

Rigorous derivation of the Kinetic/Fluid coupling involving a Kinetic layer on a toy problem

Alexis Vasseur *

Abstract: In this article, we investigate the kinetic/fluid coupling on a toy model. We obtain it rigorously from a hydrodynamical limit. The idea is that at the level of the full kinetic model, the coupling is obvious. We then investigate the coupling obtained when passing into the limit. We show that, especially in presence of a shock stuck on the interface, the coupling involves a kinetic layer known as the Milne problem. Due to this layer, the limit process is quite delicate and some blow-up techniques are needed to ensure its strong convergence.

Keywords: coupling, kinetic, fluid, blow-up techniques, strong traces, Burgers equation, hydrodynamical limit.

1 Introduction

Fluid models of mechanics, like the compressible Euler system, are known to be not valid for gas far from thermodynamical equilibrium. This is especially the case for rarefied gas, or gas in the context of Physics of high energy. In this case, it is necessary to consider the kinetic theory which involves a new 3D variable called variable of velocity, e.g., the Boltzmann equation. In a numerical point of view, this makes the system far more costly to compute. Different strategies have been developed to compute with kinetic theory only in a small region where the gas is known to be very far from equilibrium. Standard fluid schemes are then used in the rest of the space. We then have the question of the coupling between those two regions of the two kind of models.

These kind of coupling conditions have been widely studied, especially for aerospace engineering. This is the case, in particular, for the coupling of the Boltzmann equation and Navier-Stokes system (see, e.g., J.-F.Bourgat, P.Le Tallec, B.Perthame and Y.Qiu [5], P.Le Tallec et F.Mallinger [21], or S.Dellacherie [12]). Most of the methods use the thermo-equilibrium kinetic function (the Maxwellian function) to compute the in-going fluxes at the boundary.

On the other hand, in the case of the evaporation/condensation problem it is well known that a layer problem is involved. In the case of Boltzmann equation, this layer problem is called Milne problem. This problem has been extensively

*Department of Mathematics, University of Texas

studied by the team of Y.Sone and K.Aoki (see [30]). A.V. Bobylev, R. Grzhibovskis, and A. Heintz proposed a good approximation of it in [3]. Solutions of this kind of equation can be classified in different categories depending on the boundary conditions. The complete classification was obtained by F.Golse, F.Coron and C.Sulem in [8]. Finally, study of such kinetic layer can be found in asymptotic problem linked to Semi-conductors (see P.Degond and C.Schmeiser [11], or N.Ben Abdallah, P.Degond and I.M.Gamba [2]).

In this paper we consider a simplified model for which we will be able to obtain rigorously the “good” coupling by performing an hydrodynamical limit in one region. The model was introduced by B.Perthame and E.Tadmor [27]. It is a caricature of the Boltzmann equation and is given by:

$$\partial_t f + \xi \partial_x f = \mathcal{M}f - f, \quad (1)$$

where the “equilibrium function” $\mathcal{M}f$ is defined by:

$$\begin{aligned} \mathcal{M}f(t, x, \xi) &= \mathbf{1}_{\{0 \leq \xi \leq u(t, x)\}} - \mathbf{1}_{\{-u(t, x) \leq \xi \leq 0\}} \\ \text{with : } u(t, x) &= \int_{-\infty}^{+\infty} f(t, x, \xi) d\xi. \end{aligned} \quad (2)$$

The Boltzmann collision operator is replaced by the relaxation term $\mathcal{M}f - f$. This equation is a “BGK” version of the “transport-collapse” method introduced independently by Y.Brenier [6, 7] and Y.Giga and Y.Miyakawa [14] as a numerical scheme for scalar conservation laws.

Consider a function $u^0 \in L^\infty(\mathbb{R})$. We can define the associated equilibrium function $f^0 = M(u^0(x); \xi) \in L^\infty(\mathbb{R} \times \mathbb{R})$ such that:

$$\begin{aligned} \mathcal{M}f^0 &= f^0 \\ \int_{\mathbb{R}} f^0(x, \xi) d\xi &= u^0(x) \quad \text{for every } x \in \mathbb{R}. \end{aligned}$$

In [22], P.L. Lions, B.Perthame and E.Tadmor showed that the solutions $f_\varepsilon \in L^\infty(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R})$ of the rescaled equation (1)

$$\partial_t f_\varepsilon + \xi \partial_x f_\varepsilon = \frac{\mathcal{M}f_\varepsilon - f_\varepsilon}{\varepsilon}, \quad (3)$$

with initial value $M(u^0; \cdot)$ converge, when ε goes to 0, to $M(u(t, x); \xi)$, where $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ is solution to the Burgers equation

$$\begin{aligned} \partial_t u + \partial_x \frac{u^2}{2} &= 0, \\ u(t = 0) &= u^0, \end{aligned} \quad (4)$$

and verifies the entropy conditions

$$\partial_t \phi(u) + \partial_x H(u) \leq 0, \quad (5)$$

for every convex function ϕ with associated entropy flux H verifying $H'(y) = y\phi'(y)$. This equation mimics the Euler equation of the compressible gas. In this article, our aim is to derive a natural coupling condition between equations (1) and (4). We choose to obtain it from the asymptotic limit when ε converges to 0 of the following problem

$$\begin{aligned} \partial_t f_\varepsilon + \xi \partial_x f_\varepsilon &= \alpha_\varepsilon(x)(\mathcal{M}f_\varepsilon - f_\varepsilon), \quad \text{for } t \in \mathbb{R}^+, x \in \mathbb{R}, \xi \in \mathbb{R}, \\ f_\varepsilon(t=0) &= f^0, \quad \text{for } x \in \mathbb{R}, \xi \in \mathbb{R}, \end{aligned} \quad (6)$$

where the function $\alpha_\varepsilon(x)$ is 1 for $x \leq 0$ and $1/\varepsilon$ for $x > 0$. This corresponds to solving the problem (1) in the domain $x < 0$ and the problem (3) in the domain $x > 0$ with a boundary coupling of the type complete transmission at $x = 0$.

In most of the numerical coupling, the in-flux entering in the kinetic part is defined through the equilibrium function. The coupling, in this simplified model would be

$$\begin{aligned} \mathcal{F}^+(u(t, 0+)) &= \int_0^\infty \xi f(t, 0-, \xi) d\xi \\ f(t, 0-, \xi) &= M(u(t, 0+); \xi) \quad \text{for } \xi \leq 0, \end{aligned} \quad (7)$$

where \mathcal{F}^+ is the positive semi-flux of the Engquist-Osher scheme [13] (see also Degond and Jin [10] for another strategy of coupling involving a smooth transition). This provides a plausible coupling for the the system (1) (4). However, this is not the one obtained in the hydrodynamical limit. This is due to the possible production of a kinetic layer of size ε at the interface $x = 0$ (the equivalent of the one described in the context of Boltzmann equation by the Milne problem) (see A.Klar [18]). We show, in this article, that the correct coupling obtained at the limit of (6) is the following one

$$\begin{cases} \partial_t u + \partial_x \frac{u^2}{2} = 0, & \text{for } t \in \mathbb{R}^+, x \in \mathbb{R}^+, \\ u(t=0, x) = u^0(x) = \int_{-\infty}^{+\infty} f^0(x, \xi) d\xi, & \text{for } x \in \mathbb{R}^+, \\ u(t, x=0+) \stackrel{BLN}{=} \sqrt{2 \int_0^\infty \xi f(t, 0-, \xi) d\xi}, & \text{for } t \in \mathbb{R}^+, \end{cases} \quad (8)$$

$$\begin{cases} \partial_t f + \xi \partial_x f = \mathcal{M}f - f, & \text{for } t \in \mathbb{R}^+, x \in \mathbb{R}^-, \xi \in \mathbb{R}, \\ f(t=0, x, \xi) = f^0(x, \xi) & \text{for } x \in \mathbb{R}^-, \xi \in \mathbb{R}, \\ f(t, x=0-, \xi \leq 0) = F(t, y=0+, \xi \leq 0) & \text{for } t \in \mathbb{R}^+, \xi \in \mathbb{R}^-, \end{cases} \quad (9)$$

$$\begin{cases} \xi \partial_y F = \mathcal{M}F - F, & \text{for } t \in \mathbb{R}^+, y \in \mathbb{R}^+, \xi \in \mathbb{R}, \\ F(t, y=0+, \xi \geq 0) = f(t, x=0-, \xi \geq 0), & \text{for } t \in \mathbb{R}^+, \xi \in \mathbb{R}^+, \\ \int_{-\infty}^{+\infty} \xi F(t, y, \xi) d\xi = \frac{u(t, 0+)^2}{2} & \text{for } t \in \mathbb{R}^+, y \in \mathbb{R}^+. \end{cases} \quad (10)$$

Note that the limit system should verify the equality of the flux at the interface:

$$\int_{-\infty}^{+\infty} \xi f(t, x = 0-, \xi) d\xi = \frac{u(t, 0+)^2}{2}.$$

The coupling condition in (8) shows that it is possible, (as suggested in (7)), to define the boundary condition for the fluid equation only from the out-going flux of the kinetic part. It is worth noticing that this condition corresponds exactly to the condition of Bardos-Leroux-Nédélec [1], which is the natural boundary condition of Dirichlet type for the Burgers equation. It is defined in the following way

$$\begin{aligned} u & \stackrel{BLN}{=} v \\ & \text{if and only if} \\ & [u^2/2 - k^2/2] \text{Sign}(u - v) \leq 0 \quad \text{for every } k \text{ between } u \text{ and } v. \end{aligned}$$

This result, in some sense, justifies the relevance of the Milne problem in the case of evaporation/condensation problems. At the numerical point of view, the result could seem a little bit odd. The computation of the Milne problem is very costly, and is certainly not needed anyway. Indeed, we show that there exists two types of layers depending on the boundary values on each side of the interface. In the case of the rarefaction layer, we show that the boundary condition coincides with the numerical condition (7). The layer problem provides a different boundary condition only in the other case of shock layer. This case corresponds physically to a shock coming from the fluid domain that artificially sticks to the interface. In some way, this means that the position of the interface has not been wisely chosen. To get an accurate description of the physics, the full shock should be modeled by the kinetic equation. More sophisticated numerical schemes use a moving interface chosen on the fly, tracking exactly the motion of the shocks near this interface (see Goldstein and al [29]).

For this simplified model, first steps were done by A.Nouri, A.Omrane and J.P.Villa [23], C.Bourdarias, M.Gisclon and A.Omrane [4] and M.Tidiri [32, 31]. However, none of those works consider the existence of the kinetic layer. Let us also mention that F.Golse gave a detailed study of the shock profiles of (1) in the whole line (see [15]).

We give in the next section the precise statements of the results. The characterization of the coupling condition in (8) relies on the result of strong traces for scalar conservation laws [35] (see Panov [25] and [20] for new versions).

The convergence in the kinetic layer is obtained thanks to a “blow-up” method first applied in the hyperbolic context in [34, 33] (see also [9] for more elaborate results using this technique). We present this technique in section 6.1. It makes use of a Liouville’s type lemma presented in section 6.2.

Let us fix $L > 0$. From now on, we will consider initial values supported in ξ in $[-L, L]$. A maximum principle implies that for every $\varepsilon > 0$ the solutions to (10), (6), and (9) preserve this support property (see Perthame [28]). This condition coincides with solutions u of (8) that are bounded by L .

The results of this paper were announced some time ago in [36].

2 Statement of the results

We first show the well-posedness and stability of the limit problem (8) (9) (10).

Theorem 1 *Let $(f^0, u^0) \in L^\infty(\mathbb{R}^- \times [-L, L]) \times L^\infty(\mathbb{R}^+)$ be such that $0 \leq \text{Sign}(\xi)f^0 \leq 1$ and $|u^0| \leq L$. Then there exists a unique solution $(f, u, F) \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^- \times [-L, L]) \times L^\infty(\mathbb{R}^+ \times \mathbb{R}^+) \times L^\infty(\mathbb{R}^+ \times \mathbb{R}^+ \times [-L, L])$ to (8) (9)(10) verifying $0 \leq \text{Sign}(\xi)f \leq 1$ and $0 \leq \text{Sign}(\xi)F \leq 1$. Moreover, two such solutions (f_1, u_1, F_1) , (f_2, u_2, F_2) verify for any $t \in \mathbb{R}^+$:*

$$\begin{aligned} & \|f_1(t) - f_2(t)\|_{L^1(\mathbb{R}^- \times [-L, L])} + \|u_1(t) - u_2(t)\|_{L^1(\mathbb{R}^+)} \\ & \leq \|f_1^0 - f_2^0\|_{L^1(\mathbb{R}^- \times [-L, L])} + \|u_1^0 - u_2^0\|_{L^1(\mathbb{R}^+)}. \end{aligned}$$

Note that we do not claim the L^1 stability of the layer F (Indeed, as we will see later, it is not stable).

Then we consider the asymptotic limit. We prove the asymptotic limit in two different situations. The first set of initial values considered is the following. Let $0 < u^+ < L$ and consider the functions $f_\varepsilon^0 \in L^\infty(\mathbb{R} \times [-L, L])$ with $0 \leq \text{Sign}(\xi)f_\varepsilon^0 \leq 1$ verifying

$$\begin{aligned} f_\varepsilon^0(x, \xi) & \geq M(u^+; \xi) \quad \text{for } x \in \mathbb{R}^-, \xi \in [-L, L], \\ f_\varepsilon^0(x, \xi) & \geq M(-u^+; \xi) \quad \text{for } x \in \mathbb{R}^+, \xi \in [-L, L]. \end{aligned} \quad (11)$$

The second set of initial values is the following.

$$\begin{aligned} \text{Supp } f^0(x, \cdot) & \subset]-L, u^+ - \eta] \quad \text{for } x \in \mathbb{R}^-, \\ f^0(x, \xi) & = M(-u^+; \xi) \quad \text{for } x \in \mathbb{R}^+, \xi \in [-L, L]. \end{aligned} \quad (12)$$

Note that this case is more restrictive, since it is constant in the fluid region.

We then prove the following result.

Theorem 2 *Consider initial values f_ε^0 verifying (11) and converging weakly to a function $f^0 \in L^\infty(\mathbb{R} \times [-L, L])$ in $L^\infty*$. Let f_ε be the solution to (6) with initial value f_ε^0 . Then the family $(f_\varepsilon|_{\{x \leq 0\}}, u_\varepsilon = \int f_\varepsilon d\xi|_{\{x \geq 0\}})$ converges weakly to the unique solution (f, u) to (8) (9) (10), with initial value $(f^0|_{\{x \leq 0\}}, u^0 = \int f^0 d\xi|_{\{x \geq 0\}})$.*

Consider now an initial value f^0 verifying (12). Let f_ε be the solution to (6) associated to this initial value. Let F_ε be defined by

$$F_\varepsilon(t, y, \xi) = f_\varepsilon(t, \varepsilon y, \xi), \quad \text{for } t \in \mathbb{R}^+, y \in \mathbb{R}^+, \xi \in [-L, L]. \quad (13)$$

Then $(f_\varepsilon|_{\{x \leq 0\}}, u_\varepsilon = \int f_\varepsilon d\xi|_{\{x \geq 0\}}, F_\varepsilon)$ converges to (f, u, F) solution to (8) (9) (10) with initial value

$$\begin{aligned} & f^0(x, \xi) \quad \text{for } x \in \mathbb{R}^-, \xi \in [-L, L], \\ & u^0(x) = \int_{-L}^L f^0(x, \xi) d\xi, \quad \text{for } x \in \mathbb{R}^+. \end{aligned}$$

Note that most of the difficulties of this limit lie in obtaining the boundary condition on the in-flow fluxes in the kinetic domain. Indeed, the boundary conditions on the fluid part can be obtained, in a full generality of initial conditions (see Proposition 14).

Note that the initial values of type (11) can converge only weakly. This means that the oscillations on the initial values cannot propagate in the non-linear terms of the kinetic and fluid domains. However, as we will see later, the kinetic layer can be destroyed in the limit. Indeed, in this case, we do not claim convergence of the kinetic layer. But the limit functions behave as if the kinetic layer was present. For the second set of initial values, things are more tedious. In this case, the limit solution depends on the precise structure of the kinetic layer. It is then crucial to show its convergence. Let us first give the idea why such a layer should appear. Consider the function F_ε defined on $\mathbb{R}^+ \times \mathbb{R}^+ \times [-L, L]$ by (13). This function F_ε verifies

$$\begin{aligned} \varepsilon \partial_t F_\varepsilon + \xi \partial_y F_\varepsilon &= \mathcal{M}F_\varepsilon - F_\varepsilon, \quad \text{for } t \in \mathbb{R}^+, y \in \mathbb{R}^+, \xi \in [-L, L], \\ F_\varepsilon(t, y = 0, \xi) &= f_\varepsilon(t, x = 0, \xi), \quad \text{for } t \in \mathbb{R}^+, \xi \in [-L, L]. \end{aligned}$$

Passing to the limit formally, we get

$$\begin{cases} \xi \partial_y F = \mathcal{M}F - F, & \text{for } t \in \mathbb{R}^+, y \in \mathbb{R}^+, \xi \in [-L, L], \\ F(t, y = 0, \xi) = f(t, x = 0, \xi), & \text{for } t \in \mathbb{R}^+, \xi \in [-L, L], \\ F(t, y = +\infty, \xi) = M(u(t, 0+); \xi), & \text{for } t \in \mathbb{R}^+, \xi \in [-L, L]. \end{cases} \quad (14)$$

The last equation is obtained formally by continuity with the fluid domain. We will show later that, even if the limit problem can always be formulated in terms of the kinetic layer, in some cases this asymptotic limit on the layer can fail.

In order to grasp this fact, we need to get a refined study of the kinetic layer (or Milne problem). Consider the set

$$\mathcal{C} = \left\{ (V, g) \in [0, L^2/2] \times L^1(\mathbb{R}^+); 0 \leq g(\xi) \leq 1; V \geq \int_0^L \xi g(\xi) d\xi \right\}.$$

For every boundary data $(V, g) \in \mathcal{C}$, we say that $F \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ with $0 \leq \text{Sign}(\xi)F(y, \xi) \leq 1$ is a solution to the kinetic layer problem with given data (V, g) if F verifies

$$\begin{cases} \xi \partial_y F = \mathcal{M}F - F & \text{for } y \in \mathbb{R}^+, \xi \in (-L, L), \\ F(y = 0, \xi \geq 0) = g & \text{for } \xi \in (0, L), \\ \int_{-L}^L \xi F(y, \xi) d\xi = V & \text{for } y \in \mathbb{R}^+. \end{cases} \quad (15)$$

We show the following proposition which classifies the solution to (15).

Proposition 3 *For any $(V, g) \in \mathcal{C}$, there exists a unique solution to (15). Let $u_\infty \in \mathbb{R}^+$ be such that*

$$V = \frac{u_\infty^2}{2}.$$

Then there are two cases.

(a) If $V = \int_0^\infty \xi g(\xi) d\xi$, we have $F(y, \xi) = 0$ for every $\xi \leq 0$ and

$$\lim_{y \rightarrow \infty} F(y, \cdot) = M(u_\infty, \cdot) \text{ in } L^1(\mathbb{R}).$$

We call this case a “relaxation layer”.

(b) If $V > \int_0^\infty \xi g(\xi) d\xi$, we have

$$\lim_{y \rightarrow \infty} F(y, \cdot) = M(-u_\infty, \cdot) \text{ in } L^1(\mathbb{R}).$$

We call this case a “shock layer”.

Let us first explain the terminology. In the first case of layer, the first moment $\int \xi f d\xi$ of the given kinetic data at the interface coincides with the first moment of the given Maxwellian at infinity. In this case the process in the layer corresponds only to a relaxation from the kinetic data at the interface to its Maxwellian. Note that in this case, the outgoing kinetic flow from the layer is always 0.

In the second type of layer, the value at $+\infty$ is a Maxwellian supported in $\{\xi \leq 0\}$ with a flux stronger than the in-going kinetic flux at the interface. This corresponds to a shock front moving toward the interface. We can consider this kind of layer as a shock profile which “sticks” to the interface. The situations in the two cases of layer are very different. The first set of initial conditions (11) in Theorem 2 ensures that, at the limit, the layer will be always of the relaxation type. This implies in particular that, in this case, the refined structure of the layer is not needed to get the boundary condition. This is why we can still consider, in this case, weak limit on the initial data. Those oscillations can destroy the kinetic layer, but not the boundary condition. Things are completely different for the second set of initial data (12), which corresponds to shock layer. Here, the precise structure of layer is needed to get the correct boundary condition. This explains why this case is more complicated. A precise control of the function F_ε is required.

Note that the uniqueness in Proposition 3 implies that it does not exist any kinetic layer matching the Maxwellian $M(u_\infty, \xi)$ at $y = 0$ with $M(-u_\infty, \xi)$ at $y = +\infty$. This shows that we cannot have a classical shock profile in a half space. This fact was already proven by F. Golse in [15]. This situation would correspond to the critic situation between (a) and (b). Thus, we cannot go in a continuous way from a relaxation layer to a shock layer.

The last remark has an interesting consequence. Consider the following initial data (non dependent on ε):

$$\begin{aligned} f^0(x, \xi) &= M(u^+; \xi) \text{ for } x \leq 0, \xi \in [-L, L], \\ f^0(x, \xi) &= M(-u^+; \xi) \text{ for } x \geq 0, \xi \in [-L, L]. \end{aligned}$$

This initial condition verifies (11), so Theorem 2 shows that $(f_\varepsilon|_{\{x \leq 0\}}, u_\varepsilon = \int f_\varepsilon d\xi|_{\{x \geq 0\}})$ converges to the function (independent of time):

$$\begin{aligned} f(t, x, \xi) &= M(u^+; \xi) \text{ for } t \in \mathbb{R}^+, x \leq 0, \xi \in [-L, L], \\ u(t, x) &= -u^+ \text{ for } t \in \mathbb{R}^+, x \geq 0, \xi \in [-L, L]. \end{aligned}$$

This corresponds to a steady shock localized at the interface $x = 0$. But as we have seen, there is no kinetic layer matching $M(u^+; \cdot)$ to $-u^+$. Hence, F_ε cannot converge to (14). Physically, this means that at the ε scaling, the shock is swept out from the interface. At the limit we only get the relaxation layer.

In view of this pathology, the stability of the limit system could seem surprising. The stability comes from the fact that, even if the kinetic layer is not stable with respect to the boundary data, the boundary condition $f(t, 0-, \xi) \mathbf{1}_{\{\xi < 0\}}$ itself is stable.

3 Preliminary results

We begin with preliminary results on the model which will be useful later. All those results are fairly standard and can be found, for example, in [28]. For every $u \in [-L, L]$, we define $M(u, \cdot) \in L^\infty([-L, L])$ by:

$$M(u, \xi) = \text{Sign}(u) \mathbf{1}_{\{0 < \xi \text{Sign} u < |u|\}}. \quad (16)$$

Lemma 4 *For every regular function $\phi \in C^\infty([-L, L])$ and $u \in [-L, L]$ we have:*

$$\int_{-L}^L \phi'(\xi) M(u, \xi) d\xi = \phi(u) - \phi(0).$$

Let N be an integer and \mathcal{O} be an open set of \mathbb{R}^N . Then for every function $f \in L^\infty(\mathcal{O}; L^1([-L, L]))$ verifying $0 \leq \text{Sign}(\xi) f(\cdot, \xi) \leq 1$, we define $\mathcal{M}(f) \in L^\infty(\mathcal{O}; L^1([-L, L]))$ by:

$$\mathcal{M}(f)(a, \xi) = M\left(\int_{-L}^L f(a, \zeta) d\zeta, \xi\right) \quad a \in \mathcal{O}, \quad \xi \in [-L, L]. \quad (17)$$

We have the following property:

Lemma 5 *For every function $f \in L^\infty([-L, L])$ compactly supported and such that $0 \leq \text{Sign}(\xi) f(\xi) \leq 1$ and every convex function ϕ we have:*

$$\int_{-L}^L \phi'(\xi) [\mathcal{M}f - f] d\xi \leq 0.$$

Moreover, if for one strictly convex function ϕ we have the equality:

$$\int_{-L}^L \phi'(\xi) [\mathcal{M}f - f] d\xi = 0,$$

then $f = \mathcal{M}f$.

Proof. Notice that

$$\begin{aligned} \mathcal{M}f - f &\geq 0 \quad \text{on }]-L, \int f(\xi) d\xi[\\ &\leq 0 \quad \text{on }] \int f(\xi) d\xi, L[. \end{aligned}$$

Let us define

$$h(\xi) = \int_{-L}^{\xi} (\mathcal{M}f - f)(\zeta) d\zeta.$$

The function h is null at $-L$, it is nondecreasing on $] -L, \int f(\xi) d\xi[$ and non increasing on $] \int f(\xi) d\xi, L[$. From Lemma 4 and (17):

$$\int_{-L}^L M \left(\int_{-L}^L f(\zeta) d\zeta; \xi \right) d\xi = \int_{-L}^L f(\zeta) d\zeta.$$

So h is null at L . Hence $h(\xi) \geq 0$ on $[-L, L]$. Finally, integrating by parts give:

$$\int_{-L}^L \phi'(\xi)(\mathcal{M}f - f) d\xi = - \int_{-L}^L \phi''(\xi)h(\xi) d\xi \leq 0$$

if ϕ is convex.

Now consider one strictly convex function ϕ . If the inequality is an equality, then we have:

$$\int_{-L}^L \phi''(\xi)h(\xi) d\xi = 0,$$

which implies that h is null for almost every $\xi \in [-L, L]$. Because of the definition of h , this implies that $\mathcal{M}f = f$ for almost every $\xi \in [-L, L]$. \square

Lemma 6 *Let $g_n \in L^\infty(\mathcal{O}; L^1([-L, L]))$, $0 \leq \text{Sign}(\xi)g_n \leq 1$ and $u \in L^1(\mathcal{O})$ be such that g_n converges weakly to $M(u, \xi)$ in $L^\infty w*$. Then, the convergence holds strongly in $L^1_{\text{loc}}(\mathcal{O} \times [-L, L])$.*

Proof. Let Ω be a compact set of $\mathcal{O} \times [-L, L]$. g_n is bounded in L^∞ so in $L^2(\Omega)$. The convergence holds weakly in $L^2 w$. But:

$$\begin{aligned} \int_{\Omega} |M(u(x), \xi)|^2 dx d\xi &\leq \varliminf \int_{\Omega} |g_n|^2 dx d\xi \\ &\leq \varlimsup \int_{\Omega} |g_n|^2 dx d\xi \\ &\leq \lim \int_{\Omega} \text{Sign}(\xi)g_n(x, \xi) dx d\xi = \int_{\Omega} \text{Sign}(\xi)M(u(x), \xi) dx d\xi. \end{aligned}$$

Since $|M(u, \xi)|^2 = \text{Sign}(\xi)M(u, \xi)$, we conclude that the $L^2(\Omega)$ norm of g_n converges to the $L^2(\Omega)$ norm of $M(u, \xi)$. Hence g_n converges strongly in $L^2(\Omega)$ and so in $L^1_{\text{loc}}(\mathcal{O} \times [-L, L])$. \square

Lemma 7 *Consider two functions $f, g \in L^1([-L, L])$ verifying the compatibility conditions $0 \leq \text{Sign}(\xi)f(\xi) \leq 1$ and $0 \leq \text{Sign}(\xi)g(\xi) \leq 1$. Then:*

$$\int_{-L}^L \mathbf{1}_{\{f \geq g\}} [(\mathcal{M}f(\xi) - \mathcal{M}g(\xi)) - (f(\xi) - g(\xi))] d\xi \leq 0. \quad (18)$$

Moreover, if this quantity is equal to 0 then $\text{Sign}(f - g)$ is constant on $[-L, L]$. Especially, in this case, $\text{Sign}(\mathcal{M}f - \mathcal{M}g) = \text{Sign}(f - g)$.

By $\text{Sign}(f - g)$ is constant on $[-L, L]$, we mean that either $f(\xi) \leq g(\xi)$ for all $\xi \in (-L, L)$, or $f(\xi) \geq g(\xi)$ for all $\xi \in (-L, L)$.

Proof. If $\int f d\xi \leq \int g d\xi$ then $\mathcal{M}f - \mathcal{M}g \leq 0$ for every $\xi \in [-L, L]$ and the inequality is true. If, moreover, (18) is an equality, then

$$\int_{-L}^L \mathbf{1}_{\{f \geq g\}} [f(\xi) - g(\xi)] d\xi = \int_{-L}^L \mathbf{1}_{\{f \geq g\}} [\mathcal{M}f(\xi) - \mathcal{M}g(\xi)] d\xi \leq 0,$$

and so $f \leq g$ on $[-L, L]$.

Assume now that $\int f d\xi \geq \int g d\xi$. Then $\mathcal{M}f - \mathcal{M}g \geq 0$ for every $\xi \in [-L, L]$ and:

$$\begin{aligned} \int_{-L}^L \mathbf{1}_{\{f \geq g\}} (\mathcal{M}f(\xi) - \mathcal{M}g(\xi)) d\xi &\leq \int_{-L}^L (\mathcal{M}f(\xi) - \mathcal{M}g(\xi)) d\xi \\ &= \int_{-L}^L (f(\xi) - g(\xi)) d\xi \\ &\leq \int_{-L}^L \mathbf{1}_{\{f \geq g\}} (f(\xi) - g(\xi)) d\xi + \int_{-L}^L \mathbf{1}_{\{f \leq g\}} (f(\xi) - g(\xi)) d\xi \\ &\leq \int_{-L}^L \mathbf{1}_{\{f \geq g\}} (f(\xi) - g(\xi)) d\xi, \end{aligned}$$

Hence the inequality is verified. If (18) is an equality, then

$$\int_{-L}^L (f(\xi) - g(\xi)) d\xi = \int_{-L}^L \mathbf{1}_{\{f \geq g\}} (f(\xi) - g(\xi)) d\xi,$$

so $f \geq g$ on $[-L, L]$, which ends the proof. \square

4 Study of the layer

This section is devoted to the proof of Proposition 3. Notice that it is necessary for (V, g) to be in \mathcal{C} in order to have a solution to (15). Indeed, if F is solution to (15), then integrating $\xi F(0, \xi)$ with respect to ξ we find:

$$V = \int_{-L}^L \xi F(0, \xi) d\xi = \int_0^L \xi g(\xi) d\xi + \int_{-L}^0 \xi F(0, \xi) d\xi \geq \int_0^L \xi g(\xi) d\xi.$$

Indeed $\int_{-L}^0 \xi F(0, \xi) d\xi \geq 0$ because of the condition $0 \leq \text{Sign} \xi F(y, \xi)$.

As said in the introduction, in the case of a relaxation layer, the value $F(y = 0, \xi \leq 0)$ used in the limit coupled problem is 0. Hence, in this case, the precise structure of the layer is not needed. In some cases the layer function \bar{F}_ε does not converge to the corresponding layer solution. But this can occur only in the case of the relaxation layer. And we will show that, fortunately, we still recover the limit problem since the coupling value $f(t, 0-, \xi \leq 0) = 0$ is verified, which is the same that the value provided by the layer solution.

In the contrary, the precise structure of the shock layer is needed to defined the limit problem. Fortunately, in this case the layer function \overline{F}_ε converges to the layer solution.

We decompose the proof into three lemmas. The first one is about the properties of solutions to the layer problem, the second one about the uniqueness and the third one about the existence of the solution.

Lemma 8 *Let $(V, g) \in \mathcal{C}$. We set $u_\infty \in [0, L[$ such that:*

$$V = \frac{u_\infty^2}{2}.$$

Assume that $F \in L^\infty(\mathbb{R}^+ \times]-L, L[)$ verifies $0 \leq \text{Sign}(\xi)F \leq 1$ and is solution to the layer problem (15). Then, we have the following property depending on the boundary data.

(a) Relaxation layer:

if $V = \int_0^L \xi g(\xi) d\xi$ then:

$$\lim_{y \rightarrow \infty} F(y, \cdot) = M(u_\infty, \xi) \text{ in } L^1(]-L, L[).$$

Moreover $F(y, \xi) = 0$ for every $y > 0$ and $\xi \in]-L, 0[$.

(b) shock layer:

if $V > \int_0^L \xi g(\xi) d\xi$ then:

$$\lim_{y \rightarrow \infty} F(y, \cdot) = M(-u_\infty, \xi) \text{ in } L^1(]-L, L[).$$

Proof. We divide the proof into several steps.

(i) First Property of relaxation layers. We consider the case of relaxation layers, namely $V = \int_0^L \xi g(\xi) d\xi$. We have

$$V = \int_0^L \xi g(\xi) d\xi + \int_{-L}^0 \xi F(0, \xi) d\xi.$$

But $\text{Sign}(\xi)F \geq 0$ Hence $F(0, \xi) = 0$ for $\xi < 0$. Since $\mathcal{M}F$ is non positive for $\xi \leq 0$, equation (15) gives for $\xi < 0$:

$$\partial_y(\xi F) = \mathcal{M}F - F \leq -\frac{\xi F}{\xi}.$$

So, for every $y > 0, \xi < 0$:

$$\xi F(y, \xi) \leq \xi F(0, \xi) e^{-y/\xi} = 0.$$

But $\text{Sign}(\xi F) \geq 0$, hence:

$$F(y, \xi) = 0 \text{ for } y \in \mathbb{R}^+, \xi \in]-L, 0[. \quad (19)$$

(ii) *Preliminary result for shock layers.* We consider the case of shock layer, namely $V > \int_0^L \xi g(\xi) d\xi$. Multiplying equation (15) by $\mathbf{1}_{\{0 \leq \xi\}}$ we find (thanks to Lemma 5):

$$\int_0^L \xi F(y, \xi) d\xi$$

is non increasing. But

$$V = \int_{-L}^0 \xi F(y, \xi) d\xi + \int_0^L \xi F(y, \xi) d\xi.$$

Hence for $y > 0$:

$$\int_{-L}^0 \xi F(y, \xi) d\xi \geq V - \int_0^L \xi g(\xi) d\xi > 0. \quad (20)$$

(iii) *Vanishing entropy up to a subsequence.* Multiplying the first equation of (15) by $(-\xi)$ and integrating it on $[0, y] \times (-L, L)$, we find

$$\begin{aligned} & \int_0^y \int_{-L}^L (-\xi) [\mathcal{M}F(y, \xi) - F(y, \xi)] d\xi dy \\ & \leq \int_{-L}^L \xi^2 F(0, \xi) d\xi - \int_{-L}^L \xi^2 F(y, \xi) d\xi \\ & \leq 2 \frac{L^3}{3}, \end{aligned}$$

since $|F| \leq 1$. Thanks to Lemma 5, $\int (-\xi) [\mathcal{M}F(y, \xi) - F(y, \xi)] d\xi$ is nonnegative, so it is bounded in $L^1(\mathbb{R}^+)$ as a function of y . Hence, there exists a sequence $y_n \rightarrow \infty$ such that $\int (-\xi) [\mathcal{M}F(y_n, \xi) - F(y_n, \xi)] d\xi$ converges to 0.

(iv) *Convergence of $\mathcal{M}F(y_n, \cdot)$ up to a subsequence.* Let us denote $u(y) = \int_{-L}^L F(y, \xi) d\xi$. Thanks to the definition to \mathcal{M} :

$$\frac{u^2(y_n)}{2} = \int_{-L}^L \xi \mathcal{M}F(y_n, \xi) d\xi.$$

Using the definition of V and the result (iii), we find that

$$\frac{u^2(y_n)}{2} - V = \int_{-L}^L \xi [\mathcal{M}F(y_n, \xi) - F(y_n, \xi)] d\xi$$

converges to 0 when n goes to infinity. Hence, up to a subsequence, $u(y_n)$ converges to u_∞ or to $-u_\infty$. This means that, up to a subsequence, $\mathcal{M}F(y_n, \cdot)$ converges in $L^1([-L, L])$ to $M(u_\infty, \cdot)$ or to $M(-u_\infty, \cdot)$, when n goes to infinity.

(v) *Convergence of $F(y_n, \cdot)$.* Since $\|F\|_{L^\infty} \leq 1$, extracting a subsequence from the above one if necessary, $F(y_n, \cdot)$ converges weakly in L^∞ to a function F_∞

such that $0 \leq \text{Sign}(\xi)F_\infty \leq 1$. Thanks to (iii) and the strong convergence of $\mathcal{M}F(y_n, \cdot)$ stated in (iv) we find at the limit:

$$\int_{-L}^L \xi[F_\infty(\xi) - \mathcal{M}F_\infty(\xi)] d\xi = 0.$$

Thanks to Lemma 5, this implies that $F_\infty = \mathcal{M}F_\infty$. Hence $F_\infty(\xi)$ is $M(u_\infty, \cdot)$ or $M(-u_\infty, \cdot)$. Thanks to Lemma 6, this implies that the convergence holds strongly in $L^1(\cdot - L, L)$. But thanks to (i), the limit cannot be $M(-u_\infty, \cdot)$ (at least if $u_\infty \neq 0$) in the case of relaxation layers. So in this case the limit is $M(u_\infty, \cdot)$. In the case of shock layers, thanks to (ii), $M(u_\infty, \cdot)$ cannot be the limit. Hence in this case, the limit is $M(-u_\infty, \cdot)$. By the uniqueness of the limit, the entire sequence $F(y_n, \xi)$ converges strongly in $L^1(\mathbb{R})$ to $M(u_\infty, \cdot)$ in the case of relaxation layer and to $M(-u_\infty, \cdot)$ in the case of shock layer.

(vi) *Convergence for $y \rightarrow \infty$.* Consider a monotonic function h . Multiplying the first equation to (15) by this function and integrating with respect to ξ , we find thanks to Lemma 5 that $\int \xi h(\xi) F(y, \xi) d\xi$ is monotonic with respect to y . Notice that since F is bounded by 1 and compactly supported in $[-L, L]$, this function is bounded and is converging to a constant when y goes to infinity. Thanks to (v) the limit is $\int \xi h(\xi) M(u_\infty, \xi) d\xi$ in the case of relaxation layer and $\int \xi h(\xi) M(-u_\infty, \xi) d\xi$ in the case of shock layer. Since every regular function is the difference between two nondecreasing functions, this convergence holds true for every regular function h . This implies that for every $\eta > 0$, The whole family $F(y, \cdot)$ converges in $\mathcal{D}'(\cdot, \eta, L)$ and in $\mathcal{D}'(\cdot - L, -\eta)$ to the corresponding limit function. But since those functions are uniformly bounded in $L^\infty(\cdot - L, L)$, finally the convergence holds in $\mathcal{D}'(\cdot - L, L)$. But the limit function is an equilibrium function, hence, thanks to Lemma 6, the convergence holds strongly in $L^1(\cdot - L, L)$. \square

Let us now consider the uniqueness of solution to the layer problem.

Lemma 9 *For every $(V, g) \in \mathcal{C}$ there exists at most one solution F to (15) verifying $F \in L^\infty(\mathbb{R}^+ \times]-L, L])$ with $0 \leq \text{Sign}(\xi)F \leq 1$.*

Proof. Consider F_1, F_2 two solutions to problem (15) for the same condition values $(V, g) \in \mathcal{C}$. Multiplying the difference of the first equations of (15) for F_1 and F_2 by $\text{Sign}(F_1 - F_2)$, and integrating in ξ , we find:

$$\partial_y \int_{-L}^L \xi |F_1 - F_2| d\xi = \int_{-L}^L \text{Sign}(F_1 - F_2) [\mathcal{M}F_1 - \mathcal{M}F_2 - F_1 + F_2] d\xi.$$

Thanks to Lemma 7 this quantity is non-positive. Thanks to Lemma 8, we have:

$$\int_0^L \xi |F_1 - F_2| d\xi \Big|_{y=0} = 0,$$

and:

$$\lim_{y \rightarrow \infty} \int_{-L}^L \xi |F_1 - F_2| d\xi = 0.$$

Hence:

$$\int_0^\infty \int_{-L}^L \text{Sign}(F_1 - F_2) [\mathcal{M}F_1 - \mathcal{M}F_2 - F_1 + F_2] d\xi dy = 0$$

and:

$$\int_{-L}^0 \xi |F_1 - F_2| d\xi \Big|_{y=0} = 0.$$

Thanks to Lemma 7 this implies that $\partial_\xi(\text{Sign}(F_1 - F_2)) = 0$. Let us fix a $\xi < 0$. We denote $\Omega_\xi = \{y | F_1(y, \xi) > F_2(y, \xi)\}$. Notice that since $|\partial_y(F_1 - F_2)| \leq 4/|\xi|$, $(F_1 - F_2)(\cdot, \xi)$ is continuous and so Ω_ξ is an open subset. Assume that it is not empty. Denote $y > 0$ one of its elements and $y_0 = \inf\{z < y | z \in \Omega_\xi\}$. For every $z \in]y_0, y[$ and every $\zeta \in]-L, L[$ we have: $F_1(z, \xi) > F_2(z, \xi)$, hence we have $F_1(z, \zeta) \geq F_2(z, \zeta)$ too. This leads to:

$$\xi \partial_y(F_1 - F_2) + (F_1 - F_2) \geq \mathcal{M}F_1 - \mathcal{M}F_2 \geq 0, \quad \text{on }]y_0, y[.$$

Hence, we find that $y_0 = 0$ and $(F_1 - F_2)(0, \xi) \geq (F_1 - F_2)(y, \xi)e^{y/\xi} > 0$. This gives a contradiction. Hence Ω_ξ is empty and so $F_1 \leq F_2$ for $\xi < 0$. Exchanging the indices gives:

$$F_1(y, \xi) = F_2(y, \xi) \quad y \in \mathbb{R}^+, \xi \in [-L, 0].$$

Hence, for every $y > 0$

$$\int_0^L \xi |F_1 - F_2|(y, \xi) d\xi = \int_{-L}^L \xi |F_1 - F_2|(y, \xi) d\xi \leq \int_{-L}^L \xi |F_1 - F_2|(0, \xi) d\xi = 0.$$

This implies that $(F_1 - F_2)(y, \xi) = 0$ for positive ξ too. This ends the proof. \square

Let us now show the existence of the solution to the layer problem which ends the proof of Proposition 3.

Lemma 10 *For every condition values $(V, g) \in \mathcal{C}$, there exists a solution $F \in L^\infty(\mathbb{R}^+ \times]-L, L[)$ with $0 \leq \text{Sign}(\xi)F \leq 1$, solution to the layer problem (15).*

Proof. We divide the proof into several parts.

(i) *Construction of approximated solutions.*

In order to show the existence of the solution to layer problem (15), we use the method of Golse [15]. Let $(V, g) \in \mathcal{C}$. We construct by induction F_n in the following way. We set $F_1 = 0$ for the case (a) and $F_1 = M(-u_\infty, \xi)$ for the case (b) and for $n > 1$:

$$\begin{aligned} F_{n+1}(y, \xi) &= g(\xi)e^{-\frac{y}{\xi}} + \int_0^y \frac{1}{\xi} \mathcal{M}F_n(z, \xi) e^{\frac{z-y}{\xi}} dz \quad \text{for } y \geq 0, \xi \in]0, L[, \\ &= - \int_y^\infty \frac{1}{\xi} \mathcal{M}F_n(z, \xi) e^{\frac{z-y}{\xi}} dz \quad \text{for } y \geq 0, \xi \in]-L, 0[. \end{aligned}$$

We can check that F_n verifies $0 \leq \text{Sign}(\xi)F_n \leq 1$:

$$\begin{cases} \xi \partial_y F_{n+1} = \mathcal{M}F_n - F_{n+1} & y > 0, \xi \in]-L, L[, \\ F_{n+1}(y = 0, \xi \geq 0) = g & \xi \in]0, L[. \end{cases} \quad (21)$$

(ii) *Convergence when $n \rightarrow \infty$.*

Let us show by induction that for every fixed point $(y, \xi) \in \mathbb{R}^+ \times]-L, L[$, the sequence $\{F_n(y, \xi)\}$ is non decreasing. First check that

$$F_2(y, \xi) - F_1(y, \xi) = g(\xi)e^{-\frac{y}{\xi}} \mathbf{1}_{\{\xi \geq 0\}} \geq 0.$$

Assume now that $F_n(y, \xi) \geq F_{n-1}(y, \xi)$ for every $(y, \xi) \in \mathbb{R}^+ \times]-L, L[$. Then

$$\begin{aligned} & F_{n+1}(y, \xi) - F_n(y, \xi) \\ &= \int_0^y \frac{1}{\xi} [\mathcal{M}F_n(z, \xi) - \mathcal{M}F_{n-1}(z, \xi)] e^{\frac{z-y}{\xi}} dz \geq 0 \quad \text{for } \xi \in]0, L[, y \geq 0, \\ &= - \int_y^\infty \frac{1}{\xi} [\mathcal{M}F_n(z, \xi) - \mathcal{M}F_{n-1}(z, \xi)] e^{\frac{z-y}{\xi}} dz \geq 0 \quad \text{for } \xi \in]-L, 0[, y \geq 0. \end{aligned}$$

By this procedure we have shown that the sequence $\{F_n(y, \xi)\}$ is non decreasing. Moreover it is bounded by 1, hence it converges almost everywhere to a function $F(y, \xi)$ with $0 \leq \text{Sign}(\xi)F(y, \xi) \leq 1$. By Lebesgue's Theorem, $u_n(y) = \int_{-L}^L F_n(y, \xi) d\xi$ converges almost everywhere to $u_F(y) = \int_{-L}^L F(y, \xi) d\xi$. So thanks to the definition to \mathcal{M} , $\mathcal{M}F_n$ converges strongly to $\mathcal{M}F$ in $L^1_{\text{loc}}(\mathbb{R}^+ \times]-L, L[)$. Passing to the limit in the first equation to (21) shows that F is solution to the first equation to (15). The first equation of (21) shows that $\partial_y(\xi F_n)$ is bounded in $L^\infty(\mathbb{R}^+ \times]-L, L[)$. Since $F_n \in L^\infty(\mathbb{R}^+ \times]-L, L[)$ and $L^\infty(]-L, L[)$ is compactly imbedded in $H^{-1}(]-L, L[)$, from Aubin's Theorem we find that the convergence holds in $C^0(\mathbb{R}^+; H^{-1}(]-L, L[))$. Hence we retrieve at the limit $n \rightarrow \infty$

$$F(0, \xi \geq 0) = g \quad y > 0, \xi \in]0, L[.$$

(iii) *Flux condition inside the layer.*

We have now to show the last condition of (15). Let us first consider the relaxation case:

(a) $V = \int_0^L \xi g(\xi) d\xi$. First, integrating the first equation of (15) with respect to ξ we find that

$$\int_{-L}^L \xi F(y, \xi) d\xi = \int_{-L}^L \xi F(0, \xi) d\xi \quad y \geq 0.$$

Since $F_1(y, \xi) = 0$ for every $(y, \xi) \in \mathbb{R}^+ \times]-L, 0[$ and F_n is non decreasing, we have that $F(y, \xi) \geq 0$ for $(y, \xi) \in \mathbb{R}^+ \times]-L, 0[$. But at the limit $0 \leq \text{Sign}(\xi)F(y, \xi) \leq 1$ hence $F(y, \xi) = 0$ for $(y, \xi) \in \mathbb{R}^+ \times]-L, 0[$. So

$$V = \int_0^L \xi g(\xi) d\xi = \int_{-L}^L \xi F(0, \xi) d\xi = \int_{-L}^L \xi F(y, \xi) d\xi \quad y > 0.$$

Consider now the shock case:

(b) $V > \int_0^L \xi g(\xi) d\xi$. By induction we show that for every $n > 0$, $F_n(y, \cdot)$ converges to $M(-u_\infty, \cdot)$ in $L^1(]-L, L[)$ when $y \rightarrow \infty$. This is obviously true for $n = 1$ and if it is true for n then, $\mathcal{M}F_n(y, \cdot)$ converges to $M(u_\infty, \cdot)$ in $L^1(\mathbb{R})$ too. The results follows from the definition to F_{n+1} . Integrating the first equation of (21) with respect to ξ and using the nondecreasing property of $\{F_n(y, \xi)\}$ we find:

$$\begin{aligned} \partial_y \int_{-L}^L \xi F_n(y, \xi) d\xi &= \int_{-L}^L (\mathcal{M}F_{n-1}(y, \xi) - F_n(y, \xi)) d\xi \\ &\leq \int_{-L}^L (\mathcal{M}F_n(y, \xi) - F_n(y, \xi)) d\xi = 0. \end{aligned}$$

Hence using the limit at $+\infty$, we find for every y :

$$\int_{-L}^L \xi F_n(y, \xi) d\xi \geq \int_{-L}^L \xi M(-u_\infty, \xi) d\xi = V.$$

Passing to the limit we find:

$$\int_{-L}^L \xi F(y, \xi) d\xi = V_0 \geq V.$$

In particular $V_0 > \int \xi g(\xi) d\xi$ hence F is a layer of type (b). Thanks to Lemma 8:

$$\lim_{y \rightarrow \infty} F(y, \cdot) = M(-u_0, \cdot),$$

where $u_0 \in [0, L[$ verifies $V_0 = u_0^2/2$. Since $F \geq F_1$, we have $F(y, \xi) = 0$ for $\xi \leq -u_\infty$. Hence $u_0 \leq u_\infty$. So finally $V_0 \leq V$ and so $V_0 = V$. \square

We finish this section with the following Proposition which gives a property of good confinement of the layers.

Proposition 11 *For any $(V, g) \in \mathcal{C}$, the function F solution to (15) verifies*

$$\int_0^\infty \int_{-L}^L |F(y, \xi) - F_\infty(\xi)| d\xi dy < \infty,$$

where

$$F_\infty = \lim_{y \rightarrow \infty} F(y, \cdot).$$

Proof. If $V = 0$, then F is identically 0 and the result holds. If not, we have $V = (u_\infty)^2/2 > 0$. From Lemma 8,

$$F_\infty(\xi) = M(u_*, \xi),$$

with $u_* = \pm u_\infty$. Let $u(y) = \int_{-L}^L F(y, \xi) d\xi$. We have

$$\lim_{y \rightarrow \infty} u(y) = u_*,$$

and, thanks to Lemma 5,

$$(u(y))^2 - (u_*)^2 = (u(y))^2 - 2V = 2 \int_{-L}^L \xi (\mathcal{M}F(y, \xi) - F(y, \xi)) d\xi \leq 0.$$

Multiplying the layer equation by 2ξ and integrating in ξ , we find, for $0 < y < \infty$,

$$2\partial_y \int_{-L}^L \xi^2 F(y, \xi) d\xi = 2 \int_{-L}^L \xi (\mathcal{M}F(y, \xi) - F(y, \xi)) d\xi = (u(y))^2 - (u_*)^2.$$

Integrating in y gives that

$$\int_0^\infty [(u(y))^2 - (u_*)^2] dy < \infty.$$

Note that $|u_* - u(y)| = ((u_*)^2 - (u(y))^2)/|u_* + u(y)|$, and for y large enough we have $|u_* + u(y)| > |u_*| > 0$. Hence

$$\int_0^\infty |u(y) - u_*| dy < \infty.$$

Using $\mathcal{M}F(y, \xi) = M(u(y), \xi)$, we find

$$\int_0^\infty \int_{-L}^L |\mathcal{M}F(y, \xi) - F_\infty(\xi)| d\xi dy = \int_0^\infty |u(y) - u_*| dy < \infty.$$

Finally, $|F(y, \xi) - F_\infty(\xi)| = \text{Sign}(\xi - u_*)(F(y, \xi) - F_\infty(\xi))$ and

$$\begin{aligned} & \text{Sign}(\xi - u_*) \xi \partial_y F(y, \xi) \\ &= \text{Sign}(\xi - u_*)(\mathcal{M}F(y, \xi) - F_\infty(\xi)) - \text{Sign}(\xi - u_*)(F(y, \xi) - F_\infty(\xi)) \\ &= \text{Sign}(\xi - u_*)(\mathcal{M}F(y, \xi) - F_\infty(\xi)) - |F(y, \xi) - F_\infty(\xi)|. \end{aligned}$$

Integrating this expression in y and ξ gives

$$\int_0^\infty \int_{-L}^L |F(y, \xi) - F_\infty(\xi)| d\xi dy \leq \int_0^\infty \int_{-L}^L |\mathcal{M}F(y, \xi) - F_\infty(\xi)| d\xi dy + 2L^2 < \infty.$$

This ends the proof. \square

5 Well-posedness of the limit problem

This section is devoted to the proof of Theorem 1 which shows that the coupled system (8) (9)(10) is well-posed and stable with respect to the initial values.

A careful reader could be surprised that the limit problem is stable with respect to the initial conditions. Indeed, in the layer, $F_1 - F_2$ cannot be controlled by the initial conditions. However, notice that the coupling in the limit problem

depends only on the values at $y = 0$ of the layer. And this value is continuous with respect to the conditions of the problem.

In order to obtain the existence of a solution to the limit problem, we use the fixed point Theorem of Schauder applied on the trace of u at the interface $x = 0$. The result relies on the stability of some quantities at $x = 0$.

We first show a stability result in the kinetic domain.

Lemma 12 *Let $f_1^0, f_2^0 \in L^\infty((-\infty, 0) \times]-L, L[)$ and $g_1, g_2 \in L^\infty((0, \infty) \times]-L, 0[)$. Consider $f_1, f_2 \in L^\infty((0, \infty) \times (-\infty, 0) \times]-L, L[)$ solutions for $i = 1, 2$ to*

$$\begin{cases} \partial_t f_i + \xi \partial_x f_i = \mathcal{M}f_i - f_i, & \text{for } t \in \mathbb{R}^+, x \in \mathbb{R}^-, \xi \in]-L, L[, \\ f_i(t = 0, x, \xi) = f_i^0(x, \xi) & \text{for } x \in \mathbb{R}^-, \xi \in]-L, L[, \\ f_i(t, x = 0-, \xi \leq 0) = g_i(t, \xi \leq 0) & \text{for } t \in \mathbb{R}^+, \xi \in]-L, 0[. \end{cases}$$

Then, we have, for every $t > 0$

$$\begin{aligned} & \int_{-\infty}^0 \int_{-L}^L |f_1(t) - f_2(t)| d\xi dx + \int_0^t \int_0^L \xi |f_1(s, 0-, \xi) - f_2(s, 0-, \xi)| d\xi ds \\ & \leq \int_{-\infty}^0 \int_{-L}^L |f_1^0 - f_2^0| d\xi dx + \int_0^t \int_{-L}^0 (-\xi) |g_1(s, \xi) - g_2(s, \xi)| d\xi ds. \end{aligned}$$

Moreover, if f_ε^0 converges strongly in L_{loc}^1 to f^0 and g_ε converges weakly to g , then the function $f_\varepsilon(t, 0-, \xi \geq 0)$ converges *STRONGLY* to $f(t, 0-, \xi \geq 0)$ in $L_{\text{loc}}^1((0, \infty) \times]0, L[)$, where f is the solution with initial data f^0 and boundary data g .

Proof. Thanks to Lemma 7, we have

$$\partial_t \int_{-L}^L |f_1 - f_2| d\xi + \partial_x \int_{-L}^L \xi |f_1 - f_2| d\xi \leq 0 \quad \text{for } t \in \mathbb{R}^+, x \in \mathbb{R}^-.$$

Integrating in (t, x) gives the first result. For the second one, first notice that, thanks to averaging lemmas (introduced first in [16]. See [26] for our case.), $\int f_\varepsilon(t, x, \xi) d\xi$ converges strongly in $L_{\text{loc}}^1((0, \infty) \times \mathbb{R}^-)$. Hence $\mathcal{M}f_\varepsilon$ converges strongly to $\mathcal{M}f$ in $L_{\text{loc}}^1((0, \infty) \times \mathbb{R}^- \times (-L, L))$. Then the Duhamel formula gives for $\xi > 0$:

$$f_\varepsilon(t, 0-, \xi) = f_\varepsilon^0(-t\xi, \xi)e^{-t} + \int_0^t e^{s-t} \mathcal{M}f_\varepsilon(s, (s-t)\xi, \xi) ds,$$

which gives the result. \square

We now give a similar result for the coupling of the fluid part with the layer.

Lemma 13 For any $g \in L^\infty(\mathbb{R}^+ \times (0, L))$ with $0 \leq g(t, \xi) \leq 1$, and $u^0 \in L^\infty(0, \infty)$, $-L \leq u^0(x) \leq L$, there exists a unique solution (u, F) to

$$\begin{cases} \partial_t u + \partial_x \frac{u^2}{2} = 0, & \text{for } t \in \mathbb{R}^+, x \in \mathbb{R}^+, \\ u(t=0, x) = u^0(x), & \text{for } x \in \mathbb{R}^+, \\ u(t, x=0+) \stackrel{BLN}{=} \sqrt{2 \int_0^\infty \xi g(t, \xi) d\xi}, & \text{for } t \in \mathbb{R}^+, \\ \begin{cases} \xi \partial_y F = \mathcal{M}F - F, & \text{for } t \in \mathbb{R}^+, y \in \mathbb{R}^+, \xi \in (-L, L), \\ F(t, y=0+, \xi \geq 0) = g(t, \xi), & \text{for } t \in \mathbb{R}^+, \xi \in (0, L), \\ \int_{-\infty}^{+\infty} \xi F(t, y, \xi) d\xi = \frac{u(t, 0+)^2}{2} & \text{for } t \in \mathbb{R}^+, y \in \mathbb{R}^+. \end{cases} \end{cases}$$

Moreover, if (u_1, F_1) and (u_2, F_2) are two such solutions associated to g_1, u_1^0 (respectively g_2, u_2^0), then, for any $t > 0$,

$$\begin{aligned} & \int_0^\infty |u_1(t, x) - u_2(t, x)| dx + \int_0^t \int_{-L}^0 |\xi| |F_1(s, 0+, \xi) - F_2(s, 0+, \xi)| d\xi ds \\ & \leq \int_0^\infty |u_1^0(x) - u_2^0(x)| dx + \int_0^t \int_0^L \xi |g_1(s, \xi) - g_2(s, \xi)| d\xi ds. \end{aligned}$$

Proof. For any g , we can construct a solution to the initial-boundary problem of the Burgers equation (see [24] or [17], for instance). This solution constructed, the strong trace theorem [34] gives a meaning to $u(t, 0+)$. We can then consider the solution of the layer constructed in Proposition 3.

Consider now two solution (u_1, F_1) and (u_2, F_2) of this problem. From the Kruzkov's theory of the Burgers equation ([19]), and using the strong trace theorem, we find for any $t > 0$

$$\begin{aligned} & \partial_t \int_0^\infty |u_1(t, x) - u_2(t, x)| dx \\ & \leq \text{Sign}(u_1(t, 0+) - u_2(t, 0+))(u_1(t, 0+)^2 - u_2(t, 0+)^2)/2. \end{aligned}$$

Using Lemma 7, we find that

$$\partial_y \int_{-L}^L \xi |F_1(t, y, \xi) - F_2(t, y, \xi)| d\xi \leq 0.$$

Integrating in y between 0 and ∞ and using Proposition 3 give

$$\begin{aligned} & \int_{-L}^0 |\xi| |F_1(t, 0, \xi) - F_2(t, 0, \xi)| d\xi \\ & \leq \int_0^L \xi |g_1(t, \xi) - g_2(t, \xi)| d\xi - \int_{-L}^L \xi |M(u_1^\infty(t, \xi), \xi) - M(u_2^\infty(t, \xi), \xi)| d\xi, \end{aligned}$$

where

$$\lim_{y \rightarrow \infty} F_1(t, y, \xi) = M(u_1^\infty(t), \xi), \quad \lim_{y \rightarrow \infty} F_2(t, y, \xi) = M(u_2^\infty(t), \xi).$$

Note that

$$\begin{aligned} & \int_{-L}^L \xi |M(u_1^\infty(t, \xi), \xi) - M(u_2^\infty(t, \xi), \xi)| d\xi \\ &= \text{Sign}(u_1^\infty(t) - u_2^\infty(t))(u_1^\infty(t)^2 - u_2^\infty(t)^2)/2, \end{aligned}$$

and

$$u_1^2 = u_1^{\infty 2}, \quad u_2^2 = u_2^{\infty 2}.$$

Putting those results together gives

$$\begin{aligned} & \partial_t \int_0^\infty |u_1(t, x) - u_2(t, x)| dx + \int_{-L}^0 |\xi| |F_1(t, 0+, \xi) - F_2(t, 0+, \xi)| d\xi \\ & \leq \int_0^L \xi |g_1(t, \xi) - g_2(t, \xi)| d\xi \\ & \quad + (\text{Sign}(u_1(t, 0+) - u_2(t, 0+))(u_1(t, 0+)^2 - u_2(t, 0+)^2)/2) \quad (22) \\ & \quad - (\text{Sign}(u_1^\infty(t) - u_2^\infty(t))(u_1^\infty(t)^2 - u_2^\infty(t)^2)/2). \quad (23) \end{aligned}$$

Note that the two last terms have the same absolute value. Without any loss of generality, we can assume $u_1(t, 0+) \leq u_2(t, 0+)$. If $u_2(t, 0+)^2 \leq u_1(t, 0+)^2$, then (22) is negative, and the sum of (22) and (23) is non positive. Otherwise, $u_1(t, 0+)^2 \leq u_2(t, 0+)^2$, and so $u_2(t, 0+) > 0$ and $-u_1(t, 0+) \leq u_2(t, 0+)$. Then the BLN condition implies that $u_2^2/2 = \int \xi g_2 d\xi$, and so F_2 is a relaxation layer (thanks to Proposition 3). This implies that $u_2^\infty = u_2$, and so $u_2^\infty \geq u_1^\infty$. Hence (23) is negative and the sum of (22) and (23) is non positive too. This gives the result. \square

We can now prove Theorem 1.

Proof of Theorem 1. Let us first show the stability of the system. We use Lemma 13 and Lemma 12 with $(i=1,2)$

$$\begin{aligned} g_i(t, \xi \geq 0) &= f_i(t, 0-, \xi \geq 0), & \xi \in (0, L), \quad t \geq 0, \\ g_i(t, \xi \leq 0) &= F_i(t, 0+, \xi \leq 0), & \xi \in (-L, 0), \quad t \geq 0. \end{aligned}$$

Adding the two estimates of those lemmas gives for any $t > 0$

$$\begin{aligned} & \int_{-\infty}^0 \int_{-L}^L |f_1(t) - f_2(t)| dx d\xi + \int_0^\infty |u_1(t) - u_2(t)| dx \\ & \leq \int_{-\infty}^0 \int_{-L}^L |f_1^0 - f_2^0| d\xi dx + \int_0^{+\infty} |u_1^0 - u_2^0| dx. \end{aligned}$$

This gives the stability and the uniqueness.

Let us show now the existence. We fix a initial data (f^0, u^0) . We denote $H(T)$ the set of boundary condition on the interface for $\xi < 0$

$$H = \{\xi g \in L^1([0, T[\times] - L, 0[), -1 \leq g(t, \xi) \leq 0\}.$$

Note that this set is convex. We define the function \mathcal{F} from H to H in the following way. We consider the solution f on $(0, T) \times (-\infty, 0) \times (-L, L)$ to the kinetic equation with initial value f^0 and boundary condition g . Then we consider u solution to the Burgers equation in $(0, T) \times (0, \infty)$ with initial value u^0 and boundary condition (in the sense of BLN) $\sqrt{2 \int \xi f(t, 0-, \xi) d\xi}$. Finally we consider F solution to the layer problem with data $(f(t, 0-, \xi), (u(t, 0+))^2/2)$. We then set

$$\mathcal{F}(g) = \xi F(t, 0+, \xi < 0).$$

Lemma 12 and Lemma 13 ensure that \mathcal{F} is continuous from H to H .

Consider a sequence of function $g_n \in H$ converging weakly to $g \in H$. From Lemma 12, we get that $f_n(t, 0-, \xi \geq 0)$ converges strongly to $f(t, 0-, \xi \geq 0)$. Then, Lemma 13 ensures that $\mathcal{F}(g_n)$ converges strongly in H to $\mathcal{F}(g)$. Hence $\mathcal{F}(H)$ is compact.

So, using the classical Schauder fixed point theorem, we get the existence of a $g \in H$ such that $\mathcal{F}(g) = g$. The associated functions (f, F, u) is then solution to (8), (9), (10). \square

6 Asymptotic limit

This section is devoted to the proof of Theorem 2. Let us first show the following proposition. It states that the solution to (6) for $x \geq 0$ converges to the the solution to (8) with the correct boundary condition.

Proposition 14 *Let $f_\varepsilon^0 \in L^\infty(\mathbb{R} \times [-L, L])$ be such that $0 \leq \text{Sign}(\xi) f_\varepsilon^0 \leq 1$. Denote $f_\varepsilon \in L^\infty(\mathbb{R}^+ \times \mathbb{R} \times [-L, L])$ the solution to (6) with initial value f_ε^0 . Then there exists $\varepsilon_n \rightarrow 0$, two functions $f^0 \in L^\infty(\mathbb{R} \times [-L, L])$, $f \in L^\infty(\mathbb{R}^+ \times \mathbb{R} \times [-L, L])$ such that $f_\varepsilon^0, f_\varepsilon$ converge weakly to f^0, f . Moreover the function $u \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)$ defined by*

$$u(t, x) = \int_{-L}^L f(t, x, \xi) d\xi \quad \text{for } t \in \mathbb{R}^+, x \in \mathbb{R}^+,$$

is solution to (8).

To show this Proposition, we first give a kinetic formulation of the BLN condition (see [17] for an other kind of kinetic formulation).

Lemma 15 *Kinetic version of the BLN conditions. Consider $g \in L^\infty(0, L)$ with $0 \leq g(\xi) \leq 1$ and denote $v = \sqrt{2 \int_0^L \xi g(\xi) d\xi}$. We have*

$$u \stackrel{BLN}{:=} v$$

if and only if there exists $h \in L^\infty(-L, 0)$, $-1 \leq h(\xi) \leq 0$ and m nonnegative measure on $(-L, L)$ such that

$$\xi(M(u, \xi) - (g + h)(\xi)) = m'(\xi). \quad (24)$$

Proof of the lemma. Indeed, $u \stackrel{BLN}{=} v$ if and only if we have either $u = v$ or $u \leq -v$. In the case $u = v$, we get (24) with $h = 0$ and $m(\xi) = \int_{-L}^{\xi} \zeta(M(u, \zeta) - g(\zeta)) d\zeta$. The function m is equal to zero for $\xi \leq 0$ and at $\xi = L$. It is increasing on $[0, u]$ and decreasing on $[u, L]$. Hence it is nonnegative. If $u \leq -v$, we take $h(\xi) = \mathbf{1}_{[-\sqrt{u^2 - v^2}, 0]}$ and $m(\xi) = \int_{-L}^{\xi} \zeta(M(u, \zeta) - (g + h)(\zeta)) d\zeta$. We still have m equal to 0 for $\xi \leq u$, $m(L) = 0$ and m increasing for $\xi < 0$ and decreasing for $\xi > 0$, so m is still nonnegative.

Conversely, assume that u verifies (24). Noting that $\xi h \geq 0$, and integrating in ξ , we find

$$u^2/2 \geq v^2/2.$$

If $u \leq 0$, this gives $u \leq -v$. Assume that $u \geq 0$. Multiplying (24) by $(\xi)_-$ and integrating in ξ , we find

$$0 \leq \int_{-L}^0 \xi^2(-h)(\xi) d\xi = - \int_{-L}^0 m(\xi) d\xi \leq 0.$$

Hence $h = 0$, and integrating (24) in ξ we have

$$u^2/2 = v^2/2,$$

and so $u = v$. This shows that if u verifies (24) then $u \stackrel{BLN}{=} v$. \square

Proof of the Proposition. By weak compactness, there exists $\varepsilon_n \rightarrow 0$ such that $f_n = f_{\varepsilon_n}$ converges weakly to f . Using averaging lemmas, $\int f_n d\xi$ converges strongly to $\int f d\xi$, and so $\mathcal{M}f_n$ converges strongly to $\mathcal{M}f$. Passing into the limit in the kinetic domain gives that f verifies for $x \in (-\infty, 0)$

$$\partial_t f + \xi \partial_x f = \mathcal{M}f - f.$$

For the fluid domain, multiplying the equation by ε_n shows that at the limit

$$\mathcal{M}f(t, x, \xi) - f(t, x, \xi) = 0 \quad \text{for almost every } t > 0, x > 0, \xi \in (-L, L). \quad (25)$$

Thanks to Lemma 6, there exists $m_n \geq 0$ such that

$$a_{\varepsilon_n}(x)(\mathcal{M}f_n - f_n) = \partial_\xi m_n,$$

and those measures are uniformly bounded by

$$\int_0^T \int_{-\infty}^{\infty} \int_{-L}^L m_n(t, x, \xi) d\xi dx dt \leq \int_{-\infty}^{\infty} \int_{-L}^L \xi f_{\varepsilon_n}^0(x, \xi) d\xi \leq C.$$

Thus, up to a subsequence, m_n converges to nonnegative measure m (with possible concentration, especially at $x = 0$). This gives that for $x > 0$:

$$\partial_t f + \xi \partial_x f = \partial_\xi m, \quad t > 0, x > 0, -L < \xi < L.$$

This together with (25) is the kinetic formulation of the Burgers equation and so $u(t, x) = \int f(t, x, \xi) d\xi$ verifies the Burgers equation for $x > 0$. We want now to recover the boundary conditions on the Burgers equation. From the limit equation, we get also that $f \in BV(\mathbb{R}_x; W^{-1,2}(\mathbb{R}_t^+ \times (-L, L)))$ and so f has a limit from the left and a limit from the right at each point $x \in \mathbb{R}$ (in the sense of distribution). Moreover

$$\xi(f(t, 0+, \xi) - f(t, 0-, \xi)) = \partial_\xi \tilde{m}(t, 0, \xi), \quad t > 0, \xi \in (-L, L).$$

The strong trace theorem ensures that $f(t, 0+, \xi) = M(u(t, 0+), \xi)$ for $t > 0$ and $-L < \xi < L$, and the kinetic formulation of the BLN conditions (24) gives

$$u(t, 0+) \stackrel{BLN}{=} \sqrt{\int_0^L \xi f(t, 0-, \xi) d\xi}.$$

□

We now show the Theorem 2 in the case of the first set of initial values. We first consider the special initial value, for $u_0 > 0$

$$\psi^0(x, \xi) = \mathbf{1}_{\{x < 0\}} M(u_0, \xi) + \mathbf{1}_{\{x > 0\}} M(-u_0, \xi).$$

We want to show that for this initial value, at the limit $\varepsilon \rightarrow 0$ we get $f(t) = \psi^0$ for $t > 0$. For this matter, we first consider the initial value

$$\psi_\delta^0(x, \xi) = \psi^0(x - \delta, \xi).$$

Consider the limit function f_δ . We have seen that $u_\delta = \int f_\delta d\xi$ verifies the Burgers equation for $x > 0$ with initial value $u_\delta^0(x) = \mathbf{1}_{\{x < \delta\}} u_0 - \mathbf{1}_{\{x > \delta\}} u_0$. Thanks to the finite speed of propagation there exists a finite time (at least δ/u_0) such that $u(t, 0+) > 0$. Hence, the BLN conditions ensures that on this lapse of time $u_\delta(t, 0+)^2/2 = \int_0^L \xi f_\delta(t, 0-, \xi) d\xi$. But the conservation of flux at the interface ensures then that $f_\delta(t, 0-, \xi < 0) = 0$ for this lapse of time and $\xi < 0$. And so, solving the equation on the kinetic domain shows that the solution is stationary. The values at the interface does not change on time either, and so the solution of the fluid domain is constant also. Finally we find $f(t) = \psi_\delta^0$ for all $t < \delta/u_0$. iterating the argument shows that this is true for all time $t > 0$. Using Lemma 7, we can show that for any ε and $t > 0$

$$\int_{-\infty}^{\infty} \int_{-L}^L |f_\varepsilon(t) - f_\varepsilon^\delta(t)| d\xi dx \leq \int_{-\infty}^{\infty} \int_{-L}^L |\psi^0 - \psi_\delta^0| d\xi dx = 2\delta u_0.$$

up to subsequence, the functions converges strongly, and so

$$\int_{-\infty}^{\infty} \int_{-L}^L |f(t) - \psi_\delta^0| d\xi dx \leq 2\delta u_0.$$

Passing into the limit when $\delta \rightarrow 0$ gives that $f(t) = \psi^0$ for every $t > 0$.

Then, using Lemma 7, we can show that if $f_1^0 \leq f_2^0$ then for any ε , $f_{1,\varepsilon} \leq f_{2,\varepsilon}$ (see [28]). And so, at the limit $f_1 \leq f_2$. The hypothesis on the initial data are exactly $f^0 \geq \psi^0$, hence, at the limit:

$$f(t, x, \xi) \geq \psi^0(x, \xi) \quad \text{for } t > 0, x \in \mathbb{R}, \xi \in (-L, L).$$

This implies that $f(t, 0-, \xi < 0) = 0$ for $t > 0, \xi < 0$. But the condition implies also that the kinetic layer has to be of the relaxation type. The condition is then the good one.

We are now left to show the asymptotic limit for the second set of initial conditions. We need, in this case, to pass into the limit in the term F_ε . To avoid the possibility for the layer to be swept away, we need to show that the shock is well confined against the interface $x = 0$. This will be provided by Proposition 11. The hypothesis on the initial data (12), and a comparison argument gives that, uniformly with respect to ε

$$|F_\varepsilon(t, y, \xi) - M(-u^+; \xi)| \leq |F_g(y, \xi) - M(-u^+; \xi)| \text{ for } t \in \mathbb{R}^+, y \in \mathbb{R}^+, \xi \in \mathbb{R},$$

where F_g is the solution to the layer problem (15) associated to (g, V) with $g(\xi) = M(u^+ - \eta; \xi)$ and $V = (u^+)^2/2$.

We need now to get strong compactness to pass into the limit in the layer. For this, we use a blow-up technique first introduced in [33].

6.1 The “blow up” method

The equation on F_ε does not control the oscillations in time when $\varepsilon \rightarrow 0$. The idea is to get back the balanced structure of (1) doing a zoom in time of the equation. We introduce a new local variable of time $s \in \mathbb{R}$ and a new rescaled function defined by

$$\bar{F}_\varepsilon(t, s, y, \xi) = F_\varepsilon(t + \varepsilon s, y, \xi). \quad (26)$$

For almost every fixed $t > 0$, the function \bar{F}_ε verifies

$$\partial_s \bar{F}_\varepsilon + \xi \partial_y \bar{F}_\varepsilon = \mathcal{M} \bar{F}_\varepsilon - \bar{F}_\varepsilon, \quad \text{for } s \in]-t/\varepsilon, +\infty[, y \in \mathbb{R}^+, \xi \in [-L, L],$$

$$\bar{F}_\varepsilon(t, s, 0, \xi) = f_\varepsilon(t + \varepsilon s, 0, \xi) \quad \text{for } s \in]-t/\varepsilon, +\infty[, \xi \in [-L, L],$$

$$|\bar{F}_\varepsilon(t, s, y, \xi) - M(-u^+; \xi)| \leq |F_g(y, \xi) - M(-u^+; \xi)|$$

$$\text{for } s \in]-t/\varepsilon, +\infty[, y \in \mathbb{R}^+, \xi \in [-L, L].$$

(27)

Results obtained on the rescaled functions can be translated on the non-rescaled one (and vice versa) thanks to the following lemma whose proof can be found in [33]:

Lemma 16 *(From local to global) Let $F_{\varepsilon_n}, F \in L^\infty(\mathbb{R}^p \times \mathbb{R}^q)$, p, q integers. Then, F_{ε_n} converges strongly to F in $L_{\text{loc}}^1(\mathbb{R}^p \times \mathbb{R}^q)$ if and only if for every $R_1 > 0, R_2 > 0$ and $R_3 > 0$:*

$$\int_{B_p(0, R_1)} \int_{B_q(0, R_2)} \int_{B_p(0, R_3)} |F_{\varepsilon_n}(x + \varepsilon_n y, z) - F(x, z)| dy dz dx \xrightarrow{n \rightarrow +\infty} 0. \quad (28)$$

Let us first apply this lemma on $f_\varepsilon(\cdot, 0, \cdot)$ with $p = 1$ (time variable) and $q = 1$ (ξ variable). From Lemma 12, the function $f_\varepsilon(t, 0, \xi > 0)$ converges strongly to $f(t, 0, \xi > 0)$ in $L^1_{\text{loc}}(\mathbb{R}^+ \times (0, L))$. Hence for any $T > 0$, $R > 0$

$$\int_0^T \left(\int_0^R \int_0^L |\bar{F}_\varepsilon(t + \varepsilon s, 0, \xi) - f(t, 0, \xi)| d\xi ds \right) dt \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Hence, there is a subsequence ε_n such that for almost every fixed $t > 0$, the local function (restricted to positive ξ) $\bar{F}_{\varepsilon_n}(t, \cdot, 0, \cdot)$ converges strongly in $L^1_{\text{loc}}(\mathbb{R} \times (0, L))$ to $f(t, 0, \cdot)$.

Let us fix such a time $t > 0$. We pass into the limit when ε_n goes to 0. For this we first show the following proposition.

Proposition 17 *Let s_n be a sequence converging to $-\infty$ and consider $\bar{F}_n \in L^\infty((s_n, \infty) \times \mathbb{R}^+ \times (-L, L))$, with $0 \leq \text{Sign}(\xi)\bar{F}_n \leq 1$ a sequence of solutions to*

$$\partial_s \bar{F}_n + \xi \partial_y \bar{F}_n = \mathcal{M} \bar{F}_n - \bar{F}_n, \quad (29)$$

$$|\bar{F}_n(s, y, \xi) - M(-u^+; \xi)| \leq |F_g(y, \xi) - M(-u^+; \xi)| \quad (30)$$

for $s \in]s_n, +\infty[, y \in \mathbb{R}^+, \xi \in [-L, L]$.

Assume, in addition that $\bar{F}_n(\cdot, 0, \cdot)$ (restricted to positive ξ) converges strongly in $L^1_{\text{loc}}(\mathbb{R} \times (0, L))$. Then, up to a subsequence, \bar{F}_n converges strongly in $L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^+ \times (-L, L))$ to a solution F_l of (29) (30). In addition, $\bar{F}_n(\cdot, 0, \cdot)$ converges strongly in $L^1_{\text{loc}}(\mathbb{R} \times (-L, L))$ to $F_l(\cdot, 0, \cdot)$.

Proof of the proposition. The function \bar{F}_n is bounded by 1, so, up to a subsequence, it converges weakly to a function denoted F_l . We can now use the averaging lemmas on the kinetic equation (29). We deduce that $\mathcal{M} \bar{F}_n$ converges to $\mathcal{M} F_l$ in L^1_{loc} . Hence F_l verifies the same equation (29). Moreover, there exists a subsequence such that $\mathcal{M} \bar{F}_n$ and the restriction to $\xi > 0$ of the trace $\bar{F}_n(\cdot, 0, \cdot)$ converges almost everywhere. For $\xi > 0$, integrating the equation (29) along characteristics, we find that for almost any s, y and n big enough:

$$\bar{F}_n(s, y, \xi) = e^{-\frac{y}{\xi}} \bar{F}_n(s - \frac{y}{\xi}, 0, \xi) + \int_0^y \frac{e^{-\frac{z-y}{\xi}}}{\xi} \mathcal{M} \bar{F}_n(s + \frac{z-y}{\xi}, z, \xi) dz.$$

From the convergence of \bar{F}_n almost everywhere on the trace ($y = 0, \xi > 0$) and the convergence of $\mathcal{M} \bar{F}_n$ almost everywhere, we deduce the convergence for almost every $(s, y, \xi) \in \mathbb{R} \times \mathbb{R}^+ \times (0, L)$ of $\bar{F}_n(s, y, \xi)$. This function is bounded by 1. The Lebesgue's Theorem ensures the strong convergence of (the restriction for $\xi > 0$ of) \bar{F}_n in $L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^+ \times (0, L))$.

For $\xi < 0$, we use the same strategy, using (30). Thanks to Lemma 8, for any $\eta > 0$, there exists $Y > 0$ (as big as we wish) such that for any $y > Y$ we have

$$\int_{-L}^0 |F_g(y, \xi) - M(-u^+; \xi)| d\xi \leq \eta.$$

Thanks to (30), we deduce that for any $y > Y$, we have for any n, m

$$\int_{-L}^0 \sup_{s, s'} |\overline{F}_n(s, y, \xi) - \overline{F}_m(s', y, \xi)| d\xi \leq 2\eta.$$

Integrating, again, the equation along the characteristics, we find for n big enough

$$\overline{F}_n(s, y, \xi) = e^{\frac{Y-y}{\xi}} \overline{F}_n(s + (Y-y)/\xi, Y, \xi) + \int_Y^y \frac{e^{(z-y)/\xi}}{\xi} \mathcal{M}\overline{F}_n(s + \frac{z-y}{\xi}, z, \xi) dz.$$

So, for n, m big enough, we have for every s , and $y < Y$

$$\begin{aligned} & \int_{-L}^0 |\overline{F}_n(s, y, \xi) - \overline{F}_m(s, y, \xi)| d\xi \\ & \leq 2\eta + \int_{-L}^0 \int_Y^y \frac{e^{(z-y)/\xi}}{\xi} |\mathcal{M}\overline{F}_n - \mathcal{M}\overline{F}_m|(s + \frac{z-y}{\xi}, z, \xi) dz d\xi. \end{aligned}$$

using the convergence of $\mathcal{M}\overline{F}_n$ almost everywhere, and the Lebesgue's Theorem, we find that for almost every $s \in \mathbb{R}$, and every $y < Y$, the restriction for $\xi < 0$ of \overline{F}_n is Cauchy in $L^1_{\text{loc}}(-L, 0)$. Using the Lebesgue's Theorem, and the previous result for $\xi > 0$, we get the convergence of \overline{F}_n in $L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^+ \times (-L, L))$.

The result with $y = 0$ gives also the strong convergence of the trace. \square

For almost every fixed $t > 0$, we apply this proposition on $\overline{F}_{\varepsilon_n}(t, \cdot, \cdot, \cdot)$. Passing into the limit in (27), we find that, up to a subsequence, $\overline{F}_{\varepsilon_n}(t, \cdot, \cdot, \cdot)$ converges strongly to $\overline{F}_{\infty}(t, \cdot, \cdot, \cdot)$ solution to

$$\begin{aligned} \partial_s \overline{F}_{\infty} + \xi \partial_y \overline{F}_{\infty} &= \mathcal{M}\overline{F}_{\infty} - \overline{F}_{\infty}, \quad \text{for } s \in \mathbb{R}, y \in \mathbb{R}^+, \xi \in [-L, L], \\ \overline{F}_{\infty}(t, s, 0+, \xi) &= f(t, 0+, \xi) \quad \text{for } s \in \mathbb{R}, \xi \in [0, L], \\ |\overline{F}_{\infty}(t, s, y, \xi) - M(-u^+; \xi)| &\leq |F_g(y, \xi) - M(-u^+; \xi)|, \\ &\text{for } s \in \mathbb{R}, y \in \mathbb{R}^+, \xi \in [-L, L]. \end{aligned} \tag{31}$$

The following Liouville lemma (proven below) completely characterizes the limit.

Lemma 18 *There exists a unique solution $\overline{F}_{\infty}(t, \cdot, \cdot, \cdot)$ to (31). This solution does not depend on s .*

We then deduce that \overline{F}_{∞} is the unique solution F to the kinetic layer problem (10) associated to $(f(t, 0+, \xi \geq 0), V = (u^+)^2/2)$. By the uniqueness of the limit, the whole sequence is converging in L^1_{loc} .

Lemma 16 ensures the convergence of the global functions F_{ε_n} from the result of convergence of the local functions $\overline{F}_{\varepsilon_n}$.

6.2 Liouville's lemma

This subsection is dedicated to the proof of Lemma 18. Existence of a (steady) solution to (31) is given by Proposition 3 and a comparison principle. Assume that there exists two such solutions F_1 and F_2 . We first have:

$$\begin{aligned} \partial_s |F_1 - F_2| + \xi \partial_y |F_1 - F_2| &= -\mathcal{D}(F_1, F_2), \text{ for } s \in \mathbb{R}, y \in \mathbb{R}^+, \xi \in \mathbb{R}, \\ |F_1 - F_2|(s, 0+, \xi) &= 0 \text{ for } s \in \mathbb{R}, \xi \in \mathbb{R}^+, \\ |F_1 - F_2|(s, y, \xi) &\leq 2|F_g(y, \xi) - M(-u^+, \xi)| \text{ for } s \in \mathbb{R}, y \in \mathbb{R}^+, \xi \in \mathbb{R}, \end{aligned} \quad (32)$$

where

$$\int_{-L}^L \mathcal{D}(F_1, F_2) d\xi = - \int_{-L}^L \text{Sign}(F_1 - F_2) [\mathcal{M}F_1 - \mathcal{M}F_2 - F_1 + F_2] d\xi \geq 0,$$

thanks to Lemma 7. Thanks to Proposition 11, $F_g - M(-u^+; \cdot)$ is integrable, hence $|F_1 - F_2|(s, \cdot, \cdot)$ is integrable for every fixed s . Integrating the first equation in (32) with respect to y and ξ , we get:

$$\begin{aligned} &\frac{d}{ds} \int_{-L}^L \int_0^\infty |F_1 - F_2|(s, y, \xi) d\xi dy \\ &= - \int_{-L}^L \int_0^\infty \mathcal{D}(F_1, F_2)(s, y, \xi) d\xi dy + \int_{-\infty}^0 \xi |F_1 - F_2|(s, 0+, \xi) d\xi \leq 0. \end{aligned} \quad (33)$$

We can deduce that $\int_0^\infty \int_{-\infty}^{+\infty} |F_1 - F_2|(s, y, \xi) d\xi dy$ is a non increasing bounded function. We denote δ its limit at $-\infty$. Consider now the functions translated in time $F_i^n(s, \cdot, \cdot) = F_i(s - n, \cdot, \cdot)$ for $i = 1, 2$. Thanks to Proposition 17, up to a subsequence, those functions converges in L_{loc}^1 to F_1^∞, F_2^∞ solutions to (31). In particular, the quantity $|F_1^\infty - F_2^\infty|$ verifies (32). Thanks to the Lebesgue's dominated convergence theorem, we get $\int_0^\infty \int_{-\infty}^{+\infty} |F_1^\infty - F_2^\infty|(s, y, \xi) d\xi dy = \delta$ for every $s \in \mathbb{R}$. Hence, we have

$$\int_0^\infty \int_{-L}^L \mathcal{D}(F_1^\infty, F_2^\infty) d\xi dy = 0,$$

and

$$|F_1^\infty - F_2^\infty|(s, 0, \xi) = 0.$$

The proof now follows the lines of the proof of Lemma 9. Thanks to Lemma 7, we have $\partial_\xi(\text{Sign}(F_1^\infty - F_2^\infty)) = 0$. Let us fix $s \in \mathbb{R}$, $y \in \mathbb{R}^+$ and $\xi < 0$. We denote

$$\tilde{F}_i(\tau) = F_i^\infty(s + \tau, y + \tau\xi, \xi), \quad i = 1, 2, \quad \tau \in \mathbb{R},$$

and

$$\Omega_{s,y,\xi} = \{0 \leq \tau \leq -y/\xi \mid \tilde{F}_1(\tau) > \tilde{F}_2(\tau)\}.$$

Note that this set is open since $\tilde{F}_1 - \tilde{F}_2$ is Lipschitz (and so continuous). Assume that it is not empty. Denote τ one of its elements and $\tau_0 = \sup\{\lambda > \tau \mid (\tau, \lambda) \subset \Omega_{s,y,\xi}\}$. For every $\lambda \in (\tau, \tau_0)$ and every $\zeta \in (-L, L)$ we have: $F_1^\infty(s + \lambda, y +$

$\lambda\xi, \xi) > F_2^\infty(s + \lambda, y + \lambda\xi, \xi)$, hence we have $F_1^\infty(s + \lambda, y + \lambda\xi, \zeta) \geq F_2^\infty(s + \lambda, y + \lambda\xi, \zeta)$ too. This leads to:

$$\partial_\tau(\tilde{F}_1 - \tilde{F}_2) + (\tilde{F}_1 - \tilde{F}_2) \geq \mathcal{M}\tilde{F}_1 - \mathcal{M}\tilde{F}_2 \geq 0, \quad \text{on } (\tau, \tau_0).$$

Hence, we find that $\tau_0 = -y/\xi$ and $(F_1^\infty - F_2^\infty)(s - y/\xi, 0, \xi) \geq (\tilde{F}_1 - \tilde{F}_2)(\tau)e^{y/\xi} > 0$. This gives a contradiction. Hence Ω_ξ is empty and so $F_1^\infty \leq F_2^\infty$ for $\xi < 0$. Exchanging the indices gives:

$$F_1^\infty(s, y, \xi) = F_2^\infty(s, y, \xi) \quad s \in \mathbb{R}, y \in \mathbb{R}^+, \xi \in (-L, 0).$$

Hence, for every $y > 0$

$$\begin{aligned} & \frac{d}{ds} \int_{-L}^L \int_0^y |F_1^\infty - F_2^\infty|(s, z, \xi) d\xi dz \\ &= - \int_0^L \xi |F_1^\infty - F_2^\infty|(s, y, \xi) d\xi = D(s, y) \leq 0. \end{aligned} \tag{34}$$

So for any $y > 0$, $\int_0^y \int_{-L}^L |F_1^\infty - F_2^\infty|(s, z, \xi) d\xi dz$ is a non increasing bounded function. We denote $d(y)$ its limit for s goes to $-\infty$. Consider a new translation in time

$$F_i^{\infty, n}(s, \cdot, \cdot) = F_i^\infty(s - n, \cdot, \cdot),$$

for $i = 1, 2$. As before, up to a subsequence, those functions converge in L_{loc}^1 to $F_1^{\infty, \infty}, F_2^{\infty, \infty}$. They verify

$$|F_1^{\infty, \infty} - F_2^{\infty, \infty}|(s, y, \xi < 0) = 0,$$

and for any $y > 0$, and $s \in \mathbb{R}$ $\int_{-L}^L \int_0^y |F_1^{\infty, \infty} - F_2^{\infty, \infty}|(s, z, \xi) d\xi dz = d(y)$. Hence, from (34), for any $y > 0$, $s \in \mathbb{R}$:

$$\int_0^L \xi |F_1^{\infty, \infty} - F_2^{\infty, \infty}|(s, y, \xi) d\xi = 0.$$

Finally $F_1^{\infty, \infty} = F_2^{\infty, \infty}$ and $d(y) = 0$ for any y . We can deduce that $|F_1^\infty - F_2^\infty|(s, y, \xi) = 0$ everywhere, and so $\delta = 0$. This gives $F_1 = F_2$. \square

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