

On spherically symmetric motions of a viscous compressible barotropic and selfgravitating gas

Bernard Ducomet, Šárka Nečasová and Alexis Vasseur

SUMMARY. We consider the Cauchy problem for the equations of spherically symmetric motions in \mathbb{R}^3 , of a selfgravitating barotropic gas, with possibly non monotone pressure law, in two different situations: in the first one we suppose that the viscosities $\mu(\rho)$, and $\lambda(\rho)$ are density-dependent and satisfy the Bresch-Desjardins condition, in the second one we consider constant densities. In the two cases, we prove that the problem admits a global weak solution, provided that the polytropic index γ satisfy $\gamma > 1$.

Mathematics Subject Classification (2000). 76N10,35Q30.

Keywords. spherically symmetric motion, selfgravitating gas, non monotone pressure law, density-dependent viscosities.

1. Introduction

We consider the Navier-Stokes-Poisson system in \mathbb{R}^3 for a compressible isentropic gas with density-dependent viscosities

$$\left\{ \begin{array}{l} \partial_t \rho + \operatorname{div}(\rho \vec{v}) = 0, \\ \partial_t(\rho \vec{v}) + \operatorname{div}(\rho \vec{v} \otimes \vec{v}) - \operatorname{div}(2\mu(\rho)D(\vec{v})) - \nabla(\lambda(\rho)\operatorname{div}\vec{v}) + \nabla P - \rho \nabla \Phi = \vec{0}, \\ \Delta \Phi = -4\pi G \rho. \end{array} \right. \quad (1.1)$$

Here ρ is the density, \vec{v} is the velocity, Φ is the Newtonian gravitational potential ($G > 0$ is the Newton constant), P is the pressure, D is the strain tensor with $D(\vec{v}) = 1/2(\nabla \vec{v} + {}^t \nabla \vec{v})$.

The pressure $P(\rho)$ is related to the density ρ by a general barotropic constitutive law (see [4] for motivations) satisfying

$$\begin{cases} P \in C^1(\mathbb{R}_+), & P(0) = 0, \\ \frac{1}{a} z^{\gamma-1} - b \leq P'(z) \leq az^{\gamma-1} + b & \text{for all } z \geq 0, \end{cases} \quad (1.2)$$

for three constants $a > 0$, $b \geq 0$ and $\gamma > 0$.

Following Mellet and Vasseur [13], we suppose that μ and λ are two viscosity coefficients satisfying

$$\mu(\rho) = \mu_1 \Psi(\rho), \quad \lambda(\rho) = 2\mu_1 (\rho \Psi'(\rho) - \Psi(\rho)), \quad (1.3)$$

where Ψ is an increasing function of ρ .

The viscosity coefficients satisfy the constraints introduced in [13]

$$\begin{cases} \mu'(\rho) \geq m, & \mu(0) \geq 0 \\ |\lambda'(\rho)| \leq \frac{1}{\underline{m}} \mu'(\rho), \\ \underline{m}\mu(\rho) \leq 2\mu(\rho) + 3\lambda(\rho) \leq \bar{m}\mu(\rho), \end{cases} \quad (1.4)$$

for suitable positive constants μ_1, m, \underline{m} and \bar{m} .

We also assume the upper bound

$$\mu(\rho) \leq C\rho^{\frac{\gamma-1}{2}}, \quad (1.5)$$

for a positive constant C , the physical meaning of (1.5) follows from the behavior of the temperature dependence of gaseous viscosity $\mu \sim C\theta^{1/2}$ and the ‘‘barotropic equivalence’’ $\theta \sim C\rho^{\gamma-1}$ (see [3]).

If $\gamma \geq 3$, we also require that

$$\liminf_{\rho \rightarrow \infty} \frac{\mu(\rho)}{\rho^{\frac{2}{3}+\varepsilon}} > 0, \quad (1.6)$$

for some $\varepsilon > 0$.

We also define the auxiliary functions ϕ and Π

$$\rho\phi'(\rho) = \Psi'(\rho), \quad \Pi(\rho) := \rho \int_1^\rho \frac{P(z)}{z^2} dz.$$

We want to solve the Cauchy problem (1.1) for $t > 0$ with initial conditions

$$\rho|_{t=0} = \rho^0(x), \quad \rho\vec{v}|_{t=0} = \vec{m}^0(x), \quad \text{on } \mathbb{R}^3. \quad (1.7)$$

We focus on spherically symmetric solutions satisfying $\rho(\vec{x}, t) = \rho(|x|, t)$ and $\vec{v}(x, t) = v(|x|, t) \frac{\vec{x}}{|x|}$, and we study weak solutions for the above problem with

properties

$$\left\{ \begin{array}{l} \rho \in L^\infty(0, T); L^1(\mathbb{R}^3) \cap L^\gamma(\mathbb{R}^3), \\ \sqrt{\rho} \in L^\infty(0, T); H^1(\mathbb{R}^3), \quad \sqrt{\rho} \vec{v} \in (L^\infty([0, T], L^2(\mathbb{R}^3)))^3, \\ \mu(\rho) D(\vec{v}) \in (L^2(0, T); W_{loc}^{-1,1}(\mathbb{R}^3))^9, \quad \lambda(\rho) \operatorname{div} \vec{v} \in (L^2(0, T); W_{loc}^{-1,1}(\mathbb{R}^3))^9. \end{array} \right. \quad (1.8)$$

We also assume the following conditions on the initial data $\rho_0(\vec{x}) \equiv \rho_0(|\vec{x}|)$ and $\vec{m}_0(\vec{x}) = m_0(|\vec{x}|) \frac{\vec{x}}{|\vec{x}|}$:

$$\left\{ \begin{array}{l} \rho_0 \geq 0 \text{ on } \Omega, \quad \vec{m}_0 = 0 \text{ on } \{x \in \mathbb{R}^3 \mid \rho_0(x) = 0\}, \\ \rho_0 \in L^1(\mathbb{R}^3) \cap L^\gamma(\mathbb{R}^3), \quad \rho_0^{-1/2} \nabla \mu(\rho_0) \in L^2(\mathbb{R}^3), \quad \frac{\vec{m}_0^2}{\rho_0} \in L^1(\mathbb{R}^3). \end{array} \right. \quad (1.9)$$

For any $0 \leq \tau_1 \leq \tau_2$, the weak solution (ρ, v) on $\mathbb{R}^3 \times [0, T]$ must satisfy, for any test function $\zeta \in C_c^1(\mathbb{R}^3 \times [0, T])$, the weak continuity equation

$$\int_{\mathbb{R}^3} \rho \zeta \, dx \Big|_{\tau_1}^{\tau_2} = \int_{\tau_1}^{\tau_2} \int_{\mathbb{R}^3} (\rho \zeta_t + \rho \vec{v} \cdot \nabla \zeta) \, dx, \quad (1.10)$$

and for any test function $\vec{\eta} \in (C_c^2(\mathbb{R}^3 \times [0, T]))^3$ such that $\vec{\eta}(\cdot, T) = \vec{0}$, the weak momentum equation

$$\begin{aligned} & \int_{\mathbb{R}^3} \vec{m}_0 \cdot \vec{\eta}(x, 0) \, dx + \int_0^T \int_{\mathbb{R}^3} (\rho \vec{v} \cdot \vec{\eta}_t + \vec{v} \otimes \vec{v} : \nabla \vec{\eta}) \, dx \, dt + \int_0^T \int_{\mathbb{R}^3} P(\rho) \operatorname{div} \vec{\eta} \, dx \, dt \\ & + \int_0^T \int_{\mathbb{R}^3} \mu(\rho) (\vec{v} \cdot \Delta) \cdot \vec{\eta} \, dx \, dt + \int_0^T \int_{\mathbb{R}^3} 2\mu'(\rho) \sqrt{\rho} v_j \partial_i \sqrt{\rho} \partial_i \eta_j \, dx \, dt \\ & + \int_0^T \int_{\mathbb{R}^3} \mu(\rho) (\vec{v} \cdot \nabla) (\operatorname{div} \vec{\eta}) \, dx \, dt \\ & + \int_0^T \int_{\mathbb{R}^3} 2\sqrt{\rho} \mu'(\rho) v_i \partial_j \sqrt{\rho} \partial_i \zeta_j \, dx \, dt + \int_0^T \int_{\mathbb{R}^3} \lambda(\rho) (\vec{v} \cdot \nabla) (\operatorname{div} \vec{\eta}) \, dx \, dt \\ & + \int_0^T \int_{\mathbb{R}^3} \sqrt{\rho} \lambda(\rho) (\vec{v} \cdot \nabla) (\sqrt{\rho} \operatorname{div} \vec{\eta}) \, dx \, dt = \int_0^T \int_{\mathbb{R}^3} 4\pi G \rho \nabla \Delta^{-1} \rho \cdot \vec{\eta} \, dx \, dt. \end{aligned} \quad (1.11)$$

In this formula, the five last terms in the left-hand side are requested in order to provide a rigorous meaning to the viscous contribution in the equation of motion (see [18]), and the right hand side is a shortened notation for

$$\int_0^T \int_{\mathbb{R}^3} 4\pi G \rho \nabla \left(\int_{\mathbb{R}^3} \frac{\rho(y, t)}{|x - y|} \, dy \right) \cdot \vec{\eta} \, dx \, dt.$$

It is known after [4] that the Cauchy problem for the system (1.1)-(1.7) admits a (renormalized) weak solution, when the viscosity coefficients are positive constants, provided that the pressure $P(\rho)$ behaves like $C\rho^\gamma$ for large ρ , with

$\gamma > 3/2$. Let us observe that the restriction on γ is the same as in the non-gravitating case [7].

In [5] it was proved that the problem (1.1)-(1.7) admits also a weak solution when the viscosity coefficients are density dependent functions related by Bresch-Desjardins relations [1], provided that the pressure $P(\rho)$ behaves like $C\rho^\gamma$ for large ρ with a smaller $\gamma > 4/3$. In this last case, the restriction on γ was requested in order to control the gravitational term.

On the other hand, in recent works, S. Jiang and P. Zhang [11] (in the renormalized case with constant viscosities), Z. Guo, Q. Jiu, Z. Xin [9] and T. Zhang, D. Fang [18] (for density dependent viscosities) proved that the Cauchy problem for the spherically symmetric version of the non-gravitating Navier-Stokes system also admits a weak solution, in the pure barotropic case $P(\rho) = C\rho^\gamma$ when the condition $\gamma > 1$ is achieved.

In the present paper we show that the Cauchy problem for the spherically symmetric Navier-Stokes-Poisson system admits in both cases (for density dependent viscosities or in the renormalized case with constant viscosities) a weak solution, for a general pressure law given by (1.2), without “loss of γ ” (i.e. with the optimal condition $\gamma > 1$).

In the density-dependent case (Sections 2) we follow the strategy of Zhang and Fang [18] (see also [10]): first consider the problem (1.1)(1.7) when \mathbb{R}^3 is replaced by the ball $B(0, R)$ then pass to the limit $R \rightarrow \infty$. Moreover, as far as the spherical symmetry is concerned with the classical divergence at the origin, it is also requested to remove the origin, and finally one is led to solve the problem (1.1)(1.7) in the shell $B_{\varepsilon, R} := B(0, R) \setminus B(0, \varepsilon)$ and pass to the limits ($\varepsilon \rightarrow 0$, $R \rightarrow \infty$).

In the renormalized case with constant viscosities (Section 3) we adapt the proof of S. Jiang and P. Zhang [11] to the Navier-Stokes-Poisson system with non-monotone pressure.

2. The density-dependent case

The spherically symmetric version of (1.1) reads

$$\begin{cases} \rho_t + (\rho v)_r + \frac{2\rho v}{r} = 0, \\ \rho(v_t + vv_r) = \left(-P + (\lambda + 2\mu) \left(v_r + \frac{2v}{r} \right) \right)_r - 4\mu_r \frac{v}{r} + \rho f, \end{cases} \quad (2.1)$$

in the domain $[0, T] \times \mathbf{R}^+$ for the density $\rho(r, t)$ and the velocity $v(r, t)$. The Newton force is now $f(r, t) := -\Phi_r(r, t)$, where the gravitational potential is easily computed as the suitable solution of the ode

$$-\frac{1}{r^2} (r^2 \Phi_r)_r = 4\pi G \rho.$$

One gets

$$\Phi(r, t) = \int_0^\infty K(r, s) \rho(s, t) s^2 ds, \quad (2.2)$$

with the kernel

$$K(r, s) = 4\pi G \begin{cases} \frac{1}{r} & \text{for } s \leq r, \\ \frac{1}{s} & \text{for } s \geq r. \end{cases} \quad (2.3)$$

We consider Dirichlet boundary conditions

$$v(0, t) = 0, \quad \text{for } t > 0, \quad (2.4)$$

and initial conditions

$$\rho|_{t=0} = \rho_0(r), \quad \rho v|_{t=0} = m_0(r), \quad \text{on } \mathbf{R}^+. \quad (2.5)$$

Then our main result is the following

Theorem 2.1. *Suppose that the initial data satisfy (1.9) and that T is an arbitrary positive number. Then for any $\gamma > 1$, the problem (2.1)(2.4)(2.5) possesses a global weak solution satisfying (1.8) together with properties (1.10) and (1.11).*

2.1. A priori estimates for approximate solutions

We follow the strategy of [18]: to get first uniform bounds for the density in an approximate problem depending on numbers $0 < \varepsilon \ll 1$ and $R \gg 1$, then pass to the limit ($\varepsilon \rightarrow 0$, $R \rightarrow \infty$).

So we consider the perturbed problem

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \vec{v}) = 0, \\ \partial_t(\rho \vec{v}) + \operatorname{div}(\rho \vec{v} \otimes \vec{v}) - \operatorname{div}(2\mu^\varepsilon(\rho)D(\vec{v})) - \nabla(\lambda^\varepsilon(\rho)\operatorname{div}\vec{v}) + \nabla P - \rho \nabla \Phi = \vec{0}, \\ \Delta \Phi = -4\pi G \rho, \end{cases} \quad (2.6)$$

in the truncated domain $B_{\varepsilon, R} \times [0, T]$, with the perturbed viscosity coefficients

$$\mu^\varepsilon(\rho) = \mu(\rho) + \frac{1}{2} \varepsilon \rho^\theta, \quad \lambda^\varepsilon(\rho) = \lambda(\rho) + (\theta - 1) \varepsilon \rho^\theta, \quad (2.7)$$

where μ^ε and λ^ε have been adjusted in order to satisfy relation (1.3) and $\theta \in (2/3, 1)$.

We want to solve (2.6) in $B_{\varepsilon, R} \times [0, T]$, with Dirichlet boundary conditions

$$\rho \vec{v}|_{\partial B_{\varepsilon, R}} = \vec{0}, \quad (2.8)$$

for $t > 0$, and initial conditions

$$\rho|_{t=0} = \rho_0(x), \quad \rho \vec{v}|_{t=0} = \vec{m}_0(x), \quad \text{on } B_{\varepsilon, R}. \quad (2.9)$$

The spherically symmetric version of (2.6) reads now

$$\left\{ \begin{array}{l} \rho_t + (\rho v)_r + \frac{2\rho v}{r} = 0, \\ \rho(v_t + vv_r) = \left(-P + (\lambda^\varepsilon + 2\mu^\varepsilon) \left(v_r + \frac{2v}{r} \right) \right)_r - 4\mu_r^\varepsilon \frac{v}{r} - \rho f^\varepsilon, \end{array} \right. \quad (2.10)$$

in the domain $\Omega^\varepsilon \times [0, T]$, with $\Omega^\varepsilon := (\varepsilon, R)$, for the truncated Newton force

$$f^\varepsilon(r, t) := \frac{4\pi G}{r^2} \int_\varepsilon^r \rho(s, t) s^2 ds.$$

We supplement (2.10) with

$$v(\varepsilon, t) = v(R, t) = 0, \quad \text{for } t > 0, \quad (2.11)$$

and regularized initial data

$$\rho|_{t=0} = \rho_{0,\varepsilon,R} * j_\delta, \quad v|_{t=0} = v_{0,\varepsilon,R} * j_\delta, \quad \text{on } \Omega^\varepsilon, \quad (2.12)$$

where j_δ is a standard mollifier, and the truncated data are

$$\rho_{0,\varepsilon,R}(r) = \begin{cases} \rho_0(\varepsilon) + \varepsilon & \text{for } r \in [0, \varepsilon], \\ \rho_0(r) + \varepsilon & \text{for } r \in [\varepsilon, R], \\ \rho_0(R) + \varepsilon & \text{for } r \in [R, \infty), \end{cases}$$

and

$$v_{0,\varepsilon,R}(r) = \begin{cases} 0 & \text{for } r \in [0, \varepsilon + 2\delta], \\ \frac{m_0(r)}{\rho_0(r) + \varepsilon} & \text{for } r \in [\varepsilon + 2\delta, R - 2\delta], \\ 0 & \text{for } r \in [R - 2\delta, \infty). \end{cases}$$

We suppose now that $\sqrt{\varepsilon} \leq R^{-3}$ and we denote by $(\rho_{\varepsilon,R,\delta}, v_{\varepsilon,R,\delta})$ the solution of the problem (2.10)(2.11)(2.12).

Finally in order to extract convergent subsequences it is convenient to extend $(\rho_{\varepsilon,R,\delta}, v_{\varepsilon,R,\delta})$ to the whole space by setting

$$\tilde{\rho}_{\varepsilon,R,\delta}(r, t) = \begin{cases} \rho_{\varepsilon,R,\delta}(r, t) & \text{for } r \in [\varepsilon, R], \\ 0 & \text{else,} \end{cases} \quad (2.13)$$

and

$$\tilde{v}_{\varepsilon,R,\delta}(r, t) = \begin{cases} v_{\varepsilon,R,\delta}(r, t) & \text{for } r \in [\varepsilon, R], \\ 0 & \text{else,} \end{cases} \quad (2.14)$$

and in order to simplify the notations we denote in the following by (ρ_n, v_n) the approximate solution $(\tilde{\rho}_{\varepsilon_n, R_n, \delta_n}, \tilde{v}_{\varepsilon_n, R_n, \delta_n})$, for sequences $\varepsilon_n \rightarrow 0$, $R_n \rightarrow \infty$, $\delta_n \rightarrow 0$, when $n \rightarrow \infty$. In the same stroke, we write $\Omega_n := [\varepsilon_n, R_n]$, $\mu_n := \mu^{\varepsilon_n}(\rho_n)$, $\lambda_n := \lambda^{\varepsilon_n}(\rho_n)$ and $P_n := P(\rho_n)$.

In the following, for any measurable set $\omega \subset \Omega := \Omega := (0, \infty)$ and $p \geq 1$, we denote by $\mathcal{L}^p(\omega)$ the weighted Lebesgue space

$$\mathcal{L}^p(\omega) := \left\{ f \in L^p_{loc}(\omega) ; \int_\omega |f(r)|^p r^2 dr < \infty \right\},$$

and we adopt the analogous notations $\mathcal{H}^m(\omega)$ and $\mathcal{W}^{m,p}(\omega)$ for the corresponding weighted Sobolev spaces

We begin with energy-entropy estimates.

Lemma 2.2. *There exists a constant C independent on n such that, for any $t \in [0, T]$*

1.

$$\int_{\Omega} \rho_n r^2 dr \leq C. \tag{2.15}$$

2.

$$\begin{aligned} & \int_{\Omega} \left(\frac{1}{2} \rho_n (v_n)^2 + \Pi_n - \rho_n \Phi_n \right) r^2 dr \\ & + \int_0^t \int_{\Omega} \left[\frac{1}{12} (3\lambda_n + 2\mu_n) \left[\frac{1}{r} (r^2 v_n)_r \right]^2 + 12\mu_n \frac{3\lambda_n + 2\mu_n}{3\lambda_n + 10\mu_n} v_n^2 \right] dr d\tau \leq C, \end{aligned} \tag{2.16}$$

3. *If $\int_{\Omega} [\frac{1}{2} \rho_0 (v^0)^2 + \Pi(\rho_0) - \frac{1}{2} \rho_0 \Phi(\rho_0)] dx < \infty$, the following estimates hold*

$$\|\sqrt{\rho_n} v_n\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \|\rho_n\|_{L^\infty(0,T;L^\gamma(\Omega))} \leq C, \tag{2.17}$$

$$\|\sqrt{\mu_n} (v_n)_r\|_{L^2(0,T;L^2(\Omega))} \leq C, \|\sqrt{\mu_n} v_n\|_{L^2(0,T;L^2(\Omega))} \leq C, \tag{2.18}$$

$$\|\rho\|_{L^\infty(0,T;L^1(\Omega))} \leq C. \tag{2.19}$$

Proof. 1. Integrating the first equation (2.10) and using boundary conditions gives (2.15).

2. Noting $D_t := \partial_t + v\partial_r$, using the identity $\int_{\Omega} \rho_n D_t \phi r^2 dr = \frac{d}{dt} \int_{\Omega} \rho_n \phi r^2 dr$ valid for any function ϕ smooth on $\Omega \times [0, T]$, multiplying the second equation (2.10) by $v_n r^2$ and integrating by parts, we get

$$\frac{d}{dt} \int_{\Omega} \left(\frac{1}{2} \rho_n v_n^2 + \Pi_n - \frac{1}{2} \rho_n \Phi_n \right) r^2 dr = - \int_{\Omega} \left[(\lambda_n + 2\mu_n) \frac{[(r^2 v_n)_r]^2}{r^2} - 4\mu_n (r v_n^2)_r \right] dr. \tag{2.20}$$

Using the notations $w = r^2 v_n$, $a = \lambda_n + 2\mu_n$ and $b = \mu_n$ one checks that the integrand in the right-hand side rewrites

$$(\lambda_n + 2\mu_n) \frac{(r^2 v_n)_r^2}{r^2} - 4\mu_n (r v_n^2)_r = \frac{1}{r^2} \left[a w_r^2 - \frac{8b}{r} w w_r + \frac{12b}{r^2} w^2 \right].$$

Observing that the number $\zeta := \frac{3a-4b}{4a}$ is positive, we have

$$a w_r^2 - \frac{8b}{r} w w_r + \frac{12b}{r^2} w^2 = \frac{2}{3} a \zeta w_r^2 + \frac{6a\zeta(3-4\zeta)}{3-2\zeta} \frac{w^2}{r^2} + \frac{a}{3} \left[\sqrt{3-2\zeta} w_r - \frac{3(3-4\zeta)}{\sqrt{3-2\zeta}} \frac{w}{r} \right]^2,$$

where all the terms in the right hand side are positive after (1.4). Plugging into (2.20) and integrating in t , we get (2.16).

3. After (1.2) one observe that

$$\Pi(\rho_n) \geq C(\rho_n^\gamma - \rho_n \log \rho_n - \rho_n),$$

for a $C > 0$.

Let us estimate the gravitational energy.

$$\begin{aligned} \mathcal{E}_n(t) &= \frac{1}{2} \int_{\Omega} \rho_n \Phi(\rho_n) r^2 dr \\ &= \frac{1}{2} \int_0^1 \int_0^\infty s^2 \rho_n(s, t) K(r, s) r^2 \rho_n(r, t) ds dr + \frac{1}{2} \int_1^\infty \int_0^\infty s^2 \rho_n(s, t) K(r, s) r^2 \rho_n(r, t) ds dr =: A+B. \end{aligned}$$

We get first

$$\begin{aligned} B &= \frac{1}{2} \int_1^\infty r \rho_n(r, t) \int_0^1 s^2 \rho_n(s, t) ds dr + \frac{1}{2} \int_1^\infty r^2 \rho_n(r, t) \int_1^\infty s \rho_n(s, t) ds dr \\ &\leq \frac{1}{2} \int_1^\infty r^2 \rho_n(r, t) \int_0^1 s^2 \rho_n(s, t) ds dr + \frac{1}{2} \int_1^\infty r^2 \rho_n(r, t) \int_1^\infty s^2 \rho_n(s, t) ds dr \leq \|r^2 \rho_n(r, t)\|_{L^1(\Omega)}^2. \end{aligned}$$

Now using Hölder inequality, we bound A as follows

$$\begin{aligned} &\int_0^1 \int_0^\infty s^2 \rho_n(s, t) K(r, s) r^2 \rho_n(r, t) ds dr \\ &\leq \left(\int_0^1 (r^2 \rho_n(r, t))^{\frac{\gamma}{\gamma-1}} dr \right)^{\frac{\gamma-1}{\gamma}} \left(\int_0^1 \left(\int_0^\infty s^2 \rho_n(s, t) K(r, s) ds \right)^\gamma dr \right)^{\frac{1}{\gamma}}. \end{aligned}$$

By convexity, the first integral in the right-hand side is bounded by $\|\rho_n(\cdot, t)\|_{L^1(\Omega)}$.

For the second one, we note that the kernel K is homogeneous of degree -1 and satisfies, for $\gamma > 1$

$$\int_0^\infty |K(1, s)| s^{-\frac{1}{\gamma}} ds < \infty,$$

so we can apply an inequality of Hardy-Littlewood-Polya (see [16] p. 271)

$$\left(\int_0^1 \left(\int_0^\infty s^2 \rho_n(s, t) K(r, s) ds \right)^\gamma dr \right)^{\frac{1}{\gamma}} \leq C \left(\int_0^\infty s^{2\gamma} \rho_n^\gamma(s, t) ds \right)^{\frac{1}{\gamma}} \leq C \left(\int_0^\infty s^2 \rho_n^\gamma(s, t) ds \right)^{\frac{1}{\gamma}}.$$

Finally, using (2.15), we get

$$\mathcal{E}_n(t) \leq C (1 + \|\rho_n^\gamma(r, t)\|_{L^1(\Omega)}).$$

We conclude from the previous estimates that

$$\int_{\Omega} [\Pi(\rho_n) - \rho_n \Phi(\rho_n)] r^2 dr \geq C \left[\int_0^\infty s^2 \rho_n^\gamma ds - \int_0^\infty s^2 \rho_n \log \rho_n ds - 1 - \left(\int_0^\infty s^2 \rho_n^\gamma ds \right)^{\frac{1}{\gamma}} \right].$$

As the log is subdominant and $\gamma > 1$, we end with the lower bound

$$\int_{\Omega} [\Pi(\rho_n) - \rho_n \Phi(\rho_n)] r^2 dr \geq C \int_0^\infty s^2 \rho_n^\gamma ds - C. \quad (2.21)$$

Plugging (2.21) into (2.16) and taking (1.4) into account gives immediately the inequalities (2.17), (2.18) and (2.19) by inspection. \square

We have the following ‘‘spherical’’ version of the Bresch-Desjardins entropy estimate (see [1])

Lemma 2.3. *There exists a constant C independent on n such that, for any $t \in [0, T]$*

1.

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \rho_n (v_n + 2\mu_1(\phi(\rho_n))_r)^2 r^2 dr + \int_{\Omega} \Pi_n r^2 dr - \int_{\Omega} \rho_n \Phi_n r^2 dr \\ & + 2\mu_1 \int_0^t \int_{\Omega} P'(\rho_n) \Psi'(\rho_n) \frac{((\rho_n)_r)^2}{\rho_n} r^2 dr d\tau - 2\mu_1 \int_0^t \int_{\Omega} \Psi_r \Phi_r r^2 dr \\ & + 2\mu_1 \int_0^t \int_{\Omega} [r^2((v_n)_r)^2 + 4v_n^2] \Psi_n dr d\tau \leq C, \end{aligned} \quad (2.22)$$

with $\Psi_n := \Psi(\rho_n)$.

2. *If $\frac{1}{2} \int_{\Omega} \rho_0 (v_0 + 2\mu_1(\phi(\rho_0))_r)^2 r^2 dr < \infty$, the following estimates hold*

$$\|\sqrt{\rho_n} \phi_r(\rho_n)\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \quad (2.23)$$

$$\|\sqrt{\mu'_n(\rho_n) \rho_n^{\gamma-2}} (\rho_n)_r\|_{L^2(0,T;L^2(\Omega))} \leq C, \quad (2.24)$$

$$\|\sqrt{\mu_n(\rho_n)} v_n\|_{L^2(0,T;L^2(\Omega))} \leq C. \quad (2.25)$$

Proof. 1. In the proof of this estimate, we omit the explicit n dependence.

First multiplying the first equation (2.1) by ϕ' and derivating with respect to r , we get

$$\phi_{rt} + v\phi_{rr} + v_r\phi_r + \left[\rho\phi' \frac{1}{r^2} (r^2v)_r \right]_r = 0.$$

Multiplying by $r^2\rho\phi_r$, using the continuity equation and integrating in Ω , we get

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho\phi_r^2 r^2 dr + \int_{\Omega} \rho\phi_r^2 v_r r^2 dr + \int_{\Omega} \left[\phi' \rho \frac{1}{r^2} (r^2v)_r \right]_r \Psi_r r^2 dr = 0. \quad (2.26)$$

From the second equation (2.1), we have

$$\rho(v_t + vv_r) - \left[(\lambda + 2\mu) \frac{1}{r^2} (r^2v)_r \right]_r - 4\mu_r \frac{v}{r} + P_r + \rho f = 0,$$

which rewrites, by using (1.3)

$$\rho(v_t + vv_r) - 2\mu_1 \left[\Psi \frac{1}{r^2} (r^2v)_r \right]_r + 2\mu_1 \Psi_r v_r - 2\mu_1 \left[(\rho\Psi' - \Psi) \frac{1}{r^2} (r^2v)_r \right]_r + P_r + \rho f = 0.$$

Multiplying by $\rho^{-1}\Psi_r r^2$ and integrating, we obtain

$$\begin{aligned} & \int_{\Omega} (v_t + vv_r) \Psi_r r^2 dr - 2\mu_1 \int_{\Omega} \left[\Psi \frac{1}{r^2} (r^2v)_r \right]_r \frac{\Psi_r}{\rho} r^2 dr + 2\mu_1 \int_{\Omega} \left[(\Psi - \rho\Psi') \frac{1}{r^2} (r^2v)_r \right]_r \frac{\Psi_r}{\rho} r^2 dr \\ & + 2\mu_1 \int_{\Omega} \frac{\Psi_r^2}{\rho} v_r r^2 dr + \int_{\Omega} \frac{\Psi_r P_r}{\rho} r^2 dr - \int_{\Omega} \Psi_r \Phi_r r^2 dr = 0. \end{aligned} \quad (2.27)$$

Multiplying (2.26) by $2\mu_1$ and adding the result to (2.27), we get

$$\int_{\Omega} (v_t + vv_r) \Psi_r r^2 dr + \frac{\mu_1}{2} \frac{d}{dt} \int_{\Omega} 2\rho\phi_r^2 r^2 dr - 2\mu_1 \int_{\Omega} \Psi \left[\frac{1}{r^2} (r^2v)_r \right]_r \frac{\Psi_r}{\rho} r^2 dr$$

$$\begin{aligned}
& +2\mu_1 \int_{\Omega} \left[(\Psi - \rho\Psi') \frac{1}{r^2} (r^2v)_r \right]_r \frac{\Psi_r}{\rho} r^2 dr + 2\mu_1 \int_{\Omega} \left[\Psi' \frac{1}{r^2} (r^2v)_r \right]_r \Psi_r r^2 dr \\
& \quad + \int_{\Omega} P' \Psi' \frac{\rho_r^2}{\rho} r^2 dr - \int_{\Omega} \Psi_r \Phi_r r^2 dr = 0.
\end{aligned}$$

Checking the identity

$$\begin{aligned}
2\mu_1 \int_{\Omega} \Psi \left[\frac{1}{r^2} (r^2v)_r \right]_r \frac{\Psi_r}{\rho} r^2 dr &= 2\mu_1 \int_{\Omega} \left[(\Psi - \rho\Psi') \frac{1}{r^2} (r^2v)_r \right]_r \frac{\Psi_r}{\rho} r^2 dr \\
& \quad + 2\mu_1 \int_{\Omega} \left[\Psi' \frac{1}{r^2} (r^2v)_r \right]_r \Psi_r r^2 dr,
\end{aligned}$$

one gets finally

$$\int_{\Omega} (v_t + vv_r) \Psi_r r^2 dr + \frac{\mu_1}{2} \frac{d}{dt} \int_{\Omega} 2\rho\phi_r^2 r^2 dr + \int_{\Omega} P' \Psi' \frac{\rho_r^2}{\rho} r^2 dr - \int_{\Omega} \Psi_r \Phi_r r^2 dr = 0. \quad (2.28)$$

The first integral in (2.28) rewrites

$$\int_{\Omega} (v_t + vv_r) \Psi_r r^2 dr = \frac{d}{dt} \int_{\Omega} v \Psi_r r^2 dr - \int_{\Omega} v (\Psi_t)_r r^2 dr + \int_{\Omega} vv_r \Psi_r r^2 dr =: \mathcal{A}, \quad (2.29)$$

where \mathcal{A} rewrites

$$\begin{aligned}
\mathcal{A} &= \frac{d}{dt} \int_{\Omega} v \Psi_r r^2 dr - \int_{\Omega} \left[\frac{1}{r^2} (r^2v)_r \right]_r^2 \rho \Psi' r^2 dr - \int_{\Omega} (r^2v)_r v \rho_r \Psi' dr + \int_{\Omega} vv_r \Psi' \rho_r r^2 dr \\
&=: U_1 + U_2 + U_3 + U_4.
\end{aligned} \quad (2.30)$$

Computing the term U_3 , we obtain

$$U_3 = - \int_{\Omega} vv_r \Psi' \rho_r r^2 dr + 2 \int_{\Omega} (rv^2)_r \Psi dr,$$

so we get

$$\mathcal{A} = \frac{d}{dt} \int_{\Omega} v \Psi_r r^2 dr - \int_{\Omega} \left[\frac{1}{r^2} (r^2v)_r \right]_r^2 (\rho \Psi' - \Psi) r^2 dr - \int_{\Omega} (r^2v_r^2 + 2v^2) \Psi dr. \quad (2.31)$$

Plugging into (2.29), we have

$$\begin{aligned}
\int_{\Omega} (v_t + vv_r) \Psi_r r^2 dr &= \frac{d}{dt} \int_{\Omega} v \Psi_r r^2 dr - \int_{\Omega} \left[\frac{1}{r^2} (r^2v)_r \right]_r^2 (\rho \Psi' - \Psi) r^2 dr \\
& \quad - \int_{\Omega} (r^2v_r^2 + 2v^2) \Psi dr.
\end{aligned} \quad (2.32)$$

From (2.20) we have

$$\frac{d}{dt} \int_{\Omega} \left(\frac{1}{2} \rho v^2 + \Pi - \frac{1}{2} \rho \Phi \right) r^2 dr = - \int_{\Omega} \left[(\lambda + 2\mu) \frac{[(r^2v)_r]^2}{r^2} - 4\mu (rv^2)_r \right] dr,$$

so

$$\frac{d}{dt} \int_{\Omega} \rho \frac{v^2}{2} r^2 dr + 2\mu_1 \int_{\Omega} \left[\frac{1}{r^2} (r^2v)_r \right]_r^2 (\rho \Psi' - \Psi) r^2 dr + \frac{d}{dt} \int_{\Omega} \Pi r^2 dr - \frac{d}{dt} \int_{\Omega} \rho \Phi r^2 dr$$

$$= -2\mu_1 \int_{\Omega} \left[\frac{1}{r^2} (r^2 v)_r \right]^2 \Psi r^2 dr + 4\mu_1 \int_{\Omega} (rv^2)_r \Psi dr. \tag{2.33}$$

Now plugging (2.32) into (2.28) gives

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} v \Psi_r r^2 dr - \int_{\Omega} \left[\frac{1}{r^2} (r^2 v)_r \right]^2 (\rho \Psi' - \Psi) r^2 dr - \int_{\Omega} (r^2 v_r^2 + 2v^2) \Psi dr \\ & + \frac{\mu_1}{2} \frac{d}{dt} \int_{\Omega^\varepsilon} 2\rho \phi_r^2 r^2 dr + \int_{\Omega} P' \Psi' \frac{\rho_r^2}{\rho} r^2 dr - \int_{\Omega} \Psi_r \Phi_r r^2 dr = 0. \end{aligned} \tag{2.34}$$

Multiplying this relation by $2\mu_1$ gives

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} [2\mu_1^2 \rho \phi_r^2 + 2\mu_1 v \Psi_r] r^2 dr + 2\mu_1 \int_{\Omega} P' \Psi' \frac{\rho_r^2}{\rho} r^2 dr - 2\mu_1 \int_{\Omega} \Psi_r \Phi_r r^2 dr \\ & - 2\mu_1 \int_{\Omega} \left[\frac{1}{r^2} (r^2 v)_r \right]^2 (\rho \Psi' - \Psi) r^2 dr = -2\mu_1 \int_{\Omega} (r^2 v_r^2 + 2v^2) \Psi dr, \end{aligned} \tag{2.35}$$

where the integrand in the last integral is positive. Finally, adding (2.35) to (2.33), integrating in time and restoring the n dependence gives (2.22).

2. Observing that after Poisson equation

$$\int_{\Omega} \Psi_r \Phi_r r^2 dr = \int_{\Omega} \rho \Psi r^2 dr,$$

and using (1.5) together with (2.16) into (2.22), we get immediately the inequalities (2.23), (2.24) and (2.25) by inspection. \square

Lemma 2.4. *The sequence $\{\rho_n^\gamma\}_{n \in \mathbb{N}}$ is bounded in $L^{5/3}(0, T; \mathcal{L}^{5/3}(\Omega))$.*

Proof. After (2.19) and (2.24), $\rho_n^{\gamma/2}$ is bounded in $L^2(0, T; \mathcal{H}^1(\Omega_n))$ and then ρ_n^γ is bounded in $L^2(0, T; \mathcal{L}^3(\Omega_n))$. As, after (2.19), ρ_n^γ is also bounded in $L^\infty(0, T; \mathcal{L}^1(\Omega_n))$, using Hölder inequality ρ_n^γ is also bounded in $L^{5/3}(0, T; \mathcal{L}^{5/3}(\Omega_n))$, and then (see (2.13) and (2.14)) in $L^{5/3}(0, T; \mathcal{L}^{5/3}(\Omega))$. \square

Lemma 2.5. *If $\rho_0(1 + v_0^2) \log(1 + v_0^2) \in \mathcal{L}^1(\Omega)$, then the sequence $\{\rho_n v_n^2 \log(1 + v_n^2)\}_{n \in \mathbb{N}}$ is bounded in $L^\infty(0, T; \mathcal{L}^1(\Omega))$.*

Proof. Recalling the identity

$$\frac{d}{dt} \int_{\Omega_n} \rho_n F r^2 dr = \int_{\Omega_n} \rho_n (F_t + v F_r) r^2 dr,$$

and choosing $F = \frac{1}{2} (1 + v_n^2) \log(1 + v_n^2)$, we get

$$\frac{d}{dt} \int_{\Omega_n} \frac{1}{2} \rho_n (1 + v_n^2) \log(1 + v_n^2) r^2 dr = \int_{\Omega_n} v_n (1 + \log(1 + v_n^2)) \rho_n [(v_n)_t + v_n (v_n)_r] r^2 dr.$$

Then multiplying the momentum equation by $v_n (1 + \log(1 + v_n^2)) r^2$ and integrating by parts, we have

$$\frac{d}{dt} \int_{\Omega_n} \frac{1}{2} \rho_n (1 + v_n^2) \log(1 + v_n^2) r^2 dr = - \int_{\Omega_n} v_n (1 + \log(1 + v_n^2)) (P(\rho_n))_r r^2 dr$$

$$\begin{aligned}
& + \int_{\Omega_n} v_n(1+\log(1+v_n^2)) \left[(\lambda_n + 2\mu_n) \frac{1}{r^2} (r^2 v_n)_r \right]_r r^2 dr + \int_{\Omega_n} 4v_n(1+\log(1+v_n^2)) (\mu_n)_r \frac{v_n}{r} r^2 dr \\
& \quad - \int_{\Omega_n} 4\pi G v_n(1+\log(1+v_n^2)) \rho_n(\Phi_n)_r r^2 dr.
\end{aligned}$$

Using energy estimates and (1.3), we get

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega_n} \frac{1}{2} \rho_n(1+v_n^2) \log(1+v_n^2) r^2 dr + \underline{m} \int_{\Omega_n} \mu_n \left[\frac{1}{r^2} (r^2 v_n)_r \right]^2 r^2 dr \\
& \leq - \int_{\Omega_n} v_n(1+\log(1+v_n^2)) (P(\rho_n))_r r^2 dr - \int_{\Omega_n} 4\pi G v_n(1+\log(1+v_n^2)) \rho_n(\Phi_n)_r r^2 dr =: I+J.
\end{aligned} \tag{2.36}$$

Using (1.2) and Cauchy-Schwarz, the first contribution in the right-hand side of (2.36) is bounded as follows

$$\begin{aligned}
|I| & = \left| \int_{\Omega_n} v_n(1+\log(1+v_n^2)) P_r r^2 dr \right| \leq \int_{\Omega_n} \mu_n [(r^2 v_n)_r]^2 dr + \frac{1}{2} \underline{m} \int_{\Omega_n} \mu_n \left[\frac{1}{r^2} (r^2 v_n)_r \right]^2 r^2 dr \\
& \quad + C \int_{\Omega_n} (1+\log(1+v_n^2)) \frac{(P(\rho_n))^2}{\mu_n} r^2 dr,
\end{aligned}$$

and the last integral is bounded by

$$\left(\int_{\Omega_n} \rho_n [\log(1+v_n^2)]^{\frac{2}{\eta}} r^2 dr \right)^{\frac{\eta}{2}} \times \left(\int_{\Omega_n} \left(\frac{\rho_n^{-\frac{\eta}{2}} P_n^2}{\mu_n} \right)^{\frac{2}{2-\eta}} r^2 dr \right)^{\frac{2-\eta}{2}},$$

for any $0 < \eta < 2$.

One checks that the first integral is bounded after previous estimates, and the second one is also bounded provided that $2\gamma - 1 < \frac{5\gamma}{3}$ i.e. $\gamma < 3$. When $\gamma > 3$, the same result holds if condition (1.6) is satisfied. So for any $\gamma > 1$

$$|I| \leq C. \tag{2.37}$$

For the second contribution in (2.36), one has

$$|J| \leq C \int_{\Omega_n} |v_n| (1+\log(1+v_n^2)) \rho_n(r, \tau) \int_0^r \rho_n(s, \tau) s^2 ds dr \leq C \int_{\Omega_n} \rho_n v_n^2 dr,$$

so using (1.3)

$$|J| \leq \int_{\Omega_n} \mu_n v_n^2 dr. \tag{2.38}$$

Plugging (2.37) and (2.38) in (2.36), integrating on $(0, t)$ and using (2.16), we find that

$$\int_{\Omega_n} \frac{1}{2} \rho_n(1+v_n^2) \log(1+v_n^2) r^2 dr \leq C,$$

which ends the proof, by taking (2.13) and (2.14) into account. \square

2.2. Proof of Theorem 2.1

Following verbatim the strategy of [18], the proof of Theorem 2.1 is achieved in the same way as in [13] by showing first the convergence of the density and the pressure, then improving the convergence of $\rho_n v_n^2$, proving the convergence of the momentum, showing the strong convergence of $\sqrt{\rho_n} (r^2 v_n)_r$ and finally the convergence of diffusion terms.

We only sketch the proof hereafter, only stressing the specific roles played by gravitation and pressure, and we refer to [18] for complete details.

1) Convergence of the density.

Lemma 2.6. *The sequence $\{\rho_n\}$ is bounded in $L^\infty(0, T; \mathcal{L}^p(\Omega))$ for any $p \geq 1$, and up to an extracted subsequence*

$$\rho_n \rightarrow \rho \text{ strongly in } C(0, T; \mathcal{L}_{loc}^q(\Omega)), \tag{2.39}$$

for any $q \in [1, 3)$.

Proof. From Lemma 2.3, one get $\sup_{t \in [0, T]} \|\sqrt{\rho_n}\|_{\mathcal{H}^1(\Omega_n)} \leq C$, so after (2.13), the sequence $\{\sqrt{\rho_n}\}$ is bounded in $L^\infty(0, T; \mathcal{L}^\beta(\Omega))$ for $\beta \in [2, 6]$.

So $\{\rho_n\}$ is bounded in $L^\infty(0, T; \mathcal{L}^3(\Omega))$ and $\{\rho_n v_n\}$ is bounded in $L^\infty(0, T; \mathcal{L}^{3/2}(\Omega))$ after Lemma 2.2.

Using the continuity equation, $\{\partial_t \sqrt{\rho_n}\}_{\epsilon_n \leq \frac{1}{K}, R_n \leq N}$ is bounded in $L^\infty(0, T; \mathcal{W}^{-1, 3/2}(\frac{1}{K}, N))$, for any $K \geq N^6$.

As $(\rho_n)_r = 2\sqrt{\rho_n}(\sqrt{\rho_n})_r$, we find that $\{(\rho_n)_r\}$ is bounded in $L^\infty(0, T; \mathcal{L}^{3/2}(\frac{1}{K}, N))$ for $\epsilon_n \leq \frac{1}{K}$ and $R_n \leq N$.

This implies, after Aubin-Lions lemma that $\rho_n \rightarrow \rho$ strongly in $C(0, T; \mathcal{L}^{3/2}(\frac{1}{K}, N))$.

Using finally the splitting

$$\|\rho_n - \rho\|_{L^\infty(0, T; \mathcal{L}^{3/2}(0, N))} \leq \frac{C}{K} \|\rho_n - \rho\|_{L^\infty(0, T; \mathcal{L}^3(0, \frac{1}{K}))} + \|\rho_n - \rho\|_{L^\infty(0, T; \mathcal{L}^{3/2}(\frac{1}{K}, N))},$$

we get the property. □

2) Convergence of the pressure and the gravitational force.

As a direct consequence of Lemma 2.4 and 2.6, we have

Lemma 2.7. *For $\gamma > 1$*

$$\begin{aligned} P(\rho_n) & \text{ converges strongly to } P(\rho) \text{ in } L_{loc}^1((0, T); \mathcal{L}_{loc}^1(\Omega)), \\ \rho_n(\Phi_n)_r & \text{ converges strongly to } \rho\Phi_r \text{ in } L_{loc}^1((0, T); \mathcal{L}_{loc}^1(\Omega)). \end{aligned}$$

3) Convergence of the momentum.

Lemma 2.8. *Up to extracted subsequences*

1.

$$\begin{aligned} \rho_n v_n & \rightarrow m \text{ strongly in } L_{loc}^1(0, T; \mathcal{L}_{loc}^1(\Omega)) \cap L^2(0, T; \mathcal{L}_{loc}^\beta(\Omega)), \\ & \text{for any } 1 \leq \beta < 3/2. \end{aligned}$$

2.

$$\sqrt{\rho_n} v_n \rightarrow \frac{m}{\sqrt{\rho}} \text{ strongly in } L^2_{loc}(0, T; \mathcal{L}^2_{loc}(\Omega)),$$

where $\frac{m}{\sqrt{\rho}} = 0$ when $m = 0$.

In particular $m = 0$ a.e. on $\{\rho(r, t) = 0\}$, and there exists a function $v(r, t)$ such that

$$m(r, t) = \rho(r, t)v(r, t).$$

Proof. The second part of the Lemma follows directly from [13] and we omit the proof.

As $\{\sqrt{\rho_n}\}$ is bounded in $L^\infty(0, T; \mathcal{L}^2 \cap \mathcal{L}^6(\Omega))$ and $\{\sqrt{\rho_n} v_n\}$ is bounded in $L^\infty(0, T; \mathcal{L}^2(\Omega))$, we get that

$$\{\rho_n v_n\} \text{ is bounded in } L^\infty(0, T; \mathcal{L}^1 \cap \mathcal{L}^{3/2}(\Omega)). \quad (2.40)$$

Using the identity $(\rho_n v_n)_r = 2\sqrt{\rho_n} v_n (\sqrt{\rho_n})_r + \sqrt{\rho_n} \sqrt{\rho_n} (v_n)_r$ and Lemma 2.5, we see that

$$\{(\sqrt{\rho_n} v_n)_r\}_{\epsilon_n \leq \frac{1}{K}, R_n \geq N} \text{ is bounded in } L^\infty(0, T; \mathcal{L}^1(\frac{1}{K}, N)).$$

In particular the sequence $\{\rho_n v_n\}_{\epsilon_n \leq \frac{1}{K}, R_n \geq N}$ is bounded in $L^2(0, T; \mathcal{W}^{1,1}(\frac{1}{K}, N))$.

As $\{\rho_n\}_{\epsilon_n \leq \frac{1}{K}, R_n \geq N}$ is bounded in $L^\infty(0, T; \mathcal{L}^\infty(\frac{1}{K}, N))$, we see that

$$\{\partial_t(\rho_n v_n)\}_{\epsilon_n \leq \frac{1}{K}, R_n \geq N} \text{ is bounded in } L^2(0, T; \mathcal{W}^{-2,3/2}(\frac{1}{K}, N)),$$

so using the Aubin-Lions Lemma

$$\rho_n v_n \rightarrow m \text{ strongly in } L^2(0, T; \mathcal{L}^\beta(\frac{1}{K}, N)),$$

for any $\beta \in [1, 3/2)$. From (2.40), we get also that

$$\rho_n v_n \rightarrow m \text{ strongly in } L^2(0, T; \mathcal{L}^\beta(0, N)),$$

for any $\beta \in [1, 3/2)$. \square

4) Convergence in the weak formulation.

In order to complete the proof of Theorem 2.1, the last point is to check that the pair (ρ, v) is actually a weak solution of (2.10). In other words we must check that if (ρ, v) is the limit of the sequence $(\{\rho_n\}, \{v_n\})$ obtained in Lemma 2.6 and 2.8, then ρ and v satisfy relations (1.10) and (1.11).

The fact that (ρ, v) satisfy relation (1.10) is already proved in [18] (see Proposition 3.5).

Let $\psi \in C_0^\infty(\Omega \times [0, T])$ with $\psi(0, t) = 0$ and $\psi(r, T) = 0$ be a test function such that $\text{Supp } \psi(\cdot, t) \subset [0, N]$.

Multiplying the second equation (2.10) by $r^2 \psi$ and integrating on $\Omega \times [0, T]$, we have

$$\int_0^t \int_\Omega \left(\rho_n v_n \psi_t + \rho_n v_n^2 \psi_r + P(\rho_n) \left(\psi_r + \frac{2\psi}{r} \right) \right) r^2 dr dt$$

$$\begin{aligned}
 & - \int_0^t \int_{\Omega} 2\mu(\rho_n) \left((v_n)_r \psi_r + \frac{2v_n \psi}{r^2} \right) r^2 dr dt \\
 & + \int_0^t \int_{\Omega} 2\epsilon_n \rho_n^\theta \left((v_n)_r \psi + v_n \psi_r + \frac{v_n \psi}{r} \right) r dr dt \\
 & - \int_0^t \int_{\Omega} (\lambda(\rho_n) + \theta \epsilon_n P(\rho_n)) \left((v_n)_r + \frac{2v_n}{r} \right) \left(\psi_r + \frac{2\psi}{r} \right) r^2 dr dt \\
 & + \int_{\epsilon_n}^\infty \rho_{0,\epsilon_n,R_n}(r,0) v_{0,\epsilon_n,R_n}(r,0) \psi(r,0) r^2 dr \\
 & + \int_0^T [P(\rho_n) - (2\mu(\rho_n) + \lambda(\rho_n) + \theta \epsilon_n \rho_n^\theta) (v_n)_r] (\epsilon_n, t) \epsilon_n^2 \psi(\epsilon_n, t) dt \\
 & = - \int_0^t \int_{\Omega} \rho_n (\Phi(\rho_n))_r \psi r^2 dr dt, \tag{2.41}
 \end{aligned}$$

for any n such that $\epsilon_n \leq \frac{1}{K}$ and $R_n \geq N$.

Following verbatim [18] one can pass to the limit $n \rightarrow \infty$ in all of the terms of the left-hand side in (2.41). Note in particular that

$$\lim_{n \rightarrow \infty} \int_0^T [P(\rho_n) - (2\mu(\rho_n) + \lambda(\rho_n) + \theta \epsilon_n \rho_n^\theta) (v_n)_r] (\epsilon_n, t) \epsilon_n^2 \psi(\epsilon_n, t) dt = 0.$$

Moreover, after Lemma 2.7, one also gets that the same holds for the gravitational term in the right hand side

$$\int_0^t \int_{\Omega} \rho_n (\Phi(\rho_n))_r \psi r^2 dr dt \rightarrow \int_0^t \int_{\Omega} \rho (\Phi(\rho))_r \psi r^2 dr dt,$$

when $n \rightarrow \infty$, which ends the proof of Theorem 2.1 \square

3. The constant viscosities case

For constant viscosities, the spherically symmetric version of (1.1) reads

$$\begin{cases} \rho_t + (\rho v)_r + \frac{2\rho v}{r} = 0, \\ \rho(v_t + vv_r) = \left(-P + (\lambda + 2\mu) \left(v_r + \frac{2v}{r} \right) \right)_r - \rho \Phi_r, \end{cases} \tag{3.1}$$

in the domain $[0, T] \times \Omega \equiv [0, T] \times (0, \infty)$ for the density $\rho(r, t)$ and the velocity $v(r, t)$. The gravitational potential is given as previously by (2.2) and (2.3).

We consider Dirichlet boundary condition at the origin

$$v(0, t) = 0, \quad \text{for } t > 0, \tag{3.2}$$

and initial conditions

$$\rho|_{t=0} = \rho_0(r), \quad \rho v|_{t=0} = m_0(r), \quad \text{on } \mathbf{R}^+. \tag{3.3}$$

We recall that (ρ, v) is a weak solution of (3.1)(3.2)(3.3) if

1. $\rho \geq 0$ a.e. and for any $T > 0$ (see (1.8))

$$\left\{ \begin{array}{l} \rho \in L^\infty(0, T; \mathcal{L}^\gamma(\Omega)), \quad \rho v^2 \in L^\infty(0, T; \mathcal{L}^1(\Omega)), \\ v_r \in L^\infty(0, T; \mathcal{L}^1(\Omega)), \quad \frac{v}{r} \in L^\infty(0, T; \mathcal{L}^1(\Omega)), \\ \rho \in C\left([0, T]; \left(\mathcal{L}_{\text{loc}}^\gamma\right)_w(\Omega_0)\right), \quad \rho v \in C\left([0, T]; \left(\mathcal{L}_{\text{loc}}^{\frac{2\gamma}{\gamma+1}}\right)_w(\Omega_0)\right), \end{array} \right. \quad (3.4)$$

where $\Omega_0 \equiv [0, \infty)$, and the index w means that the corresponding \mathcal{L}^p space is endowed with the weak topology.

2. For any $t_2 \geq t_1 \geq 0$ and any test function $\eta \in C_0^1(\mathbb{R} \times \Omega)$ (see (1.10) and (1.11))

$$\int_{\Omega} \rho \eta r^2 dr \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_{\Omega} (\rho \eta_t + \rho v \eta_r) r^2 dr = 0. \quad (3.5)$$

$$\begin{aligned} & \int_{\Omega} \rho v \eta r^2 dr \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \int_{\Omega} \left(\rho v \eta_t + \rho v^2 \eta_r + P \left(\eta_r + \frac{2\eta}{r} \right) \right) r^2 dr dt \\ & + (\lambda + 2\mu) \int_{t_1}^{t_2} \int_{\Omega} \left(v_r \eta_r + \frac{2v\eta}{r^2} \right) r^2 dr dt \\ & + 4\pi G \int_{t_1}^{t_2} \int_{\Omega} \eta(r, t) \rho(r, t) \int_0^r \rho(s, t) s^2 ds r^2 dr dt = 0. \end{aligned} \quad (3.6)$$

The following result holds

Theorem 3.1. *Suppose that T is an arbitrary positive number and that the initial data satisfy*

$$0 \leq \rho_0 \in \mathcal{L}^1(\Omega) \cap \mathcal{L}^\gamma(\Omega), \quad (3.7)$$

and

$$m_0 \in \mathcal{L}^{\frac{2\gamma}{\gamma+1}}(\Omega), \quad \frac{m_0^2}{\rho_0} \in \mathcal{L}^1(\Omega). \quad (3.8)$$

Then for any $\gamma > 1$, the problem (3.1)(3.2)(3.3) possesses a global weak solution satisfying (3.4) together with properties (3.5) and (3.6).

In the constant viscosities case, extra estimates on the density gradient using Bresch-Desjardins inequality are no more available and one must rely on the original Lions-Feireisl method [12] [8] [15] (see [4]).

Following [11], we consider approximate solutions in the spirit of Section 3 and pass to the limit. As in the density dependent case, we only concentrate on the role of gravitation and pressure, referring to the paper of Jiang and Zhang [11] for further details.

3.1. Approximate problem and priori estimates

Adapting the approximating scheme of Hoff [10] to the self-gravitating situation, we define in $\Omega_\epsilon \equiv (\epsilon, \infty)$ the mollified data

$$(\rho_0)_\epsilon(r) := \frac{1}{r^{2/\gamma}} \left[(r^{2/\gamma} \rho_0) * j_{\epsilon/2} \right] (r) + \epsilon e^{-r^2}, \tag{3.9}$$

$$(m_0)_\epsilon(r) := \frac{\sqrt{(\rho_0)_\epsilon}}{r^{2/\gamma}} \left[\left(r \frac{m_0}{\sqrt{\rho_0}} \right) * j_{\epsilon/2} \right] (r), \tag{3.10}$$

and

$$(v_0)_\epsilon(r) := \left(\chi_\epsilon \frac{(m_0)_\epsilon}{(\rho_0)_\epsilon} \right) (r), \tag{3.11}$$

where $\epsilon > 0$, j is the standard Friedrich mollifier and $\chi_\epsilon \in C_0^\infty(\Omega)$ is a cut-off such that

$$\chi_\epsilon(r) = \begin{cases} 0 & \text{if } r \leq \epsilon, \\ 1 & \text{if } r \geq 2\epsilon. \end{cases}$$

Then denoting by δ_ϵ and $\delta_{\epsilon,R}$ the step functions

$$\delta_\epsilon(r) = \begin{cases} 0 & \text{if } r \leq \epsilon, \\ 1 & \text{if } r > \epsilon. \end{cases}, \quad \delta_{\epsilon,R}(r) = \begin{cases} 0 & \text{if } r \leq \epsilon \text{ or } r \geq R, \\ 1 & \text{if } \epsilon < r < R, \end{cases}$$

for a $R > 0$, one checks that, uniformly in R

$$\left\{ \delta_\epsilon \left((\rho_0)_\epsilon - \rho_0 - \epsilon e^{-r^2} \right) \right\} \rightarrow 0 \text{ in } \mathcal{L}^\gamma(\Omega),$$

$$\left\{ \delta_\epsilon \left((\rho_0)_\epsilon - \rho_0 - \epsilon e^{-r^2} \right) \right\} \rightarrow 0 \text{ in } \mathcal{L}^1(\Omega),$$

and

$$\left\{ \delta_{\epsilon,R} \left((\rho_0)_\epsilon (v_0)_\epsilon - m_0 \right) \right\} \rightarrow 0 \text{ in } \mathcal{L}^{\frac{2\gamma}{\gamma+1}}(\Omega),$$

when $\epsilon \rightarrow 0$, for any $\gamma > 0$.

We consider the approximate problem

$$\begin{cases} (\rho_\epsilon)_t + (\rho_\epsilon v)_r + \frac{2\rho_\epsilon v_\epsilon}{r} = 0, \\ \rho_\epsilon \left((v_\epsilon)_t + v_\epsilon (v_\epsilon)_r \right) = \left(-P(\rho_\epsilon) + (\lambda + 2\mu) \left((v_\epsilon)_r + \frac{2v_\epsilon}{r} \right) \right)_r - \rho_\epsilon (\Phi_\epsilon)_r, \end{cases} \tag{3.12}$$

in the domain $[0, T] \times \Omega_\epsilon$ for the density ρ_ϵ and the velocity v_ϵ . The approximate gravitational potential is

$$\Phi_\epsilon(r, t) = \int_\epsilon^\infty K(r, s) \rho_\epsilon(s, t) s^2 ds, \tag{3.13}$$

with K is given by (2.3).

We consider Dirichlet boundary condition

$$v_\epsilon(0, \epsilon) = 0, \quad \text{for } t > 0, \tag{3.14}$$

and initial conditions

$$\rho_\epsilon|_{t=0} = (\rho_0)_\epsilon(r), \quad \rho_\epsilon v_\epsilon|_{t=0} = (m_0)_\epsilon(r), \quad \text{on } \Omega_\epsilon. \quad (3.15)$$

Then we have

Proposition 3.2. *Suppose that ϵ and T are arbitrary positive numbers and that the initial data satisfy*

$$0 \leq \rho_0 \in \mathcal{L}^1(\Omega) \cap \mathcal{L}^\gamma(\Omega), \quad (3.16)$$

and

$$m_0 \in \mathcal{L}^{\frac{2\gamma}{\gamma+1}}(\Omega), \quad \frac{m_0^2}{\rho_0} \in \mathcal{L}^1(\Omega). \quad (3.17)$$

Then for any $\gamma > 1$, the exterior problem (3.12)(3.14)(3.15) possesses a weak solution such that

$$\rho_\epsilon > 0, \quad \rho_\epsilon \in C(0, T; \mathcal{L}_{loc}^\gamma(\Omega_\epsilon)), \quad v_\epsilon \in C(0, T; \mathcal{L}^2(\Omega_0)) \cap C(0, T; \mathcal{H}_0^1(\Omega)),$$

and

$$\int_{\Omega_\epsilon} \left(\frac{1}{2} \rho_\epsilon (v_\epsilon)^2 + \frac{1}{2} \Pi(\rho_\epsilon) \right) r^2 dr + \frac{3\lambda + 2\mu}{2} \int_0^t \int_{\Omega_\epsilon} [r^2 ((v_\epsilon)_r)^2 + v_\epsilon^2] dr d\tau \leq C, \quad (3.18)$$

where $C > 0$ does not depend on ϵ .

The proof is a direct consequence of [10]: the gravitational contribution is bounded a priori due to the core presence, and the subdominant contribution in the pressure (see (1.2)) is controlled as above by mass conservation.

In order to get precompactness of the sequences $\{\rho_\epsilon\}$, $\{v_\epsilon\}$, as we have no more direct information on the gradient of ρ_ϵ , we need improved $L^p(\mathcal{L}^q)$ estimates on ρ_ϵ itself for p, q large enough, which is the matter of the following result.

Lemma 3.3. 1. *For any test function $\phi \in C_0^\infty(\Omega_0)$ with $\phi(r) = O(r^3)$ as $r \rightarrow 0$, there exists a constant C independent on ϵ such that*

$$\int_0^T \int_{\Omega_\epsilon} (\rho_\epsilon)^{2\gamma} \phi^4 dr dt \leq C, \quad (3.19)$$

2. *Let $\gamma > 3/2$. For any $T > 0$ and for any test function $\psi \in C_0^\infty(\Omega_0)$ with $\phi \geq 0$ and $\phi(r) = O(r^3)$ as $r \rightarrow 0$, there exists a constant C independent on ϵ such that*

$$\int_0^T \int_{\Omega_\epsilon} (\rho_\epsilon)^{\gamma+\theta} \psi dr dt \leq C, \quad (3.20)$$

for any $0 < \theta < \gamma$.

Proof. In the proof of these estimates, we omit the explicit ϵ dependence.

1. Multiplying by ϕ and integrating on (r, ∞) the momentum equation

$$(\rho v)_t + \frac{1}{r^2} (r^2 \rho v^2)_r + P_r = (\lambda + 2\mu) \left(\frac{1}{r^2} (r^2 v)_r \right)_r - \rho \Phi_r = 0, \quad (3.21)$$

with $\Phi_r(\rho) = \frac{4\pi G\rho}{r^2} \int_\epsilon^r \rho(s, \tau) s^2 ds$, we get

$$\begin{aligned} \phi P &= (\lambda + 2\mu) \frac{1}{r^2} (r^2 v)_r \phi - \rho v^2 \phi + \partial_t \int_r^\infty \rho v \phi ds - \int_r^\infty (P + \rho v^2) \phi_s ds \\ &+ \int_r^\infty \frac{2\rho v^2 \phi}{s} ds + (\lambda + 2\mu) \int_r^\infty \frac{1}{s^2} (s^2 v)_s \phi ds + \int_r^\infty \rho \Phi_s \phi ds. \end{aligned} \quad (3.22)$$

Multiplying this equality by ρ^γ and observing after [11] that

$$(\rho^\gamma)_t + (v\rho^\gamma)_r + \frac{2\gamma v\rho^\gamma}{r} = (1 - \gamma)v_r\rho^\gamma,$$

holds in $\mathcal{D}'((0, T) \times \Omega_\epsilon)$ which implies, by multiplying by $\int_r^\infty \rho \phi v ds$, that

$$\begin{aligned} \rho^\gamma \left(\int_r^\infty \rho \phi v ds \right)_t &= \left(\rho^\gamma \int_r^\infty \rho \phi v ds \right)_t + \left(v\rho^\gamma \int_r^\infty \rho \phi v ds \right)_r \\ &+ \left[(\gamma - 1)v_r\rho^\gamma + \frac{2\gamma v\rho^\gamma}{r} \right] \int_r^\infty \rho \phi v ds + v^2\rho^{\gamma+1}\phi, \end{aligned}$$

we get

$$\begin{aligned} \phi P \rho^\gamma &= (\lambda + 2\mu) \frac{1}{r^2} (r^2 v)_r \phi \rho^\gamma + \left(\rho^\gamma \int_r^\infty \rho \phi v ds \right)_t + \left(v\rho^\gamma \int_r^\infty \rho \phi v ds \right)_r \\ &+ \left[(\gamma - 1)v_r\rho^\gamma + \frac{2\gamma v\rho^\gamma}{r} \right] \int_r^\infty \rho \phi v ds - \rho^\gamma \int_r^\infty (P + \rho v^2) \phi_s ds + \rho^\gamma \int_r^\infty \frac{2\rho v^2 \phi}{s} ds \\ &+ (\lambda + 2\mu) \rho^\gamma \int_r^\infty \frac{1}{s^2} (s^2 v)_s \phi_s ds + \rho^\gamma \int_r^\infty \rho \Phi_s \phi ds. \end{aligned}$$

Then multiplying by ϕ^3 and integrating on $(0, T) \times \Omega_\epsilon$, we obtain

$$\begin{aligned} \int_0^T \int_{\Omega_\epsilon} \phi P \rho^\gamma \phi^3 dr dt &= (\lambda + 2\mu) \int_0^T \int_{\Omega_\epsilon} \phi^3 \frac{1}{r^2} (r^2 v)_r \phi \rho^\gamma dr dt \\ &+ \int_0^T \int_{\Omega_\epsilon} \phi^3 \left(\rho^\gamma \int_r^\infty \rho \phi v ds \right)_t dr dt + \int_0^T \int_{\Omega_\epsilon} \phi^3 \left(v\rho^\gamma \int_r^\infty \rho \phi v ds \right)_r dr dt \\ &+ \int_0^T \int_{\Omega_\epsilon} \phi^3 \left[(\gamma - 1)v_r\rho^\gamma + \frac{2\gamma v\rho^\gamma}{r} \right] \int_r^\infty \rho \phi v ds dr dt \\ &- \int_0^T \int_{\Omega_\epsilon} \phi^3 \int_r^\infty \rho v^2 \phi_s ds dr dt + \int_0^T \int_{\Omega_\epsilon} \phi^3 \rho^\gamma \int_r^\infty \frac{2\rho v^2 \phi}{s} ds dr dt \\ &+ (\lambda + 2\mu) \int_0^T \int_{\Omega_\epsilon} \phi^3 \rho^\gamma \int_r^\infty \frac{1}{s^2} (s^2 v)_s \phi_s ds dr dt \\ &- \int_0^T \int_{\Omega_\epsilon} \phi^3 \rho^\gamma \int_r^\infty P \phi_s ds dr dt + \int_0^T \int_{\Omega_\epsilon} \phi^3 \rho^\gamma \int_r^\infty \rho \Phi_s \phi ds dr dt =: \sum_{k=1}^9 J_k. \end{aligned} \quad (3.23)$$

After [11] one knows that for any $\eta > 0$

$$\sum_{k=1}^7 |J_k| \leq \eta \int_0^T \int_{\Omega_\epsilon} \phi \rho^{2\gamma} \phi^4 \, dr \, dt + C_\eta. \quad (3.24)$$

Using (1.2) one gets for a $R > 0$ such that $\text{Supp } \phi \subset [0, R]$

$$|J_8| \leq a \int_0^T \int_{\Omega_\epsilon} \phi^3 \rho^\gamma \int_r^R \rho^\gamma s^2 \, ds \, dr \, dt + b \int_0^T \int_{\Omega_\epsilon} \phi^3 \rho^\gamma \int_r^R \rho s^2 \, ds \, dr \, dt \leq C. \quad (3.25)$$

In the same stroke

$$\begin{aligned} |J_9| &\leq \left| 4\pi G \int_0^T \int_{\Omega_\epsilon} \phi^3 \rho^\gamma \int_r^\infty \rho \phi \frac{1}{s^2} \int_\epsilon^s \rho(\alpha, t) \alpha^2 d\alpha \, ds \, dr \, dt \right| \\ &\leq \frac{\eta}{2} \int_0^T \int_{\Omega_\epsilon} \phi \rho^{2\gamma} \phi^4 \, dr \, dt + \frac{1}{2\eta} \|\rho\|_{\mathcal{L}^1(\Omega)} \int_0^T \int_{\Omega_\epsilon} \phi^2 \int_r^\infty \rho |\phi| \frac{1}{s} \, ds \, dr \, dt \\ &\leq \frac{\eta}{2} \int_0^T \int_{\Omega_\epsilon} \phi \rho^{2\gamma} \phi^4 \, dr \, dt + C_\eta. \end{aligned} \quad (3.26)$$

After (1.2) the left-hand side of (3.23) can be estimated as follows

$$\int_0^T \int_{\Omega_\epsilon} \phi P \rho^\gamma \phi^3 \, dr \, dt \geq \frac{1}{a\gamma} \int_0^T \int_{\Omega_\epsilon} \rho^{2\gamma} \phi^4 \, dr \, dt - b \int_0^T \int_{\Omega_\epsilon} \rho^{\gamma+1} \phi^4 \, dr \, dt.$$

As $x \leq \eta x^\gamma + C_\eta$, we get

$$\int_0^T \int_{\Omega_\epsilon} \phi P \rho^\gamma \phi^3 \, dr \, dt \geq \frac{1}{a\gamma} \int_0^T \int_{\Omega_\epsilon} \rho^{2\gamma} \phi^4 \, dr \, dt - b\eta \int_0^T \int_{\Omega_\epsilon} \rho^{2\gamma} \phi^4 \, dr \, dt - C_\eta.$$

Plugging this last inequality together with (3.25) and (3.26) into (3.23) and choosing η such that $(1 + \frac{b}{2})\eta = \frac{1}{2a\gamma}$ gives (3.19).

2. The proof follows the same scheme as the previous one.

Let $\phi \in C_0^\infty(\Omega_0)$ be a test function such that $\phi \equiv 1$ on $\text{Supp } \psi$. Multiplying (3.22) by $\psi \rho^\theta$ for $\theta > 0$ and observing after [11] that

$$(\rho^\theta)_t + (v\rho^\theta)_r + \frac{2\theta v\rho^\theta}{r} = (1 - \theta)v_r \rho^\theta,$$

holds in $\mathcal{D}'((0, T) \times \Omega_\epsilon)$ for any $\theta > 0$, we get

$$\begin{aligned} \psi P \rho^\theta &= (\lambda + 2\mu) \frac{1}{r^2} (r^2 v)_r \psi \rho^\theta + \left(\rho^\theta \psi \int_r^\infty \rho \phi v \, ds \right)_t + \left(v \rho^\theta \psi \int_r^\infty \rho \phi v \, ds \right)_r \\ &+ \left[(\theta - 1)v_r \rho^\theta \psi + \frac{2\theta v \rho^\theta \psi}{r} \right] \int_r^\infty \rho \phi v \, ds - \rho^\theta \int_r^\infty (P + \rho v^2) \phi_s \, ds + \rho^\theta \psi \int_r^\infty \frac{2\rho v^2 \phi}{s} \, ds \\ &+ (\lambda + 2\mu) \rho^\theta \psi \int_r^\infty \frac{1}{s^2} (s^2 v)_s \phi_s \, ds + \rho^\theta \psi \int_r^\infty \rho \Phi_s \phi \, ds. \end{aligned}$$

Then integrating on $(0, T) \times \Omega_\epsilon$, we obtain

$$\begin{aligned}
 & \int_0^T \int_{\Omega_\epsilon} \psi P \rho^\theta \, dr \, dt = (\lambda + 2\mu) \int_0^T \int_{\Omega_\epsilon} \frac{1}{r^2} (r^2 v)_r \psi \rho^\theta \, dr \, dt \\
 & + \int_0^T \int_{\Omega_\epsilon} \psi \left(\rho^\theta \int_r^\infty \rho \phi v \, ds \right)_t \, dr \, dt + \int_0^T \int_{\Omega_\epsilon} \psi \left(v \rho^\theta \int_r^\infty \rho \phi v \, ds \right)_r \, dr \, dt \\
 & \quad + \int_0^T \int_{\Omega_\epsilon} \psi \left[(\theta - 1)v_r \rho^\theta + \frac{2\theta v \rho^\theta}{r} \right] \int_r^\infty \rho \phi v \, ds \, dr \, dt \\
 & - \int_0^T \int_{\Omega_\epsilon} \psi \int_r^\infty \rho v^2 \phi_s \, ds \, dr \, dt + \int_0^T \int_{\Omega_\epsilon} \psi \rho^\theta \int_r^\infty \frac{2\rho v^2 \phi}{s} \, ds \, dr \, dt \\
 & \quad + (\lambda + 2\mu) \int_0^T \int_{\Omega_\epsilon} \psi \rho^\theta \int_r^\infty \frac{1}{s^2} (s^2 v)_s \phi_s \, ds \, dr \, dt \\
 & - \int_0^T \int_{\Omega_\epsilon} \psi \rho^\theta \int_r^\infty P \phi_s \, ds \, dr \, dt + \int_0^T \int_{\Omega_\epsilon} \psi \rho^\theta \int_r^\infty \rho \Phi_s \phi \, ds \, dr \, dt \\
 & \qquad \qquad \qquad =: \sum_{k=1}^9 I_k. \tag{3.27}
 \end{aligned}$$

After [11] one knows that for any $\eta > 0$

$$\sum_{k=1}^7 |I_k| \leq \eta \int_0^T \int_{\Omega_\epsilon} \psi \rho^{\gamma+\theta} \, dr \, dt + C_\eta. \tag{3.28}$$

Using (1.2) one gets for a $R > 0$ such that $\text{Supp } \phi \subset [0, R]$

$$|I_8| \leq a \int_0^T \int_{\Omega_\epsilon} \psi \rho^\gamma \int_r^R \rho^\gamma s^2 \, ds \, dr \, dt + b \int_0^T \int_{\Omega_\epsilon} \psi \rho^\gamma \int_r^R \rho s^2 \, ds \, dr \, dt \leq C. \tag{3.29}$$

In the same stroke

$$\begin{aligned}
 |I_9| & \leq \left| 4\pi G \int_0^T \int_{\Omega_\epsilon} \psi \rho^\theta \int_r^\infty \rho \phi \frac{1}{s^2} \int_\epsilon^s \rho(\alpha, t) \alpha^2 d\alpha \, ds \, dr \, dt \right| \\
 & \leq \frac{\eta}{2} \int_0^T \int_{\Omega_\epsilon} \phi \rho^{2\gamma} \psi \, dr \, dt + \frac{1}{2\eta} \|\rho\|_{\mathcal{L}^1(\Omega)} \int_0^T \int_{\Omega_\epsilon} \psi \int_r^\infty \rho |\phi| \frac{1}{s} \, ds \, dr \, dt \\
 & \leq \frac{\eta}{2} \int_0^T \int_{\Omega_\epsilon} \psi \rho^{\gamma+\theta} \, dr \, dt + C_\eta. \tag{3.30}
 \end{aligned}$$

After (1.2) the left-hand side of (3.27) can be estimated as previously

$$\int_0^T \int_{\Omega_\epsilon} \psi P \rho^\theta \, dr \, dt \geq \frac{1}{a\gamma} \int_0^T \int_{\Omega_\epsilon} \psi \rho^{\gamma+\theta} \, dr \, dt - b \int_0^T \int_{\Omega_\epsilon} \psi \rho^{\theta+1} \, dr \, dt.$$

As $x \leq \eta x^\gamma + C_\eta$, we get

$$\int_0^T \int_{\Omega_\epsilon} \psi P \rho^\theta \, dr \, dt \geq \frac{1}{a\gamma} \int_0^T \int_{\Omega_\epsilon} \psi \rho^{\gamma+\theta} \, dr \, dt - b\eta \int_0^T \int_{\Omega_\epsilon} \psi \rho^{\gamma+\theta} \, dr \, dt - C_\eta.$$

Plugging this last inequality together with (3.29) and (3.30) into (3.27) and choosing η such that $(1 + \frac{b}{2})\eta = \frac{1}{2a\gamma}$ gives (3.20). \square

3.2. Proof of Theorem 3.1

As in Section 2, we consider the extension to Ω the pair $(\rho_\epsilon, v_\epsilon)$ defined as

$$\tilde{\rho}_\epsilon(r, t) = \begin{cases} \rho_\epsilon(r, t) & \text{for } r \in [\epsilon, \infty), \\ 0 & \text{else,} \end{cases} \quad (3.31)$$

and

$$\tilde{v}_\epsilon(r, t) = \begin{cases} v_\epsilon(r, t) & \text{for } r \in [\epsilon, \infty), \\ 0 & \text{else,} \end{cases} \quad (3.32)$$

and we denote by $(\rho_\epsilon, v_\epsilon)$ this extension, by simplicity. After the energy inequality, there exist extracted sequences such that

$$\rho_\epsilon \rightharpoonup \rho \text{ weak} - * \text{ in } L^\infty(0, T; \mathcal{L}_{loc}^\gamma(\Omega)),$$

$$\rho_\epsilon \rightharpoonup \rho \text{ weak in } L^{2\gamma}(0, T; \mathcal{L}_{loc}^{2\gamma}(\Omega)),$$

$$v_\epsilon \rightharpoonup v \text{ weak} - * \text{ in } L^2(0, T; \mathcal{H}_{loc}^1(\Omega)),$$

moreover after energy inequality

$$\rho \in L^\infty(0, T; \mathcal{L}^\gamma(\Omega)) \cap L^{2\gamma}(0, T; \mathcal{L}_{loc}^{2\gamma}(\Omega)),$$

$$v, v_r \in L^2(0, T; \mathcal{L}^2(\Omega)).$$

In order to complete the proof of Theorem 3.1, we have finally to prove that the couple (ρ, v) satisfies the system (3.1).

Using Lemma 3.3 and proceeding exactly as Jiang and Zhang in [11] one proves the following properties of the weak limits

1. For any $0 < \theta < \gamma$, one has

$$\overline{\rho^\theta P(\rho)} - \overline{\rho^\theta} \overline{P(\rho)} + (3\lambda + 2\mu)\overline{\rho^\theta} v_r = w - \lim_{\epsilon \rightarrow 0} \left\{ (3\lambda + 2\mu)\overline{\rho_\epsilon^\theta} (v_\epsilon)_r \right\}, \quad (3.33)$$

2. For any $0 < \theta < 1$ such that $\frac{1}{2} (1 - \theta + \sqrt{1 + 6\theta + \theta^2}) \leq \gamma$

$$\rho^\theta - \overline{\rho^\theta} \in L^{\frac{2}{\theta}}(0, T; \mathcal{L}^{\frac{2}{\theta}}(0, 1)), \quad (3.34)$$

for any $0 < \theta < \gamma$.

3. The limits ρ and v satisfy the energy inequality

$$\int_\Omega \left(\frac{1}{2} \rho v^2 + \frac{1}{2} \Pi(\rho) \right) r^2 dr + \frac{3\lambda + 2\mu}{2} \int_0^t \int_\Omega [r^2 (v_r^2 + v^2)] dr d\tau \leq C, \quad (3.35)$$

moreover

$$\rho, \left(\overline{\rho^\theta} \right)^{\frac{1}{\theta}} \in L^\infty(0, T; \mathcal{L}^1(\Omega)). \quad (3.36)$$

In particular, passing to the limit into (3.12), we find that (ρ, v) satisfies in $\mathcal{D}'((0, T) \times \Omega)$ the system

$$\begin{cases} \rho_t + (\rho v)_r + \frac{2\rho v}{r} = 0, \\ (\rho v)_t + [\rho v^2 + \bar{P}]_r + \frac{2\rho v^2}{r} = (\lambda + 2\mu) \left(\left(v_r + \frac{2v}{r} \right) \right)_r - \rho \bar{\Phi}_r. \end{cases} \quad (3.37)$$

As property (3.34) excludes the concentration of mass at the origin, the end of the proof is verbatim the same as that of Theorem 1.1 in [11].

Acknowledgment *Š. N. was supported by the Grant Agency of the Czech Republic n. 201/08/0012 and by the Academy of Sciences of the Czech Republic, Institutional Research Plan N. AV0Z10190503. The last part of paper was done during her stay in Cea, Arpajon. She would like to thank for hospitality of Prof. Ducomet and his colleagues during her stay there. A. V. was partially supported by Nečas Center for Mathematical Modelling LC06052 financed by MŠMT.*

References

- [1] D. Bresch, B. Desjardins. Some diffusive capillary models of Korteweg type. *C. R. Acad. Sci. Paris, Section Mécanique* 2004; **332**:881–886.
- [2] D. Bresch, B. Desjardins, D. Gérard-Varet. On compressible Navier-Stokes equations with density dependent viscosities in bounded domains. *J. Math. Pures Appl.* 2007; **87**:227–235.
- [3] S. Chapman, T.G. Cowling. *The mathematical theory of non-uniform gases*, Cambridge University Press, 1995.
- [4] B. Ducomet, E. Feireisl, A. Petzeltová, I. Straškraba. Global in time weak solutions for compressible barotropic self-gravitating fluid, *Discrete and Continuous Dynamical Systems* 2004; **11**:113–130.
- [5] B. Ducomet, Š. Nečasová, A. Vasseur. On global motions of a compressible barotropic and self-gravitating gas with density-dependent viscosities, *Submitted*.
- [6] B. Ducomet, A. Zlotnik, Viscous compressible barotropic symmetric flows with free boundary under general mass force, Part I: Uniform-in-time bounds and stabilization, *Mathematical Methods in the Applied Sciences* 28 (2005) 827–863.
- [7] E. Feireisl. Compressible Navier-Stokes Equations with a Non-Monotone Pressure Law, *Journal of Diff. Equ.* 2002; **184**:97–108.
- [8] E. Feireisl. Viscous and/or heat conducting compressible fluids, in *“Handbook of mathematical fluid dynamics, Vol 1, S. Friedlander and D. Serre Ed., North-Holland, 2002.*
- [9] Z. Guo, Q. Jiu, Z. Xin. Spherically symmetric isentropic compressible flows with density-dependent viscosity coefficients. *SIAM J. Math. Anal.* 2008; **39**:1402–1427.
- [10] D. Hoff. Spherically symmetric solutions of the Navier-Stokes equations for compressible, isothermal flow with large discontinuous initial data. *Indiana Univ. Math. J.* 1992; **41**:1225–1302.

- [11] S. Jiang, P. Zhang. On spherically symmetric solutions of the compressible isentropic Navier-Stokes equations. *Comm. Math. Phys.* 2001; **215**:559–581
- [12] P.L. Lions. *Mathematical topics in fluid mechanics, Vol 2: Compressible models*, Oxford University Press, 1998.
- [13] A. Mellet, A. Vasseur. On the barotropic compressible Navier-Stokes equations. *Comm. Partial Differential Equations* 2007; **32**:431–452.
- [14] A. Mellet, A. Vasseur. Existence and uniqueness of global strong solutions for one-dimensional compressible Navier-Stokes equations. *SIAM J. Math. Anal.* 2008; **39**:1344–1365.
- [15] A. Novotný, I. Straškraba. *Introduction to the mathematical theory of compressible flow*, Oxford University Press, 2004.
- [16] E. Stein. *Singular integrals and differentiability properties of functions*, Princeton University Press, 1970.
- [17] T. Zhang, D. Fang. Global behavior of spherically symmetric Navier-Stokes equations with density-dependent viscosity. *J. Differential Equations* 2007; **236**:293–341.
- [18] T. Zhang, D. Fang. A note on spherically symmetric isentropic compressible flows with density-dependent viscosity coefficients. *Nonlinear Analysis: Real World Applications* (in press)

Bernard Ducomet
CEA, DAM, DIF
F-91297 Arpajon, France
e-mail: bernard.ducomet@cea.fr

Šárka Nečasová
Mathematical Institute AS ČR
Žitna 25,
115 67 Praha 1,
Czech Republic
e-mail: matus@math.cas.cz

Alexis Vasseur
Department of Mathematics,
University of Texas at Austin
University Station C1200
TX, 78712-0257
USA
e-mail: vasseur@math.utexas.edu