\|_{L^1(\mathbb{R}^{D+1})}$ where $C(\eta)$ depends only on the test function. Remark that $\|pQ\|_{L^1(\mathbb{R}^{D+1})}$ is bounded because of assumption (B5) and of the fact that Q_σ has a compact support with respect to σ , cf. Lemma 4.5.

In the same way we have:

$$Q_-^h = \Lambda_\tau(\xi)\eta(t, x, \xi) + h\tau^2 r_-^h, \quad \Lambda_\tau(\xi) = \int_{\mathbb{R}^D} q_\tau(\xi', \xi) d\xi' \tag{67}$$

with $\|r_-^h\|_{L^2(\mathbb{R}^{2D})} \leq C(\eta) \|Q\|_{L^1(\mathbb{R}^{D+1})}$. We conclude the first step of the proof of Theorem 4.2 by the following proposition.

Proposition 4.5. *Under assumptions (B), for any $\eta \in C_c^\infty(\mathbb{R}^{2D})$ we have:*

$$L_t^h(\eta) = Q_\tau(\eta) + r_t^h \quad \text{with } \|r_t^h\|_{L^2(\mathbb{R}^{2D})} \leq C(\eta)(h\tau^2 + \varepsilon\tau).$$

The constant $C(\eta)$ does not depend on time t and on the parameters h, ε, τ . The diffusive collision operator Q_τ is given by

$$Q_\tau(\eta) = \int_{\mathbb{R}^D} q_\tau(\xi, \xi')\eta(x, \xi') d\xi' - \Lambda_\tau(\xi)\eta(x, \xi),$$

where the cross sections q_τ and Λ_τ are given in Definition 4.3.

Proof. Using the decomposition (63) of Lemma 4.9 together with the estimates (64), (66) and (67) we obtain:

$$L_t^h(\eta) = Q_\tau(\eta) + O(\varepsilon\tau + h\tau^2) \quad \text{in } L^\infty(\mathbb{R}; L^2(\mathbb{R}^{2D}))$$

with q_τ defined by (65). It gives the desired result. \square

Second step: limit equations. We first assume that τ is fixed and $hM(\varepsilon) \rightarrow 0, \varepsilon \rightarrow 0$. Using Proposition 4.3 we obtain that up to subsequences we have:

$$\langle w_h \rangle \rightarrow f \quad \text{in } L^\infty(\mathbb{R}; L^2(\mathbb{R}^{2D}))\text{-weak}^*,$$

where f is a nonnegative function. By Lemma 4.8 and using (56) we obtain for any test in $C_c^\infty(\mathbb{R}^{2D})$ with $t > 2h\tau$,

$$\int_{\mathbb{R}^{2D}} \left(\frac{\partial}{\partial t} \langle w_h \rangle + \xi \cdot \nabla_x \langle w_h \rangle \right) \eta \, dx \, d\xi = \int_{\mathbb{R}^{2D}} \langle w_h(t) \rangle L_t^h(\eta) \, dx \, d\xi + o(1),$$

where the $o(1)$ is uniform with respect to time. In view of Proposition 4.5, we obtain:

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^{2D}} \langle w_h \rangle \eta \, dx + \int_{\mathbb{R}^{2D}} \xi \cdot \nabla_x \langle w_h \rangle \eta \, dx \, d\xi \\ &= \int_{\mathbb{R}^{2D}} \langle w_h(t) \rangle Q_\tau(\eta) \, dx \, d\xi + o(1), \quad \forall t > 2h\tau. \end{aligned} \tag{68}$$

It remains to pass to the limit and to remark that Q_τ is self-adjoint to get that the limit f is a distribution solution of the linear Boltzmann equation.

In the second case $\tau \rightarrow \infty$ with $\varepsilon\tau \rightarrow 0$ and $h\tau^4 M(\varepsilon)^2 \rightarrow 0$ we obtain as previously (68). It remains to compute the limit of $Q_\tau(\eta)$ in $L^2(\mathbb{R}^{2D})$ strong to get the result. This is done in the following lemma.

Lemma 4.10. *Let $\eta \in L^\infty(\mathbb{R}^{2D})$ be a test function with a compact support then $Q_\tau(\eta)$ tends in $L^2(\mathbb{R}^{2D})$ towards*

$$Q_\infty(\eta) = \int_{\mathbf{S}_{D-1}} k(\xi, \nu) \eta(x, |\xi|\nu) \, d\nu - \Sigma(\xi) \eta(x, \xi),$$

where the cross sections k and Σ are given in Definition 4.3.

Proof. Because x plays only the role of a parameter, we consider only test functions depending on ξ with support in $B_R = \{\xi, |\xi| \leq R\}$. The extension to the case where η depends also on x is obvious. By using Definition 4.3 and the change of variable $\omega = \tau(|\xi'|^2 - |\xi|^2)$ we have:

$$Q_\tau^+(\eta) := \int_{\mathbb{R}^D} q_\tau(\xi, \xi') \eta(\xi') \, d\xi' = \pi \int_{-\tau|\xi|^2}^{+\infty} \int_{\mathbf{S}_{D-1}} \mathcal{R}(\omega, \xi - \xi') \eta(\xi') |\xi'|^{D-2} \, d\omega \, d\nu$$

with $\xi' = \sqrt{|\xi|^2 + \omega/\tau} \nu$. On the support of η we have $|\xi'| \leq R$. It follows $|\xi|^2 \leq R^2 + \omega/\tau$. Let γ be the exponent of the assumption (B'), for $\tau \geq 1$ we have $(1 + R^2 + |\omega|)^\gamma / (1 + |\xi|^2)^\gamma \geq 1$. We obtain:

$$|\mathcal{R}(\omega, \xi - \xi') \eta(\xi') |\xi'|^{D-2}| \leq \frac{\|\eta\|_{L^\infty(\mathbb{R}^D)} R^{D-2}}{(1 + |\xi|^2)^\gamma} (1 + R^2 + |\omega|)^\gamma \sup_{p \in \mathbb{R}^D} |\mathcal{R}(\omega, p)|.$$

Remark that $(1 + |\xi|^2)^{-\gamma}$ belongs to $L^2(\mathbb{R}^D)$. Therefore using twice the dominated convergence theorem we get $\lim_{\tau \rightarrow \infty} Q_\tau^+(\eta) = Q_\infty^+(\eta)$ in $L^2(\mathbb{R}^D)$, where

$$\begin{aligned} Q_\infty^+(\eta) &:= \pi \int_{\mathbf{S}_{D-1}} \left(\int_{\mathbb{R}} \mathcal{R}(\omega, \xi - |\xi|v) d\omega \right) |\xi|^{D-2} \eta(|\xi|v) dv, \\ &= \pi \int_{\mathbf{S}_{D-1}} Q_0(\xi - |\xi|v) \eta(|\xi|v) dv. \end{aligned}$$

We proceed in the same way for the loss term,

$$Q_\tau^-(\eta) = \pi \int_{-\tau|\xi|^2}^{+\infty} \int_{\mathbf{S}_{D-1}} \mathcal{R}(\omega, \xi - \xi') \eta(\xi) |\xi'|^{D-2} d\omega dv.$$

For $D \geq 2$ and $\tau \geq 1$ we have $|\xi'|^{D-2} \leq C(|\xi|^{D-2} + |\omega|^{(D-2)/2})$. It follows

$$|\mathcal{R}(\omega, \xi - \xi') \eta(\xi) |\xi'|^{D-2}| \leq C(R^{D-2} + |\omega|^{(D-2)/2}) \sup_{p \in \mathbb{R}^D} |\mathcal{R}(\omega, p)| \eta(\xi).$$

We conclude to the convergence of $Q_\tau^-(\eta)$ in $L^2(\mathbb{R}^D)$. \square

Third step: uniform in time convergence. At this stage we have that for a subsequence

$$\langle w_h \rangle \rightarrow f \quad \text{in } L^\infty((0, \infty); L^2(\mathbb{R}^{2D}))\text{-weak}^*,$$

where f is a weak solution of a linear Boltzmann equation. We want to obtain a uniform convergence in time for the weak topology of $L^2(\mathbb{R}^{2D})$. In view of (55) we have:

$$\| \langle S_{-2h\tau} w_h(t + 2h\tau) - w_h(t) \rangle \|_{L^2(\mathbb{R}^{2D})} \leq 4\sqrt{C_0} \sqrt{h} \tau M(\varepsilon)^{1/3}.$$

Let $\eta \in C_c^\infty(\mathbb{R}^{2D})$ be a test function, we get

$$\begin{aligned} & \left| \int_{\mathbb{R}^{2D}} \langle w_h(t + 2h\tau) \rangle \eta dx d\xi - \int_{\mathbb{R}^{2D}} \langle w_h(t) \rangle \eta dx d\xi \right| \\ &= \left| \int_{\mathbb{R}^{2D}} \langle w_h(t + 2h\tau) \rangle (\eta - S_{2h\tau} \eta) dx d\xi + \int_{\mathbb{R}^{2D}} \langle S_{-2h\tau} w_h(t + 2h\tau) - w_h(t) \rangle \eta dx d\xi \right| \\ &\leq \sqrt{C_0} \|\eta - S_{2h\tau} \eta\|_{L^2(\mathbb{R}^{2D})} + 4\sqrt{C_0} \sqrt{h} \tau M(\varepsilon)^{1/3}. \end{aligned}$$

It follows that in both asymptotic regimes of Theorem 4.2, $\langle w_h(t + 2h\tau) \rangle - \langle w_h(t) \rangle$ converges uniformly to 0 in the sense of distributions. Therefore it is enough to prove the uniform convergence of $\langle w_h(t + 2h\tau) \rangle$. Thanks to (68) we have for $t > 0$,

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^{2D}} \langle w_h(t + 2h\tau) \rangle \eta \, dx &= \int_{\mathbb{R}^{2D}} \langle w_h(t + 2h\tau) \rangle \xi \cdot \nabla_x \eta \, dx \, d\xi \\ &+ \int_{\mathbb{R}^{2D}} \langle w_h(t + 2h\tau) \rangle Q_\tau(\eta) \, dx \, d\xi + o(1), \quad \forall t > 0, \end{aligned}$$

where the $o(1)$ is uniform on \mathbb{R}^+ in $L^2(\mathbb{R}^{2D})$. As a consequence the functions $t \rightarrow \int_{\mathbb{R}^{2D}} \langle w_h \rangle(t + 2h\tau) \eta \, dx \, d\xi$ are bounded in $W^{1,\infty}(\mathbb{R}^+)$. Ascoli theorem implies that they converge uniformly on any interval $[0, T]$, $T > 0$, towards $t \rightarrow \int_{\mathbb{R}^{2D}} f(t) \eta \, dx \, d\xi$. By an $\varepsilon/3$ argument and using the density of $C_c^\infty(\mathbb{R}^{2D})$ in $L^2(\mathbb{R}^{2D})$ we conclude that $\langle w_h \rangle(t + 2h\tau)$ and then also $\langle w_h \rangle(t)$ converge towards $f(t)$ in $C^0([0, T]; L^2(\mathbb{R}^{2D})$ -weak). In particular it gives a sense to the Cauchy data condition,

$$f(0) = f^I = L^2_{\text{weak}}\text{-}\lim_{h \rightarrow 0} \langle w_h^I \rangle.$$

Fourth step: uniqueness and convergence of the concentration. In this last step we focus on the convergence of the concentration:

$$n_h(t, x) = \sum_{n=1}^\infty \lambda_{n,h} |\psi_{n,h}(t, x)|^2 = \int_{\mathbb{R}^D} \langle w_h \rangle(t, x, \xi) \, d\xi.$$

We recall that the general theory of Wigner measures, cf. [10], implies that $f \in L^\infty((0, \infty); L^1(\mathbb{R}^{2D}))$ and that the weak* limit n of n_h , in $L^\infty((0, \infty); \mathcal{M}_1(\mathbb{R}^D))$ satisfies

$$\int_{\mathbb{R}^D} f(t, x, \xi) \, d\xi \leq n(t, x), \quad \int_{\mathbb{R}^D} n(t, x) \, dx \leq 1. \tag{69}$$

The uniqueness of the limit will imply that the whole sequence $\langle w_h \rangle$ converges if the initial data converges. The conservation of mass for the limit equation will give us the argument to conclude to the equality in (69).

Lemma 4.11. *The weak solutions of the Cauchy problem for the linear Boltzmann equation (47) or (48) are unique in the class $C^0(\mathbb{R}^+; L^2(\mathbb{R}^{2D})$ -weak). If moreover, they belong to $L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^{2D}))$, then they satisfy the following conservation law*

$$\int_{\mathbb{R}^{2D}} f(t, x, \xi) \, dx \, d\xi = \int_{\mathbb{R}^{2D}} f(0, x, \xi) \, dx \, d\xi. \tag{70}$$

Proof. Although the uniqueness result seems to be well known in the context of transport equation, the lack of a precise reference motivates the proof below. Let $g(t, x, \xi) = f(t, x + \xi t, \xi)$ we have:

$$\frac{\partial}{\partial t} g = Q(f)(t, x + \xi t, \xi) \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^{2D})).$$

Therefore we can use the chain rule to obtain

$$\frac{\partial}{\partial t} g^2 = 2g(t, x, \xi)Q(f)(t, x + \xi t, \xi) = 2fQ(f)(t, x + \xi t, \xi),$$

where Q is one of the linear Boltzmann operator. We use that for any $f \in L^2(\mathbb{R}^{2D})$ we have $\int_{\mathbb{R}^D} fQ(f) d\xi \leq 0$ to obtain that $\|f\|_{L^2(\mathbb{R}^{2D})}^2$ is a decreasing function of $W^{1,1}(0, T)$. So f is indeed a continuous non increasing function of time for the strong topology of $L^2(\mathbb{R}^{2D})$. In particular, if the initial data vanishes, f also vanishes.

We prove now the mass conservation. Let $\eta \in C_c^\infty(\mathbb{R}^{2D})$ be a function satisfying $0 \leq \eta \leq 1$, $\eta(x, \xi) = 1$ if $|x| \leq 1$ and $|\xi| \leq 1$, $\eta(x, \xi) = 0$ if $|x| \geq 2$ or $|\xi| \geq 2$, η depends only on $(x, |\xi|)$. We put $\eta_k(x, \xi) = \eta(kx, k\xi)$ for $k \geq 1$. For the pure elastic case we have $Q_\infty(\eta) = 0$ therefore we obtain:

$$\frac{d}{dt} \int_{\mathbb{R}^{2D}} f(t)\eta_k dx d\xi + \int_{\mathbb{R}^{2D}} f(t, x, \xi) \frac{\xi}{k} \cdot \nabla_x \eta\left(\frac{x}{k}, \frac{\xi}{k}\right) dx d\xi = 0.$$

The functions $\frac{\xi}{k} \nabla_x \eta(x/k, \xi/k)$ are uniformly bounded and vanish for $|x| + |\xi| \geq 4k$ therefore

$$\left| \int_{\mathbb{R}^{2D}} f(t) \frac{\xi}{k} \cdot \nabla_x \eta\left(\frac{x}{k}, \frac{\xi}{k}\right) dx d\xi \right| \leq C \int_{|x|+|\xi| \geq 4k} |f(t, x, \xi)| dx d\xi \rightarrow 0, \quad \text{a.e. as } k \rightarrow \infty.$$

Therefore in the limit $k \rightarrow \infty$ we obtain the mass conservation. In the “diffusive” case we have:

$$\left| \int_{\mathbb{R}^{2D}} f(t) Q_\tau(\eta_k) dx d\xi \right| \leq \int_{\mathbb{R}^{3D}} q_\tau(\xi, \xi') |f(t, x, \xi)| |\eta_k(x, \xi) - \eta_k(x, \xi')| d\xi' dx d\xi.$$

The integrand vanishes for $|x| + |\xi| + |\xi'| \leq k$, then the dominated convergence theorem implies that the above integrals vanish as $k \rightarrow \infty$. Then the mass conservation is obtained by the same argument as for the elastic case. \square

The previous lemma implies that if $\int_{\mathbb{R}^{2D}} f^I(x, \xi) dx d\xi = 1$ we have also

$$\int_{\mathbb{R}^{2D}} f(t, x, \xi) dx d\xi = 1, \quad \text{for every } t > 0.$$

This fact together with (69) yields the equality $\int_{\mathbb{R}^D} f(t, x, \xi) d\xi = n(t, x)$ which concludes the proof of Theorem 4.2. \square

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