Study of a generalized fragmentation model for sprays

Nicholas LEGER[1], and Alexis VASSEUR[1]

[1] Department of Mathematics
University of Texas at Austin
nleger@math.utexas.edu, vasseur@math.utexas.edu

Abstract

We study a mathematical model for sprays which takes into account particle break-up due to drag forces. In particular, we establish the existence of global weak solutions to a system of incompressible Navier-Stokes equations coupled with a Boltzmann-like kinetic equation. We assume the particles initially have bounded radii and bounded velocities relative to the gas, and we show that those bounds remain as the system evolves. One interesting feature of the model is the apparent accumulation of particles with arbitrarily small radii. As a result, there can be no nontrivial hydrodynamical equilibrium for this system.

Key words. Fluid-particles interaction. Vlasov-Stokes equation. Hydrodynamic limits.

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1 Introduction

In this paper, we study a fragmentation model for a general class of sprays. For our purposes, a spray can be described as an ensemble of liquid particles interacting with a gas. We describe the distribution of particles through a density function $f(t,x,v,r) \geq 0$, so that, at time $t \geq 0$, the integral

$$\int_\Omega \int_V \int_R f(t,x,v,r)\, dr\, dv\, dx,$$

represents the expected number of particles contained in the set $\Omega \subset \mathbb{R}^3$, with velocity in $V \subset \mathbb{R}^3$, and with radius in $R \subset \mathbb{R}^+$. The evolution of f is determined by the kinetic equation

$$\partial_t f + \nabla_x \cdot (vf) + \nabla_v \cdot (Af) = \Gamma(f), \tag{1.1}$$

where A represents the acceleration of a particle due to drag forces, and the fragmentation operator, Γ, is given by

$$\begin{aligned}\Gamma(f)(t,x,v,r) = &-\nu(r,v)f(t,x,v,r) \\ &+ \int_{\mathbb{R}^+}\int_{\mathbb{R}^3} \nu(r^*,v^*)h(r,v,r^*,v^*)f(t,x,v^*,r^*)\,dv^*\,dr^*.\end{aligned} \tag{1.2}$$

Here, $\nu(r,v)$ is the break-up frequency, and $h(r,v,r^*,v^*)$ is the probability density describing the redistribution of particles after break-up. More precisely, $\int_V \int_R h(r,v,r^*,v^*)\,dr\,dv$ is the probability that fragmentation of a particle with radius r^* and velocity v^* will produce a particle with radius $r \in R$ and velocity $v \in V$.

The fragmentation operator was introduced by Hylkema and Villedieu in [13] and has been previously studied by Dufour et al. (in [10]) and Baranger (in [1]), among others. In these contexts, ν and h were determined experimentally for a gas with constant velocity field u_g. In attempt to preserve the structure of Γ for general velocity fields, we introduce a change of variables. Namely, we express ν and h as functions of the relative velocity $w = v - u(t,x)$. More accurately, we assume the existence of ν and h which are independent of the velocity of the gas and depend only on the relative velocities of the particles and their radii. This is quite natural when one considers the direct dependence of fragmentation on drag forces. In these coordinates, we shall consider, instead, the density

$$g(t,x,w,r) = g(t,x,v-u(t,x),r) := f(t,x,v,r), \tag{1.3}$$

which satisfies the PDE

$$\left\{\begin{aligned} &\partial_t g + \nabla_x \cdot ((u+w)g) + \nabla_w \cdot (Ag) \\ &\qquad + \nabla_w \cdot [(-\partial_t u - u \cdot \nabla_x u)g] + \nabla_w \cdot [(-w \cdot \nabla_x u)g] = \Gamma(g). \end{aligned}\right. \tag{1.4}$$

In order to simplify the mathematical analysis, we drop the last two terms on the left hand side of (1.4); that is, we assume the gradient of $u(t,x)$ is small, so that g is given, approximately, by

$$\begin{cases} \partial_t g + \nabla_x \cdot ((u+w)g) + \nabla_w \cdot (Ag) = -\nu(r,w)g(t,x,w,r) \\ \qquad + \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \nu(r^*,w^*)h(r,w,r^*,w^*)g(t,x,w^*,r^*)\,dw^*\,dr^*. \end{cases} \tag{1.5}$$

As for the gas, we require that the velocity field $u(t,x) \in \mathbb{R}^3$ verifies the incompressible Navier-Stokes equations:

$$\begin{cases} \partial_t u + \mathrm{Div}_x(u \otimes u) - \Delta_x u + \nabla_x p = \mathfrak{F}, \\ \mathrm{div}_x(u) = 0. \end{cases} \tag{1.6}$$

Here, $p(t,x), \mathfrak{F}(t,x) \in \mathbb{R}^3$ represent the pressure and external force, respectively. Recall that $u \otimes u$ represents the $n \times n$ matrix with entries $(u \otimes u)_{ij} = u_i u_j$, so that componentwise u satisfies

$$\partial_t u_j + \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(u_i u_j) - \Delta_x u_j + \frac{\partial}{\partial x_j} p = \mathfrak{F}_j.$$

Finally, equations (1.5) and (1.6) are coupled through drag forces. We will assume that the force on a particle of radius r moving at a velocity w relative to the fluid is approximated by Stokes' Law; that is, $F(r,w) = -rw$. (We set all constants equal to 1.) Accordingly, we take the acceleration of a particle to be $A(r,w) = -\frac{w}{r^2}$. Integrating the density of drag forces exerted on the fluid, we obtain

$$\mathfrak{F} = -\int_{\mathbb{R}^+} \int_{\mathbb{R}^3} Fg\,dw\,dr = \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} rwg\,dw\,dr. \tag{1.7}$$

The goal of the paper is to establish the existence of weak solutions to the coupled equations (1.5) and (1.6) when the initial distribution of particles, g^0, has bounded support with respect to the variables r and w. As our main result, we prove the following theorem.

Theorem 1 *Assume $g^0 \in L^\infty \cap L^1(\mathbb{R}^6 \times \mathbb{R}^+)$. Fix R, $W > 0$ and suppose* $\mathrm{supp}(g^0) \subset \mathbb{R}^3_x \times \Omega$ *where* $\Omega = \{(r,w) \in \mathbb{R}^+ \times \mathbb{R}^3 : 0 < r \le R, 0 \le |w| \le W\}$. *Further suppose $u^0 \in \mathbb{P}L^2(\mathbb{R}^3)$, where $\mathbb{P}$ denotes the Leray projector. Then there exists a weak solution to the coupled system of initial value problems*

$$\begin{cases} \partial_t g + \nabla_x \cdot ((u+w)g) + \nabla_w \cdot (-\frac{w}{r^2} g) = \Gamma(g), \\ g(0,x,w,r) = g^0(x,w,r), \end{cases} \tag{1.8}$$

$$
\begin{cases}
\partial_t u + \mathrm{Div}_x(u \otimes u) - \Delta_x u + \nabla_x p = \displaystyle\int_{\mathbb{R}^+}\int_{\mathbb{R}^3} rwg\,dw\,dr, \\
\mathrm{div}_x(u) = 0, \\
u(x,0) = u^0(x).
\end{cases} \tag{1.9}
$$

such that

$$
\mathrm{supp}(g(t)) \subset \mathbb{R}^3_x \times \Omega, \text{ for a.e. } t \in [0,T], \tag{1.10}
$$

$$
g \in L^\infty(0,T; L^\infty_{loc}(\mathbb{R}^6 \times \mathbb{R}^+)) \cap L^\infty(0,T; L^1(\mathbb{R}^6 \times \mathbb{R}^+)), \tag{1.11}
$$

with g *weakly continuous from* $[0,T]$ *into* $L^2_{loc}(\mathbb{R}^6 \times \mathbb{R}^+)$*, and*

$$
u \in L^2(0,T; H^1_0(\mathbb{R}^3)) \cap L^\infty(0,T; L^2(\mathbb{R}^3)), \tag{1.12}
$$

with u *weakly continuous from* $[0,T]$ *into* $\mathbb{P}L^2(\mathbb{R}^3)$*.*

We say that g is a weak solution on $[0,T]$ if (1.8) holds in the sense of distributions, that is, for any $\varphi \in \mathcal{C}^\infty([0,T] \times \mathbb{R}^6 \times \mathbb{R}^+)$ with $\varphi(\cdot,T) = 0$ and $\varphi(\cdot,t)$ compactly supported for all $t \in [0,T]$, then

$$
\int_0^T \int_{\mathbb{R}^6\times\mathbb{R}^+} g[\partial_t\varphi + (u+w)\cdot\nabla_x\varphi - \frac{w}{r^2}\cdot\nabla_w\varphi] + \varphi\Gamma(g)\,dr\,dw\,dx\,dt
$$
$$
+ \int_{\mathbb{R}^6\times\mathbb{R}^+} g^0\varphi(0,x,w,r)\,dr\,dw\,dx = 0.
$$

Fluid/particle models of this type have been well-studied in recent years, in part, due to a growing list of industrial applications ranging from sedimentation analysis (see Berres et al. [3] and Gidaspow [11]) to combustion theory (see Williams [17] and [18]). The simplest of such models, describing two-phase flow in one spatial dimension, was investigated by Domelevo and Roquejoffre in [9], where the authors prove the global existence and uniqueness of smooth solutions to a viscous Burgers equation coupled with a Vlasov equation. The existence and uniqueness of classical solutions to the IVP for a system of Vlasov-Fokker-Planck equations (which take into account the Brownian motion of particles) coupled with Poisson's equation was established by F. Bouchut in [7]. In a subsequent paper, [8], J. Carrillo studied the initial-boundary-value problems associated to the Poisson-Vlasov-Fokker-Planck system. The existence of global weak solutions for a system of compressible Navier-Stokes equations coupled with the Vlasov-Fokker-Planck equation is verified by Mellet and Vasseur in [15], and prior to that result, the existence of global weak solutions to the Vlasov-Stokes equations was established by Hamdache in [12]. Finally, coupling of the Vlasov equation with the compressible Euler equations in the context of sprays is investigated by Baranger and Desvillettes in [2]. In this paper, the authors prove the existence and uniqueness of classical solutions for small time in the case of smooth initial data.

In addition to work related to well-posedness, there have been a number of recent results addressing the asymptotic behavior of kinetic models, including in many cases the characterization of steady-state solutions. An interesting feature of the present model is the apparent accumulation of mass near $w = 0$ and $r = 0$. Indeed, the only functions for which the fragmentation operator vanishes involve a dirac mass centered at $w = 0$. This type of phenomenon is seen, for example, in the inelastic Boltzmann models considered by Bobylev et al. (in [4], [6], and [5]), which admit (without additional forcing terms) only dirac masses as steady solutions. It is the goal of ongoing research to explore in more detail these asymptotic properties of the model.

Let us now present a brief outline of the paper. In section 2, we discuss some of the basic assumptions related to ν and h and prove that the kinetic equation is mass-preserving. Section 3 is devoted to the study of the kinetic equation. In particular, we show that for $u(t,x) \in L^2(0,T;W^{1,\infty}(\mathbb{R}^3))$, there exists a unique weak solution to the initial value problem (1.8). In the final section, we couple the kinetic equation with the incompressible Navier-Stokes equations and prove Theorem 1 by means of a fixed point argument.

2 Hypotheses and estimates related to Γ

Let $\Phi = \mathbb{R}^+ \times (\mathbb{R}^3 \setminus \{0\})$ where $\mathbb{R}^+ = \{r \in \mathbb{R} : r > 0\}$. We will assume the following:

(i) $\nu \in C^1(\overline{\mathbb{R}^+ \times \mathbb{R}^3})$, $\nu \geq 0$.

(ii) $\nu(r,w) = 0$ if and only if $w = 0$.

(iii) $h \in C^1(\overline{\Phi} \times \Phi)$, $h \geq 0$.

(iv) $h(r,w,r^*,w^*) = 0$ if $r \geq r^*$ or $|w| \geq |w^*|$.

(v) $\exists\, C > 0 : \nu(r^*, w^*) h(r, w, r^*, w^*) \le C$ for all $(r, w, r^*, w^*) \in \overline{\Phi} \times \Phi$.

(vi) $$\int_a^b \int_{\mathbb{R}^3} h(r, w, r^*, w^*)\, dw\, dr = \int_{R(b)}^{R(a)} \int_{\mathbb{R}^3} h(r, w, r^*, w^*)\, dw\, dr,$$

where $R(r) = \sqrt[3]{r^{*3} - r^3}$ and $0 \le a \le b \le \frac{r^*}{\sqrt[3]{2}}$.

(vii) $$\int_0^{\frac{r^*}{\sqrt[3]{2}}} \int_{\mathbb{R}^3} h(r, w, r^*, w^*)\, dw\, dr = \int_{\frac{r^*}{\sqrt[3]{2}}}^{r*} \int_{\mathbb{R}^3} h(r, w, r^*, w^*)\, dw\, dr = 1.$$

First, let us point out that we assume in (i) and (iii) that ν and h are continuous up to the boundary $\{r = 0\} \times \{w = 0\}$, while h may be unbounded near $\{r^* = 0\} \times \{w^* = 0\}$ (we will say more about this shortly). Furthermore, we will assume, as we shall explain in the appendix, that the product νh which appears in the operator Γ verifies (v). Condition (ii) expresses the fact that particles will break up if and only if the drag force is non-zero. Next, we assume that the size and relative velocity of a particle do not increase after break-up; this is condition (iv). It is also reasonable to assume that fragmentation of a particle produces exactly two particles with complementary radii ($r^3 + R^3 = r^{*3}$), and this is expressed in (vi). Additionally, the probability that fragmentation will produce a particle with volume less (or greater, respectively) than or equal to half the volume of the original particle should be equal to one.

It should be noted that hypotheses (vi) and (vii) are not typically found in the literature (cf. [10] and [1]). Instead, it is standard to assume that h verifies the relation: $\int_{\mathbb{R}^3} \int_{\mathbb{R}^+} r^3 h(r, w, r^*, w^*)\, dw\, dr = r^{*3}$, for all $r^* > 0$. This property is associated with mass conservation, and rightly so, however we prefer to include this as a lemma which follows from the fundamental assumptions above. In any case, taking this property into account, observe that (for fixed $r^* > 0$) as $w^* \to 0$, condition (iv) requires that the support of $r^3 h(r, w, r^*, w^*)$, with respect to r and w, has measure decreasing to zero. Therefore, if mass is conserved, h must blow-up. Further details related to ν and h are given in the appendix.

Lemma 1 *Consider a function h satisfying $(iii) - (vii)$. Then h verifies $\int_{\mathbb{R}^+} \int_{\mathbb{R}^3} r^3 h(r, w, r^*, w^*)\, dw\, dr = r^{*3}$, for every $(r^*, w^*) \in (0, \infty) \times (\mathbb{R}^3 \setminus \{0\})$.*

Proof. Fix $r^* > 0$ and $w^* \in \mathbb{R}^3 \setminus \{0\}$ and set $H(r) = \displaystyle\int_{\mathbb{R}^3} h(r, w, r^*, w^*)\, dw$. Then,

$$\int_a^b H(r)\, dr = \int_{R(b)}^{R(a)} H(s)\, ds = \int_{R^{-1}(R(a))}^{R^{-1}(R(b))} H\left(\sqrt[3]{r^{*3} - r^3}\right) \frac{r^2}{\left(\sqrt[3]{r^{*3} - r^3}\right)^2}\, dr.$$

Since this holds for all $0 \le a \le b \le \frac{r^*}{\sqrt[3]{2}}$, we conclude that

$$H(r) = \frac{r^2}{\left(\sqrt[3]{r^{*3} - r^3}\right)^2} H\left(\sqrt[3]{r^{*3} - r^3}\right), \text{ a.e.}. \tag{2.1}$$

Finally, we have

$$\begin{aligned}
\int_0^{\frac{r^*}{\sqrt[3]{2}}} r^3 H(r)\, dr &= \int_0^{\frac{r^*}{\sqrt[3]{2}}} r^3 \frac{r^2}{\left(\sqrt[3]{r^{*3} - r^3}\right)^2} H\left(\sqrt[3]{r^{*3} - r^3}\right)\, dr \\
&= \int_{R(0)}^{R(\frac{r^*}{\sqrt[3]{2}})} (r^{*3} - s^3) H(s)(-1)\, ds = \int_{\frac{r^*}{\sqrt[3]{2}}}^{r^*} (r^{*3} - s^3) H(s)\, ds \\
&= r^{*3} - \int_{\frac{r^*}{\sqrt[3]{2}}}^{r^*} s^3 H(s)\, ds,
\end{aligned}$$

which, together with (iv) ends the proof. Note that (vii) has been used in the last equality. ∎

Finally, thanks to the structure of Γ, we are able to show that the kinetic equation (1.8) is mass preserving.

Lemma 2 *Suppose u is smooth and bounded and let g be a regular solution to (1.8). Further, suppose $r^3 g^0 \in L^1(\mathbb{R}^6 \times \mathbb{R}^+)$. Then $r^3 g \in L^\infty(0, T; L^1(\mathbb{R}^6 \times \mathbb{R}^+))$. Furthermore, $\|r^3 g(t)\|_{L^1(\mathbb{R}^6 \times \mathbb{R}^+)} = \|r^3 g^0\|_{L^1(\mathbb{R}^6 \times \mathbb{R}^+)}$ for a.e. $t > 0$.*

Proof. We multiply (1.5) by r^3 and integrate with respect to x, w, and r. Due to the divergence form of the equation we get

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^6 \times \mathbb{R}^+} r^3 g(t, x, w, r)\, dr\, dw\, dx = - \int_{\mathbb{R}^6 \times \mathbb{R}^+} r^3 \nu(r, w) g(t, x, w, r)\, dr\, dw\, dx \\
&+ \int_{\mathbb{R}^6 \times \mathbb{R}^+} \nu(r^*, w^*) g(t, x, w^*, r^*) \left\{ \int_{\mathbb{R}^3 \times \mathbb{R}^+} r^3 h(r, w, r^*, w^*)\, dr\, dw \right\} dr^*\, dw^*\, dx,
\end{aligned}$$

where we use Fubini's Theorem on the last term. By Lemma 1, we conclude:

$$\frac{d}{dt}\|r^3 g(t)\|_{L^1(\mathbb{R}^6\times\mathbb{R}^+)} = 0. \tag{2.2}$$

∎

3 Study of the kinetic equation

The goal of this section is to prove the following proposition of existence and uniqueness of weak solutions to the initial value problem (1.8).

Proposition 1 *Assume* $g^0 \in L^\infty(\mathbb{R}^6 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^6 \times \mathbb{R}^+)$ *and suppose* $\mathrm{supp}(g^0) \subset \mathbb{R}^3_x \times \Omega$. *Further suppose* $u \in L^2(0,T;W^{1,\infty}(\mathbb{R}^3_x))$. *Then there exists a unique weak solution to the initial value problem*

$$\begin{cases} \partial_t g + \nabla_x \cdot ((u+w)g) + \nabla_w \cdot (-\frac{w}{r^2} g) = \Gamma(g), \\ g(0,x,w,r) = g^0(x,w,r), \end{cases} \tag{3.1}$$

where g has the following properties:

(i) $g \in L^\infty(0,T;L^1(\mathbb{R}^6 \times \mathbb{R}^+))$.

(ii) $g \in L^\infty(0,T;L^\infty_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$.

(iii) $g(t,x,w,r) \le C(\|g^0\|_{L^\infty}) e^{\frac{3t}{r^2}}$.

(iv) $g(t,x,w,r) \le C(\|g^0\|_{L^\infty}) \frac{1}{|w|^3}$.

(v) $g \in \mathcal{C}([0,T];W^{-1,p}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$, *for* $1 \le p < \infty$.

(vi) $\mathrm{supp}(g(t)) \subset \mathbb{R}^3_x \times \Omega$ *for a.e.* $t \in [0,T]$.

We begin by proving existence of solutions to (3.1) when g^0 and u are smooth and bounded, and when g^0 is compactly supported in r, w. The idea is to construct a sequence of solutions verifying

$$\begin{cases} \partial_t g_n + \nabla_x \cdot ((u+w)g_n) + \nabla_w \cdot (-\frac{w}{r^2} g_n) = -\nu(r,w) g_n(t,x,w,r) \\ \qquad + \int_{\mathbb{R}^+}\int_{\mathbb{R}^3} \nu(r^*,w^*) h(r,w,r^*,w^*) g_{n-1}(t,x,w^*,r^*)\, dw^*\, dr^*, \\ g_n(0,x,w,r) = g^0(x,w,r), \\ g_0(t,x,w,r) = 0. \end{cases} \tag{3.2}$$

Given g_{n-1} we can solve for g_n in (3.2) using the method of characteristics. We then establish a priori bounds for the sequence g_n and show that $g = \lim_{n\to\infty} g_n$ solves the original PDE.

Consider the following trajectories (in phase space):

$$\begin{cases} \dot{x}(t) = u(t, x(t)) + w(t), \\ \dot{w}(t) = -\frac{1}{r^2} w(t), \\ x(t_0) = x_0, \\ w(t_0) = w_0. \end{cases} \tag{3.3}$$

If u is smooth and bounded, the Cauchy-Lipshitz theorem for ODEs ensures a smooth solution denoted $(x(t, t_0, x_0, w_0), w(t, t_0, w_0))$. Along trajectories (3.2) reduces to the following ODE:

$$\begin{cases} \frac{d}{dt} g_n(t, x(t), w(t), r) = \left(-\nu(r, w(t)) + \frac{3}{r^2}\right) g_n(t, x(t), w(t), r) \\ \quad + \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \nu(r^*, w^*) h(r, w(t), r^*, w^*) g_{n-1}(t, x(t), w^*, r^*)\, dw^*\, dr^*, \\ g_n(0) = g^0. \end{cases} \tag{3.4}$$

Solving for g_n in (3.4) using an integrating factor and setting $t = t_0$, $x = x_0$ and $w = w_0$, we obtain

$$\begin{aligned} g_n(t, x, w, r) = e^{-\int_0^t \left(\nu(r, w(s,t,w)) - \frac{3}{r^2}\right) ds} & g^0(x(0, t, x, w), w(0, t, w), r) \\ + \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^+} e^{-\int_\tau^t \left(\nu(r, w(s,t,w)) - \frac{3}{r^2}\right) ds} & \nu h(r, w(\tau, t, w), r^*, w^*) \times \\ & \times g_{n-1}(\tau, x(\tau, t, x, w), w^*, r^*)\, dw^*\, dr^*\, d\tau. \end{aligned} \tag{3.5}$$

Note that since the trajectories are continuous with respect to the initial value parameters (again by Cauchy-Lipshitz), it is easy to show by induction that each g_n is continuous provided that g^0 is continuous.

Lemma 3 *Let $\Omega = \{(r, w) \in \mathbb{R}^+ \times \mathbb{R}^3 : 0 < r \leq R, 0 \leq |w| \leq W\}$ for fixed positive constants R and W. Suppose $g^0 \in \mathcal{C}^\infty(\mathbb{R}^6 \times \mathbb{R}^+) \cap L^\infty(\mathbb{R}^6 \times \mathbb{R}^+)$ with supp$(g^0) \subset \mathbb{R}^3 \times \Omega$. Then supp$(g_n(t)) \subset \mathbb{R}^3 \times \Omega$ for all $t \geq 0$ and for all n.*

Proof. First, observe that the norm of the relative velocity decreases as we move foward in time along trajectories ($w(t) = e^{-\frac{(t-t_0)}{r^2}} w(t_0)$). Therefore we have $|w(\tau,t,w)| \geq |w|$ when $\tau \leq t$. Now we will prove the statement by induction. The previous observation implies

$$g_1(t,x,w,r) = e^{-\int_0^t \left(\nu(r,w(s,t,w)) - \frac{3}{r^2}\right) ds} g^0(x(0,t,x,w), w(0,t,w), r)$$

vanishes when $|w| \geq W$ or $r \geq R$ for all $t \geq 0$. Now assume the statement holds for $n-1$. Then, for $|w| \geq W$, $h(r, w(\tau,t,w), r^*, w^*) = 0$ unless $|w^*| \geq W$ (this is one of our hypotheses for h). Since $g_{n-1}(\tau, x(\tau,t,x,w), w^*, r^*) = 0$ in this case, the integrand vanishes and as before we find $g_n(t,x,w,r) = 0$. A similar argument works for $r > R$. ∎

Remark 1 *As we shall see, the previous lemma will allow us to consider only those measurable $g \geq 0$ such that $supp(g(t)) \subset \mathbb{R}^3 \times \Omega$. In that case, it is convenient to write*

$$\Gamma(g)(t,x,w,r) = -\chi_\Omega \nu(r,w) g(t,x,w,r) + \int_{\Omega_r} \nu h(r,w,r^*,w^*) g(t,x,w^*,r^*)\, dw^*\, dr^*, \tag{3.6}$$

where

$$\Omega_r = \left\{(r^*,w^*) \in \mathbb{R}^+ \times \mathbb{R}^3 : r \leq r^* \leq R, 0 \leq |w^*| \leq W\right\}. \tag{3.7}$$

Note that by assumption, $h(r,w,r^*,w^*) = 0$ if $r \geq r^*$. Therefore, under the conditions above, it suffices to integrate over the compact set Ω_r. Also notice that the negative part of Γ is unchanged when multiplied by $\chi_\Omega(r,w)$.

In order to show that $g = \lim_{n\to\infty} g_n$ is a weak solution to (3.1), we need to verify that Γ is weakly continuous in following sense:

Lemma 4 *Consider the set*

$$K^p = \left\{g \in L^p_{loc}([0,T] \times \mathbb{R}^6 \times \mathbb{R}^+) : supp(g) \subset [0,T] \times \mathbb{R}^3_x \times \Omega\right\},$$

where $1 \leq p < \infty$. Suppose g_n is bounded in K^p. Then, $\Gamma(g_n)$ is bounded in K^p. Further, suppose that $g_n \xrightarrow{L^p_{loc}} g \in K^p$. Then, $\Gamma(g_n) \xrightarrow{L^p_{loc}} \Gamma(g) \in K^p$.

Proof. First let us show that for $g \in K^p$ and for $r_0, R_0 > 0$ we have

$$\|\Gamma(g)\|_{L^p([0,T]\times[-R_0,R_0]^3\times\Omega_{r_0})} \leq C_{r_0}\|g\|_{L^p([0,T]\times[-R_0,R_0]^3\times\Omega_{r_0})}. \tag{3.8}$$

Indeed, let $\tilde{\Omega} = [0,T] \times [-R_0, R_0]^3 \times \Omega_{r_0}$. Then,

$$\begin{aligned}
\|\Gamma(g)\|_{L^p(\tilde{\Omega})} &\leq \|\chi_\Omega \nu g\|_{L^p(\tilde{\Omega})} + \left\| \int_{\Omega_r} \nu h(r,w,r^*,w^*) g(t,x,w^*,r^*)\, dr^*\, dw^* \right\|_{L^p(\tilde{\Omega})} \\
&\leq \|\chi_\Omega \nu g\|_{L^p(\tilde{\Omega})} \\
&\qquad + \left\| \int_{\Omega_{r_0}} \nu h(r,w,r^*,w^*) g(t,x,w^*,r^*)\, dr^*\, dw^* \right\|_{L^p(\tilde{\Omega})} \\
&\leq \|\nu\|_{L^\infty(\Omega)} \|g\|_{L^p(\tilde{\Omega})} + \left\| \|\nu h(r,w)\|_{L^{p'}(\Omega_{r_0})} \|g(t,x)\|_{L^p(\Omega_{r_0})} \right\|_{L^p(\tilde{\Omega})} \\
&\leq \|\nu\|_{L^\infty(\Omega)} \|g\|_{L^p(\tilde{\Omega})} + \|\nu h\|_{L^\infty(\Omega_{r_0}; L^{p'}(\Omega_{r_0}))} |\Omega_{r_0}|^{\frac{1}{p}} \|g\|_{L^p(\tilde{\Omega})} \\
&\leq \|\nu\|_{L^\infty(\Omega)} \|g\|_{L^p(\tilde{\Omega})} + \|\nu h\|_{L^\infty} |\Omega_{r_0}|^{\frac{1}{p'}} |\Omega_{r_0}|^{\frac{1}{p}} \|g\|_{L^p(\tilde{\Omega})} \\
&\leq \left(\|\nu\|_{L^\infty(\Omega)} + \|\nu h\|_{L^\infty} |\Omega_{r_0}| \right) \|g\|_{L^p(\tilde{\Omega})}.
\end{aligned}$$

Note that in the fifth inequality we used our assumption that νh is bounded. Also, note carefully that we have shown Γ is a bounded linear operator only when the domain is restricted to sets of the form $\tilde{\Omega}$ and when $g \in K^p$.

Now suppose g_n is bounded in K^p and consider $\Omega_0 \subset\subset [0,T] \times \mathbb{R}^6 \times \mathbb{R}^+$. We want to show that $\|\Gamma(g_n)\|_{L^p(\Omega_0)}$ is bounded uniformily for all g_n. Taking into account condition (iv) from section 2, it is easy to verify that $supp(\Gamma(g_n)) \subset [0,T] \times \mathbb{R}^3_x \times \Omega$. Therefore, if we choose $r_0, R_0 > 0$ such that $\Omega_0 \cap \{[0,T] \times \mathbb{R}^3_x \times \Omega\} \subset \tilde{\Omega} = [0,T] \times [-R_0, R_0]^3 \times \Omega_{r_0}$, we have

$$\|\Gamma(g_n)\|_{L^p(\Omega_0)} \leq \|\Gamma(g_n)\|_{L^p(\tilde{\Omega})} \leq C\|g_n\|_{L^p(\tilde{\Omega})}.$$

Since $\tilde{\Omega} \subset\subset [0,T] \times \mathbb{R}^6 \times \mathbb{R}^+$, the right hand side is bounded, and we conclude that $\Gamma(g_n)$ is bounded in K^p.

Next let us show that $\Gamma(g_n) \xrightarrow{L^p(\Omega_0)} \Gamma(g)$ for all compact subsets Ω_0. As before, we have $\Omega_0 \cap \{[0,T] \times \mathbb{R}^3_x \times \Omega\} \subset \tilde{\Omega}$. By assumption, $g_n \xrightarrow{L^p(\tilde{\Omega})} g$.

Therefore using (3.8), we have $\Gamma(g_n) \xrightarrow{L^p(\tilde{\Omega})} \Gamma(g)$, which then gives us that $\Gamma(g_n) \rightharpoonup \Gamma(g)$ in $L^p(\Omega_0 \cap \{[0,T] \times \mathbb{R}^3_x \times \Omega\})$ Now, let $\varphi \in L^{p'}(\Omega_0)$. Then,

$$\int_{\Omega_0} \varphi(\Gamma(g_n) - \Gamma(g))\, dx\, dw\, dr\, dt = \int_{\Omega_0 \cap \{[0,T]\times\mathbb{R}^3_x\times\Omega\}} \varphi(\Gamma(g_n) - \Gamma(g))\, dx\, dw\, dr\, dt,$$

which goes to zero as $n \to \infty$ by the preceding observation. ∎

Next we establish two important bounds on the sequence g_n.

Lemma 5 *Assume $g^0 \in \mathcal{C}^\infty(\mathbb{R}^6 \times \mathbb{R}^+) \cap L^\infty(\mathbb{R}^6 \times \mathbb{R}^+)$ with $supp(g^0) \subset \mathbb{R}^3 \times \Omega$. Then, the sequence g_n given by (3.4)-(3.5) is increasing. Furthermore, the sequences $|w|^3 g_n$ and $e^{\frac{-3t}{r^2}} g_n$ are uniformily bounded in $L^\infty(0,T; L^\infty(\mathbb{R}^6 \times \mathbb{R}^+))$. More precisely, there exists $C > 0$, depending only on Ω, such that*

(i) $\| |w|^3 g_n(t) \|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)} \le \| |w|^3 g^0 \|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)} + C(\| |w|^3 g^0 \|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)}\, t e^{Ct})$,

(ii) $\left\| e^{\frac{-3t}{r^2}} g_n(t) \right\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)} \le \|g^0\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)} + C\left(\|g^0\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)}\, t e^{Ct} \right)$.

Proof. First we will show g_n increases pointwise with respect to n. By (3.5) $g_1 - g_0 = g_1 \ge 0$. Now assume $g_{n-1} - g_{n-2} \ge 0$. Then, due to Lemma 3 and Remark 1, (3.5) implies

$$g_n - g_{n-1} = \int_0^t \int_{\Omega_r} e^{-\int_\tau^t \left(\nu - \frac{3}{r^2}\right) ds} \nu h(g_{n-1} - g_{n-2})\, dw^*\, dr^*\, d\tau \ge 0.$$

So, by induction g_n is increasing. Now, using $|w(t)|^3 = e^{-3\frac{(t-t_0)}{r^2}} |w(t_0)|^3$ as an integrating factor in (3.4), we obtain

$$\begin{aligned} \frac{d}{dt}\left(|w(t)|^3 g_n(t, x(t), w(t), r)\right) &= -\chi_\Omega \nu(r, w(t)) \left(|w(t)|^3 g_n(t, x(t), w(t), r)\right) \\ &\quad + \int_{\Omega_r} \nu h(r, w(t), r^*, w^*) |w(t)|^3 g_{n-1}(t, x(t), w^*, r^*)\, dw^*\, dr^* \\ &\le \int_{\Omega_r} \nu h(r, w(t), r^*, w^*) |w(t)|^3 g_n(t, x(t), w^*, r^*)\, dw^*\, dr^*. \end{aligned}$$

Integrating in time and setting $t = t_0$, $x = x_0$ and $w = w_0$ we obtain

$$|w|^3 g_n(t,x,w,r) \le |w(0,t,w))|^3 g^0(x(0,t,x,w), w(0,t,w), r) \; + \tag{3.9}$$
$$\int_0^t \int \nu h(r, w(\tau,t,w), r^*, w^*) |w(\tau,t,w)|^3 g_n(\tau, x(\tau,t,x,w), w^*, r^*)\, dw^*\, dr^*\, d\tau.$$

Now, since $|w(\tau,t,w)| \le |w^*|$ when $h(r,w(\tau,t,w),r^*,w^*)$ is non-zero, we have

$$|w|^3 g_n(t,x,w,r) \le |w(0,t,w))|^3 g^0(x(0,t,x,w),w(0,t,w),r)$$
$$+ \int_0^t \int_{\Omega_r} \nu h(r,w(\tau,t,w),r^*,w^*)|w^*|^3 g_n(\tau,x(\tau,t,x,w),w^*,r^*)\, dw^*\, dr^*\, d\tau.$$

Therefore

$$\left\||w|^3 g_n(t)\right\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)} \le \left\||w|^3 g^0\right\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)}$$
$$+ \|\nu h\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^+;L^1(\Omega_r))} \int_0^t \left\||w|^3 g_n(\tau)\right\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)}\, d\tau.$$

Since νh is bounded, we have

$$\|\nu h\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^+;L^1(\Omega_r))} \le |\Omega_r|\|\nu h\|_{L^\infty} \equiv C.$$

Finally by Gronwall's Lemma we conclude

$$\left\||w|^3 g_n(t)\right\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)} \le \left\||w|^3 g^0\right\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)} + C(\left\||w|^3 g^0\right\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)}\, te^{Ct}).$$

On the other hand, since $|w(\tau,t,w)|^3 = e^{-3\frac{(\tau-t)}{r^2}}|w|^3$, we deduce from (3.9) that

$$g_n(t,x,w,r) \le e^{\frac{3t}{r^2}} g^0(x(0,t,x,w),w(0,t,w),r)$$
$$+ \int_0^t \int_{\Omega_r} e^{\frac{3}{r^2}(t-\tau)} \nu h(r,w(\tau,t,w),r^*,w^*) g_n(\tau,x(\tau,t,x,w),w^*,r^*)\, dw^*\, dr^*\, d\tau.$$

Now since $e^{\frac{-3\tau}{r^2}} \le e^{\frac{-3\tau}{r^{*2}}}$ for $r \le r^*$, we have

$$e^{\frac{-3t}{r^2}} g_n(t,x,w,r) \le g^0(x(0,t,x,w),w(0,t,w),r)$$
$$+ \int_0^t \int_{\Omega_r} \nu h(r,w(\tau,t,w),r^*,w^*) \left[e^{\frac{-3\tau}{r^{*2}}} g_n(\tau,x(\tau,t,x,w),w^*,r^*)\right] dw^*\, dr^*\, d\tau.$$

Proceeding exactly as before, we conclude that

$$\left\|e^{\frac{-3t}{r^2}} g_n(t)\right\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)} \le \|g^0\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)} + C\left(\|g^0\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)}\, te^{Ct}\right).$$

∎

Proof of Proposition 1.

Step 1. We first assume $g^0 \in \mathcal{C}^\infty(\mathbb{R}^6 \times \mathbb{R}^+) \cap L^\infty(\mathbb{R}^6 \times \mathbb{R}^+)$ with $\mathrm{supp}(g^0) \subset \mathbb{R}^3 \times \Omega$, and we assume that u is smooth and bounded. We claim that $g = \lim_{n\to\infty} g_n$ is a weak solution to the initial value problem (3.1) with properties $(ii) - (vi)$.

Proof. According to Lemma 5, g_n (given by (3.5)) is an increasing sequence of measurable functions uniformly bounded above by $\phi_1(t,x,w,r) = C(\|g^0\|_{L^\infty})e^{\frac{3t}{r^2}} \in L^\infty(0,T;L^\infty_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$. Therefore, by the monotone convergence theorem, $g(t,x,w,r) = \lim_{n\to\infty} g_n(t,x,w,r)$ is measurable, $g_n \to g$ in $L^p(0,T;L^p_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$ for $1 \le p < \infty$, and (by the bound above) $g \in L^\infty(0,T;L^\infty_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$. This establishes properties (ii), (iii) and (vi). Also, since g_n is uniformly bounded by $\phi_2(t,x,w,r) = C(\||w|^3 g^0\|_{L^\infty})\frac{1}{|w|^3}$, we have property (iv). (Note that $\||w|^3 g^0\|_{L^\infty} \le W^3\|g^0\|_{L^\infty}$ since $|w| \le W$ on the support of g^0.)

We now show that g satisfies (3.1) in the sense of distributions. We know that g_n given by the Duhamel formula (3.5), is a classical solution to (3.2). Therefore g_n satisfies

$$\partial_t g_n = -\nabla_x \cdot ((u+w)g_n) - \nabla_w \cdot \left(-\frac{w}{r^2}g_n\right) - \nu(g_n - g_{n-1}) + \Gamma(g_{n-1}). \quad (3.10)$$

Given that $g_n \xrightarrow{L^p(L^p_{loc})} g$ for $1 \le p < \infty$, it follows from Lemma 4 that $\Gamma(g_{n-1}) \xrightarrow{L^p(L^p_{loc})} \Gamma(g)$. Also, since multiplication by a smooth function and differentiation are continuous operations in $\mathcal{D}'((0,T) \times \mathbb{R}^6 \times \mathbb{R}^+)$, passing to the limit shows that g satisfies (3.1) in the sense of distributions.

Now it suffices to verify that $g(t) \xrightarrow{\mathcal{D}'} g^0$ as $t \to 0$. In fact, for $1 \le p < \infty$, we will show $g \in \mathcal{C}([0,T];W^{-1,p}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$ so that $g(t) \xrightarrow{W^{-1,p}_{loc}} g^0$. From (3.10) we can establish that $\partial_t g_n$ is bounded in $L^p(0,T;W^{-1,p}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$. Indeed, since, $u+w$ and $-\frac{w}{r^2}$ are bounded on compact subsets of $[0,T] \times \mathbb{R}^6 \times \mathbb{R}^+$, it is clear that $-\nabla_x \cdot ((u+w)g_n) - \nabla_w \cdot (-\frac{w}{r^2}g_n)$ is bounded in $L^p(0,T;W^{-1,p}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$. From Lemma 4, it follows that $-\nu(g_n - g_{n-1}) + \Gamma(g_{n-1})$ is bounded in $L^p(0,T;L^p_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$ which injects continuously into $L^p(0,T;W^{-1,p}_{loc}(\mathbb{R}^6 \times$

$\mathbb{R}^+)$). Now, since $L^p_{loc}(\mathbb{R}^6 \times \mathbb{R}^+) \xhookrightarrow{compact} W^{-1,p}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+)$ by the Rellich-Kondrakov Theorem, we conclude, using the Aubin lemma (see Theorem 3.2.1 in [16]), that g_n is relatively compact in $\mathcal{C}([0,T]; W^{-1,p}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$. Therefore, there exists a convergent subsequence g_{n_k}, the limit of which must be equal to g for a.e. $t \in [0,T]$. (since g_n converges to g pointwise). This gives us property (v). Also, it follows that $g(t) \to g^0$ in $W^{-1,p}_{loc}$ as $t \to 0$ and this completes the proof of the claim in Step 1. ∎

Step 2. Now suppose $g^0 \in L^\infty(\mathbb{R}^6 \times \mathbb{R}^+)$ with $\mathrm{supp}(g^0) \subset \mathbb{R}^3 \times \Omega$, and assume that u is smooth and bounded. Then there exists a weak solution to the initial value problem (3.1) with properties $(ii) - (vi)$.

Proof. By mollification, we can construct $g^0_\delta \xrightarrow{\delta \to 0} g^0$ in $L^\infty(\mathbb{R}^6 \times \mathbb{R}^+)$ such that $\|g^0_\delta\|_{L^\infty(\mathbb{R}^6 \times \mathbb{R}^+)} \leq \|g^0\|_{L^\infty(\mathbb{R}^6 \times \mathbb{R}^+)}$ for all $\delta > 0$. By Step 1, there exists a sequence of regularized (weak) solutions, g_δ, satisfying

$$\begin{cases} \partial_t g_\delta + \nabla_x \cdot ((u+w) g_\delta) + \nabla_w \cdot (-\frac{w}{r^2} g_\delta) = \Gamma(g_\delta) \\ g_\delta(0,x,w,r) = g^0_\delta(x,w,r), \end{cases} \tag{3.11}$$

Since $g_\delta(t,x,w,r) \leq C(\|g^0_\delta\|_{L^\infty}) e^{\frac{3t}{r^2}} \leq C(\|g^0\|_{L^\infty}) e^{\frac{3t}{r^2}}$, we conclude that g_δ is bounded in $L^\infty(0,T; L^\infty_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$. Also, as in the proof of Step 1, $\partial_t g_\delta$ is bounded in $L^p(0,T; W^{-1,p}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$, for $1 \leq p < \infty$. Now, consider a sequence $\delta_n \to 0$. By the Aubin lemma, g_{δ_n} is relatively compact in $\mathcal{C}([0,T]; W^{-1,p}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$. In particular, there exists a subsequence $g_{\delta_{n_k}}$ and a function g, such that

$$g_{\delta_{n_k}} \rightharpoonup g \quad \text{weakly in } L^p(0,T; L^p_{loc}(\mathbb{R}^6 \times \mathbb{R}^+)),$$

$$g_{\delta_{n_k}} \to g \quad \text{strongly in } \mathcal{C}([0,T]; W^{-1,p}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+)).$$

In particular, g has properties (iii)(which implies (ii)), (iv), (v) and (vi). We claim that g is a weak solution to (3.1). Indeed, following the proof of Step 1, the former convergence implies that g satisfies (3.1) in the sense of distributions and it is easy to show that the latter convergence implies $g(t) \to g^0$ in $W^{-1,p}_{loc}$ as $t \to 0$. ∎

Step 3. Finally, suppose $g^0 \in L^\infty(\mathbb{R}^6 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^6 \times \mathbb{R}^+)$ with $\mathrm{supp}(g^0) \subset \mathbb{R}^3 \times \Omega$, and assume that $u \in L^2(0,T; W^{1,\infty}(\mathbb{R}^3_x))$. Then there exists a unique weak solution to the initial value problem (3.1) with properties $(i) - (vi)$.

Proof. By mollification, we can construct $u^\varepsilon \to u$ in $L^2(0,T;W^{1,\infty}(\mathbb{R}^3_x))$ such that $\|u^\varepsilon\|_{L^2(0,T;W^{1,\infty}(\mathbb{R}^3_x))} \le \|u\|_{L^2(0,T;W^{1,\infty}(\mathbb{R}^3_x))}$ for all $0<\varepsilon\le\varepsilon_0$. By Step 2, there exists a sequence of regularized (weak) solutions, g^ε, satisfying

$$\begin{cases} \partial_t g^\varepsilon + \nabla_x\cdot((u^\varepsilon + w)g^\varepsilon) + \nabla_w\cdot(-\frac{w}{r^2}g^\varepsilon) = \Gamma(g^\varepsilon), \\ g^\varepsilon(0,x,w,r) = g^0(x,w,r). \end{cases} \tag{3.12}$$

As before, property (iii) implies g^ε is bounded in $L^\infty(0,T;L^\infty_{loc}(\mathbb{R}^6\times\mathbb{R}^+))$. Now we will show $\partial_t g^\varepsilon$ is bounded in $L^2(0,T;W^{-1,p}_{loc}(\mathbb{R}^6\times\mathbb{R}^+))$ for $1\le p<+\infty$. The main point is to show $\nabla_x\cdot(u^\varepsilon g^\varepsilon)$ is bounded in the aforementioned space. Given $\Omega_0 \subset\subset \mathbb{R}^6\times\mathbb{R}^+$, we have

$$\begin{aligned} \|\nabla_x\cdot(u^\varepsilon g^\varepsilon)\|_{L^2(0,T;W^{-1,p}(\Omega_0))} &\le C\|u^\varepsilon g^\varepsilon\|_{L^2(0,T;L^p(\Omega_0))} \\ &\le C\|u^\varepsilon\|_{L^2(0,T;L^\infty(\Omega_0))}\|g^\varepsilon\|_{L^\infty(0,T;L^p(\Omega_0))} \\ &\le C\|u\|_{L^2(0,T;L^\infty(\mathbb{R}^3_x))}\|g^\varepsilon\|_{L^\infty(0,T;L^p(\Omega_0))} \\ &\le C_2\|u\|_{L^2(0,T;W^{1,\infty}(\mathbb{R}^3_x))}. \end{aligned}$$

The remaining terms in (3.12) are bounded in $L^2(0,T;W^{-1,p}_{loc}(\mathbb{R}^6\times\mathbb{R}^+))$ for $1\le p<+\infty$. Finally, using the Aubin lemma, we extract a subsequence g^{ε_k} and pass to the limit exactly as before, thereby obtaining a weak solution to (3.1) with properties $(ii)-(vi)$.

Now, let us show that $g\in L^\infty(0,T;L^1(\mathbb{R}^6\times\mathbb{R}^+))$. Integrating (3.1) with respect to x, w, and r we get

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^3}\int_\Omega g(t,x,w,r)\,dr\,dw\,dx &= -\int_{\mathbb{R}^3}\int_\Omega \nu(r,w)g(t,x,w,r)\,dr\,dw\,dx \\ &\quad + \int_{\mathbb{R}^3}\int_{\Omega_r}\nu(r^*,w^*)g(t,x,w^*,r^*)\left\{\int_\Omega h(r,w,r^*,w^*)\,dr\,dw\right\}\,dr^*\,dw^*\,dx \\ &= -\int_{\mathbb{R}^3}\int_\Omega \nu(r,w)g(t,x,w,r)\,dr\,dw\,dx \\ &\quad + 2\int_{\mathbb{R}^3}\int_{\Omega_r}\nu(r^*,w^*)g(t,x,w^*,r^*)\,dr^*\,dw^*\,dx \\ &\le \int_{\mathbb{R}^3}\int_\Omega \nu(r,w)g(t,x,w,r)\,dr\,dw\,dx \le \|\nu\|_{L^\infty(\Omega)}\int_{\mathbb{R}^3}\int_\Omega g(t,x,w,r)\,dr\,dw\,dx. \end{aligned}$$

Note that condition (vii) from section 2 implies $\int_\Omega h(r,w,r^*,w^*)\,dr\,dw = 2$, since $(w^*,r^*)\in\Omega_r\subset\Omega$. The result follows from Gronwall's Lemma.

Finally, we prove that the weak solution g is unique. Suppose g_1 and g_2 are two weak solutions. Since (3.1) is a linear PDE, the difference $g_- = g_1 - g_2$ solves (3.1) with initial condition $g^0 = 0$. Therefore we have

$$\partial_t g_- + \nabla_x \cdot ((u+w)g_-) + \nabla_w \cdot (-\frac{w}{r^2} g_-) = \Gamma(g_-).$$

Multipying by $r^3 sgn(g_-)$ and integrating with respect to x, w, and r we find

$$\begin{aligned}\int_{\mathbb{R}^6\times\mathbb{R}^+} \partial_t(r^3|g_-|)\, dr\, dw\, dx + \int_{\mathbb{R}^6\times\mathbb{R}^+} \nabla_x \cdot ((w+u)r^3|g_-|)\, dr\, dw\, dx \\ + \int_{\mathbb{R}^6\times\mathbb{R}^+} \nabla_w \cdot (-wr|g_-|)\, dr\, dw\, dx \\ = \int_{\mathbb{R}^6\times\mathbb{R}^+} -r^3\nu\, |g_-|\; dr\, dw\, dx \\ + \int_{\mathbb{R}^6\times\mathbb{R}^+} \left\{ r^3 sgn(g_-) \int_{\mathbb{R}^6\times\mathbb{R}^+} \nu h g_-\, dr^*\, dw^* \right\} dr\, dw\, dx.\end{aligned}$$

We refer the reader to [14] (Lemma 2.3) for details of this type of analysis and in particular for a proof of the (formal) equality $sgn(g_-)\nabla_x \cdot (u(g_-)) = \nabla_x \cdot (u|g_-|)$ when $u \in L^2(0,T;W^{1,\alpha}(\mathbb{R}^3_x))$ and $g_- \in L^\infty(0,T;L^\beta(\mathbb{R}^3_x))$ with $\frac{1}{\alpha} + \frac{1}{\beta} \le 1$. (We take $\alpha = \infty$ and $\beta = 1$.) Now, integrating by parts on the left hand side yields

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^6\times\mathbb{R}^+} r^3|g_-|\, dr\, dw\, dx \le \int_{\mathbb{R}^6\times\mathbb{R}^+} -r^3\nu\, |g_-|\; dr\, dw\, dx \\ + \int_{\mathbb{R}^6\times\mathbb{R}^+} \left|r^3 sgn(g_-)\right| \left| \int \nu h g_-\, dr^*\, dw^* \right| dr\, dw\, dx \\ \le \int_{\mathbb{R}^6\times\mathbb{R}^+} -r^3\nu\, |g_-|\; dr\, dw\, dx + \int_{\mathbb{R}^6\times\mathbb{R}^+} r^3 \left\{ \int_{\mathbb{R}^6\times\mathbb{R}^+} \nu h|g_-|\, dr^*\, dw^* \right\} dr\, dw\, dx \\ = \int_{\mathbb{R}^6\times\mathbb{R}^+} r^3\Gamma(|g_-|)\, dr\, dw\, dx = 0.\end{aligned}$$

Since $g_-(0) = 0$, we conclude that $\int_{\mathbb{R}^6\times\mathbb{R}^+} r^3|g_-|\, dr\, dw\, dx = 0$, for almost every $t \in [0,T]$. Therefore $g_1 = g_2$ a.e.. This completes the proof. ∎

3.1 The Incompressible Navier-Stokes Equations and the Coupled Problem

The previous section established the existence and uniqueness of solutions to the kinetic equation (3.1) for velocity fields $u(t,x) \in L^2(0,T;W^{1,\infty}(\mathbb{R}^3))$. Now, we turn our attention to the fluid equation. The goal of this section is to find a solution to the coupled problem by means of a fixed point argument. First, we need an existence and uniqueness result for the associated Galerkin solutions of the incompressible Navier-Stokes equations:

$$\begin{cases} \partial_t u + \mathrm{Div}_x(u \otimes u) - \Delta_x u + \nabla_x p = \mathfrak{F}, \\ \mathrm{div}_x(u) = 0, \\ u(0,x) = u^0(x) \end{cases} \tag{3.13}$$

Specifically, following the framework detailed by Temam in [16], we consider the vector space $\mathcal{V} = \left\{ v \in \{\mathcal{D}(\mathbb{R}^3)\}^3,\ \mathrm{div}(v) = 0 \right\}$ and define the Hilbert spaces V and H to be the closure of $\mathcal{V}$ with respect to the $\{H_0^1(\mathbb{R}^3)\}^3$ and $\{L^2(\mathbb{R}^3)\}^3$ inner products, respectively. Assume $\{w_i\}_{i=1}^{\infty} \subset \mathcal{V}$ is an orthogonal basis of V and an orthonormal basis of H, and let $V_m = \mathrm{span}\{w_1, ..., w_m\} \subset V$ and $H_m = \mathrm{span}\{w_1, ..., w_m\} \subset H$. Finally, we denote the H_0^1 inner product by $((\cdot,\cdot))$ and the L^2 inner product by $(\cdot,\cdot)$, and consider the trilinear and continuous form on V defined by

$$b(u,v,w) = \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} u_i (D_i v_j) w_j.$$

Now, fix $m \in \mathbb{N}$. We say that $u_m \in L^2(0,T;V_m)$ is an approximate solution of (3.13), for a given $\mathfrak{F} \in L^2(0,T;L^2(\mathbb{R}^3))$ and $u^0 \in H$, if for all $j = 1, ..., m$, u_m verifies

$$\begin{aligned} &- \int_0^T (u_m(t), \psi'(t) w_j)\ dt + \int_0^T ((u_m(t), \psi(t) w_j))\ dt \\ &\quad + \int_0^T b\left(u_m(t), u_m(t), \psi(t) w_j\right)\ dt = \left(u^0, w_j\right) \psi(0) + \int_0^T (\mathfrak{F}(t), \psi(t) w_j)\ dt, \end{aligned} \tag{3.14}$$

for all $\psi \in \mathcal{C}^1([0,T])$ with $\psi(T) = 0$.

Theorem 2 *Given $\mathfrak{F} \in L^2(0,T;L^2(\mathbb{R}^3))$ and $u^0 \in H$, there exists a unique function u_m solving the variational problem (3.14) with the property*

$$u_m \in L^2(0,T;V_m) \cap L^\infty(0,T;H_m). \tag{3.15}$$

Moreover, by construction, we have

$$u_m \in \mathcal{C}([0,T];V_m), \tag{3.16}$$

and u_m has the following properties:

$$\|u_m\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \leq C(\|u_0\|_{L^2(\mathbb{R}^3)}; \|\mathfrak{F}\|_{L^2(0,T;L^2(\mathbb{R}^3))}), \tag{3.17}$$

$$\|u_m\|_{L^2(0,T;H_0^1(\mathbb{R}^3))} \leq C(\|u_0\|_{L^2(\mathbb{R}^3)}; \|\mathfrak{F}\|_{L^2(0,T;L^2(\mathbb{R}^3))}). \tag{3.18}$$

Proof. The existence of $u_m = \sum_{i=1}^m g_{im}(t)w_i$ verifying (3.14)-(3.18) is well-known and follows from Galerkin's method. We will sketch the proof of uniqueness for solutions with property (3.15). The argument follows closely the uniqueness proof for solutions of (3.13) in dimension 2. First, note that (3.14) implies

$$\frac{d}{dt}\left(u_m(t),\cdot\right) = -\left(\left(u_m(t),\cdot\right)\right) - b\left(u_m(t),u_m(t),\cdot\right) + \left(\mathfrak{F}(t),\cdot\right), \tag{3.19}$$

in the scalar distribution sense, as linear operators on V_m, and it can be shown that the right hand side represents an element of $L^2(0,T;V_m')$. In particular, this defines the weak time derivative $u_m' \in L^2(0,T;V_m')$, and according to Lemma 3.1.2 in [16], u_m is almost everywhere equal to a function (absolutely) continuous from [0,T] to H_m, and

$$\frac{d}{dt}|u_m|_{L^2}^2 = 2\langle u_m', u_m\rangle,$$

in $\mathcal{D}'((0,T))$. Assume $u_m^1, u_m^2 \in L^2(0,T;V_m)\cap L^\infty(0,T;H_m)$ are two functions verifying (3.14). Then, for a.e. $t \in [0,T]$, $v = u_m^1 - u_m^2$ verifies

$$\frac{d}{dt}|v(t)|_{L^2}^2 + \|v(t)\|_{H_0^1} \leq \|\nabla u_m^2(t)\|_{L^\infty}|v(t)|_{L^2}^2.$$

Based on the continuity established above, we conclude using Gronwall's lemma, that $|v(t)|_{L^2}^2 \leq C|v(0)|_{L^2}^2$.

It remains to show that $|v(0)|_{L^2}^2 = 0$. Since $t \mapsto (u_m(t), w_j)$ is absolutely continuous, a legitimate integration by parts applied to the first term in (3.14) leads to the equality $(u_m(0), w_j) = (u^0, w_j)$ for all $j = 1, ..., m$. Therefore, for any approximate solution with property (3.15), $u_m(0)$ is the orthogonal projection in $L^2(\mathbb{R}^3)$ of u^0 onto H_m. Hence, $|u_m^1(0) - u_m^2(0)|_{L^2}^2 = 0$, which finishes the proof. ∎

We now prove that the force given by (1.7) is bounded in the appropriate space.

Lemma 6 *Suppose $u(t,x) \in L^2(0,T; W^{1,\infty}(\mathbb{R}^3))$. Let g be the unique weak solution to (3.1). Then, $\mathfrak{F} = \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} rwg\, dw\, dr \in L^\infty(0,T; L^2(\mathbb{R}_x^3))$. Furthermore, $\|\mathfrak{F}\|_{L^\infty(0,T;L^2(\mathbb{R}_x^3))} \leq C(\|g^0\|_{L^\infty}; \|g^0\|_{L^1})$.*

Proof. According to Proposition 1, $g \in L^\infty(0,T; L^1(\mathbb{R}^6 \times \mathbb{R}^+))$. Since $|rwg| \leq RWg$ on the support of g, it follows that $\mathfrak{F}(t,x) = \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} rwg\, dw\, dr \in L^\infty(0,T; L^1(\mathbb{R}_x^3))$. Also, by Proposition 1, $g(t,x,w,r) \leq \frac{C}{|w|^3}$. Therefore, $|\int_{\mathbb{R}^+} \int_{\mathbb{R}^3} rwg\, dw\, dr| \leq \int_\Omega |rwg|\, dw\, dr \leq \int_\Omega R \frac{C}{|w|^2}\, dw\, dr \leq +\infty$ which implies that $\mathfrak{F} \in L^\infty(0,T; L^\infty(\mathbb{R}_x^3))$. We conclude that $\mathfrak{F} \in L^\infty(0,T; L^2(\mathbb{R}_x^3))$ and based on the properties of g given by Proposition 1, the norm of $\mathfrak{F}$ is bounded by a constant depending only on $\|g^0\|_{L^\infty}$ and $\|g^0\|_{L^1}$. ∎

Lemma 7 *Assume $g^0 \in L^\infty(\mathbb{R}^6 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^6 \times \mathbb{R}^+)$ and $u^0 \in L^2(\mathbb{R}^3)$. Then, the operator $\mathcal{T}_m : L^2(0,T; H_m) \to L^2(0,T; H_m)$ given by $u \mapsto g \mapsto u_m$, where g is the solution to (3.1) with velocity u, and u_m is the solution to (3.14) with $\mathfrak{F} = \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} rwg\, dw\, dr$, is well-defined and continuous.*

Proof. First we show that for $u \in L^2(0,T; H_m)$, we can apply Proposition 1 to find a unique $g \in L^\infty(0,T; L^1(\mathbb{R}^6 \times \mathbb{R}^+))$; that is, we must show $L^2(0,T; H_m) \subset L^2(0,T; W^{1,\infty})$. Assume, $u \in L^2(0,T; H_m)$. Since we can write $u = \sum_{i=1}^m d_i(t) w_i$, and $\{w_i\}_{i=1}^\infty$ is orthonormal in H, we have

$$\|u\|_{L^2(0,T;H_m)}^2 = \int_0^T \|u(t)\|_{L^2}^2\, dt = \int_0^T \|d_1(t)w_1 + ... + d_m(t)w_m\|_{L^2}^2\, dt$$
$$= \int_0^T |d_1(t)|^2 + ... + |d_m(t)|^2\, dt \leq +\infty$$

On the other hand,

$$\begin{aligned}\|u\|^2_{L^2(0,T;W^{1,\infty})} &= \int_0^T \|u(t)\|^2_{W^{1,\infty}}\,dt = \int_0^T \|d_1(t)w_1 + ... + d_m(t)w_m\|^2_{W^{1,\infty}}\,dt \\ &\le \int_0^T \left(\|d_1(t)w_1\|_{W^{1,\infty}} + ... + \|d_m(t)w_m\|_{W^{1,\infty}}\right)^2\,dt \\ &\le \int_0^T \left(|d_1(t)|\|w_1\|_{W^{1,\infty}} + ... + |d_m(t)|\|w_m\|_{W^{1,\infty}}\right)^2\,dt \\ &\le C^2 \int_0^T \left(\sum_{i=1}^m |d_i(t)|\right)^2\,dt \le C^2 m \int_0^T \sum_{i=1}^m |d_i(t)|^2 \le +\infty\end{aligned}$$

where $C = \max_{1\le i\le m} \|w_i\|_{W^{1,\infty}}$. Therefore $u \in L^2(0,T;W^{1,\infty})$ and Proposition 1 applies. This gives $g \in L^\infty(0,T;L^1(\mathbb{R}^6\times\mathbb{R}^+))$, and according to Lemma 6, $\mathfrak{F} = \int_{\mathbb{R}^+}\int_{\mathbb{R}^3} rwg\,dw\,dr \in L^\infty(0,T;L^2(\mathbb{R}^3_x))$ so that Theorem 2 applies and gives us a unique $u_m \in L^2(0,T;V_m)\cap L^\infty(0,T;H_m) \subset L^2(0,T;H_m)$. This proves that $\mathcal{T}_m$ is well-defined.

Now we will show that $\mathcal{T}_m$ is continuous with respect to the $L^2(0,T;L^2(\mathbb{R}^3))$ norm. Consider $u_n \xrightarrow{L^2_t(L^2_x)} u$. We want to show $\tilde{u}_n = \mathcal{T}(u_n) \xrightarrow{L^2_t(L^2_x)} \mathcal{T}(u) = \tilde{u}$. It is equivalent to show that for every subsequence $\tilde{u}_{n_k}$ there exists a sub-subsequence $\tilde{u}_{n_{k_l}} \xrightarrow{L^2_t(L^2_x)} \tilde{u}$. Let g_n be the unique solution to

$$\begin{cases} \partial_t g_n + \nabla_x\cdot((u_n+w)g_n) + \nabla_w\cdot(-\frac{w}{r^2}g_n) = \Gamma(g_n), \\ g_n(0,x,w,r) = g^0(x,w,r). \end{cases} \tag{3.20}$$

By Proposition 1, g_{n_k} is bounded in $L^\infty(0,T;L^\infty_{loc}(\mathbb{R}^6\times\mathbb{R}^+))$ and it is easy to verify that $\partial_t g_{n_k}$ is bounded in $L^2(0,T;W^{-1,2}_{loc}(\mathbb{R}^6\times\mathbb{R}^+))$. By the compactness lemma there exists $g_{n_{k_l}}$ and g such that

$$g_{n_{k_l}} \rightharpoonup g \;\text{ weakly in } L^p(0,T;L^p_{loc}(\mathbb{R}^6\times\mathbb{R}^+)),$$

$$g_{n_{k_l}} \to g \;\text{ strongly in } \mathcal{C}([0,T];W^{-1,2}_{loc}(\mathbb{R}^6\times\mathbb{R}^+)).$$

Following the proof of Proposition 1, it is easy to verify that g is the unique solution of (3.1) with velocity $u = \lim_{n\to\infty} u_n$. (The main difference in Proposition 1 is that $u_n \xrightarrow{L^2_t(L^\infty_x)} u$. However, it is enough to have, as in this case,

$u_n \xrightarrow{L^2_t(L^2_x)} u$). Also, we have

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^3} rwg_{n_{k_l}}\,dw\,dr \rightharpoonup \int_{\mathbb{R}^+}\int_{\mathbb{R}^3} rwg\,dw\,dr \quad \text{weakly in } L^2(0,T;L^2_{loc}(\mathbb{R}^3)). \tag{3.21}$$

To see this, fix $\Omega_x \subset\subset \mathbb{R}^3$ and consider $\phi \in L^2(0,T;L^2(\Omega_x))$. Then,

$$\begin{aligned}
&\int_0^T \int_{\Omega_x} \phi \left\{ \int_{\mathbb{R}^+}\int_{\mathbb{R}^3} rw(g_{n_{k_l}} - g)\,dw\,dr \right\} dx\,dt \\
&\qquad = \int_0^T \int_{\Omega_x} \phi \left\{ \int_{\Omega} rw(g_{n_{k_l}} - g)\,dw\,dr \right\} dx\,dt \\
&\qquad = \int_0^T \int_{\Omega_x \times \Omega} \phi rw(g_{n_{k_l}} - g)\,dw\,dr\,dx\,dt \to 0,
\end{aligned}$$

since $\phi rw \in L^2(0,T;L^2(\Omega_x \times \Omega))$ and $g_{n_{k_l}} \rightharpoonup g$ in $L^2(0,T;L^2_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$.

Now, according to Theorem 2, there exists a unique $\tilde{u}_n \in L^2(0,T;V_m) \cap L^\infty(0,T;H_m)$ verifying (3.14) with $\mathfrak{F} = \mathfrak{F}_n = \int_{\mathbb{R}^+}\int_{\mathbb{R}^3} rwg_n\,dw\,dr$. Since Lemma 6 implies $\|\mathfrak{F}_n\|_{L^2(0,T;L^2(\mathbb{R}^3_x)} \le C$, it follows from (3.17)-(3.18) that $\|\tilde{u}_n\|_{L^2(0,T;H^1_0(\mathbb{R}^3))} \le C$ and $\|\tilde{u}_n\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \le C$. Therefore, given any subsequence $\tilde{u}_{n_k}$ we can extract a subsequence $\tilde{u}_{n_{k_{l_p}}}$ such that (3.21) holds and

$$\tilde{u}_{n_{k_{l_p}}} \rightharpoonup u^* \quad \text{weakly in } L^2(0,T;V_m),$$

$$\tilde{u}_{n_{k_{l_p}}} \overset{*}{\rightharpoonup} u^* \quad \text{weak} - \text{star in } L^\infty(0,T;H_m).$$

Finally, using standard compactness results (for example, Theorem 3.2.2 in [16]), we can choose the subsequence above so that

$$\tilde{u}_{n_{k_{l_p}}} \to u^* \quad \text{strongly in } L^2(0,T;L^2(\mathbb{R}^3)).$$

Passing to the limit in (3.14) shows that u^* is the unique solution to (3.14) with $\mathfrak{F} = \int_{\mathbb{R}^+}\int_{\mathbb{R}^3} rwg\,dw\,dr$. That is, $\tilde{u}_{n_{k_{l_p}}} \xrightarrow{L^2_t(L^2_x)} u^* = \tilde{u}$. Therefore, $\mathcal{T}_m$ is continuous with respect to the $L^2(0,T;L^2(\mathbb{R}^3))$ norm. ∎

We are looking for g and u which solve (1.8) and (1.9) simultaneously. The idea is to find approximate solutions g_m and u_m, which correspond to a fixed point of $\mathcal{T}_m$, and then using previous estimates, extract a subsequence which converges to a solution of the original problem. With this goal in mind, we will now show that $\mathcal{T}_m$ has a fixed point.

Lemma 8 *$\mathcal{T}_m$ has a fixed point.*

Proof. It suffices to show, by the Shauder fixed point theorem, that $\mathcal{T}_m$ maps some closed convex set compactly into itself. Indeed, according to Theorem 2,

$$\|\mathcal{T}_m(u)\|_{L^2(0,T;L^2(\mathbb{R}^3))} \leq \|\mathcal{T}_m(u)\|_{L^2(0,T;H_0^1(\mathbb{R}^3))} \leq C(\|u_0\|_{L^2(\mathbb{R}^3)}; \|\mathfrak{F}\|_{L^2(0,T;L^2(\mathbb{R}^3))})$$

where, by Lemma 6, $\|\mathfrak{F}\|_{L^2(0,T;L^2(\mathbb{R}^3_x)} \leq C(\|g^0\|_{L^\infty}; \|g^0\|_{L^1})$. Therefore

$$\|\mathcal{T}_m(u)\|_{L^2(0,T;L^2(\mathbb{R}^3))} \leq C(\|u_0\|_{L^2(\mathbb{R}^3)}; \|g^0\|_{L^\infty}; \|g^0\|_{L^1}). \tag{3.22}$$

Since $\mathcal{T}_m(u)$ is bounded in $L^2(0,T;L^2(\mathbb{R}^3))$ uniformily in u, we consider the convex closed set $B(0,R)$, where R is the constant in (3.22). Clearly $\mathcal{T}_m(B(0,R)) \subset B(0,R)$. The inclusion is also compact. Indeed, standard a priori estimates and compactness results (see [16]) ensure that solutions of (3.14) are contained in a subspace of $L^2(0,T;H_0^1(\mathbb{R}^3)) \cap L^\infty(0,T;L^2(\mathbb{R}^3))$ which embeds compactly into $L^2(0,T;L^2(\mathbb{R}^3))$. Thus, the Shauder fixed point theorem applies and we obtain a fixed point of $\mathcal{T}_m$. ∎

We can now prove our main result.

Proof of Theorem 1.

As a result of Lemma 8, there exist approximate solutions g_m and u_m verifying (3.1) and (3.14) simultaneously. By Proposition 1, the sequence g_m is bounded in $L^\infty(0,T;L^\infty_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$. It is easy to check that $\partial_t g_m$ is bounded in $L^\infty(0,T;W^{-1,2}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$. Applying the Aubin lemma, there exists a subsequence g_{m_k} and a function g, such that

$$g_{m_k} \rightharpoonup g \quad \text{weakly in } L^p(0,T;L^p_{loc}(\mathbb{R}^6 \times \mathbb{R}^+)),$$

$$g_{m_k} \to g \quad \text{strongly in } \mathcal{C}([0,T];W^{-1,2}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+)).$$

Moreover, since $g \in L^\infty(0,T;L^2_{loc}(\mathbb{R}^6 \times \mathbb{R}^+)) \cap \mathcal{C}([0,T];W^{-1,2}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$, it follows from Lemma 3.1.4 in [16] that g is weakly continuous from $[0,T]$ into $L^2_{loc}(\mathbb{R}^6 \times \mathbb{R}^+)$.

On the other hand, the bounds given by (3.17) and (3.18) allow us to apply compactness results, as in [16], and extract a subsequence $u_{m_{k_l}}$ and a function u, such that

$$u_{m_{k_l}} \rightharpoonup u \ \text{ weakly in } L^2(0,T;H_0^1(\mathbb{R}^3)),$$

$$u_{m_{k_l}} \overset{*}{\rightharpoonup} u \ \text{ weak} - \text{star in } L^\infty(0,T;L^2(\mathbb{R}^3)),$$

$$u_{m_{k_l}} \to u \ \text{ strongly in } L^2(0,T;L^2(\mathbb{R}^3)).$$

Also, according to (3.21), $\int\int rwg_{m_k}\,dw\,dr \xrightarrow{L^2(L^2_{loc})} \int\int rwg\,dw\,dr$. Finally, given the information above, we claim that passing to the limit in (3.1) and (3.14) yields (1.8)-(1.9). Passing to the limit in the fluid equation is straightfoward (cf. [16]) and we obtain (1.12). Next, for the kinetic equation, the former convergences imply, as in the proof of Proposition 1, that $g(t) \xrightarrow{W^{-1,2}_{loc}} g^0$. Now, it suffices to show (1.8) holds in the sense of distributions. Let us show that $\nabla_x \cdot (u_{m_{k_l}} g_{m_{k_l}}) \xrightarrow{\mathcal{D}'} \nabla_x \cdot (ug)$. For $2 \le p < +\infty$, we have $u_{m_{k_l}} \xrightarrow{L^2(L^2)} u$ and $g_{m_{k_l}} \xrightarrow{L^p(L^p_{loc})} g$. Therefore $u_{m_{k_l}} g_{m_{k_l}} \rightharpoonup ug$ in $L^{\frac{2p}{2+p}}(0,T;L_{loc}^{\frac{2p}{2+p}}(\mathbb{R}^6 \times \mathbb{R}^+))$, which implies the statement above. It follows from previous arguments that the remaining terms converge in the sense of distributions to their respective terms in (3.1). Thus, g is a weak solution to (3.1) with velocity u. As before, integrating (3.1) shows $g \in L^\infty(0,T;L^1(\mathbb{R}^6 \times \mathbb{R}^+))$. This completes the proof.

∎

4 Appendix: The redistribution density

We conclude with a few remarks about the redistribution density function, h. Specifically, we would like to describe one case in which we can expect condition (v) from Section 2. Recall that for non-constant flows, $u(x,t)$, we have chosen to express h as a function of the relative velocities of the particles and their radii. While a precise specification of h should depend, in some complex way, on the material properties of the particles as well as the gas, we will assume h has reasonably nice structure which is at least faithful to important qualitative features of the model such as mass conservation.

Let us assume first that

$$h(r,w,r^*,w^*) = H(r,r^*)G(w,w^*). \tag{4.1}$$

That is, we assume the size and relative velocity of derivative particles are determined independently according to the probabilities H and G. Surely this is a simplifying assumption, since one would expect some interdependence of those quantities, especially when taking momentum transfer into account. Next, it is reasonable to assume that both H and G satisfy a self-similarity (homogeneity) property, namely

$$H(r,r^*) = C(r^*)H\left(\frac{r}{r^*},1\right), \tag{4.2}$$

$$G(w,w^*) = C(|w^*|)G\left(\frac{w}{|w^*|},\frac{w^*}{|w^*|}\right). \tag{4.3}$$

Additionally, G should be invariant under rotations of the pair (w,w^*). Under these assumptions, it suffices to know the behavior of H and G for fixed values of the starred variables, e.g. $r^*=1$ and $w^*=(1,0,0)$. In fact, let us assume that both $H(\cdot,1)$ and $G(\cdot,\frac{w^*}{|w^*|})$ are bounded; that is, assume there exist $C_1,C_2>0$ such that

(i) $H(s,1) \le C_1$ for all $0<s\le 1$, and

(ii) $G(z,\eta) \le C_2$ for all $|\eta|=1$ and for all $z\in B(0,1)$.

If this was not the case, then after mollification one could find suitable approximations which do have the indicated bounds. Now, from the previous assumptions together with (vii) in section 2, we deduce the following:

$$\begin{aligned}
2 &= \int_0^{r^*}\int_{B(0,|w^*|)} H(r,r^*)G(w,w^*)\,dw\,dr \\
&= C(r^*)C(|w^*|)\int_0^{r^*}\int_{B(0,|w^*|)} H\left(\frac{r}{r^*},1\right)G\left(\frac{w}{|w^*|},\frac{w^*}{|w^*|}\right)\,dw\,dr \\
&= C(r^*)C(|w^*|)\int_0^{1}\int_{B(0,1)} H\left(s,1\right)G\left(z,\frac{w^*}{|w^*|}\right)r^*|w^*|^3\,dz\,ds \\
&= r^*|w^*|^3C(r^*)C(|w^*|)\int_0^{1}\int_{B(0,1)} H\left(s,1\right)G\left(z,\frac{w^*}{|w^*|}\right)\,dz\,ds \\
&= r^*|w^*|^3C(r^*)C(|w^*|)\times 2.
\end{aligned}$$

Therefore, $C(r^*) = \frac{1}{r^*}$ and $C(|w^*|) = \frac{1}{|w^*|^3}$, and we have using $(i)-(ii)$

$$h(r,w,r^*,w^*) = H(r,r^*)G(w,w^*) \leq \frac{C_1 C_2}{r^* |w^*|^3}$$

Finally, we assume there exists $C>0$ such that $\nu(r,w) \leq Cr|w|^3$. Notice, in particular, that we do not require $\nu(r,w)=0$ for r and w sufficiently small (cf. [10]). Combining the previous two inequalities establishes the bound on νh required in section 2.

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