ASYMPTOTIC LIMIT TO A SHOCK FOR BGK MODELS USING RELATIVE ENTROPY METHOD

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Abstract. We consider the hydrodynamical limit of a kinetic BGK model in one space dimension, to a scalar conservation law with a strict convex flux. In this paper, we provide sharp estimates for the asymptotic limit to shocks, where layers, called Knudsen layers, may appear. The used method is based on the relative entropy. It follows a previous work of Choi and Vasseur [11] where similar sharp estimates were obtained for the inviscid limit of a viscous conservation laws to a shock.

Contents

1. Introduction

We consider the following BGK model in one dimensional spaces $\mathbb{R}$, first introduced by Perthame and Tadmor [24], which is based on the "transport-collapse" method of Brenier [6, 7]

$$\partial_t f + a(v)\cdot\partial_x f = \frac{Mf - f}{\epsilon} \tag{1}$$

where the flux $A''(v) := a'(v) \geq c$ for some constant $c > 0$, the local density of particle is defined by

$$U = \int_{\mathbb{R}} f(v)\mathrm{dv}, \tag{2}$$

and the Maxwellian Mf is defined as follows

$$Mf = \begin{cases} \mathbf{1}_{\{0\leq v\leq u\}}, & \text{if } v \geq 0,\\ -\mathbf{1}_{\{u\leq v\leq 0\}}, & \text{if } v < 0,\\ 0, & \text{otherwise.}\end{cases}$$

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The existence of global unique weak solutions of (1) and the hydrodynamic limit of the BGK model, as $\epsilon \to 0$ is proved in [24]. When ε converges to 0, the function U converges to a solution of the scalar conservation laws:

$$\partial_t U + \partial_x A(U) = 0. \tag{3}$$

In this paper, we are interested in getting sharp estimates for the asymptotic limit close to a shock. Let us consider the shock solutions of the scalar conservation laws (3) with the initial data

$$S_0(x) = \begin{cases} C_L \text{ if } x < 0, \\ C_R \text{ if } x \geq 0. \end{cases} \tag{4}$$

with two constants $C_L > C_R$. Then, the Rankine-Hugoniot condition ensures that the function

$$S_0(x - \sigma t) \qquad \text{with} \qquad \sigma := \frac{A(C_L) - A(C_R)}{C_L - C_R}, \tag{5}$$

is a solution to the equation (3). Notice that the condition $C_L > C_R$ implies that they verify the entropy conditions, that is:

$$\partial_t \eta(U) + \partial_x G(U) \leq 0, \qquad t > 0,\ x \in \mathbb{R},$$

for any convex functions η, and $G' = \eta' A'$. Hydrodynamic limit to a shock may exhibit the formation of layer, called Knudsen layer, which have a typical length of ε. An easy dimensional analysis shows that, because of those layers, we may have in general

$$\|U(t) - S(\cdot - \sigma t)\|_{L^2}^2 \geq C\varepsilon,$$

which means that the L^2 stability for two solutions U, S does not hold.We are interested in large perturbation (that is for any initial value without smallness condition with respect to ε). In this case, the layer study is not adequate, since it may not converge due to the large amount of energy at the scale ε (see [11]). We show however, that even in this case, the error cannot be bigger than $\varepsilon \log(1/\varepsilon)$. The main result is as follows.

Theorem 1.1. *Let $C_L > C_R$, $L > 0$ with $L > \sup(|C_L|, |C_R|)$, and $T > 0$ be any number. Then there exists $\alpha > 0$, $C > 0$, and $\varepsilon_0 > 0$, such that if U is defined in (2) by the solution f of the equation (1) with the initial value $f_0 \in L^1(\mathbb{R} \times [-L, L])$ verifying $0 \leq \mathrm{sgn}(v) f_0(x, v) \leq 1$ and $\int f_0\, dv = U_0$ with*

$$(U_0 - S_0) \in L^2(\mathbb{R}) \qquad \text{and} \qquad \|\partial_x f_0\|_{L^1(\mathbb{R}\times[-L,L])} \leq C_L - C_R + \alpha,$$

we have the following result:

- *For any $0 < \varepsilon \leq \varepsilon_0$, there exists a Lipschitz curve $X \in L^\infty(0, T)$ such that $X(0) = 0$ and for any $0 < t < T$:*

$$\begin{aligned} \|U(t) - S(t)\|_{L^2(\mathbb{R})}^2 &\leq \|U_0 - S_0\|_{L^2(\mathbb{R})}^2 \\ &\quad + \|\int_{\mathbb{R}} 2v f_0(v) \mathrm{dv} - U_0^2\|_{L^2(\mathbb{R})}^2 + C\varepsilon \log(1/\varepsilon), \end{aligned} \tag{6}$$

  where $S(t, x) := S_0(x - X(t))$, and S_0 is defined by (4).
- *Moreover, this curve satisfies*

$$|\dot{X}(t)| \leq C \qquad \textit{and} \tag{7}$$

$$\begin{aligned} |X(t) - \sigma t|^2 &\leq C t^{2/3} \Big(\|U_0 - S_0\|_{L^2(\mathbb{R})}^2 \\ &\quad + \|\int_{\mathbb{R}} 2v f_0(v) \mathrm{dv} - U_0^2\|_{L^2(\mathbb{R})}^2 + \varepsilon \log(1/\varepsilon) \Big). \end{aligned} \tag{8}$$

Remark 1. *In Theorem 1.1, the given constant ϵ_0 depends on $\alpha, C_L, C_R, \|A\|_{L^\infty}$ and C depends on $\alpha, C_L, C_R, \|A\|_{L^\infty}$ and T.*

Our method is based on the method developed in Leger and Vasseur [20, 21]. Note that the estimate is uniform with respect to ε. So, for $\varepsilon = 0$, we recover the result of [20], which is a contraction property for shocks of conservation laws in L^2, up to shift. The result cannot be true without shift (see [20]).

The relative entropy method introduced by Dafermos [12] and Diperna [14] provides an efficient tool to study the stability and asymptotic limits among thermomechanical theories, which is related to the second law of thermodynamics. They showed, in particular, that if $\overline{U}$ is a Lipschitzian solution of a suitable conservation law on a lapse of time $[0, T]$, then for any bounded weak entropic solution U it holds:

$$\int_{\mathbb{R}} |U(t) - \overline{U}(t)|^2\, dx \le C \int_{\mathbb{R}} |U(0) - \overline{U}(0)|^2\, dx, \tag{9}$$

for a constant C depending on $\overline{U}$ and T. Since Dafermos [12] and Diperna [14]'s works, there have been many recent progress as applications of the relative entropy method. For incompressible limits, see Bardos, Golse, Levermore [2, 3], Lions and Masmoudi [22], Saint Raymond et al. [18, 26, 27, 28] have studied incompressible limit problems. With the relative entropy idea, some authors, for example, Feireisl, Novotny [15], Masmoudi [25], have studied inviscid incompressible limit problems for the multidimensional compressible Navier-Stokes systems. There are also many recent results of the weak-uniqueness for the compressible Navier Stokes equations together with using relative entropy by Germain [17], Feireisl, Novotny [16]. For the relaxation there is an application for compressible models by Lattanzio, Tzavaras [19, 29] and we can also see Berthelin, Tzavaras, Vasseur [4, 5] as some applications of hydrodynamical limit problems. However, in all those cases, the method works as long as the limit solution has a good regularity such that the solution is Lipschitz. This is due to the fact that strong stability as (9) is not true when $\overline{U}$ has a discontinuity. It has been proven in [20, 21], however, that some shocks are strongly stable up to a shift (see also related works from Chen and Frid [8, 9] and Chen, Frid and Li [10]). Choi and Vasseur [11] have recently used this stability property to study sharp estimates for the inviscid limit of viscous scalar conservation laws to a shock. This paper is dedicated to the application of those ideas to study sharp estimates of hydrodynamical limits to shocks. This result shows a rate of convergence slightly worse than ε (to the log), for the hydrodynamic limit to a shock, measured via the L^2 norm (squared).

The outline of this article is as follows: In section 2, we recall some known properties of the solutions of equation (1). In Section 3 we introduce the relative entropy and some properties used in Leger [20]. In Section 4, we derive some estimates for the hyperbolic part of the relative entropy. The section 5 is devoted to estimate the dissipation terms. Finally, in section 6, we give the proof of Theorem 1.1 together with combining the estimates in section 4, 5.

2. Properties of solutions of the BGK equation

We recall the following lemma which can be found as Theorem 3.6.1 in Perthame [23].

Lemma 2.1. *Let $f_0 \in L^\infty(\mathbb{R} \times [-L, L])$ verify $0 \le \mathrm{sgn}(v) f_0(x, v) \le 1$ for almost every x, v, and $a \in L^\infty_{\mathrm{loc}}(\mathbb{R})$, then there exists a unique solution to (1). Moreover, it satisfies:*

$$|f(t, x, v)| = \mathrm{sgn}\ (v) f(t, x, v) \le 1,$$

and for every $h > 0$, $U(t,x) = \int f(t,x,v)\,dv$ *verifies* $|U(t,x)| \le L$ *for almost every* t, x, *and*

$$\|U(t,\cdot+h) - U(t,\cdot)\|_{L^1(\mathbb{R})} \le \|f_0(\cdot+h,v) - f_0(\cdot,v)\|_{L^1(\mathbb{R}^2)}.$$

This provides the following proposition.

Proposition 2.1. *Let* $u_0 \in L^1(\mathbb{R})$ *and* U *be the local density of a solution of* f *in* (1) *such that* $\|\partial_x f_0\|_{L^1}$ *is bounded. Then, for all* $t > 0, h \in \mathbb{R}$, *we have*

$$\|U(\cdot+h,t) - U(\cdot,t)\|_{L^1(\mathbb{R})} \le h\|\partial_x f_0\|_{L^1}. \tag{10}$$

Especially, $U(t)$ *is bounded in* BV *for every* $t > 0$.

From the proposition 2.1, we can show the following lemma 2.2.

Lemma 2.2. *Let* $\|\partial_x f_0\|_{L^1} \le C_L - C_R + \alpha$, *and* U *be the local density of a solution of* f *in* (1). *Then, for all* $t \in (0,T)$, *the following inequality holds:*

$$\int_{-\infty}^{\infty} (\partial_x U(t))_+ \mathrm{dx} \le \alpha. \tag{11}$$

Proof. Let $a < b$. Integrating U for space variable gives, for $t \in (0,T)$,

$$U(t,a+) - U(t,b-) = -\int_a^b \partial_x U \mathrm{dx} \le \int_a^b |(\partial_x U)_-| \mathrm{dx},$$

which implies

$$C_L - C_R \le \int_{-\infty}^{\infty} |(\partial_x U)_-| \mathrm{dx},$$

where $a \to -\infty,\ b \to \infty$. This, together with proposition 2.1, proves (11)

$$\int_{-\infty}^{\infty} |(\partial_x U)_-| \mathrm{dx} + \int_{-\infty}^{\infty} |(\partial_x U)_+| \mathrm{dx} = \int_{-\infty}^{\infty} |(\partial_x U)| \mathrm{dx}$$

$$\le \|\partial_x f_0\|_{L^1} \le C_L - C_R + \alpha.$$

Hence

$$\int |(\partial_x U)_+|\, dx \le \alpha.$$

□

3. Relative entropy and some properties

In this section we introduce a special drift function $X(t), t \in (0,T)$, defined in Leger [20] and the evolution form of the relative entropy. To begin with we need some notations and properties provided in Leger [20]. Fix any strictly convex function $\eta \in C^2$, we first define the normalized relative entropy flux $g(\cdot,\cdot)$ by

$$g(x,y) := \frac{F(x,y)}{\eta(x|y)}$$

where the associated relative entropy functional $\eta(\cdot|\cdot)$ is given by

$$\eta(x|y) := \eta(x) - \eta(y) - \eta'(y)(x-y)$$

and the flux of the relative entropy $F(\cdot,\cdot)$ is defined by

$$F(x,y) := G(x) - G(y) - \eta'(y)(A(x) - A(y)). \tag{12}$$

Note that for any fixed y and any weak entropic solution of (3), we have

$$\partial_t \eta(u|y) + \partial_x F(u,y) \le 0.$$

Hence, g can be seen as a typical velocity associated to the relative entropy $\eta(\cdot,y)$.

Using the strict convexity of the function η, Leger showed in [20] the following lemma.

Lemma 3.1. *Let $x, y \in [-L, L]$ for any $L > 0$. There exists a constant $\Lambda > 0$, such that we have*

- $\frac{1}{\Lambda} \leq \eta''(x) \leq \Lambda$,
- $\frac{1}{2\Lambda}(x-y)^2 \leq \eta(x|y) \leq \frac{1}{2}\Lambda(x-y)^2$,
- $|F(x,y)| \leq \Lambda(x-y)^2$,
- $0 \leq (\partial_x g)(x,y) \leq \Lambda$,
- $\frac{1}{\Lambda} \leq (\partial_y g)(x,y)$.

In the spirit of Leger [20], we consider the solution of the following differential equation in order to define the shift function X

$$\left\{ \begin{array}{l} \dot{X}(t) = g\Big(U(t,X(t)), \frac{C_L+C_R}{2}\Big) \\ X(0) = 0 \end{array} \right. . \tag{13}$$

The existence and uniqueness of X comes from the Cauchy-Lipschitz theorem.

First, X is Lipschitz, since we have from Lemma 3.1

$$|\dot{X}(t)| \leq \frac{\Big|F\Big(U(t,X(t)), \frac{C_L+C_R}{2}\Big)\Big|}{\eta\Big(U(t,X(t))\Big|\frac{C_L+C_R}{2}\Big)} \leq 2\Lambda^2 \tag{14}$$

where we used the fact $\|U(t)\|_{L^\infty} \leq L$ for $t > 0$.

The idea of the proof is to study the evolution of the relative entropy of the solution with respect to the shock, outside of a small region centered at $X(t)$ (this small region corresponds to the layer localization):

$$\begin{aligned} &\int_{-\infty}^{X(t)-\delta\varepsilon} \Big(\eta(U(t,x)|C_L) + \int_L^L \eta'(v) f(t,x,v)\mathrm{dv} - \eta(U(t,x))\Big)\mathrm{dx} \\ &\qquad + \int_{X(t)+\delta\varepsilon}^{\infty} \Big(\eta(U(t,x)|C_R) + \int_L^L \eta'(v) f(t,x,v)\mathrm{dv} - \eta(U(t,x))\Big)\mathrm{dx}. \end{aligned} \tag{15}$$

Note that, since U is bounded, for any $\eta > 0$

$$\int_{\{|x-X(t)|\leq C\eta\}} \eta(U(t,x)|S(t,x))\mathrm{dx} \leq C|\{|x-X(t)| \leq C\eta\}| \leq C\eta. \tag{16}$$

for any $\eta > 0$. From now on we will take a reasonable $\delta > 0$ and it will be fixed in (25) later.

For the rigorous proof, we define the evolution of the following quantity

$$\begin{aligned} \mathcal{H}_\epsilon^\delta(t) := \int_{-\infty}^{\infty} \Big[\phi_\delta(|x-X(t)|/\varepsilon)\Big]^2 \Big(&\eta(U(t,x)|S(t,x)) \\ &+ \int_{-\infty}^{\infty} \eta'(v) f(t,x,v)\mathrm{dv} - \eta(U(t,x))\Big)\mathrm{dx} \end{aligned} \tag{17}$$

for any fixed $\delta > 0$ and $X \in C^1([0,T])$ where an increasing function ϕ_δ is defined by

$$\phi_\delta(x) = \left\{ \begin{array}{lll} 0 & \text{if} & x \leq 0, \\ 1 & \text{if} & x \geq \delta. \end{array} \right.$$

From now on we drop the δ from the notation of ϕ_δ. Thus, the derivative of $\mathcal{H}_\epsilon^\delta(t)$ implies the following lemma

Lemma 3.2. *The function* $\mathcal{H}_\epsilon^\delta(t)$*, defined in* (17)*, satisfies the following on* $(0,T)$

$$
\begin{aligned}
&\frac{d\mathcal{H}_\epsilon^\delta(t)}{dt}(t)\\
&=\int_{X(t)-\delta\varepsilon}^{X(t)} \frac{2}{\varepsilon}\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)\phi'\Big(\frac{-x+X(t)}{\varepsilon}\Big)\Big[\dot X(t)\eta(U|C_L)-F(U,C_L)\Big]\mathrm{dx}\\
&+\int_{X(t)-\delta\epsilon}^{X(t)}\int_{-\infty}^{\infty}\frac{2}{\epsilon}\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)\phi'\Big(\frac{-x+X(t)}{\varepsilon}\Big)(Mf(v)-f(v))\Phi(t,x,v)\mathrm{dvdx}\\
&+\int_{-\infty}^{X(t)}\int_{-\infty}^{\infty}\frac{1}{\epsilon}\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)^2(Mf(v)-f(v))\eta'(v)\mathrm{dvdx}\\
&-\int_{X(t)}^{X(t)+\delta\varepsilon}\frac{2}{\varepsilon}\phi\Big(\frac{x-X(t)}{\varepsilon}\Big)\phi'\Big(\frac{x-X(t)}{\varepsilon}\Big)\Big[\dot X(t)\eta(U|C_R)-F(U,C_R)\Big]\mathrm{dx}\\
&-\int_{X(t)}^{X(t)+\delta\varepsilon}\int_{-\infty}^{\infty}\frac{2}{\epsilon}\phi\Big(\frac{x-X(t)}{\varepsilon}\Big)\phi'\Big(\frac{x-X(t)}{\varepsilon}\Big)(Mf(v)-f(v))\Phi(t,x,v)\mathrm{dvdx}\\
&+\int_{X(t)}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\epsilon}\phi\Big(\frac{x-X(t)}{\varepsilon}\Big)^2(Mf(v)-f(v))\eta'(v)\mathrm{dvdx}\\
&:=L_1+L_2+L_3+R_1+R_2+R_3
\end{aligned}
\tag{18}
$$

where Φ *is defined as follows*

$$\Phi(t,x,v)=-\dot X(t)\eta'(v)-a(v)(\eta'(v)-\eta'(S)).$$

Proof. Let us first denote h by

$$h(t,x,v):=\eta(U|S)+\int_{-\infty}^{\infty}\eta'(v)f(v)\mathrm{dv}-\eta(U).$$

Since the function S is constant on both the set $\{x<X(t)\}$ and the set $\{x>X(t)\}$, we have for $U=\int f(v)\,dv$ and $x\neq X(t)$

$$
\begin{aligned}
h_t&=\int_{-\infty}^{\infty}\eta'(v)f_t(v)\mathrm{dv}-\eta'(S)U_t\\
&=\int_{-\infty}^{\infty}\eta'(v)\frac{Mf(v)-f(v)}{\epsilon}\mathrm{dv}-\partial_x\int_{-\infty}^{\infty}a(v)(\eta'(v)-\eta'(S))f(v)\mathrm{dv}\\
&=-\partial_x F(U,S)+\int_{-\infty}^{\infty}\eta'(v)\frac{Mf(v)-f(v)}{\epsilon}\mathrm{dv}\\
&\qquad+\partial_x\int_{-\infty}^{\infty}a(v)(\eta'(v)-\eta'(S))(Mf(v)-f(v))\mathrm{dv}.
\end{aligned}
\tag{19}
$$

Now, we spit H_ε^δ into two parts depending on the sign of $x-X(t)$

$$\begin{aligned}\mathcal{H}_\epsilon^\delta(t) &= \int_{-\infty}^{\infty}\Big[\phi\Big(\frac{|x-X(t)|}{\varepsilon}\Big)\Big]^2 h(t)\mathrm{dx}\\ &= \int_{-\infty}^{\infty}\Big(\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)^2+\phi\Big(\frac{x-X(t)}{\varepsilon}\Big)^2\Big)h(t)\mathrm{dx}\\ &:= \mathcal{H}_{\epsilon,l}^\delta(t)+\mathcal{H}_{\epsilon,r}^\delta(t).\end{aligned}$$

We differentiate $\mathcal{H}_{\epsilon,l}^\delta(t)$ with respect to time to find

$$\begin{aligned}&\frac{d}{dt}\mathcal{H}_{\epsilon,l}^\delta(t)=\\ &\int_{-\infty}^{\infty}\frac{2}{\epsilon}\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)\phi'\Big(\frac{-x+X(t)}{\varepsilon}\Big)\dot{X}(t)h(t,x)\mathrm{dx}\\ &+\int_{-\infty}^{\infty}\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)^2 h_t(t,x)\mathrm{dx}\\ &=\int_{X(t)-\delta\varepsilon}^{X(t)}\frac{2}{\varepsilon}\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)\phi'\Big(\frac{-x+X(t)}{\varepsilon}\Big)\Big[\dot{X}(t)\eta(U|C_L)-F(U,C_L)\Big]dx\\ &+\int_{X(t)-\delta\varepsilon}^{X(t)}\int_{-\infty}^{\infty}\frac{2}{\varepsilon}\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)\phi'\Big(\frac{-x+X(t)}{\varepsilon}\Big)\dot{X}(t)(f(v)-Mf(v))\eta'(v)\mathrm{dvdx}\\ &-\int_{X(t)-\delta\varepsilon}^{X(t)}\int_{-\infty}^{\infty}\frac{2}{\varepsilon}\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)\phi'\Big(\frac{-x+X(t)}{\varepsilon}\Big)\\ &\qquad\times a(v)(\eta'(v)-\eta'(S))(Mf(v)-f(v))\mathrm{dvdx}\\ &+\int_{-\infty}^{X(t)}\int_{-\infty}^{\infty}\frac{1}{\epsilon}\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)\eta'(v)(Mf(v)-f(v))\mathrm{dvdx}.\end{aligned}$$

We obtain the result for $\mathcal{H}_{\epsilon,r}^\delta(t)$ in the same way. □

4. HYPERBOLIC TERM

This section is devoting to prove that the hyperbolic part L_1+R_1 in the equality (18) is strictly negative. Applying the key proposition 2.1 and lemma 2.2, we are able to show the main proposition for this section.

Proposition 4.1. *Let L_1 and R_1 be as in Lemma 3.2. Then, there exists a constant $\theta>0$ such that, for any ε,δ satisfying*

$$\varepsilon\delta\le\theta,$$

we have

$$L_1+R_1\le-\frac{\theta}{\varepsilon}\int_{X(t)-\delta\varepsilon}^{X(t)+\delta\varepsilon}\phi\left(\frac{|x-X(t)|}{\varepsilon}\right)\phi'\left(\frac{|x-X(t)|}{\varepsilon}\right)\eta(U|S)\mathrm{dx}.$$

Proof. Let us start with proving that L_1 is strictly negative. The proof of R_1 is similar. With the definition of $X(t)$, we write L_1 as

$$L_1=\int_{X(t)-\delta\varepsilon}^{X(t)}\frac{2}{\varepsilon}\phi\left(\frac{-x+X(t)}{\varepsilon}\right)\phi'\Big(\frac{-x+X(t)}{\varepsilon}\Big)\cdot\eta(U|C_L)\cdot G(t,x)dx$$

where $G(t,x) := \Big[g\Big(U(t,X(t)),\frac{C_L+C_R}{2}\Big) - g(U(t,x),C_L)\Big]$. Using Lemma 2.2 we find

$$U(t,X(t)) - U(t,x) = \int_x^{X(t)} (\partial_x U)(t,y)\mathrm{dy} \le \int_x^{X(t)} (\partial_x U)_+(t,y)\mathrm{dy} \le \alpha. \tag{20}$$

We next observe that $G(t,x), t\in(0,T), x\in\mathbb{R}$, is strictly negative. To do this, we rewrite the function G as

$$\begin{aligned} G(t,x) &= g\Big(U(t,X(t)),\frac{C_L+C_R}{2}\Big) - g\Big(U(t,x),\frac{C_L+C_R}{2}\Big) \\ &\quad + g\Big(U(t,x),\frac{C_L+C_R}{2}\Big) - g(U(t,x),C_L). \end{aligned}$$

Then, thanks to Lemma 3.1 and the inequality (20), we get

$$\begin{aligned} G(t,x) &\le g\Big(U(t,x)+\alpha,\frac{C_L+C_R}{2}\Big) - g\Big(U(t,x),\frac{C_L+C_R}{2}\Big) \\ &\quad + g\Big(U(t,x),\frac{C_L+C_R}{2}\Big) - g(U(t,x),C_L) \\ &\le \alpha\Lambda - \frac{C_L-C_R}{2\Lambda} \le -\theta/2 < 0 \end{aligned}$$

for $\alpha>0$ small enough. Note that the smallness of α depends only on the shock C_L, C_R, the flux A, and the L^∞ norm of U which is controlled by the BV norm of U_0. Since $\phi(\cdot),\phi'(\cdot)$ and $\eta(\cdot|\cdot)\ge 0$, we get

$$L_1 \le -\theta\int_{X(t)-\delta\varepsilon}^{X(t)} \frac{1}{\varepsilon}\phi\left(\frac{-x+X(t)}{\varepsilon}\right)\phi'\Big(\frac{-x+X(t)}{\varepsilon}\Big)\eta(U|C_L)\mathrm{dx}.$$

Similarly, we also obtain that

$$R_1 \le -\theta\int_{X(t)}^{X(t)+\delta\varepsilon} \frac{1}{\varepsilon}\phi\left(\frac{x-X(t)}{\varepsilon}\right)\phi'\Big(\frac{x-X(t)}{\varepsilon}\Big)\eta(U|C_R)\mathrm{dx}.$$

Consequently, combining the two last inequalities gives the desired result. □

5. Dissipation term

In this section we are going to estimate the dissipation term, $L_2+L_3+R_2+R_3$. The main purpose of this section is to show the proof of the following proposition.

Proposition 5.1. *Let $L_i, R_i, i=1,2$, be as in Lemma 3.2. Then, there exists constants $C>0, C^*>0$ such that the following inequality holds:*

$$L_2+L_3+R_2+R_3 \le \frac{C}{\varepsilon}\int_{X(t)-\delta\varepsilon}^{X(t)+\delta\varepsilon} \Big[\phi'\Big(\frac{|x-X(t)|}{\varepsilon}\Big)\Big]^2 \chi_{\{\phi'-C^*\phi>0\}}\mathrm{dx}. \tag{21}$$

Proof. Let us first control the estimates L_2+L_3. We use the following representation of $Mf-f$ from Brenier [6] (see also [23]): there exists a nonnegative bounded function, supported in $\mathbb{R}^+\times\mathbb{R}\times[-L,L]$ such that

$$Mf - f = \partial_v m. \tag{22}$$

Indeed, the function m is defined by

$$m(v) = \int_{-L}^{v} Mf - f\,dv.$$

Since f and Mf have the same mass, $m(L) = 0$ holds. We can check that m is nondecreasing on $[-L, \int f\, dv]$ and non increasing on $[\int f\, dv, L]$, ensuring the non negativity of m. Finally, since f and Mf are bounded by 1,

$$\|m\|_{L^\infty} \leq 2L.$$

We estimate $L_2 + L_3$ as follows:

$L_2 + L_3 :=$

$$\int_{X(t)-\delta\epsilon}^{X(t)} \int_{-L}^{L} \frac{2}{\epsilon} \phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)\phi'\Big(\frac{-x+X(t)}{\varepsilon}\Big)(Mf(v)-f(v))\Phi(t,x,v)\mathrm{dvdx}$$

$$+\int_{-\infty}^{X(t)} \int_{-L}^{L} \frac{1}{\epsilon}\Big[\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)\Big]^2 (Mf(v)-f(v))\eta'(v)\mathrm{dvdx}$$

where

$$\Phi(t,x,v) = -\dot{X}(t)\eta'(v) - a(v)(\eta'(v)-\eta'(S)).$$

Integrating by parts in v and using (22), we find to get

$$\begin{aligned}
&L_2 + L_3 \\
&\leq \int_{X(t)-\delta\epsilon}^{X(t)} \int_{-L}^{L} \frac{2}{\epsilon} \phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)\phi'\Big(\frac{-x+X(t)}{\varepsilon}\Big)\Big((\dot{X}(t)+a(v))\eta''(v) \\
&\qquad + a'(v)(\eta'(v)-\eta'(S))\Big)m(x,v)\mathrm{dvdx} \\
&\quad - \int_{-\infty}^{X(t)} \int_{-L}^{L} \frac{1}{\epsilon}\Big[\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)\Big]^2 \eta''(v) m(x,v)\mathrm{dvdx} \\
&\leq \int_{-L}^{L} \Bigg(\int_{X(t)-\delta\epsilon}^{X(t)} \frac{C}{\epsilon}\Big[\phi\phi'\Big(\frac{-x+X(t)}{\varepsilon}\Big)\Big]\mathrm{dx} \\
&\qquad - \int_{X(t)-\delta\epsilon}^{X(t)} \frac{1}{\epsilon\Lambda}\Big[\phi\Big(\frac{-x+X(t)}{\varepsilon}\Big)\Big]^2\mathrm{dx}\Bigg) m(x,v)\mathrm{dv} \\
&\leq \frac{4L^2 C}{C^*\varepsilon} \int_{X(t)-\delta\varepsilon}^{X(t)} \Big[\phi'\Big(\frac{|x-X(t)|}{\varepsilon}\Big)\Big]^2 \chi_{\{\phi'-C^*\phi>0\}}\mathrm{dx},
\end{aligned}$$

for $C^* = 1/(C\Lambda)$.

Similarly, we get the estimate for $R_2 + R_3$.

$$R_2 + R_3 \leq \frac{C}{\varepsilon} \int_{X(t)}^{X(t)+\delta\varepsilon} \Big[\phi'\Big(\frac{x-X(t)}{\varepsilon}\Big)\Big]^2 \chi_{\{\phi'-C^*\phi>0\}}\mathrm{dx}.$$

Combining the two last inequalities, we obtain the result.

□

6. Proof of Theorem1.1

From Lemma 3.2, Proposition 4.1, and Proposition (5.1), we get

$$\frac{d\mathcal{H}^\delta_\epsilon(t)}{dt} \leq \frac{C}{\varepsilon} \int_{X(t)-\delta\varepsilon}^{X(t)+\delta\varepsilon} \Big[\phi'\Big(\frac{|x-X(t)|}{\varepsilon}\Big)\Big]^2 \chi_{\{\phi'-C^*\phi>0\}}\mathrm{dx}. \tag{23}$$

Applying the change of variables $z = (x - X(t))/\varepsilon$, and changing θ by $\inf(\theta, C^*)$ if necessary, we find:

$$\begin{aligned} \frac{d\mathcal{H}^{\delta}_{\epsilon}(t)}{dt} &\le \frac{C}{\varepsilon} \int_{X(t)-\delta\varepsilon}^{X(t)+\delta\varepsilon} \left[(\phi')^2 \chi_{\{\phi'-\theta\phi>0\}}\right] \left(\frac{|x - X(t)|}{\varepsilon}\right) \mathrm{dx} \\ &\le C \int_0^{\delta} (\phi')^2(z) \chi_{\{\phi'-\theta\phi>0\}}(z) \mathrm{dz}. \end{aligned} \tag{24}$$

To get good estimate, we take a specific ϕ_δ. For any $\delta \ge 1/\theta$, we now fix the function ϕ_δ in the following explicit way.

$$\phi_\delta(x) = \begin{cases} \theta e^{1-\theta\delta} x, & \text{for } x \in [0, 1/\theta), \\ e^{\theta(x-\delta)}, & \text{for } x \in [1/\theta, \delta]. \end{cases} \tag{25}$$

We use the computation:

$$\int_0^{\delta} (\phi_\delta'(x))^2 \chi_{\{\phi_\delta' > \theta\phi_\delta\}} \, dx = C_\theta \cdot e^{-2\theta\delta}. \tag{26}$$

For the proof of (I), we integrate the estimate of Proposition 5.1 between 0 and $t \in (0, T)$ such that, for any ε, δ with $\frac{1}{\theta} \le \delta$ and $\varepsilon\delta \le \theta$, where θ is the constant from Proposition 5.1, it follows that

$$\begin{aligned} &\int_{\{|x-X(t)|\ge\delta\varepsilon\}} \eta(U(t,x)|S(t,x))\mathrm{dx} + \int_{\mathbb{R}} \int_{-L}^{L} \eta'(v) f(t,x,v)\mathrm{dv} - \eta(U(t,x))\mathrm{dx} \\ &\le \mathcal{H}^{\delta}_{\epsilon}(0) + \int_0^t \frac{d}{dt}\mathcal{H}^{\delta}_{\epsilon}(s)\mathrm{ds} \\ &\le \int_{\mathbb{R}} \eta(U_0|S_0)\mathrm{dx} + \int_{\mathbb{R}} \int_{-\infty}^{\infty} \eta'(v) f_0(x,v)\mathrm{dv} - \eta(U_0(x))\mathrm{dx} + CTe^{-\theta\delta} \end{aligned}$$

where we have used (24), (26) and the following fact

$$\int_{-\infty}^{\infty} \eta'(v) f(t,x,v)\mathrm{dv} - \eta(U(t,x)) = \int_{-\infty}^{\infty} \eta''(v) m(v)\mathrm{dv} \ge 0.$$

By taking $\varepsilon_0 := \theta^2$, we have for any $\varepsilon \le \beta \le \varepsilon_0$,

$$\begin{aligned} &\int_{\{|x-X(t)|\ge\beta/\theta\}} \eta(U(t,x)|S(t,x))\mathrm{dx} + \int_{\mathbb{R}} \int_{-\infty}^{\infty} \eta'(v) f(t,x,v)\mathrm{dv} - \eta(U(t,x))\mathrm{dx} \\ &\le \int_{\mathbb{R}} \eta(U_0|S_0)\mathrm{dx} + \int_{\mathbb{R}} \int_{-\infty}^{\infty} \eta'(v) f_0(x,v)\mathrm{dv} - \eta(U_0(x))\mathrm{dx} + CTe^{-\beta/\varepsilon}. \end{aligned} \tag{27}$$

Observe that

$$\int_{\mathbb{R}} \eta(U|S)\mathrm{dx} = \int_{\{|x-X(t)|\ge C\alpha\}} \eta(U|S)\mathrm{dx} + \int_{\{|x-X(t)|<C\alpha\}} \eta(U|S)\mathrm{dx}. \tag{28}$$

Consequently, using the inequlaities (16), (27), (28) and taking $\beta = \varepsilon\log(1/\varepsilon)$, we get, for any $t \in (0, T)$,

$$\begin{aligned} &\int_{\mathbb{R}} \eta(U|S)\mathrm{dx} + \int_{\mathbb{R}} \int_{-\infty}^{\infty} \eta'(v) f(t,x,v)\mathrm{dv} - \eta(U(t,x))\mathrm{dx} \\ &\le \int_{\mathbb{R}} \eta(U_0|S_0)\mathrm{dx} + \int_{\mathbb{R}} \int_{-\infty}^{\infty} \eta'(v) f_0(x,v)\mathrm{dv} - \eta(U_0(x))\mathrm{dx} + C\varepsilon\log(1/\varepsilon) \end{aligned} \tag{29}$$

for any $\varepsilon \le \varepsilon_0$,which proves (6) by taking $\eta(v) = v^2$.

We now prove (8). Let us define the function ψ by

$$\psi(x) := \begin{cases} 0 \text{ if } |x| > 2, \\ 1 \text{ if } |x| \le 1 \\ 2 - |x| \text{ if } 1 < |x| \le 2. \end{cases}$$

Let $s \in (0,t)$ and $R > 0$. Multiplying $\Psi_R(s,x) := \psi(\frac{x - X(s)}{R})$ to the equation (1) and integrating in x, we get

$$\begin{aligned} 0 = & -\frac{d}{ds}\int \Psi_R \cdot U dx + \int \partial_x(\Psi_R) A(U) dx + \int \partial_t(\Psi_R) U \mathrm{dx} \\ & + \int \Psi_R \cdot \partial_x \int a(v)(Mf(v) - f(v)) \mathrm{dvdx} \\ = & \underbrace{-\frac{d}{ds}\int \psi(\frac{x - X(s)}{R}) \cdot U(s,x) dx}_{(I)} \\ & + \underbrace{\frac{1}{R}\int \psi'(\frac{x - X(s)}{R}) \cdot \Big(A(U(s,x)) - \dot{X}(s) U(s,x)\Big) dx}_{(II)} \\ & \underbrace{-\frac{1}{R}\int \psi'(\frac{x - X(s)}{R}) \cdot \int a(v)(Mf(v) - f(v)) \mathrm{dvdx}}_{(III)}. \end{aligned}$$

By using the above observation, we have

$$\begin{aligned} (\sigma - \dot{X}(s)) &= \frac{1}{C_L - C_R}\Big(A(C_L) - A(C_R) - (C_L - C_R)\dot{X}(s)\Big) \\ &= \frac{1}{C_L - C_R}\Big(A(C_L) - A(C_R) - (C_L - C_R)\dot{X}(s) - (II) + (I) + (III)\Big). \end{aligned}$$

Then we integrate the above equation in time on $[0,t]$ to get:

$$\begin{aligned} |\sigma t - X(t)| \le C\Big(t \cdot \max_{s \in (0,t)} & \underbrace{\Big|A(C_L) - A(C_R) - (C_L - C_R)\dot{X}(s) - (II)\Big|}_{(II')} \\ & + \Big|\int_0^t (I) ds\Big| + t \cdot \max_{s \in (0,t)} \Big|(III)\Big|\Big). \end{aligned} \tag{30}$$

From the result of Choi and Vasseur [11], we already know the following results:

$$(II')^2 \le \frac{C}{R} \cdot \int_{\mathbb{R}} \eta(U(s)|S(s)) dx. \tag{31}$$

and

$$\Big|\int_0^t (I) ds\Big|^2 \le CR\Big(\int_{\mathbb{R}} \eta(U(t)|S(t)) dx + \int_{\mathbb{R}} \eta(U_0|S_0) dx\Big). \tag{32}$$

Before we prove the estimate of (III), we first need to control

$$\begin{aligned} \int_{\mathbb{R}} \int_{-\infty}^{\infty} a(v) f_0(v) \mathrm{dv} - A(U_0) \mathrm{dx} &= \int_{\mathbb{R}} \int_{-\infty}^{\infty} a'(v) \mu \mathrm{dvdx} \\ \le 2C \int_{\mathbb{R}} \int_{-\infty}^{\infty} \mu \mathrm{dvdx} &= C \int_{\mathbb{R}} \int_{-\infty}^{\infty} 2v f_0(v) \mathrm{dv} - U_0^2 \mathrm{dx} \end{aligned} \tag{33}$$

where μ is the nonnegative measure such that

$$Mf - f_0 = \partial_v \mu.$$

Following (29), it is easily seen to show

$$\begin{aligned}\Big|(III)\Big| &= \frac{1}{R}\Big|\int \psi'(\frac{x-X(s)}{R})\cdot\int_{\mathbb{R}} a(v)(Mf(v)-f(v)\mathrm{dvdx}\Big|\\ &\le \frac{C}{R}\Big(\int_{\mathbb{R}}|U_0-S_0|^2\mathrm{dx}+\epsilon\log\frac{1}{\epsilon}+\int_{\mathbb{R}}\int_{-\infty}^{\infty}a(v)f_0(v)\mathrm{dv}-A(U_0)\mathrm{dx}\Big).\end{aligned} \tag{34}$$

Finally, by using (29), we combine (31), (32) and (34) with (30) to get, for any $t\in(0,T)$,

$$\begin{aligned}&|\sigma t-X(t)|^2\\ &\le C\Big(\frac{t^2}{R}+\frac{t^2}{R^2}+R\Big)\cdot\Big(\int_{\mathbb{R}}|U_0-S_0|^2dx+\varepsilon\log(\frac{1}{\varepsilon})\Big)\\ &+\frac{C\cdot\cdot t^2}{R^2}\int_{\mathbb{R}}\int_{-\infty}^{\infty}a(v)f_0(v)\mathrm{dv}-A(U_0)\mathrm{dx}.\end{aligned} \tag{35}$$

Since the above estimate holds for any $0<R<\infty$, the inequality (33) and (35) provide the estimate (8) by taking $R:=t^{2/3}$.

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