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EXISTENCE AND PROPERTIES OF SEMIDISCRETE SHOCK PROFILES FOR THE ISENTROPIC GAS DYNAMIC SYSTEM WITH $\gamma = 3$*

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Abstract. We consider a semidiscrete kinetic scheme for the isentropic gas dynamic system with $\gamma = 3$ which corresponds exactly to the "transport-collapse method" of Brenier. We show the existence of shock profiles for every admissible limit values. In addition we give properties of such profiles.

Key words. conservation law, kinetic formulation, shock profile, semidiscrete scheme

AMS subject classifications. 65M06, 65M12, 35L65

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1. Introduction. We consider in this paper the isentropic gas dynamic system with $\gamma = 3$

$$\partial_t \rho + \partial_x P = 0, \tag{1.1}$$

$$\partial_t P + \partial_x \left(\frac{P^2}{\rho} + \frac{\rho^3}{12} \right) = 0, \tag{1.2}$$

where ρ is the density and P the momentum of the gas. We introduce an extra variable $v \in [-D, D]$ and call microscopic every nonnegative function $f(t, x, v) \geq 0$. We say that $f(t, x, v) \in L^\infty(\mathbb{R}^+ \times \mathbb{R} \times] - D, D[)$ is an equilibrium function if there exists $a(t, x)$ and $b(t, x)$ such that $f(t, x, v) = \mathbf{1}_{\{a \leq v \leq b\}}$. Then we can reformulate the system (1.1), (1.2) as

$$\partial_t \int_{-D}^{D} v^j f(t, x, v) dv + \partial_x \int_{-D}^{D} v^{j+1} f(t, x, v) dv = 0 \quad j = 0, 1 \tag{1.3}$$

with the constraint that f is an equilibrium function. Indeed, if we denote $\rho = b - a$ and $P = \frac{b^2 - a^2}{2}$, we find $\frac{b^3 - a^3}{3} = \frac{P^2}{\rho} + \frac{\rho^3}{12}$. In fact, Lions, Perthame, and Tadmor in [4] have shown that we can give a rigorous kinetic formulation of (1.1), (1.2) by this way. Therefore, we can define a natural semidiscrete kinetic scheme in the following way. First we fix a time step Δt. For $t \in](n-1)\Delta t, n\Delta t[$ $f_{\Delta t}$ just evolves by free streaming, namely,

$$\partial_t f_{\Delta t} + v \partial_x f_{\Delta t} = 0.$$

At time $n\Delta t$, we substitute for the values of f the values of the equilibrium function $\mathbf{1}_{\{a(x) \leq v \leq b(x)\}}$ defined such that

$$b(x) - a(x) = \int_{-D}^{D} f(n\Delta t, x, v) dv,$$

$$\frac{b(x)^2 - a(x)^2}{2} = \int_{-D}^{D} v f(n\Delta t, x, v) dv.$$

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We can notice that $f_{\Delta t}$ is a solution of (1.3). Therefore, if at the limit $\Delta t \to 0$, f is an equilibrium function, then ($\rho = \int f dv, P = \int v f dv$) is a solution of (1.1), (1.2). This has been proved in [8]. Notice that this scheme coincides exactly with the "transport-collapse" method proposed by Y. Brenier for systems [2]. The aim of this article is to investigate how faithful this scheme is in the presence of a discontinuous solution of a system. Especially the CFL number ($\Delta t/\Delta x$) of this scheme is infinite ($\Delta x = 0$ since there is no discretization in space). Since a large CFL number is well known to induce oscillations, it is a great deal to describe the perturbations induced on solutions of (1.1), (1.2). A classical way is to study the existence and properties of nontrivial shock profiles (see [7]). We restrict our study to the system (1.1), (1.2) since the scheme is simpler to handle than the more physical ones with $\gamma < 3$. Despite its common features with scalar conservation laws, it leads to new difficulties. For instance, the uniqueness is still unknown and the time regularity of the solutions has been proven only recently (see [9]). Let us emphasize that we consider here shock profiles for a time-discrete scheme with a continuous space variable. Thus our concept of shock profiles differs from all the recent works on discrete shocks [6], [5]. The existence of time-discrete shock profiles for convex scalar conservation laws has been proved by Brenier in [1]. Notice that Golse, also for scalar conservation laws, has studied a similar problem for the so-called BGK approximation in [3]. Let us introduce the rescaled time iteration operator T defined in the following way. To $f(x, v)$ we associate the unique equilibrium function $\mathbf{1}_{\{a(x)\leq v\leq b(x)\}}$ such that

$$b(x) - a(x) = \int_{-D}^{D} f(x - v, v)dv,$$

$$\frac{b(x)^2 - a(x)^2}{2} = \int_{-D}^{D} v f(x - v, v)dv.$$

We say that $f(x, v)$ is a stationary shock profile if $T(f) = f$. If f does not depend on x ($f = \mathbf{1}_{\{a\leq v\leq b\}}$ with a, b constant), we say that f is a trivial shock profile. Let us consider a stationary shock of system (1.1), (1.2). We denote (ρ_1, P_1) the left-hand state, (ρ_2, P_2) the right-hand state, and $a_i = \frac{P_i}{\rho_i} - \frac{\rho_i}{2}$, $b_i = \frac{P_i}{\rho_i} + \frac{\rho_i}{2}$ the two Riemann invariants for $i = 1, 2$. Then the Rankine–Hugoniot (R.–H.) conditions ensure that

$$\frac{b_1^2 - a_1^2}{2} = \frac{b_2^2 - a_2^2}{2}, \tag{1.4}$$

$$\frac{b_1^3 - a_1^3}{3} = \frac{b_2^3 - a_2^3}{3}. \tag{1.5}$$

We will say that (a_1, b_1, a_2, b_2) is R.–H. if (1.4) and (1.5) are satisfied.

In section 2 we give some properties of the kinetic scheme. Then in section 4 we show the following consistency theorem.

THEOREM 1.1. *Let $f(x, v) = \mathbf{1}_{\{a(x)\leq v\leq b(x)\}} \in L^\infty(\mathbb{R}\times]-D, D[)$ be a nontrivial stationary shock profile. The function $\bar{f}$ defined by $\bar{f}(x, v) = f(-x, -v)$ is also a nontrivial stationary shock profile. Then, replacing f with $\bar{f}$ if necessary, there exists a_1, b_1, a_2, $b_2 \in [-D, D]$ and $x_0 \in \mathbb{R}$ such that (a_1, b_1, a_2, b_2) is R.–H. $(a_1, b_1) \neq (a_2, b_2)$ and $a_1 > 0$, $f(x, v) = \mathbf{1}_{\{a_1\leq v\leq b_1\}}$ on $]-\infty, x_0[\times]-D, D[$, and $a(x) \to a_2$, $b(x) \to b_2$ when x goes to $+\infty$, in the sense that*

$$\lim_{X\to+\infty} \left(\|a(x) - a_2\|_{L^\infty(]X,+\infty[)} + \|b(x) - b_2\|_{L^\infty(]X,+\infty[)}\right) = 0.$$

This will be useful to show in section 5 the following existence theorem.

THEOREM 1.2. *For every $a_1, b_1, a_2, b_2 \in]-D, D[$ such that (a_1, b_1, a_2, b_2) is R.–H. with $(a_1, b_1) \neq (a_2, b_2)$, permuting (a_1, b_1) with (a_2, b_2) if necessary, there exists a nontrivial stationary shock profile denoted $f(x, v) = \mathbf{1}_{\{a(x) \leq v \leq b(x)\}}$ such that $(a(x), b(x))$ converges to (a_1, b_1) for $x \to -\infty$ and to (a_2, b_2) for $x \to +\infty$.*

Notice that if (a_1, b_1, a_2, b_2) is R.–H. with $(a_1, b_1) \neq (a_2, b_2)$, then there is only one admissible choice between (a_1, b_1) and (a_2, b_2) for the limit at $-\infty$. Indeed, the other case does not provide a shock but a rarefaction. One of the main tools is a Liouville type argument (section 3, Lemmas 3.2 and 3.3).

2. Properties of the kinetic scheme. In this section we show some basic properties of the scheme. First, Lemma 2.1 is a property of compactness, then Lemma 2.2 is a property on the structure of equilibrium functions. Finally Lemma 2.3 is a kind of maximum principle.

LEMMA 2.1. *For all equilibrium functions f, let us denote $\bar{\rho}(x) = \int T(f)(x, v)dv$ and $\bar{P}(x) = \int vT(f)(x, v)dv$. Then $\bar{\rho}$ and $\bar{P}$ belong to a bounded subset of $BV_{\text{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Therefore, we can extract from every sequence of equilibrium functions f_n a subsequence f_p such that the two first moments of $T(f_p)$ converge strongly in $L^1_{\text{loc}}(\mathbb{R})$.*

Proof. Let us first notice that $0 \leq \bar{\rho}(x) \leq 2D$ and $|\bar{P}(x)| \leq D^2$; thus $\bar{\rho}$ and $\bar{P}$ are bounded in $L^\infty(\mathbb{R})$. Next for every $R > 0$ and every $h > 0$ we have

$$\bar{\rho}(x+h) - \bar{\rho}(x) = \int_{\mathbb{R}} [f((x+h) - v, v) - f((x+h) - v, v - h)]\, dv.$$

Therefore, if we integrate with respect to x, we find

$$\int_{-R}^{R} |\bar{\rho}(x+h) - \bar{\rho}(x)| dx \leq \int_{\mathbb{R}} \int_{-R-D}^{R+D} |f(x, v) - f(x, v-h)| dx dv.$$

Since f is an equilibrium function, however,

$$\int_{\mathbb{R}} |f(x, v) - f(x, v-h)| dv \leq 2h;$$

therefore

$$\int_{-R}^{R} |\bar{\rho}(x+h) - \bar{\rho}(x)| dx \leq 4h(R + D).$$

In the same way,

$$\bar{P}(x+h) - \bar{P}(x) = h\bar{\rho}(x) + \int_{\mathbb{R}} v\,[f(x - v + h, v) - f(x - v + h, v - h)]\, dv.$$

This leads to

$$\int_{-R}^{R} |\bar{P}(x+h) - \bar{P}(x)| dx \leq 6D(D + R)h.$$

Thus $(\bar{\rho}, \bar{P})$ belong to a bounded subset of $BV_{\text{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ which is compact in $L^1_{\text{loc}}(\mathbb{R})$. ∎

LEMMA 2.2. *Let $f_n \in L^\infty(\mathbb{R}\times]-D,D[)$ be a sequence of equilibrium functions such that their two first moments (ρ_n, P_n) defined by*

$$\rho_n(x) = \int_{-D}^{D} f_n(x,v)dv,$$
$$P_n(x) = \int_{-D}^{D} v f_n(x,v)dv$$

converge strongly in $L^1_{\text{loc}}(\mathbb{R})$ to (ρ, P). Then f_n converges strongly in $L^1_{\text{loc}}(\mathbb{R}\times]-D,D[)$ to an equilibrium function whose two first moments are (ρ, P).

Notice that the crucial point is that the limit of f_n is still an equilibrium function.

Proof. First, there exists a subsequence such that $(\rho_p, P_p)(x)$ converges for almost every $x \in \mathbb{R}$ to $(\rho, P)(x)$. If $\rho(x) \neq 0$, then, for p large enough $f_p(x,v) = \mathbf{1}_{\{a_p(x)\le v\le b_p(x)\}}$ with

$$a_p(x) = \frac{P_p(x)}{\rho_p(x)} - \frac{\rho_p(x)}{2},$$

$$b_p(x) = \frac{P_p(x)}{\rho_p(x)} + \frac{\rho_p(x)}{2}.$$

Therefore $f_p(x,\cdot)$ converges strongly in $L^1(]-D,D[)$ to $\mathbf{1}_{\{a(x)\le v\le b(x)\}}$ with

$$a(x) = \frac{P(x)}{\rho(x)} - \frac{\rho(x)}{2},$$

$$b(x) = \frac{P(x)}{\rho(x)} + \frac{\rho(x)}{2}.$$

Next, if $\rho(x) = 0$, then $f_p(x,\cdot)$ converges strongly in $L^1(]-D,D[)$ to 0. Therefore, using Lebesgue theorem, f_p converges strongly in $L^1_{\text{loc}}(\mathbb{R}\times]-D,D[)$ to the equilibrium function associated with (ρ, P). Finally, since the limit is unique, the entire sequence f_n converges to the equilibrium function associated with (ρ, P). ∎

LEMMA 2.3. *Let $f \in L^\infty(\mathbb{R}\times]-D,D[)$ be an equilibrium function, and let $V \in]-D,D[$. For almost every $x \in \mathbb{R}$ such that $f(x-\cdot,\cdot) = 0$ on $]-D,V[$, we also have $T(f)(x,\cdot) = 0$ on $]-D,V[$. In the same way if $f(x-\cdot,\cdot) = 0$ on $]V,D[$, then $T(f)(x,\cdot) = 0$ also on $]V,D[$.*

Proof. Let us show only the first assertion since the proof is similar for the second. Let us denote

$$\bar\rho(x) = \int_{-D}^{D} T(f)(x,v)dv = \int_{-D}^{D} f(x-v,v)dv.$$

For $v \in]-D,V[$ we have $f(x-v,v) = 0$. Therefore, remembering that f is valued in $\{0,1\}$,

$$\bar P(x) = \int_{-D}^{D} vT(f)(x,v)dv = \int_{-D}^{D} v f(x-v,v)dv$$
$$\ge \int_{V}^{V+\bar\rho(x)} v dv = \frac{\bar\rho(x)^2}{2} + V\bar\rho(x).$$

Since $T(f)$ is an equilibrium function $T(f)(x,v) = \mathbf{1}_{\{a(x)\le v\le b(x)\}}$ with

$$a(x) = \frac{\bar{P}(x)}{\bar{\rho}(x)} - \frac{\bar{\rho}(x)}{2}.$$

Thus $a(x) \ge V$ and therefore, $T(f)(x,\cdot) = 0$ on $]-D,V[$. ∎

3. Properties of shock profiles. In this section, we define a notion of asymptotic of shock profile. Then we will show a "Liouville type" result concerning those asymptotic functions (Lemma 3.3).

DEFINITION 3.1. *Let f be a stationary shock profile. For $X > 0$ we define $f_X(x,v) = f(x+X,v)$. Then we call asymptotic at $+\infty$ every function $g \in L^\infty(\mathbb{R}\times]-D,D[)$ such that there exists a sequence X_n which converges to $+\infty$ and*

$$f_{X_n} \overset{L^1_{\mathrm{loc}}(\mathbb{R}\times]-D,D[)}{\longrightarrow} g.$$

We define a notion of asymptotic at $-\infty$ in the same manner.

In order to show that every shock profile has an asymptotic at $+\infty$ and $-\infty$ which are equilibrium functions and do not depend on x, let us show the following lemma.

LEMMA 3.2. *Let f be a stationary shock profile; then for almost every $x \in \mathbb{R}$*

$$\int_{-D}^{D} v^2[f(x-v,v) - f(x,v)]dv \ge 0 \tag{3.1}$$

with equality only if $f(x-\cdot,\cdot) = f(x,\cdot)$ almost everywhere in v. Moreover, there exists $C > 0$ such that

$$\int_{-\infty}^{+\infty}\int_{-D}^{D} v^2[f(x-v,v) - f(x,v)]dvdx < C. \tag{3.2}$$

Notice that $\int_{-D}^{D} v^2 f dv = \frac{P^2}{\rho} + \frac{\rho^3}{12}$ which is an entropy for the system (1.1), (1.2). The first statement is due to the fact that the scheme is entropic.

Proof. Since $T(f) = f$, we have

$$\int_{-D}^{D} f(x-v,v)dv = \int_{-D}^{D} f(x,v)dv = \rho(x), \tag{3.3}$$

$$\int_{-D}^{D} vf(x-v,v) = \int_{-D}^{D} vf(x,v)dv = P(x). \tag{3.4}$$

If $\rho(x) = 0$ the first statement is clear, or else let us define $u(x) = \frac{P(x)}{\rho(x)}$. Then, using (3.3), (3.4) and remembering that f is an equilibrium function,

$$\int_{-D}^{D} v^2[f(x-v,v) - f(x,v)]dv \tag{3.5}$$

$$= \int_{-D}^{D} (v-u(x))^2 \left[f(x-v,v) - \mathbf{1}_{\{u(x)-\frac{\rho(x)}{2}\le v\le u(x)+\frac{\rho(x)}{2}\}}\right] dv. \tag{3.6}$$

Finally, since f is valued in $\{0,1\}$ and using (3.3), Lebesgue's measure of $\{v \in]u(x)-\rho(x)/2, u(x)+\rho(x)/2[/ f(x-v,v) = 0\}$ is equal to Lebesgue's measure of $\{v \notin]u(x)-\rho(x)/2, u(x)+\rho(x)/2[/ f(x-v,v) = 1\}$; therefore

$$\int_{-D}^{D} v^2[f(x-v,v) - f(x,v)]dv \ge 0$$

with equality only if $f(x-\cdot,\cdot) = f(x,\cdot)$ almost everywhere in v. Finally, for every $X > 0$, doing the change of variable $y = x - v$ for the first term of the integral,

$$\begin{aligned}&\int_{-X}^{X}\int_{-D}^{D} v^2[f(x-v,v) - f(x,v)]dvdx\\ &= -\int_{C(X)} v^2 \mathrm{sgn}(v) f(x,v)dvdx\\ &\quad + \int_{C(-X)} v^2 \mathrm{sgn}(v) f(x,v)dvdx\\ &\leq 2D^4,\end{aligned}$$

where $C(X) = \{(x,v)/|v| \leq D \text{ and } \mathrm{sgn}(x-X)(x-X+v) \leq 0\}$. ∎

LEMMA 3.3. *Let g be a stationary shock profile such that*

$$\int_{\mathbb{R}}\int_{-D}^{D} v^2[g(x-v,v) - g(x,v)]dvdx = 0;$$

then g does not depend on x.

Proof. By Lemma 3.2 we know that for almost every $(x,v) \in \mathbb{R}\times]-D,D[$

$$g(x,v) = g(x-v,v). \tag{3.7}$$

The two functions $g_1(x,v) = \mathbf{1}_{\{v\geq 0\}} g(x,v)$ and $g_2(x,v) = \mathbf{1}_{\{v\geq 0\}} g(-x,-v)$ are equilibrium functions and verify (3.7). Since $g(x,v) = g_1(x,v) + g_2(-x,-v)$, if g_1 and g_2 do not depend on x, then it is the same for g. Therefore it is sufficient to consider function $g(x,v) = \mathbf{1}_{\{a(x)\leq v\leq b(x)\}}$ such that $a(x) \geq 0$.

Let us denote $\Omega = \{(x,v)\ /\ g(x,v) = 1\}$. According to Lemma 2.1, $\rho = \int g dv$ and $P = \int v g dv$ are BV_{loc} functions. Therefore, for all x_0 such that $\rho(x_0 - 0) > 0$ $a(x)$ and $b(x)$ have a left-hand side limit, and $\{x_0\}\times[a,b] \subset \Omega$ where $a = a(x_0-0)$ and $b = b(x_0 - 0)$. (Indeed we can define pointwise Ω as the union of the two sets $\{(x,v)\ /\ \rho(x\pm 0) > 0,\ a(x\pm 0) \leq v \leq b(x\pm 0)\}$ which is compatible with (3.7).) We denote

$$K(x_0) = \{(x,v)\ /\ 0 \leq v \leq D \text{ and } 0 \leq x - x_0 \leq v\}.$$

Notice that the knowledge of g on $K(x_0)$ gives the values of g everywhere. Moreover, for every $n \in \mathbb{N}$, segments $I_n = [(x_0 + na, a), (x_0 + nb, b)]$ are included in Ω. Since Ω is convex in v (g is an equilibrium function), this ensures that the set K_n of (x,v) such that $x \in [x_0 + (n+1)a, x_0 + nb]$ and which are between the two segment I_n and I_{n+1}, is included in Ω. Using the property that $g(x,v) = g(x - nv, v)$, we find that the quadrilateral K'_n whose vertices are $(x_0 + a, a)$, $(x_0, a(n+1)/n)$, (x_0, b), and $(x_0 + bn/(n+1), bn/(n+1))$ is included in Ω. Therefore the Lebesgue measure of the set of (x,v) in $K(x_0)\cap(\mathbb{R}\times[a,b])$ such that $f(x,v) = 0$ is lesser or equal to

$$\frac{a^2}{2n} + \frac{b^2}{2(n+1)}.$$

When n goes to $+\infty$ we find that $K(x_0)\cap(\mathbb{R}\times[a,b])$ is in Ω, and finally,

$$\mathbb{R}\times[a,b] \subset \Omega.$$

Thus Ω contains $\mathbb{R} \times [a(x_0 - 0), b(x_0 - 0)]$ for all x_0 such that $\rho(x_0 - 0) > 0$. Then there exists a set $I \subset]-D, D[$ such that $\Omega = \mathbb{R} \times I$. Finally, since Ω is convex with respect to v, Ω must be a strip $\mathbb{R} \times [a, b]$. $\square$

Now, for every sequence X_n which converges to $+\infty$ or $-\infty$, let us consider the sequence (f_{X_n}). Of course those states are still fixed points of T, so, according to Lemmas 2.1 and 2.2 there exists a subsequence still denoted (X_n) such that f_{X_n} converges strongly in $L^1_{\text{loc}}(\mathbb{R}\times]-D, D[)$ to a stationary shock profile g. For every $R > 0$, the function

$$\mathbf{1}_{\{-R \le x \le R\}} \int_{-D}^{D} v^2 [f_{X_n}(x - v, v) - f_{X_n}(x, v)] dv$$
$$= \mathbf{1}_{\{X_n - R \le x \le X_n + R\}} \int_{-D}^{D} v^2 [f(x - v, v) - f(x, v)] dv$$

converges to 0 for almost every $x \in \mathbb{R}$. Using Lemma 3.2 and Lebesgue theorem, we find

$$\int_{-R}^{R} \int_{-D}^{D} v^2 [g(x - v, v) - g(x, v)] dv dx = 0$$

and then

$$\int_{\mathbb{R}} \int_{-D}^{D} v^2 [g(x - v, v) - g(x, v)] dv dx = 0.$$

Therefore, thanks to Lemma 3.3, the following proposition follows.

PROPOSITION 3.4. *Let f be a stationary shock profile. Then for every sequence X_n which converges to $+\infty$ or $-\infty$, there exists a subsequence and an associated asymptotic function which is an equilibrium function and constant with respect to x.*

Remark. In particular there exist asymptotic functions both at $+\infty$ and $-\infty$, and every asymptotic function is an equilibrium function and constant with respect to x.

4. Other properties of shock profiles. This section is devoted to the proof of Theorem 1.1. Notice that those properties will be useful in the next section in order to enforce that the solution we construct is a stationary shock profile which is not trivial. First we show an easy result on the condition R.–H. Then we see some regularizing effect of T. Finally we prove Theorem 1.1.

LEMMA 4.1. *Let $P \in \mathbb{R}$ and $Q \in \mathbb{R}^+$; then there exist at most two values of (a, b) such that $a < b$ and*

$$\frac{b^2 - a^2}{2} = P, \tag{4.1}$$

$$\frac{b^3 - a^3}{3} = Q. \tag{4.2}$$

Moreover, if there are two distinct values, then one is such that $ab > 0$ and the other such that $ab < 0$.

Proof. Let us denote $\rho = b - a > 0$. Then (a, b) verifies (4.1) and (4.2) if and only if $Q = P^2/\rho + \rho^3/12$, or

$$G(\rho) = \frac{\rho^4}{12} - \rho Q + P^2 = 0.$$

But G decreases for $\rho \leq (3Q)^{\frac{1}{3}}$ and increases for $\rho \geq (3Q)^{\frac{1}{3}}$. Thus there are at most two values of ρ such that $G(\rho) = 0$. If there are two distinct values, then one is smaller than $(3Q)^{\frac{1}{3}}$ and the other one is bigger. Since

$$a = P/\rho - \rho/2,$$
$$b = P/\rho + \rho/2,$$

using that $Q = P^2/\rho + \rho^3/12$, we find that those cases correspond, respectively, to $ab > 0$ and $ab < 0$. ∎

We consider in the following some properties in bounded domains $]-R, R[$ which will be useful in the next section. Notice that to define $T(f)$ on $]-R, R[$ we need to define f on $]-R-D, R+D[$.

LEMMA 4.2. *Let $R \in]0, +\infty]$, and $f \in L^\infty(]-R-D, R+D[\times]-D, D[)$ be an equilibrium function such that $T(f) = f$ on $]-R, R[\times]-D, D[$. Let $x_0 \in]-R, R[$ and $V \geq 0$. If $f(x, v) = 0$ on $]x_0 - D, x_0[\times]V, D[$, then $f(x, v) = 0$ on $]x_0 - D, R[\times]V, D[$.*

In the same way, if $f(x, v) = 0$ on $]x_0, x_0 + D[\times]-D, -V[$, then $f(x, v) = 0$ on $]-R, x_0 + D[\times]-D, -V[$.

Proof. We show only the first assertion since the second one can be proved in the same way. Let us first assume that $V > 0$. We show by induction that for every $n \in \mathbb{N}$ $f(x, v) = 0$ on $(]x_0 - D, x_0 + nV[\cap]-R-D, R[)\times]V, D[$. For $n = 0$, this is exactly the hypothesis of the lemma. If the property is verified at step n, then for almost every $(x, v) \in (]x_0, x_0 + (n+1)V[\cap]-R, R[)\times]V, D[$, we have $(x - v, v) \in (]x_0 - D, x_0 + nV[\cap]-R-D, R[)\times]V, D[$, so $f(x - v, v) = 0$. Then using Lemma 2.3, we find that $f(x, v) = 0$ on $(]x_0, x_0 + (n+1)V[\cap]-R, R[)\times]V, D[$. The case $V = 0$ can be proven by considering a sequence $V_n \to 0$ and passing to the limit. ∎

The next lemma is very simple. Two corollaries will follow in order to show its interest.

LEMMA 4.3. *Let X_1, $X_2 \in \mathbb{R}$ be such that $X_1 < X_2$, and let $f \in L^\infty(]X_1 - D, X_2 + D[\times]-D, D[)$ be an equilibrium function such that $T(f) = f$ on $]X_1, X_2[\times]-D, D[$. Then*

$$\int_{C(X_1)} \operatorname{sgn}(v) f(x, v)\, dv dx = \int_{C(X_2)} \operatorname{sgn}(v) f(x, v)\, dv dx, \tag{4.3}$$

$$\int_{C(X_1)} |v| f(x, v)\, dv dx = \int_{C(X_2)} |v| f(x, v)\, dv dx, \tag{4.4}$$

where

$$C(X) = \{(x, v)/|v| \leq D \text{ and } \operatorname{sgn}(x - X)(x - X + v) \leq 0\}. \tag{4.5}$$

Proof. Since $T(f) = f$ on $]X_1, X_2[\times]-D, D[$,

$$\int_{-D}^{D} \int_{X_1}^{X_2} f(x, v)\, dv dx = \int_{-D}^{D} \int_{X_1}^{X_2} f(x - v, v)\, dv dx,$$
$$\int_{-D}^{D} \int_{X_1}^{X_2} v f(x, v)\, dv dx = \int_{-D}^{D} \int_{X_1}^{X_2} v f(x - v, v)\, dv dx.$$

Making the change of variable $y = x - v$ in the right-hand side term of the equalities leads to the desired results. ∎

Corollary 4.4. *Let $a, b \in \mathbb{R}$ with $-D < a \leq b < D$ and $f \in L^\infty(\mathbb{R}\times]-D,D[)$ be a stationary shock profile such that $f(x,v) = 0$ on $\mathbb{R} \times (]-D,a[\cup]b,D[)$. Assume that*

$$\int_{C(0)} |v| f(x,v)dvdx = \int_{C(0)} |v| \mathbf{1}_{\{a\leq v\leq b\}} dvdx = \frac{b^3 - a^3}{3};$$

then $f(x,v) = \mathbf{1}_{\{a\leq v\leq b\}}$ on $\mathbb{R}\times]-D,D[$.

Proof. For every $x \in \mathbb{R}$, using Lemma 4.3 (with $X_1 = 0$, $X_2 = x$ if $x > 0$ and $X_1 = x$, $X_2 = 0$ if $x < 0$), we find

$$\int_{C(x)} |v| f(y,v)dvdy = \int_{C(0)} |v| \mathbf{1}_{\{a\leq v\leq b\}} dvdy$$
$$= \int_{C(x)} |v| \mathbf{1}_{\{a\leq v\leq b\}} dvdy$$

since $\mathbf{1}_{\{a\leq v\leq b\}}$ does not depend on x. However, $f \leq 1$ and is equal to 0 on $\mathbb{R} \times (]-D,a[\cup]b,D[)$, so $f(y,v) = \mathbf{1}_{\{a\leq v\leq b\}}$ on $C(x)$. We conclude noticing that the union of $C(x)$ for $x \in \mathbb{R}$ is $\mathbb{R}\times]-D,D[$. $\square$

Corollary 4.5. *Let $a,b \in \mathbb{R}$ with $0 \leq a \leq b < D$, and let $f \in L^\infty(]-R-D,R+D[\times]-D,D[)$ be an equilibrium function such that $T(f) = f$ on $]-R,R[\times]-D,D[$ and $f(x,v) = 0$ on $]-R-D,R[\times]b,D[$. Moreover, we assume that there exists $x_0 \in]-R,R[$ such that $f(x,v) = 0$ on $]x_0,x_0+D[\times]-D,0[$, and*

$$\int_{C(x_0)} \operatorname{sgn}(v) f(x,v)dvdx = \frac{b^2 - a^2}{2}, \tag{4.6}$$

$$\int_{C(x_0)} |v| f(x,v)dvdx = \frac{b^3 - a^3}{3}. \tag{4.7}$$

Then $f(x,v) = \mathbf{1}_{\{a\leq v\leq b\}}$ on $]-R,x_0[\times]-D,D[$.

Proof. First, using the second part of Lemma 4.2 with $V = 0$, we have $f(x,v) = 0$ on $]-R,x_0+D[\times]-D,0[$. Thus

$$\operatorname{sgn}(v) f(x,v) = f(x,v) \tag{4.8}$$

on $]-R,x_0+D[\times]-D,D[$. Using (4.6), (4.7), (4.8) and Lemma 4.3 with $X_1 = x$, $X_2 = x_0$, we get for every $x \in]-R,x_0[$

$$\int_{C(x)} [f(x,v) - \mathbf{1}_{\{a\leq v\leq b\}}]dvdx = 0, \tag{4.9}$$

$$\int_{C(x)} |v|[f(x,v) - \mathbf{1}_{\{a\leq v\leq b\}}]dvdx = 0. \tag{4.10}$$

Then, since f is valued in $\{0,1\}$ and $f(x,v) = 0$ for $v \geq b$, (4.9), (4.10) ensures that $f(x,v) = \mathbf{1}_{\{a\leq v\leq b\}}$ on $C(x)$. Indeed,

$$\int_{C(x)} \mathbf{1}_{\{v\leq a\}} \mathbf{1}_{\{f=1\}} dxdv = \int_{C(x)} \mathbf{1}_{\{a\leq v\leq b\}} \mathbf{1}_{\{f=0\}} dxdv$$

and

$$\int_{C(x)} |v|[f(x,v) - \mathbf{1}_{\{a\leq v\leq b\}}]dvdx$$
$$= \int_{C(x)} |v| \mathbf{1}_{\{v\leq a\}} \mathbf{1}_{\{f=1\}} dxdv - \int_{C(x)} |v| \mathbf{1}_{\{a\leq v\leq b\}} \mathbf{1}_{\{f=0\}} dxdv.$$

Finally, since the union of $C(x)$ for $x \in]-R, x_0[$ includes $]-R, x_0[$, $f(x,v) = \mathbf{1}_{\{a \leq v \leq b\}}$ for x in this interval. ∎

PROPOSITION 4.6. *Let* $(\rho_0, P_0) \in \mathbb{R}^+ \times \mathbb{R}$, $\rho_0 > 0$, $x_0 \in \mathbb{R}$ *and* $f \in L^\infty(]x_0 - D, x_0 + 2D[\times]-D, D[)$ *be such that* $T(f) = f$ *on* $]x_0, x_0 + D[\times]-D, D[$. *Let us denote* $(\rho, P) \in [L^\infty(\mathbb{R})]^2$ *the two first moments of* f *with respect to* v. *If*

$$\int_{x_0 - D}^{x_0 + 2D} (|\rho(x) - \rho_0| + |P(x) - P_0|)\, dx \leq \epsilon^2, \tag{4.11}$$

then for almost every $x \in [x_0, x_0 + D]$

$$|\rho(x) - \rho_0| + |P(x) - P_0| \leq M\epsilon, \tag{4.12}$$

where M *depends only on* D *and* ρ_0.

Proof. Since f is an equilibrium function we can write $f(x,v) = \mathbf{1}_{\{a(x) \leq v \leq b(x)\}}$. Let us denote Ω the set of $x \in [x_0 - D, x_0 + 2D]$ such that

$$|\rho(x) - \rho_0| + |P(x) - P_0| \leq \epsilon.$$

Notice that for every $x \in \Omega$

$$|a(x) - a_0| + |b(x) - b_0| \leq \bar{M}\epsilon,$$

where $\bar{M} = \frac{1}{2} + \frac{D+1}{\rho_0}$, $a_0 = P_0/\rho_0 - \rho_0/2$, and $b_0 = P_0/\rho_0 + \rho_0/2$. So for every $v \in [a_0 + \bar{M}\epsilon, b_0 - \bar{M}\epsilon]$ such that $x - v \in \Omega$, we have $v \in [a(x-v), b(x-v)]$, which implies that $f(x-v, v) = 1$. Because of (4.11), however, the Lebesgue measure of the complementary of Ω in $[x_0 - D, x_0 + 2D]$ is less than ϵ. Therefore for almost every $x \in]x_0, x_0 + D[$

$$\begin{aligned} \rho(x) \geq & \int_{(x-\Omega)\cap[-D,D]} f(x-v, v)dv \\ & \geq \rho_0 - 2\bar{M}\epsilon - \epsilon \\ & \geq \rho_0 - 2\epsilon(\bar{M} + 1). \end{aligned}$$

In the same manner we obtain

$$P(x) \geq P_0 - 2D(\bar{M} + 1)\epsilon.$$

Moreover for every v such that $x - v \in \Omega$, if $f(x-v, v) = 1$, then $v \in [a_0 - \bar{M}\epsilon, b_0 + \bar{M}\epsilon]$. Therefore

$$\rho(x) \leq \rho_0 + 2(\bar{M} + 1)\epsilon.$$

In the same manner we obtain

$$P(x) \leq P_0 + 2D(\bar{M} + 1)\epsilon.$$

Therefore, we obtain the desired result with $M = 2(D+1)(\bar{M}+1) = M(\rho_0, D)$. ∎

We are now able to show Theorem 1.1.

Proof. First, according to Proposition 3.4 there exist asymptotic functions both at $+\infty$ and $-\infty$, and they are equilibrium functions and constant with respect to x. Let us denote one $\mathbf{1}_{\{a \leq v \leq b\}}$ (which could be at $+\infty$ or at $-\infty$). Then there exists a sequence X_n such that $f(\cdot + X_n, \cdot)$ converges to this asymptotic function in $L^1_{\mathrm{loc}}(\mathbb{R} \times [-D, D])$. For every $x_0 \in \mathbb{R}$, using Lemma 4.3 for f at points x_0 and X_n and using the definition of f_X we find

$$\int_{C(0)} \operatorname{sgn}(v) f_{X_n}(x,v)\,dv\,dx = \int_{C(x_0)} \operatorname{sgn}(v) f(x,v)\,dv\,dx,$$
$$\int_{C(0)} |v| f_{X_n}(x,v)\,dv\,dx = \int_{C(x_0)} |v| f(x,v)\,dv\,dx.$$

Then, passing to the limit when n tends to $+\infty$ we find

$$\int_{C(0)} \operatorname{sgn}(v) \mathbf{1}_{\{a \le v \le b\}}\,dv\,dx = \int_{C(x_0)} \operatorname{sgn}(v) f(x,v)\,dv\,dx, \tag{4.13}$$
$$\int_{C(0)} |v| \mathbf{1}_{\{a \le v \le b\}}\,dv\,dx = \int_{C(x_0)} |v| f(x,v)\,dv\,dx. \tag{4.14}$$

Since f is assured to be a nontrivial shock profile, the right-hand side of (4.14) must be different from 0 at least for an $x_0 \in \mathbb{R}$ and therefore a must be different from b. For $x_0 = 0$, we find

$$\frac{b^2 - a^2}{2} = \int_{C(0)} \operatorname{sgn}(v) f(x,v)\,dv\,dx = P,$$
$$\frac{b^3 - a^3}{3} = \int_{C(0)} |v| f(x,v)\,dv\,dx = Q.$$

Therefore, thanks to Lemma 4.1, there are at most two admissible values (a, b) of asymptotic function, and those values are such that $a < b$.

Let us now consider two asymptotic functions $\mathbf{1}_{\{a^- \le v \le b^-\}}$ and $\mathbf{1}_{\{a^+ \le v \le b^+\}}$, respectively, at $-\infty$ and $+\infty$ (which could be equal). Then there exist X_n^- and X_n^+ which converge, respectively, to $-\infty$ and $+\infty$ such that $f(\cdot + X_n^-, \cdot)$ and $f(\cdot + X_n^+, \cdot)$ converge to the associated asymptotic functions in $L^1_{\mathrm{loc}}(\mathbb{R} \times [-D, D])$.

Since a^+ and b^+ are constant, we can use Proposition 4.6 (with $\rho_0 = b^+ - a^+$ and $P_0 = ({b^+}^2 - {a^+}^2)/2$) to deduce that $\int_{-D}^{D} f(\cdot + X_n^+, v)\,dv$ and $\int_{-D}^{D} v f(\cdot + X_n^+, v)\,dv$ converge strongly in $L^\infty_{\mathrm{loc}}(\mathbb{R})$. This means that, if we define $a(x)$ and $b(x)$ such that $f(x,v) = \mathbf{1}_{\{a(x) \le v \le b(x)\}}$, $a(\cdot + X_n^+)$ and $b(\cdot + X_n^+)$ converge in $L^\infty_{\mathrm{loc}}(\mathbb{R})$, respectively, to a^+ and b^+. Using the second statement of Lemma 4.2 (with $x_0 = X_n^+$, $R = \infty$ and $V = -\inf\{0, a(y),\ y \in]X_n^+, X_n^+ + D[\}$), we find that for $x \in]-\infty, X_n^+ + D[$

$$a(x) \ge \inf_{y \in]X_n^+, X_n^+ + D[} \{a(y), 0\}.$$

Then, letting n going to $+\infty$, we find that $a(x) \ge \inf(a^+, 0)$ on $\mathbb{R}$. After doing the same thing for $-\infty$, we have obtained that

$$a(x) \ge \inf(a^+, 0) \text{ and } b(x) \le \sup(b^-, 0). \tag{4.15}$$

We have seen that there are at most two asymptotic functions. First let us assume that we have only one denoted $\mathbf{1}_{\{a \le v \le b\}}$. We will show that in this case f is a trivial shock. If $ab \le 0$, it means that $a \le 0 \le b$, so, because of (4.15) we have $f(x,v) = 0$ on $\mathbb{R} \times (]-D, a[\cup]b, D[)$. Therefore, since (4.14), thanks to Corollary 4.4, f is constant with respect to x. Let us now assume $ab > 0$. Since $\mathbf{1}_{\{-b \le v \le -a\}}$ is an asymptotic function of $\bar{f}$ defined by $\bar{f}(x,v) = f(-x,-v)$, replacing f with $\bar{f}$ if necessary, we can assume without loss of generality that $0 < a < b$. Then (4.15) ensures us that $f(x,v) = 0$ on $\mathbb{R} \times (]-D,0[\cup]b,D[)$. Therefore, using Corollary 4.5 with (4.13), (4.14), we find again that f is a trivial shock. We conclude that we have exactly two distinct asymptotic functions. Thanks to Lemma 4.1, there is one such that $a_1b_1 > 0$. Replacing as above f with $\bar{f}$ if necessary, we can assume without loss of generality that $b_1 > a_1 > 0$. Thanks to Lemma 4.1, the other value is such that $a_2b_2 < 0$. Therefore $a_2 < 0 < b_2$. Finally $b_2 < b_1$ (because $b_2^3 - a_2^3 = b_1^3 - a_1^3$). Now assume that $\mathbf{1}_{\{a_1 \le v \le b_1\}}$ is an asymptotic function at $+\infty$, then according to (4.15) $f(x,v) = 0$ for $v \le 0$, which is impossible since $\mathbf{1}_{\{a_2 \le v \le b_2\}}$ is an asymptotic function and $a_2 < 0$. Therefore $\mathbf{1}_{\{a_2 \le v \le b_2\}}$ is the unique asymptotic function at $+\infty$, and we can set $a^+ = a_2$ and $b^+ = b_2$. In the same way, if $\mathbf{1}_{\{a_2 \le v \le b_2\}}$ is an asymptotic function at $-\infty$, then according to (4.15) $f(x,v) = 0$ for $v \ge b_2$ which contradicts that $\mathbf{1}_{\{a_1 \le v \le b_1\}}$ is an asymptotic function with $b_1 > b_2$. Therefore $\mathbf{1}_{\{a_1 \le v \le b_1\}}$ is the unique asymptotic function at $-\infty$ and we can set $a^- = a_1$ and $b^- = b_1$. From Proposition 3.4, we deduce that for every sequence X_n which converges to $+\infty$ or $-\infty$, f_{X_n} converges in $L^1_{\mathrm{loc}}(\mathbb{R}\times]-D,D[)$ to the associated unique asymptotic function. Next, using Proposition 4.6 we find

$$\lim_{X\to-\infty} \left(\|a(x) - a_1\|_{L^\infty(]-\infty,X[)} + \|b(x) - b_1\|_{L^\infty(]-\infty,X[)}\right) = 0,$$
$$\lim_{X\to+\infty} \left(\|a(x) - a_2\|_{L^\infty(]X,+\infty[)} + \|b(x) - b_2\|_{L^\infty(]X,+\infty[)}\right) = 0.$$

Since $a_1 > 0$, however, this implies that there exists $X < 0$ large enough such that for $(x,v) \in]-\infty,X[\times]-D,0[$ we have $f(x,v) = 0$. Since (4.15) implies that $f(x,v) = 0$ on $\mathbb{R}\times]b_1,D[$, we deduce from (4.13), (4.14), using Corollary 4.5 (with $x_0 = X - D$, $R = \infty$, $a = a_1$, $b = b_1$) that $f(x,v) = \mathbf{1}_{\{a_1 \le v \le b_1\}}$ on $]-\infty,X-D[\times]-D,D[$. ∎

5. Existence of stationary shock profiles. In this section we show Theorem 1.2. Thanks to Theorem 1.1 and Lemma 4.1, we can restrict ourselves to show the existence of a stationary shock profile which is not constant and has a left-hand side limit $\mathbf{1}_{\{a_1 \le v \le b_1\}}$ such that $0 < a_1 < b_1$. Therefore we fix for all this section a left-hand state (a_1,b_1) with $0 < a_1 < b_1$, and we denote $\rho_1 = b_1 - a_1$, and $P_1 = (b_1^2 - a_1^2)/2$.

In order to show Theorem 1.2, we use the Schauder fixed point theorem for the two first moments of f on a bounded domain $]-R,R[$. But as R goes to $+\infty$ we may get a trivial shock profile (constant with respect to x). Therefore we construct our solution on a bounded domain with a property which ensures us that the limit is not trivial. In fact we use the following property which is a corollary of Theorem 1.1.

PROPOSITION 5.1. *Let f be a stationary shock profile such that*

$$f(x,v) = 0 \text{ on } \mathbb{R}\times]b_1,D[, \tag{5.1}$$

$$\int_{C(0)} \mathrm{sgn}(v) f(x,v)\,dx dv = \frac{b_1^2 - a_1^2}{2}, \tag{5.2}$$

$$\int_{C(0)} |v| f(x,v)\,dx dv = \frac{b_1^3 - a_1^3}{3}, \tag{5.3}$$

where $C(0)$ is defined by (4.5). *Assume that*

$$\frac{\epsilon}{2} \le \int_{-D}^{2D} |\rho(x) - \rho_1|\,dx + \int_{-D}^{2D} |P(x) - P_1|\,dx \le \epsilon \tag{5.4}$$

for some $\epsilon > 0$ small enough, with $\rho(x) = \int f(x,v)dv$ and $P(x) = \int vf(x,v)dv$. Then $f(x,v) = \mathbf{1}_{\{a_1 \le v \le b_1\}}$ for $x \le 0$, $f(x,\cdot)$ converges to $\mathbf{1}_{\{a_2 \le v \le b_2\}}$ when x converges to $+\infty$, and $(a_2,b_2) \neq (a_1,b_1)$. In particular f is a nontrivial shock profile.

Proof. Thanks to Proposition 4.6 and the second inequality of (5.4), for $x \in]0,D[$ we have $a(x) \ge a_1 - M\sqrt{\epsilon}$ for some constant M depending only on ρ_1 and D. Therefore, for ϵ small enough we have $a(x) \ge 0$ on $]0,D[$, since $a_1 > 0$. Using Corollary 4.5 we find that $f(x,v) = \mathbf{1}_{\{a_1 \le v \le b_1\}}$ for $x \le 0$. Moreover, the lower bound in (5.4) ensures us that $f(x,\cdot)$ is not constant with respect to x and therefore, thanks to Theorem 1.1, $f(x,\cdot)$ converges to $\mathbf{1}_{\{a_2 \le v \le b_2\}}$ when x converges to $+\infty$ with $(a_2,b_2) \neq (a_1,b_1)$. ◻

We will apply the Schauder theorem on the set of $(\rho = \int f dv, P = \int vf dv)$. Therefore we introduce the closed convex subset $\mathcal{L}_R$ of $L^1(]-R,R[)$ of all $(\rho \ge 0, P)$ such that

$$\forall x \in]-R,R[\;\; P(x) + \frac{\rho(x)^2}{2} \le D\rho(x),$$

$$P(x) - \frac{\rho(x)^2}{2} \ge -D\rho(x),$$

(which implies $\rho(x) \le 2D$, $|P(x)| \le 2D^2$). Moreover, in order to enforce property (5.4) we introduce F_ϵ the function defined on $\mathbb{R}_+$ by

$$F_\epsilon(y) = -D \text{ if } 0 \le y \le \frac{\epsilon}{2},$$
$$F_\epsilon(y) = -2D\left(1 - \frac{y}{\epsilon}\right) \text{ if } \frac{\epsilon}{2} \le y \le \epsilon,$$
$$F_\epsilon(y) = 0 \text{ if } y \ge \epsilon.$$

Then we define $\mathcal{A}_\epsilon : L^1(]-R,R[)^2 \longrightarrow \mathbb{R}$ by

$$\mathcal{A}_\epsilon(\rho,P) = F_\epsilon\left(\int_{-D}^{2D} |\rho(x) - \rho_1|dx + \int_{-D}^{2D} |P(x) - P_1|dx\right). \tag{5.5}$$

Notice that ϵ is fixed as in Proposition 5.1 and does not depend on R. In order to define the operator T_R, we associate with $(\rho,P) \in \mathcal{L}_R$ the equilibrium function f defined on $\mathbb{R}\times]-D,D[$

$$f(x,v) = \mathbf{1}_{\{a_1 \le v \le b_1\}} \text{ if } x < -R, \tag{5.6}$$

$$f(x,v) = \mathbf{1}_{\{P(x) - \frac{\rho(x)^2}{2} \le \rho(x)v \le P(x) + \frac{\rho(x)^2}{2}\}} \text{ if } x \in]-R,R[, \tag{5.7}$$

$$f(x,v) = \mathbf{1}_{\{\mathcal{A}_\epsilon(\rho,P) \le v \le 0\}} \text{ if } x > R. \tag{5.8}$$

Then $T_R(\rho,P) = (\rho',P')$ is defined on $]-R,R[$ by

$$\rho'(x) = \int_{-D}^{D} f(x-v,v)dv,$$
$$P'(x) = \int_{-D}^{D} vf(x-v,v)dv.$$

Let us explain the idea of this construction. Assume that $(\rho,P) = (\rho_R,P_R)$ is a fixed point of T_R. First, the corresponding function $f = f_R$ is such that $T(f_R) = f_R$ on

$]-R,R[\times]-D,D[$. Therefore we will be able to show, using results of section 4, that $\mathcal{A}_\epsilon(\rho_R,P_R)$ cannot be equal to $-D$ nor 0. Therefore, because of the definition of F_ϵ, it means that (ρ_R,P_R) (which are the two first moments with respect to v of f_R on $]-R,R[$ by definition of f_R) verifies property (5.4). Then when R goes to $+\infty$, thanks to Lemma 2.2, we will find a subsequence $R_n\to\infty$ such that f_{R_n} converges to a new function f which is an equilibrium function, $T(f)=f$ on $\mathbb{R}\times]-D,D[$ and verifies property (5.4). Finally, thanks to Proposition 5.1, we will be able to show that f is a nontrivial stationary shock profile connecting (a_1,b_1) and (a_2,b_2). Let us now show it in details.

LEMMA 5.2. *The set $\mathcal{L}_R$ is a closed convex subset of $L^1(]-R,R[)^2$, stable under T_R. Moreover, the set $T_R(\mathcal{L}_R)$ is relatively compact in $L^1(]-R,R[)^2$.*

Proof. Lemma 2.3 ensures that $\mathcal{L}_R$ is stable under T_R and thanks to Lemma 2.1, $T_R(\mathcal{L}_R)$ is relatively compact in $\left[L^1(]-R,R[)\right]^2$. ∎

LEMMA 5.3. *Operator T_R is continuous from $\mathcal{L}_R$ to $\mathcal{L}_R$ in $\left[L^1(]-R,R[)\right]^2$.*

Proof. Since ϵ is fixed, it is clear that $\mathcal{A}_\epsilon$ is continuous from $\mathcal{L}_R$ to $\mathbb{R}$. Therefore, thanks to Lemma 2.2, the operator which gives the equilibrium function f from (ρ,P) is continuous from $\mathcal{L}_R$ to $L^1(]-R-D,R+D[\times]-D,D[)$. So T_R is continuous from $L^1(]-R,R[)$ to $L^1(]-R,R[)$. ∎

We are now able to show Theorem 1.2 with (a_1,b_1) as left-hand side values.

Proof. Thanks to Lemmas 5.3 and 5.2, we can use the Schauder theorem to show the existence of an equilibrium function $f_R\in L^1(\mathbb{R}\times]-D,D[)$ such that

$$\int_{-D}^{D} f_R(x,v)dv=\int_{-D}^{D} f_R(x-v,v)dv=\rho_R(x),$$
$$\int_{-D}^{D} vf_R(x,v)dv=\int_{-D}^{D} vf_R(x-v,v)dv=P_R(x)$$

for $x\in]-R,R[$, $f_R(x,v)=\mathbf{1}_{\{a_1\le v\le b_1\}}$ for $x\le -R$ and $f_R(x,v)=\mathbf{1}_{\{a_2^R\le v\le 0\}}$ for $x\ge R$, where

$$a_2^R=F_\epsilon\left(\int_{-D}^{2D}|\rho_R(x)-\rho_1|dx+\int_{-D}^{2D}|P_R(x)-P_1|dx\right).$$

We have $f_R(x,v)=0$ on $]-R-D,-R[\times]b_1,D[$, so Lemma 4.2 (with $x_0=-R$, $V=b_1$) ensures us that

$$f_R(x,v)=0 \text{ on }]-R-D,R[\times]b_1,D[. \tag{5.9}$$

Let us show that (ρ_R,P_R) verifies (5.4). This is fulfilled by the definition of a_2^R. Indeed, if (5.4) was not true, from (5.5) we would get $a_2^R=0$ or $a_2^R=-D$. We will see that because of the conservation of $\int_{-D}^{2D}\rho_R(x)\,dx$ and $\int_{-D}^{2D}P_R(x)\,dx$ by T_R, the first case would imply that $(\rho_R,P_R)\equiv(\rho_1,P_1)$ on $]0,D[$ and the second case that $(\rho_R,P_R)\not\equiv(\rho_1,P_1)$ on $]0,D[$ which are not compatible with definitions of a_2^R and F_ϵ. More precisely, assume that

$$\int_{-D}^{2D}|\rho_R(x)-\rho_1|dx+\int_{-D}^{2D}|P_R(x)-P_1|dx\ge\epsilon. \tag{5.10}$$

By definition (5.5) $\mathcal{A}_\epsilon(\rho_R,P_R)=0$ and, by (5.8), $f_R(x,v)=0$ on $]R,R+D[\times]-D,0[$. Using the second part of Lemma 4.2 (with $x_0=R$, $V=0$)

$$f_R(x,v)=0 \text{ on }]-R,R+D[\times]-D,0[. \tag{5.11}$$

Thus, for R larger than D, we deduce from definition (5.6) that

$$\int_{C(-R)} \mathrm{sgn}(v) f_R(x,v)\,dxdv = \frac{b_1^2 - a_1^2}{2},$$
$$\int_{C(-R)} |v| f_R(x,v)\,dxdv = \frac{b_1^3 - a_1^3}{3}$$

(where $C(-R)$ is defined by (4.5)). Therefore, using Lemma 4.3 (with $X_1 = -R$, $X_2 = R$) we find

$$\int_{C(R)} \mathrm{sgn}(v) f_R(x,v)\,dxdv = \frac{b_1^2 - a_1^2}{2},$$
$$\int_{C(R)} |v| f_R(x,v)\,dxdv = \frac{b_1^3 - a_1^3}{3};$$

therefore we can use Corollary 4.5 (with $a = a_1$, $b = b_1$, $x_0 = R$, using (5.9), (5.11)), and find that $f(x,v) = \mathbf{1}_{\{a_1 \le v \le b_1\}}$ on $]-R, R[\times]-D, D[$. For $R > 2D$, this leads to a contradiction with (5.10). We conclude that

$$\int_{-D}^{2D} |\rho_R(x) - \rho_1|dx + |P_R(x) - P_1|dx \le \epsilon \tag{5.12}$$

for $R > 2D$. Thanks to Proposition 4.6, for $x \in]0, D[$, we have $a(x) \ge a_1 - M\sqrt{\epsilon}$, where M depends only on D and ρ_1. Therefore, since $a_1 > 0$, for ϵ small enough we have $a(x) \ge 0$ on $]0, D[$. Using the second part of Lemma 4.2 (with $V = 0, x_0 = 0$) we find that $a(x) \ge 0$ on $]-R, D[$. Therefore, as above, if $R > D$

$$\int_{C(-R)} \mathrm{sgn}(v) f_R(x,v)\,dxdv = \frac{b_1^2 - a_1^2}{2},$$
$$\int_{C(-R)} |v| f_R(x,v)\,dxdv = \frac{b_1^3 - a_1^3}{3},$$

and using Lemma 4.3 (with $X_1 = -R$, $X_2 = 0$)

$$\int_{C(0)} \mathrm{sgn}(v) f_R(x,v)\,dxdv = \frac{b_1^2 - a_1^2}{2}, \tag{5.13}$$

$$\int_{C(0)} |v| f_R(x,v)\,dxdv = \frac{b_1^3 - a_1^3}{3}. \tag{5.14}$$

Now assume that

$$\int_{-D}^{2D} |\rho_R(x) - \rho_1|dx + \int_{-D}^{2D} |P_R(x) - P_1|dx \le \frac{\epsilon}{2}.$$

By (5.5) and (5.8), using Lemma 4.3 (with $X_1 = 0$, $X_2 = R$), we find

$$\int_{C(0)} \mathrm{sgn}(v) f_R(x,v)\,dxdv = \frac{b_1^2 - a_1^2}{2}$$
$$= \int_{C_-(R)} f_R(x,v)\,dxdv - \frac{D^2}{2} \le 0$$

for $R > 2D$, where $C_-(R) = \{(x,v) || v| \leq D,\ x \leq R,\ x - R + v \geq 0\}$. Notice that the inequality holds true because the Lebesgue measure of $C_-(R)$ is smaller than $\frac{D^2}{2}$ and $f_R \leq 1$. But this is impossible since $0 < a_1 < b_1$. Therefore, taking (5.12) into account, we have obtained

$$\frac{\epsilon}{2} \leq \int_{-D}^{2D} |\rho_R(x) - \rho_1| dx + \int_{-D}^{2D} |P_R(x) - P_1| dx \leq \epsilon \tag{5.15}$$

for all $R > 2D$. Finally, thanks to Lemmas 5.2 and 2.2, there exists a function $f \in L^\infty(\mathbb{R}\times] - D, D[)$ and a sequence $R_n \to +\infty$ such that f_{R_n} converges to f in $L^1_{\text{loc}}(\mathbb{R}\times] - D, D[)$. Since the convergence is strong, f is an equilibrium function (Lemma 2.2), and so $T(f) = f$. Passing to the limit in (5.9), (5.13), (5.14), and (5.15) we find (5.1), (5.2), (5.3), and (5.4). Proposition 5.1 concludes the proof. ∎

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