Strong traces for solutions to scalar conservation laws with general flux

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Abstract

In this paper we consider bounded weak solutions u of scalar conservation laws, not necessarily of class BV, defined in a subset $\Omega \subset \mathbb{R}^+ \times \mathbb{R}$. We define a strong notion of trace at the boundary of Ω reached by L^1 convergence for a large class of functionals of u, $G(u)$. Those functionals G depend on the flux function of the conservation law and on the boundary of Ω. The result holds for general flux function and general subset.

Keywords: conservation laws, trace theorem, kinetic formulation, boundary value problem, averaging lemma.

AMS Subject Classification: 35L65, 35B05, 35B65.

1 Introduction

In this article we consider an open subset $\Omega \subset \mathbb{R}^+ \times \mathbb{R}$ with $\mathcal{C}^2$ boundary and functions $u \in L^\infty(\Omega)$, solutions to the scalar conservation law:

$$\partial_t u + \partial_x A(u) = 0, \quad (t,x) \in \Omega, \tag{1}$$

where the flux function is assumed to be regular, $A \in C^2(\mathbb{R})$. As usual, we deal only with entropy solutions, namely those that fulfill in the sense of distributions

$$\partial_t \eta(u) + \partial_x H(u) \leq 0, \quad (t,x) \in \Omega, \tag{2}$$

for every convex function η and related entropy flux defined by

$$H' = A'\eta'. \tag{3}$$

The initial value problem in the half space $\mathbb{R}^+ \times \mathbb{R}$ is well known since the works of Kruzkov [10], where the existence and uniqueness is obtained whenever the initial value is reached strongly in $L^1_{\text{loc}}(\mathbb{R})$. Hence, the function u could be, for instance, the restriction to Ω of such a solution of Kruzkov. But it could be a solution of the conservation law with boundary values on $\partial\Omega$, or other. We investigate in this paper the behavior of the function u on the boundary of Ω. Our aim is to show the existence of trace of some quantities depending on u reached in a strong topology (i.e. without oscillations).

The question of strong traces arose initially in the context of limit of hyperbolic relaxation towards a scalar conservation law in the case $\Omega = (0, +\infty) \times \mathbb{R}$, that is the trace at $t = 0$ (see for instance Natalini [13], Tzavaras [16]). The question was whether the limit obtained is the one defined by Kruzkov since the strong continuity at $t = 0$ is not naturally preserved at the limit. To avoid any misunderstanding, let us recall that the uniqueness is well known to hold if we incorporate the initial value in the inequality (2) in the following way:

$$\int_0^\infty \int_{-\infty}^\infty (\partial_t \phi(t,x)\eta(u(t,x)) + \partial_x \phi(t,x) H(u(t,x)))\, dx\, dt \\ + \int_{-\infty}^\infty \phi(0,x)\eta(u_0(x))\, dx \geq 0,$$

for any non negative test function $\phi \in \mathcal{D}([0,\infty) \times \mathbb{R})$. However this condition gives that $\eta(u(t,\cdot))$ converges to $\eta(u_0)$ when t goes to 0 (at least weakly), which implies the strong convergence of $u(t,\cdot)$ at $t = 0$. Hence, putting the initial value in the entropy inequality is exactly equivalent to assuming the existence of a strong trace at $t = 0$ (i.e. reached by a strong topology). Now, if we don't have either of those assumptions but only the weak form:

$$\lim_{t \to 0} \int_{\mathbb{R}} (u(t,x) - u_0(x))\phi(x)\, dx = 0,$$

for every test function ϕ, the question is: does this solution have a strong trace at $t = 0$, and so is it the unique Kruzkov solution ?

The first result of this kind has been proven in [17] for the system of isentropic gas with $\gamma = 3$ which has a lot of similarities with the scalar case. It involves the introduction of blow-up techniques and the use of the theory of kinetic formulation of conservation laws introduced by Lions, Perthame, and Tadmor in [11, 12] which allows using the so-called averaging lemmas ([1, 7, 9, 15]). This blow-up method is inherited from techniques widely used for parabolic equations (see for instance [8]). Up to our knowledge this is the first time that it has been used in the context of hyperbolic conservation laws. The method has been generalized, in [18], in the case of trace for arbitrary domain Ω of the multidimensional scalar conservation laws involving a "non degenerate" flux function verifying:

$$\mathcal{L}(\{\xi \mid \tau + \zeta . A'(\xi) = 0\}) = 0, \qquad \text{for every} \quad (\tau, \xi) \neq (0,0). \tag{4}$$

where $\mathcal{L}$ is the Lebesgue measure. This can be seen as a non-degeneracy property since it avoids flux functions whose restriction to an open subset is linear. This assumption permits to use the averaging lemmas in this case. Let us cite also the result of Chen and Rascle [4] where the strong trace at $t = 0$ is proven for the mono-dimensional case using compensated compactness techniques with a slightly different hypothesis of non-degenerate flux. Those results can be seen as a regularization effect at the boundary induced by the non-degeneracy of A. The case of trace at $t = 0$ has been solved recently by Panov [14] for general flux. He shows that any solution verifying (1) (2) in $(0, \infty) \times \mathbb{R}^d$ has a trace reached by the strong topology at $t = 0$. The method involves the blow-up method with refined techniques of H-measures. Let us also mention that those blow-up methods in the framework of a kinetic formulation of conservation laws have been used later in different context with tools coming from geometrical measure theory (see for instance the result of De Lellis, Otto and Westdickenberg [6]). For a complete review of this kind of results, we refer to the book of Dafermos [5].

In this paper we want to generalize the trace results in the case of general domain Ω and general flux. The situation is significantly different that the case of the trace at $t = 0$. Notice that in the characteristic case, we can have situations where no trace can be defined. A trivial example is obtained with equation (1) with $A = 0$ and $\Omega = \mathbb{R} \times (0, \infty)$. The function $u(t, x) = sin(1/x)$ is clearly solution and has no trace (even in a weak sense) at $x = 0$. The aim is to find a large class of functional of u for which we can define a trace reached by a strong topology. Those functionals depend on the flux A and the boundary $\partial\Omega$. Recently in [2], Bürger, Frid and Karlsen have used the strong trace result [18] with "non degenerate" flux verifying (4) to study the Initial Boundary Problem with zero-flux condition at the boundary. One of the motivation of our paper is to be able to extend this kind of result to the case of general flux functions.

To define traces on the boundary, we use the framework of "regular deformable boundary" (see for instance Chen and Frid in [3], where they consider only Lipschitz boundaries). Indeed for any domain Ω with $\mathcal{C}^2$ boundary, There exists at least one $\partial\Omega$ regular deformation, where, for K open subset of $\partial\Omega$, we call K regular deformation every function $\psi : [0,1] \times K \to \bar{\Omega}$, $\mathcal{C}^1$ diffeomeomorphism over its image which verifies $\psi(0,\cdot) \equiv I_K$, where I_K is the identity map over K.

Let us denote ν the unit outward normal field of $\partial\Omega$, and ν_s the unit outward normal field of the approximated boundary $\psi(\{s\} \times \partial\Omega)$. Notice that ν_s converges strongly to ν when s goes to 0.

More precisely, we want to study the behavior of $u(\psi(s,\cdot))$ when s goes to zero for such $\partial\Omega$ regular deformation. Our main theorem is the following:

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^2$ be a regular open set with $\mathcal{C}^2$ boundary and the flux function A lie in $C^2(\mathbb{R})$. For any function $\eta \in W^{1,1}(\mathbb{R})$ we consider $\overline{H}_\eta = (\eta, H)$ with the flux H verifying (3). Consider any function $u \in L^\infty(\Omega)$ which verifies (1)and(2) in Ω. Then, for any $\eta \in W^{1,1}(\mathbb{R})$, there exists $[\overline{H}_\eta(u)]^\tau \in [L^\infty(\partial\Omega)]^2$ such that, for every $\partial\Omega$ regular deformation ψ and every compact set $K \subset\subset \partial\Omega$:*

$$\operatorname*{esslim}_{s\to 0} \int_K |\overline{H}_\eta(u(\psi(s,\hat{z}))) \cdot \nu_s(\hat{z}) - \overline{H}_\eta(u)^\tau(\hat{z}) \cdot \nu(\hat{z})| \, d\hat{z} = 0. \qquad (5)$$

In particular, for any η the trace $\overline{H}_\eta(u)^\tau \cdot \nu$ is unique and for any function $F \in C^0(\mathbb{R})$, $F(\overline{H}_\eta(u) \cdot \nu_s)$ has also a trace and:

$$\left[F(\overline{H}_\eta(u) \cdot \nu_s)\right]^\tau = F\left([\overline{H}_\eta(u)]^\tau \cdot \nu\right).$$

This theorem means that for any entropy function η (convex or not) the related flux of entropy through the boundary has a trace which is reached in a strong topology. It is well known (see for instance Chen and Frid [3]) that this quantity has a weak trace (i.e. reached by a weak topology). All the interest of the result lies in the fact that the trace is reached strongly in L^1_{loc}. As mentioned above we cannot expect to have a trace for u itself, in opposition to the "non-degenerated flux" case or the case of trace at $t = 0$. Hence the result can be roughly summarized in the following way: For all the quantity of the form $\overline{H}(u) \cdot \nu_s$ for which a trace is known to exist weakly for any solution u, the trace is indeed reached strongly.

2 Reformulation of the problem

First we split the boundary into a countable quantity of subsets on which we can consider a local map. Indeed, for each $\hat{z} = (\hat{t}, \hat{x}) \in \partial\Omega$, there exists

$r_{\hat{z}} > 0$, a $\mathcal{C}^2$ mapping $\gamma_{\hat{z}} : \mathbb{R} \to \mathbb{R}$ and an isometry for the Euclidean norm $\mathcal{R}_{\hat{z}} : \mathbb{R}^2 \to \mathbb{R}^2$ such that: upon rotating, relabeling and translating the coordinate axes $(y_0, y_1) = \mathcal{R}_{\hat{z}}(t, x)$ if necessary,

$$\begin{aligned}&\mathcal{R}_{\hat{z}}(\hat{z}) = 0,\\ &\mathcal{R}_{\hat{z}}(\Omega) \cap]-r_{\hat{z}}, r_{\hat{z}}[^2 = \{y \in]-r_{\hat{z}}, r_{\hat{z}}[^2 \mid y_0 > \gamma_{\hat{z}}(y_1)\}.\end{aligned}$$

We have:

$$\partial\Omega \subset \bigcup_{\hat{z}\in\partial\Omega} \mathcal{R}_{\hat{z}}^{-1}(]-r_{\hat{z}}, r_{\hat{z}}[^2).$$

Since $\partial\Omega \cap [-n, n]^2$ is a compact subset for every integer n, there exists a finite set I_n such that:

$$\left(\partial\Omega \cap [-n, n]^2\right) \subset \bigcup_{\alpha\in I_n} \mathcal{R}_{\hat{z}_\alpha}^{-1}(]-r_{\hat{z}_\alpha}, r_{\hat{z}_\alpha}[^2).$$

So, $I = \cup I_n$ is a countable set such that:

$$\partial\Omega = \bigcup_{\alpha\in I} \Gamma_\alpha,$$

where Γ_α is defined by:

$$\Gamma_\alpha = \mathcal{R}_\alpha^{-1}(\{y \in]-r_\alpha, r_\alpha[^2 \mid y_0 = \gamma_\alpha(y_1)\}).$$

In order to simplify the notation we write α instead of $\hat{z}_\alpha$ in the indices, and we denote in the same way

$$\Omega_\alpha = \{y \in]-r_\alpha, r_\alpha[^2 \mid y_0 > \gamma_\alpha(y_1)\}.$$

From now on we work in Ω_α and in the new y coordinates. We denote $A_\alpha(\xi) = \mathcal{R}_\alpha(\xi, A(\xi))$ and $H_\alpha(\xi) = \mathcal{R}_\alpha(\eta(\xi), H(\xi))$. We define $u_\alpha : \Omega_\alpha \to \mathbb{R}$ by $u_\alpha(y) = u(\mathcal{R}_\alpha^{-1}(y))$. In the y coordinates, (1) and (2) correspond in Ω_α to:

$$\text{div}_y A_\alpha(u_\alpha) = 0 \tag{6}$$

$$\text{div}_y H_\alpha(u_\alpha) \leq 0. \tag{7}$$

We now introduce the kinetic formulation due to Lions, Perthame, and Tadmor. In order to do so we set $L = \|u\|_{L^\infty(\Omega)}$ and introduce a new variable $\xi \in]-L, L[$ and denote for every $v \in]-L, L[$:

$$\begin{aligned}\chi(v, \xi) &= \mathbf{1}_{\{0\leq\xi\leq v\}} \text{ if } v \geq 0\\ &= -\mathbf{1}_{\{v\leq\xi\leq 0\}} \text{ if } v < 0.\end{aligned}$$

Then we introduce new functions called microscopic functions which depend on ξ and on a variable z which can lie on Ω_α, on Γ_α or on a local space as we will see later. We will especially consider the following ones:

Definition 2.1 *Let N be an integer, $\mathcal{O}$ be an open set of $\mathbb{R}^N$, $I =]a, b[$ for $-L \leq a < b \leq L$, and the microscopic function $f \in L^\infty(\mathcal{O} \times I)$ be such that $0 \leq \mathrm{sgn}(\xi) f(z, \xi) \leq 1$ for almost every (z, ξ). We say that f is a χ-function if there exists $u \in L^\infty(\mathcal{O})$ such that for almost every $z \in \mathcal{O}$ and $\xi \in I$:*

$$f(z, \cdot) = \chi(u(z), \cdot).$$

Notice that, if $0 \in I$ and f is a χ-function then we can choose u by the formula $u(z) = \int_I f(z, \xi)\, d\xi$.

Lions, Perthame, and Tadmor have shown in [11] the following theorem:

Theorem 2.1 *A function $u \in L^\infty(\Omega_\alpha)$ with $|u| \leq L$ is solution of (6)and(7) in Ω_α if and only if there exists a nonnegative measure $m \in \mathcal{M}^+(\Omega_\alpha \times] - L, L[)$ such that the related χ-function f defined by $f(y, \cdot) = \chi(u(y), \cdot)$ for almost every $(y, \xi) \in (\Omega_\alpha \times] - L, L[)$ verifies:*

$$a(\xi).\nabla_y f = \partial_\xi m \tag{8}$$

in $\Omega_\alpha \times] - L, L[$ with $a(\xi) = {A_\alpha}'(\xi) = (a^0(\xi), a^1(\xi))$.

In addition, this structure of χ-function can be used to characterize strong convergence.

Lemma 2.1 *Let $\mathcal{O}$ be an open set of $\mathbb{R}^N$, $I =]a, b[$ be an interval such that $-L \leq a < b \leq L$, and $f_n \in L^\infty(\mathcal{O} \times I)$ be a sequence of χ-functions converging weakly to $f \in L^\infty(\mathcal{O} \times I)$. We denote $u_n(\cdot) = \int_I f_n(\cdot, \xi)\, d\xi$ and $u(\cdot) = \int_I f(\cdot, \xi)\, d\xi$. Then for almost every $z \in \mathcal{O}$, the function $f(z, \cdot)$ lies in $BV(I)$. Moreover, the three following propositions are equivalent:*

- *f_n converges strongly to f in $L^1_{\mathrm{loc}}(\mathcal{O} \times I)$,*
- *u_n converges strongly to u in $L^1_{\mathrm{loc}}(\mathcal{O})$,*
- *f is a χ-function.*

Proof of Lemma 2.1: Since f_n is a χ-function we have

$$\partial_\xi f_n = \delta(\xi) - \delta(\xi - u_n) \qquad \text{in } I. \tag{9}$$

So at the limit $\partial_\xi f \in L^\infty(\mathcal{M}(I))$, which means that for almost every $z \in \mathcal{O}$, $f(z,\cdot)$ lies in $BV(I)$.

Now, if f_n converges strongly, the same holds for u_n. If u_n converges strongly, then its young measure $\delta(\xi - u_n)$ converges to $\delta(\xi - u)$. Hence:

$$\partial_\xi f = \delta(\xi) - \delta(\xi - u) \qquad \text{in } \ I.$$

This ensures that f is a χ-function. Finally, if f is a χ-function, in particular, $\mathrm{sgn}(\xi) f = f^2$ so $\|f_n\|_{L^2_{\mathrm{loc}}(\mathcal{O}\times I)}$ converges to $\|f\|_{L^2_{\mathrm{loc}}(\mathcal{O}\times I)}$. This provides the strong convergence of f_n in $L^2_{\mathrm{loc}}(\mathcal{O}\times I)$ and then in $L^1_{\mathrm{loc}}(\mathcal{O}\times I)$. □

For every fixed α, we will consider the set Ω_α, and the χ-function f associated to u_α. Since Γ_α is parametrized by y_1, for every Γ_α regular deformation ψ and every $y_1 \in]-r_\alpha, r_\alpha[$ we set:

$$\begin{aligned} \tilde{\psi}(s, y_1) &= \psi(s, \mathcal{R}_\alpha^{-1}(\gamma_\alpha(y_1), y_1)), \\ f_\psi(s, y_1, \xi) &= f(\tilde{\psi}(s, y_1), \xi). \end{aligned}$$

To simplify the notations we keep denoting ν_s and ν the normal vectors as functions of y_1.

The next sections are dedicated to the proof of the following theorem:

Theorem 2.2 *For every fixed α, There exists a unique trace function $(a\cdot\nu)f^\tau \in L^\infty(]-r_\alpha, r_\alpha[\times]-L, L[)$ (which does not depend on the deformation ψ) such that:*

$$\operatorname*{esslim}_{s\to 0} (a\cdot\nu_s) f_\psi(s,\cdot,\cdot) = (a\cdot\nu) f^\tau \ \text{ in } L^1(]-r_\alpha, r_\alpha[\times]-L, L[).$$

We finish this section showing that this theorem implies the main Theorem 1.1.

Proof of Theorem 1.1 using Theorem 2.2: Let us consider the χ-function f associated to the solution u to (1) (2) through Theorem 2.1. From the definition of $\overline{H}_\eta$ and the structure of χ-functions we have for almost every $(t,x) \in \Omega$:

$$\overline{H}_\eta(u(t,x)) = \int_{-L}^{L} \eta'(\xi) a(\xi) f(t,x,\xi)\, d\xi.$$

Hence, for $\mathcal{R}_\alpha \hat{z} = (\gamma_\alpha(y_1), y_1))$, we have:

$$\overline{H}_\eta(u(\psi(s,\hat{z}))) \cdot \nu_s(\hat{z}) = \int_{-L}^{L} \eta'(\xi)(a(\xi)\cdot\nu_s(y_1)) f_\psi(s, y_1, \xi)\, d\xi.$$

Let us define for $\mathcal{R}_\alpha \hat{z} = (\gamma_\alpha(y_1), y_1)$):

$$\overline{H}_\eta(u)^\tau(\hat{z}) \cdot \nu(\hat{z}) = \int_{-L}^{L} \eta'(\xi)(a(\xi) \cdot \nu(y_1)) f^\tau(y_1, \xi)\, d\xi.$$

So we have:

$$\begin{aligned} &\int_{\Gamma_\alpha} |\overline{H}_\eta(u)(\psi(s,\hat{z})) \cdot \nu_s(\hat{z}) - \overline{H}_\eta(u)^\tau(\hat{z}) \cdot \nu(\hat{z})|\, d\hat{z} \\ &\leq C \int_{-r_\alpha}^{r_\alpha} \int_{-L}^{L} |\eta'(\xi)||(a(\xi) \cdot \nu_s(y_1)) f_\psi(s) - (a(\xi) \cdot \nu(y_1)) f^\tau)|\, d\xi\, dy_1, \end{aligned} \tag{10}$$

where the constant C depends on the Jacobian of the transformation. Theorem 2.2 show that $(a \cdot \nu_s) f_\psi(s) - (a \cdot \nu) f^\tau$ converges strongly in L^1 to 0. So, up to a subsequence $s_n \to 0$, it converges to 0 almost everywhere. Notice that

$$|\eta'(\xi)||(a(\xi) \cdot \nu_s) f_\psi(s) - (a(\xi) \cdot \nu) f^\tau| \leq 2|\eta'(\xi)|\|a\|_{L^\infty}$$

where the righthand side is integrable ($\eta \in W^{1,1}(\mathbb{R})$) and does not depend on s_n. So from Lebesgue's Theorem, the righthand side of (10) converges to 0 when $s_n \to 0$. By uniqueness of the limit, the left hand side of (10) converges to 0 when $s \to 0$. This holds for every α. Finally for every compact set $K \subset\subset \partial\Omega$, $\{\Gamma_\alpha\}$ is a covering of K with open sets of $\partial\Omega$ so there exists a finite set I_0 such that $K \subset \bigcup_{\alpha \in I_0} \Gamma_\alpha$ and so

$$\begin{aligned} &\int_K |\overline{H}_\eta(u)(\psi(s,\hat{z})) \cdot \nu_s(\hat{z}) - \overline{H}_\eta(u)^\tau(\hat{z}) \cdot \nu(\hat{z})|\, d\hat{z} \\ &\leq \sum_{\alpha \in I_0} \int_{\Gamma_\alpha} |\overline{H}_\eta(u)(\psi(s,\hat{z})) \cdot \nu_s(\hat{z}) - \overline{H}_\eta(u)^\tau(\hat{z}) \cdot \nu(\hat{z})|\, d\hat{z}. \end{aligned}$$

which ends the proof. □

3 Construction of the weak trace

From now on we fix α, and the associated Ω_α and χ-function f associated to u_α. In this section we will first show that $(a \cdot \nu_s) f_\psi$ has a (at least weak) trace at $s = 0$ which does not depend on the deformation ψ, namely:

Proposition 3.1 *Let f be a solution of (8) in $\Omega_\alpha \times]-L, L[$. Then there exists $f^\tau \in L^\infty(]-r_\alpha, r_\alpha[\times]-L, L[)$ such that for all Γ_α regular deformation ψ:*

$$\operatorname*{esslim}_{s \to 0} (a \cdot \nu_s) f_\psi(s, \cdot, \cdot) = (a \cdot \nu) f^\tau \text{ in } H^{-1}(]-r_\alpha, r_\alpha[\times]-L, L[).$$

Moreover $(a \cdot \nu) f^\tau$ is uniquely defined.

This shows the existence of a weak trace on Γ_α of $(a\cdot\nu_s)f$ which does not depend on the way chosen to reach the boundary. This result is an extension to proposition 1 in [18]. We give the proof for the sake of completeness.

Proof of Proposition 3.1: Since $\|f_\psi(s,\cdot,\cdot)\|_{L^\infty}\leq 1$, by weak compactness and Sobolev imbedding, for every regular deformation ψ and every sequence s^n which tends to 0 there exists a subsequence n_p and a function $g^\tau_\psi \in L^\infty(]-r_\alpha,r_\alpha[\times]-L,L[)$ such that

$$f_\psi(s^{n_p},\cdot,\cdot) \overset{H^{-1}\cap L^\infty W*}{\longrightarrow} g^\tau_\psi \qquad \text{when } n_p\to+\infty. \tag{11}$$

Let us now show that $(a\cdot\nu)g^\tau_\psi$ does not depend on ψ, on the sequence s^n and s^{n_p}. In order to do so, let us first consider the entropy flux associated with entropy η:

$$\overline{H}_\eta(y)=\int_{-L}^{L} a(\xi)\eta'(\xi)f(y,\xi)\,d\xi. \tag{12}$$

Multiplying (8) by $\eta'(\xi)$ and integrating it with respect to ξ we find:

$$\mathrm{div}_y\overline{H}_\eta=-\int_{-L}^{L}\eta''(\xi)m(y,d\xi)\in\mathcal{M}(]-r_\alpha,r_\alpha[^2).$$

We can now use the following Theorem proved by Chen and Frid in [3]:

Theorem 3.1 *Let Ω be an open set with regular boundary $\partial\Omega$ and $F\in[L^\infty(\Omega)]^2$ be such that* div_yF *is a bounded measure. Then there exists $F.\nu\in L^\infty(\partial\Omega)$ such that for every ψ $\partial\Omega$ regular deformation:*

$$\operatorname*{esslim}_{s\to0}F(\psi(s,\cdot)).\nu_s(\cdot)=F.\nu \text{ in } L^\infty(\partial\Omega)\ w*,$$

where ν_s is a unit outward normal field of $\psi(\{s\}\times\partial\Omega)$.

This theorem insures that there exists $\overline{H}^\tau_\eta.\nu\in L^\infty(]-r_\alpha,r_\alpha[)$ which does not depend on ψ such that

$$\overline{H}_\eta(\tilde\psi(s,\cdot)).\nu_s(\cdot)\underset{s\to0}{\overset{\mathcal{D}'(]-r_\alpha,r_\alpha[)}{\longrightarrow}}\overline{H}^\tau_\eta.\nu, \tag{13}$$

for every regular deformation ψ. The function ν_s converges strongly in $L^1(]-r_\alpha,r_\alpha[)$ to ν, unit outward normal field of Γ_α. So, using (12) and (11), (13) leads to:

$$\int_{-r_\alpha}^{r_\alpha}\int_{-L}^{L}\varphi(y_1)\eta'(\xi)a(\xi).\nu(y_1)g^\tau_\psi(y_1,\xi)\,d\xi\,dy_1=\int_{-r_\alpha}^{r_\alpha}\overline{H}^\tau_\eta.\nu(y_1)\varphi(y_1)\,dy_1$$

for every test functions $\varphi \in \mathcal{D}(]-r_\alpha, r_\alpha[)$. The right-hand side of this equation is independent of ψ, sequence s^n and subsequence s^{n_p} so $(a.\nu)g_\psi^\tau$ does not depend on those quantities too. The result is obtained from the uniqueness of the limit. $\square$

To prove Therorem 2.2, we just need to prove that $(a \cdot \nu)f^\tau$ is reached strongly by L^1 convergence. We will first prove it for the range of ξ where $a(\xi)$ is constant on an interval (section 4), and then on range of ξ such that $a(\xi)$ is strictly nonlinear on an interval (section 5). We will then show that the general case can be reduced to those in the last section.

In the two next sections, we will begin to show that the convergence holds strongly for the special Γ_α deformation ψ_0 defined by:

$$\tilde{\psi}_0(s, y_1) = (s + \gamma_\alpha(y_1), y_1), \tag{14}$$

and then show that this holds for any deformation ψ. In order to simplify the notations, we will denote:

$$\tilde{f}(s, y_1, \xi) = f_{\psi_0}(s, y_1, \xi) = f(\tilde{\psi}_0(s, y_1), \xi), \tag{15}$$

when we work with this special Γ_α deformation (14). Notice that $\tilde{\psi}_0(s, y_1) \in \Omega_\alpha$ if and only if $y_1 \in]-r_\alpha, r_\alpha[$ and $0 < s < r_\alpha$. From (8) we find that $\tilde{f}$ is a solution of:

$$\tilde{a}^0(y_1, \xi)\partial_s \tilde{f} + a^1(\xi)\partial_{y_1}\tilde{f} = \partial_\xi \tilde{m}, \tag{16}$$

and

$$\begin{aligned} \tilde{a}^0(y_1, \xi) &= a^0(\xi) - \gamma_\alpha'(y_1).a^1(\xi) && (17)\\ &= \lambda(y_1)a(\xi).\nu(y_1) && (18) \end{aligned}$$

with a $\lambda(y_1) \neq 0$ and $\tilde{m}(s, y_1, \xi) = m(\tilde{\psi}_0(s, y_1), \xi)$.

4 Degenerated range

In this section we consider any interval I in ξ such that a is constant on I, namely, there exists $a_0 \in \mathbb{R}^2$ with:

$$a(\xi) = a_0 \qquad \xi \in I. \tag{19}$$

This section is devoted to the proof of the following proposition:

Proposition 4.1 *Consider an interval I verifying (19). Then for any deformation ψ:*

$$\lim_{s\to 0}\int_I\int_{-r_\alpha}^{r_\alpha} |a(\xi)\cdot\nu_s(y_1)f_\psi(s,y_1,\xi) - a(\xi)\cdot\nu(y_1)f^\tau(y_1,\xi)|\, dy_1\, d\xi = 0.$$

This means that for those values of ξ, the trace is reached strongly in L^1.

Proof. We first fix the particular Γ_α deformation (14) and use the notation (15). We split the proof into four parts.

(i) points of the boundary for which $a_0\cdot\nu(\cdot)=0$. Let us denote $\mathcal{E}_1$ the measurable subset of $]-r_\alpha, r_\alpha[$ consisting of points y_1 such that $a_0\cdot\nu(y_1)=0$. Since the boundary is regular, for every $y_1\in\mathcal{E}_1$, we have $a_0\cdot\nu_s(y_1)$ which converges to 0 when s converges to 0. Notice that $|a_0\cdot\nu_s(y_1)\tilde{f}|\le|a_0|$, so by dominated convergence theorem we have:

$$\lim_{s\to 0}\int_I\int_{\mathcal{E}_1}\left|a_0\cdot\nu_s(y_1)\tilde{f}(s,y_1,\xi) - a_0\cdot\nu(y_1)f^\tau(y_1,\xi)\right| dy_1\, d\xi = 0.$$

(ii) points of the boundary for which $a_0\cdot\nu(\cdot)\neq 0$. Since the boundary is regular, for any y_1^0 such that $a_0\cdot\nu(y_1^0)\neq 0$, there exists a interval $I_{y_1^0}$ centered on y_1^0 such that for every $y_1\in I_{y_1^0}$ either $a_0\cdot\nu(y_1)>0$ or $a_0\cdot\nu(y_1)<0$. We can assume without loss of generality that $a_0\cdot\nu(y_1)>\varepsilon$ on $I_{y_1^0}$ for an $\varepsilon>0$. For any non negative regular function $\rho\in\mathcal{D}(I)$ we define for s small enough and $y_1\in I_{y_1^0}$:

$$u_\rho(s,y_1) = \int_I \tilde{f}(s\tilde{a}^0(y_1), y_1+sa^1,\xi)\,\rho(\xi)\,d\xi,$$

where $\tilde{a}^0$ is defined by (17). Notice that for s small enough $(s\tilde{a}^0(y_1), y_1+sa^1)$ lies in $]0,r_\alpha]\times]-r_\alpha,r_\alpha[$ and so the function u_ρ is well defined. The functions $\tilde{a}^0$ and a^1 do not depend on ξ since $a(\xi)=a_0$. Hence, integrating (16) along the characteristic lines gives:

$$\frac{\partial}{\partial s}\tilde{f}(s\tilde{a}^0(y_1), y_1+sa^1,\xi) = \partial_\xi\tilde{m}(s\tilde{a}^0(y_1), y_1+sa^1,\xi).$$

Multiplying by $\rho(\xi)$ and integrating with respect to ξ we find:

$$\partial_s u_\rho(s,y_1) = -\int_I \rho'(\xi)\tilde{m}(s\tilde{a}^0(y_1), y_1+sa^1)\,d\xi.$$

We obtain that for any $s_1 \leq s_2$ small enough:

$$\begin{aligned}
&\int_{I_{y_1^0}} |u_\rho(s_1, y_1) - u_\rho(s_2, y_1)| \, dy_1 \\
&\leq \int_{I_{y_1^0}} \int_{s_1}^{s_2} \int_I |\tilde{m}(s\tilde{a}^0(y_1), y_1 + sa^1, \xi)||\rho'(\xi)| \, d\xi \, ds \, dy_1 \\
&\leq C|\tilde{m}|(I_{y_1^0} \times]0, s_2] \times I).
\end{aligned}$$

Since the intersection of the sets $I_{y_1^0} \times]0, s_2] \times I$ for $s_2 > 0$ is empty, the last term converges to 0 when s_2 tends to 0. This provides that $u_\rho(s, \cdot)$ is a Cauchy family indexed by s in $L^1(I_{y_1^0})$. Thus $u_\rho(s, \cdot)$ converges strongly in $L^1(I_{y_1^0})$ to its limit which is $\int_I f^\tau(\cdot, \xi)\rho(\xi) \, d\xi$. Then for any $\varepsilon > 0$, there exists a ρ_ε non-negative regular function such that

$$\int_I |\rho_\varepsilon(\xi) - 1| \, d\xi \leq \varepsilon.$$

Then (since $|\tilde{f}| \leq 1$):

$$\begin{aligned}
&\int_{I_{y_1^0}} \left| \int_I \tilde{f}(s\tilde{a}^0(y_1), y_1 + sa^1, \xi) \, d\xi - \int_I f^\tau(y_1, \xi) \, d\xi \right| \, dy_1 \\
\leq &\int_{I_{y_1^0}} \left| \int_I \tilde{f}(s\tilde{a}^0(y_1), y_1 + sa^1, \xi)\rho_\varepsilon(\xi) \, d\xi - \int_I f^\tau(y_1, \xi)\rho_\varepsilon(\xi) \, d\xi \right| \, dy_1 \\
&+2|I_{y_1^0}|\varepsilon.
\end{aligned}$$

This can be bounded by $3|I_{y_1^0}|\varepsilon$ for s small enough. This gives that

$$\int_I \tilde{f}(s\tilde{a}^0(y_1), y_1 + sa^1, \xi) \, d\xi$$

converges strongly in $L^1(I_{y_1^0})$ to $\int_I f^\tau(y_1, \xi) \, d\xi$. We conclude that $\tilde{f}(s, \cdot, \cdot)$ converges strongly to f^τ in $L^1(I_{y_1^0} \times I)$ using Lemma 2.1.

(iii) General case. For every $\varepsilon > 0$, there exists O_ε open set containing $\mathcal{E}_1$ such that:

$$\mathcal{L}(O_\varepsilon \setminus \mathcal{E}_1) \leq \varepsilon,$$

where $\mathcal{L}$ denotes the Lebesgue's measure. The set $[-r_\alpha, r_\alpha] \cap O_\varepsilon^c$ is a compact set covered by the open set $I_{y_1^0}$ for $y_1^0 \in O_\varepsilon^c$. So there is a finite set of y_1^i, $i = 1, \cdot\cdot, N$ such that:

$$[-r_\alpha, r_\alpha] \cap O_\varepsilon^c \subset \cup_{i=1}^N I_{y_1^i}.$$

From *(i)* and *(ii)*, there exists s_0 small enough such that for $s < s_0$:

$$\int_I \int_{\mathcal{E}_1 \cup O_\varepsilon^c} \left| a_0 \cdot \nu_s(y_1) \tilde{f}(s, y_1, \xi) - a_0 \cdot \nu(y_1) f^\tau(y_1, \xi) \right| \, dy_1 \, d\xi \le \varepsilon.$$

But since:

$$|a_0 \cdot \nu_s(y_1) \tilde{f}(s, y_1, \xi) - a_0 \cdot \nu(y_1) f^\tau(y_1, \xi)| \le 2|a_0|,$$

we also have :

$$\int_I \int_{O_\varepsilon \setminus \mathcal{E}_1} \left| a_0 \cdot \nu_s(y_1) \tilde{f}(s, y_1, \xi) - a_0 \cdot \nu(y_1) f^\tau(y_1, \xi) \right| \, dy_1 \, d\xi \le 2|a_0||I|\varepsilon.$$

Hence, for every $\varepsilon > 0$, there exists s_0 such that for every $s \le s_0$ we have:

$$\int_I \int_{-r_\alpha}^{r_\alpha} \left| a_0 \cdot \nu_s(y_1) \tilde{f}(s, y_1, \xi) - a_0 \cdot \nu(y_1) f^\tau(y_1, \xi) \right| \, dy_1 \, d\xi \le (1 + 2|a_0||I|)\varepsilon.$$

This shows the proposition for this special deformation function ψ_0.

(iv) General deformation. In this part we want to show that the proposition is true for any deformation. For this we need the following lemma. We state it in a general framework since we will use it also in the next section.

Lemma 4.1 *Let $I =]a, b[$ be an interval such that $-L \le a < b \le L$, and $g \in L^\infty(]0,1[\times]-r_\alpha, r_\alpha[\times I)$ be a χ-function. We consider $\beta \in C^0(]0,1[\times]-r_\alpha, r_\alpha[\times I)$ and denote $\beta_0 = \beta(0,\cdot,\cdot)$. We assume that there exists $\beta_0 g^\tau \in L^\infty(]-r_\alpha, r_\alpha[\times I)$ such that $\beta(s)g(s)$ converges weakly to $\beta_0 g^\tau$. Then the following two propositions are equivalent:*

- *$\beta(s)g(s)$ converges strongly to $\beta_0 g^\tau$ in $L^1_{\mathrm{loc}}(]-r_\alpha, r_\alpha[\times I)$ when $s \to 0$,*
- *For almost every $(y_1, \xi) \in]-r_\alpha, r_\alpha[\times I$, $\beta_0(y_1,\xi) g^\tau(y_1,\xi)$ is equal to $\mathrm{sign}(\xi)\beta_0(y_1,\xi)$ or 0.*

Proof of Lemma 4.1: Assume that $\beta(s)g(s)$ converges strongly to $\beta_0 g^\tau$ in $L^1_{\mathrm{loc}}(]-r_\alpha, r_\alpha[\times I)$ when $s \to 0$. Then, there exists a sequence s_n converging to 0 such that $\beta(s_n, y_1, \xi) g(s_n, y_1, \xi)$ converges to $\beta_0(y_1,\xi) g^\tau(y_1,\xi)$ for almost every $(y_1,\xi) \in]-r_\alpha, r_\alpha[\times I$. Let us fix such a point (y_1,ξ). If $\beta_0(y_1,\xi)$ is different from 0 then, since β is continuous, $g(s_n, y_1, \xi)$ converges to $g^\tau(y_1,\xi)$. But g is a χ-function, so for every n, $\mathrm{sign}(\xi) g(s_n, y_1, \xi)$ is equal to 1 or 0. So its limit is 1 or 0. This shows that for almost every $(y_1,\xi) \in]-r_\alpha, r_\alpha[\times I$ we have $\beta_0(y_1,\xi) g^\tau(y_1,\xi)$ is equal to $\mathrm{sign}(\xi)\beta_0(y_1,\xi)$ or 0.

Conversely, assume that for almost every (y_1, ξ) in $]-r_\alpha, r_\alpha[\times I$, we have $\beta_0(y_1,\xi)g^\tau(y_1,\xi)$ equal to $\mathrm{sign}(\xi)\beta_0(y_1,\xi)$ or 0. Since g is a χ-function, we have:

$$\int_I \int_{r_\alpha}^{r_\alpha} |\beta(s,y_1,\xi)g(s,y_1,\xi)|^2\, dy_1\, d\xi$$
$$= \int_I \int_{r_\alpha}^{r_\alpha} |\beta(s,y_1,\xi)|^2 \mathrm{sign}(\xi) g(s,y_1,\xi)\, dy_1\, d\xi.$$

But β is continuous, so for every (y_1,ξ) in $]-r_\alpha, r_\alpha[\times I$, $\beta(s,y_1,\xi)\mathrm{sign}(\xi)$ converges to $\beta_0(y_1,\xi)\mathrm{sign}(\xi)$. Since $\beta(s)g(s)$ converges weakly to $\beta_0 g^\tau$ we have that:

$$\lim_{s\to 0} \int_I \int_{r_\alpha}^{r_\alpha} |\beta(s,y_1,\xi)|^2 \mathrm{sign}(\xi) g(s,y_1,\xi)\, dy_1\, d\xi$$
$$= \int_I \int_{r_\alpha}^{r_\alpha} |\beta_0(y_1,\xi)|^2 \mathrm{sign}(\xi) g^\tau(y_1,\xi)\, dy_1\, d\xi.$$

From the hypothesis:

$$\beta_0^2 \mathrm{sign}(\xi) g^\tau = \beta_0^2 |g^\tau|^2.$$

Indeed, the equality is trivial at the points where $\beta_0(y_1,\xi) = 0$. And if $\beta_0(y_1,\xi) \neq 0$, then the hypothesis gives that $g^\tau(y_1,\xi)$is equal to $\mathrm{sign}(\xi)$ or 0 at this point. The equality is verified for both cases. Altogether, this shows that:

$$\lim_{s\to 0} \int_I \int_{r_\alpha}^{r_\alpha} |\beta(s,y_1,\xi)g(s,y_1,\xi)|^2\, dy_1\, d\xi = \int_I \int_{r_\alpha}^{r_\alpha} |\beta_0(y_1,\xi)g^\tau(y_1,\xi)|^2\, dy_1\, d\xi.$$

Hence $\beta(s)g(s)$ converges weakly in L^2 to $\beta_0 g^\tau$ and $\|\beta(s)g(s)\|_{L^2}$ converges to $\|\beta_0 g^\tau\|_{L^2}$. Hence the convergence holds strongly. □

Let us apply this lemma with $g = \tilde{f}$, $\beta = a \cdot \nu_s$. This gives the following proposition:

Proposition 4.2 *Consider an interval I on which a is constant. Then for almost every $]-r_\alpha, r_\alpha[\times I$, $(a\cdot\nu(y_1))f^\tau(y_1,\xi)$ is equal to* $\mathrm{sign}(\xi)(a\cdot\nu(y_1))$ *or 0.*

Now for any regular deformation ψ, $(a\cdot\nu_s)f_\psi(s)$ converges weakly to $(a\cdot\nu)f^\tau$ from Proposition 3.1. Then Proposition 4.2 and Lemma 4.1 gives Proposition 4.1. □

5 The fully non-linear range

In this section we consider any interval I such that a verifies the non linear non-degeneracy:

$$\mathcal{L}(\{\xi|\zeta \cdot a(\xi) = 0\}) = 0, \qquad \forall \zeta \neq 0. \tag{20}$$

We show in this section that $f^\tau(y_1, \cdot)$ is a χ-function for almost every $y_1 \in\,]-r_\alpha, r_\alpha[$, $\xi \in I$. From Lemma 2.1, this will ensure that $f^\tau \mathbf{1}_I$ is reached by L^1_{loc} convergence for any deformation ψ. We fix the particular Γ_α deformation on Ω_α defined by (14). The proof is the same as in [18]. We give it for the sake of completeness. Before introducing the notion of rescaled solution, let us state two lemmas. For the sake of completeness their proof are provided in the appendix.

Lemma 5.1 *There exists a sequence δ_n which converges to 0 and a set $\mathcal{E} \subset\,]-r_\alpha, r_\alpha[$ with $\mathcal{L}(]-r_\alpha, r_\alpha[\, \setminus \mathcal{E}) = 0$ such that for every $y_1 \in \mathcal{E}$ and every $R > 0$*

$$\lim_{n\to\infty} \frac{1}{\delta_n} \tilde{m}(]0, R\delta_n[\times (y_1 +]-R\delta_n, R\delta_n[) \times I) = 0.$$

Lemma 5.2 *There exists a subsequence still denoted δ_n and a subset $\mathcal{E}'$ of $]-r_\alpha, r_\alpha[$ such that $\mathcal{E}' \subset \mathcal{E}$, $\mathcal{L}(]-r_\alpha, r_\alpha[\, \setminus \mathcal{E}') = 0$, and for every $y_1 \in \mathcal{E}'$ and every $R > 0$:*

$$\lim_{\delta_n \to 0} \int_I \int_{]-R,R[} |f^\tau(y_1, \xi) - f^\tau(y_1 + \delta_n \underline{y}_1, \xi)|\, d\underline{y}_1\, d\xi = 0,$$

$$\lim_{\delta_n \to 0} \int_I \int_{]-R,R[} |\tilde{a}^0(y_1, \xi) - \tilde{a}^0(y_1 + \delta_n \underline{y}_1, \xi)|\, d\underline{y}_1\, d\xi = 0.$$

We are now able to introduce the localization method. We denote

$$\Omega^\delta_\alpha =]0, r_\alpha/\delta[\times]-r_\alpha/\delta, r_\alpha/\delta[.$$

We want to show that for every $y_1 \in \mathcal{E}'$, $f^\tau(y_1, \cdot)$ is a χ-function. From now on we fix such a $y_1 \in \mathcal{E}'$. Then, we rescale the $\tilde{f}$ function by introducing a new function $\tilde{f}_\delta$ which depends on new variables $(\underline{s}, \underline{y}_1) \in \Omega^\delta_\alpha$ and which is defined by:

$$\tilde{f}_\delta(\underline{s}, \underline{y}_1, \xi) = \tilde{f}(\delta \underline{s}, y_1 + \delta \underline{y}_1, \xi). \tag{21}$$

This function depends obviously on y_1 but, since it is fixed all along this section, we skip it in the notation. Function $\tilde{f}_\delta$ is still a χ-function and we can notice that:

$$\tilde{f}_\delta(0, \underline{y}_1, \xi) = f^\tau(y_1 + \delta \underline{y}_1, \xi). \tag{22}$$

Hence we expect to gain some knowledge on $f^\tau(y_1, \cdot)$ itself by studying the limit of $\tilde{f}_\delta$ when $\delta \to 0$. We define in the same way:

$$\tilde{a}^0_\delta(\underline{y}_1, \xi) = \tilde{a}^0(y_1 + \delta \underline{y}_1, \xi),$$

and we get from (16):

$$\tilde{a}^0_\delta(\underline{y}_1, \xi)\partial_{\underline{s}}\tilde{f}_\delta + a^1(\xi)\partial_{\underline{y}_1}\tilde{f}_\delta = \partial_\xi \tilde{m}_\delta, \tag{23}$$

where $\tilde{m}_\delta$ is the nonnegative measure defined for every real $R_1^i < R_2^i, L_1 < L_2$ by:

$$\tilde{m}_\delta(\prod_{0\le i\le 1}[R_1^i, R_2^i]\times[L_1, L_2]) = \frac{1}{\delta}\tilde{m}(\prod_{0\le i\le 1}[y_i+\delta R_1^i, y_i+\delta R_2^i]\times[L_1, L_2]). \tag{24}$$

We now pass to the limit when δ goes to 0 in the scaling.

Proposition 5.1 *There exists a sequence δ_n which goes to 0, and a χ-function $\tilde{f}_\infty \in L^\infty(\mathbb{R}^+ \times \mathbb{R} \times I)$ such that $\tilde{f}_{\delta_n}$ converges strongly to $\tilde{f}_\infty$ in $L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R} \times I)$ and:*

$$\tilde{a}^0(y_1, \xi)\partial_{\underline{s}}\tilde{f}_\infty + a^1(\xi).\partial_{\underline{y}_1}\tilde{f}_\infty = 0. \tag{25}$$

Notice that $\tilde{a}^0(y_1, \xi)$ does not depend on the local variable $\underline{y}$. In fact, we have $\tilde{a}^0(y_1, \xi) = \lambda(y_1)a(\xi).\nu(y_1)$ where y_1 is the fixed point of the localization.

Proof of Proposition 5.1: We consider the sequence δ_n of Lemma 5.2. By weak compactness, there exists a function $\tilde{f}_\infty \in L^\infty(\mathbb{R}^+\times\mathbb{R}\times I)$ such that, up to extraction, $\tilde{f}_{\delta_n}$ converges weakly in $L^\infty*$ to $\tilde{f}_\infty$. Thanks to Lemma 5.1, $\tilde{m}_{\delta_n}$ converges to 0 in the sense of measure. Thanks to Lemma 5.2, $\tilde{a}^0_{\delta_n}$ converges strongly in $L^1_{\text{loc}}(\mathbb{R} \times I)$ to $\tilde{a}^0(y_1, \cdot)$, so passing to the limit in (23) gives (25). The strong convergence is an application of averaging lemmas. Here we use the following one which is a particular case of the version of Perthame and Souganidis (see [15]):

Theorem 5.1 *Let N be an integer, f_n bounded in $L^\infty(\mathbb{R}^{N+1})$ and $\{h_n^1, h_n^2\}$ be relatively compact in $\left[L^p(\mathbb{R}^{N+1})\right]^{2N}$ with $1 < p < +\infty$ solutions of the transport equation:*

$$a(\xi).\nabla_y f_n = \partial_\xi(\nabla_y.h_n^1) + \nabla_y.h_n^2,$$

where $a \in \left[C^2(\mathbb{R})\right]^N$ verifies the non-degeneracy condition (20). Let $\phi \in \mathcal{D}(\mathbb{R})$, then the average $u_n^\phi(y) = \int_{\mathbb{R}} \phi(\xi) f_n(y,\xi)\, d\xi$ is relatively compact in $L^p(\mathbb{R}^N)$.

First we localize in $\underline{y}, \xi$. For any $R > 0$ big enough, we consider Φ_1, Φ_2 with values in $[0,1]$ such that $\Phi_1 \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R})$, $\Phi_2 \in \mathcal{D}(\mathbb{R})$, and $\mathrm{Supp}(\Phi_1) \subset]1/(2R), 2R[\times] - 2R, 2R[$, $\mathrm{Supp}(\Phi_2) \subset I$. Moreover $\Phi_1(\underline{y}) = 1$ for $\underline{y} \in]1/R, R[\times] - R, R[$ and $\Phi_2(\xi) = 1$ for $\xi \in]a + 1/R, b - 1/R[$ where $I =]a, b[$. Hence for $\delta < r_\alpha/(2R)$, we can define on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$:

$$\tilde{f}_\delta^R = \Phi_1 \Phi_2 \tilde{f}_\delta,$$

(where $\tilde{f}_\delta^R = 0$ if $\tilde{f}_\delta$ is not defined). On $]1/R, R[\times] - R, R[\, \times I$ we have $\tilde{f}_\delta^R = \tilde{f}_\delta$. So, if we denote $a_{y_1}(\xi) = (\tilde{a}^0(y_1,\xi), a^1(\xi))$ (which depends only on ξ since y_1 is fixed), from (23) we get:

$$\begin{aligned}
a_{y_1}(\xi).\nabla_{\underline{s},\underline{y}_1} \tilde{f}_\delta^R &= \partial_\xi(\Phi_1\Phi_2\tilde{m}_\delta) - \Phi_1\Phi_2'\tilde{m}_\delta \\
&\quad + a_{y_1}(\xi).\nabla_{\underline{s},\underline{y}_1}\Phi_1\Phi_2\tilde{f}_\delta + \partial_{\underline{s}}[(\tilde{a}^0(y_1,\xi) - \tilde{a}_\delta^0(\underline{y}_1,\xi))\tilde{f}_\delta^R] \\
&= \partial_\xi\mu_{1,\delta} + \mu_{2,\delta} + \partial_{\underline{s}}[(\tilde{a}^0(y_1,\xi) - \tilde{a}_\delta^0(\underline{y}_1,\xi))\tilde{f}_\delta^R],
\end{aligned}$$

where μ_{1,δ_n} and μ_{2,δ_n} are measures uniformly bounded with respect to n. Moreover thanks to Lemma 5.2 $a_{y_1}^0 - \tilde{a}_{\delta_n}^0(\underline{y}_1, \xi)$ converges to 0 in $L^1_{\mathrm{loc}}(\mathbb{R} \times I)$. So it converges to 0 in L^p_{loc} for every $1 \le p < \infty$ since these functions are bounded in L^∞. Since the measures are compactly imbedding in $W^{-1,p}(\mathbb{R}^3)$ for $1 \le p < 3/2$, we can apply Theorem 5.1 with $N = 2$, $y = (\underline{s}, \underline{y}_1)$ $f_n = \tilde{f}_{\delta_n}^R$, $\phi(\xi) = \Phi_2(\xi)$ and $a(\xi) = a_{y_1}(\xi)$. It follows that $\int \tilde{f}_\delta^R \Phi_2(\xi)\, d\xi$ is compact in L^p for $1 \le p < 3/2$. And so by uniqueness of the limit, $\int \tilde{f}_{\delta_n}(\cdot,\xi)\, d\xi$ converges strongly to $\int \tilde{f}_\infty(\cdot,\xi)\, d\xi$ in $L^1_{\mathrm{loc}}(\mathbb{R}^2)$. Lemma 2.1 ensures us that $\tilde{f}_{\delta_n}$ converges strongly to $\tilde{f}_\infty$ in $L^1_{\mathrm{loc}}(\mathbb{R}^2 \times I)$ and that $\tilde{f}_\infty$ is a χ-function.□

We now turn to characterize the limit function $\tilde{f}_\infty$.

Proposition 5.2 *For every $y_1 \in \mathcal{E}'$ we have the following equality:*

$$\tilde{f}_\infty(0, \underline{y}_1, \xi) = f^\tau(y_1, \xi)$$

for almost every $(\underline{y}_1, \xi) \in \mathbb{R} \times I$.

Notice that this implies that $\tilde{f}_\infty(0,\cdot,\cdot)$ does not depend on $\underline{y}_1$.

Proof of Proposition 5.2: Let us introduce

$$h_\Phi^\delta(\underline{s}) = \int_I \int_{\mathbb{R}} [\tilde{a}_\delta^0(\underline{y}_1,\xi)\tilde{f}_\delta(\underline{s},\underline{y}_1,\xi) - \tilde{a}^0(y_1,\xi)\tilde{f}_\infty(\underline{s},\underline{y}_1,\xi)]\Phi(\underline{y}_1,\xi)\, d\underline{y}_1\, d\xi$$

for every test function $\Phi \in C_0^\infty(\mathbb{R} \times I)$. We have:

$$|h_\Phi^\delta(0)| \le C(\|h_\Phi^\delta\|_{L^1(]0,1[)} + \|\partial_{\underline{s}} h_\Phi^\delta\|_{\mathcal{M}(]0,1[)}),$$

and from (23) and (25) h_Φ^δ is a BV function and:

$$\|\partial_{\underline{s}} h_\Phi^\delta\|_{\mathcal{M}} \le C_\Phi(\|\tilde{f}_\delta - \tilde{f}_\infty\|_{L^1_{\text{loc}}} + \|\tilde{m}_\delta\|_{\mathcal{M}})$$
$$\|h_\Phi^\delta\|_{L^1(]0,1[)} \le C_\Phi(\|\tilde{f}_\delta - \tilde{f}_\infty\|_{L^1_{\text{loc}}} + \|\tilde{a}_\delta^0 - \tilde{a}^0\|_{L^1_{\text{loc}}}).$$

Lemma 5.1 and definition (24) ensure that the term $\|\tilde{m}_{\delta_n}\|_{\mathcal{M}}$ converges to 0, Lemma 5.2 that $\|\tilde{a}_{\delta_n}^0 - \tilde{a}^0\|_{L^1_{\text{loc}}}$ converges to 0, and Proposition 5.1 that $\|\tilde{f}_{\delta_n} - \tilde{f}_\infty\|_{L^1_{\text{loc}}}$ converges to 0. So $h_\Phi^{\delta_n}(0)$ converges to 0 when n converges to $+\infty$. Remembering (22), thanks to Lemma 5.2, $\tilde{f}_{\delta_n}(0,\cdot,\cdot)$ converges strongly to f^τ, and so, $\tilde{a}_{\delta_n}^0(\cdot,\cdot)\tilde{f}_{\delta_n}(0,\cdot,\cdot)$ converges strongly to $\tilde{a}^0(y_1,\cdot)f^\tau$. Then

$$\tilde{a}^0(y_1,\xi)\tilde{f}_\infty(0,\underline{y}_1,\xi) = \tilde{a}^0(y_1,\xi)f^\tau(y_1,\xi)$$

for almost every $(\underline{y}_1,\xi) \in \mathbb{R} \times I$. □

From (25) we deduce that:

$$\tilde{f}_\infty(\tilde{a}^0(y_1,\xi)\underline{s},\underline{y}_1 + a^1(\xi)\underline{s},\xi) = f^\tau(y_1,\xi),$$

for almost every $\underline{s} > 0$. But $\tilde{a}^0(y_1,\xi) \ne 0$ (thanks to (20)) for almost every ξ so

$$\tilde{f}_\infty(\underline{s},\underline{y}_1,\xi) = f^\tau(y_1,\xi)$$

for almost every $(\underline{s},\underline{y}_1,\xi) \in \mathbb{R}^2 \times I$, which is constant with respect to $(\underline{s},\underline{y}_1)$. Finally since $\tilde{f}_\infty$ is a χ-function for almost every $(\underline{s},\underline{y}_1)$, we deduce that for every $y_1 \in \mathcal{E}'$ function $f^\tau(y_1,\cdot)$ is a χ-function in I. This leads to the following proposition

Proposition 5.3 *Consider an interval I on which a verifies the non-degeneracy (20). Then f^τ is a χ-function on $]-r_\alpha,r_\alpha[\times I$.*

6 General case

We say that a function $g^\tau \in L^\infty(]-r_\alpha,r_\alpha[\times]-L,L[)$ is suitable at a point $(y_1,\xi) \in]-r_\alpha,r_\alpha[\times]-L,L[$ if at this point $(a(\xi)\cdot\nu(y_1))g^\tau(y_1,\xi)$ is equal to $\text{sign}(\xi)a(\xi)\cdot\nu(y_1)$ or 0. Notice that χ-functions are suitable at almost every point $(y_1,\xi) \in]-r_\alpha,r_\alpha[\times]-L,L[$. We want to show in the general case that f^τ is suitable at almost every point (y_1, ξ). Then we will be able to conclude thanks to Lemma 4.1. We split the proof into several parts.

(i) Let us consider $\mathcal{A}_1$ the set of maximal open intervals such that $a'(\xi) = 0$. Notice that those intervals are disjoints so:

$$\sum_{I \in \mathcal{A}_1} |I| \leq 2r_\alpha.$$

Hence $\mathcal{A}_1$ is at most countable. We want now to construct a countable covering of intervals of the set $\{\xi | a'(\xi) \neq 0\}$. notice that

$$\{\xi | a'(\xi) \neq 0\} = \cup_{n=1}^{\infty} \{\xi \mid |a'(\xi)| \geq 1/n\} = \cup_{n=1}^{\infty} B_n.$$

For each $\xi \in B_n$, there exists an interval I_ξ such that a' is far from zero globally on I_ξ. Since $B_n \subset \cup_{\xi \in B_n} I_\xi$ and B_n is compact, there is a finite number of I_ξ covering B_n. Let us denote $\mathcal{A}_2$ the union of those covering for all n. This gives a countable covering of intervals of the set $\{\xi | a'(\xi) \neq 0\}$.

(ii) For any interval I belonging to $\mathcal{A}_1$, Proposition 4.2 ensures that f^τ is suitable at almost every point $(y_1, \xi) \in]-r_\alpha, r_\alpha[\times I$. We want to prove the same property for any interval $I \in \mathcal{A}_2$. Consider such an interval. Let us show that a verifies the non-degeneracy condition (20) on I. The vector valued function $a(\xi)$ is defined from the real valued flux function A in (1) through the rotation $\mathcal{R}_\alpha$ by:

$$a(\xi) = \mathcal{R}_\alpha(1, A'(\xi)).$$

Since $a'(\xi) \neq 0$ on I, we also have $A''(\xi) \neq 0$ on I and so A' is one to one on I. For every $\zeta \in \mathbb{R}^2$ different from 0, let us denote $\overline{\zeta} = \mathcal{R}_\alpha^{-1} \zeta$. We have

$$a(\xi) \cdot \zeta = 0,$$

if and only if:

$$\overline{\zeta}_1 + A'(\xi) \overline{\zeta}_2 = 0.$$

If $\overline{\zeta}_2 = 0$, then $\overline{\zeta}_1 \neq 0$ so:

$$\{\xi \mid a(\xi) \cdot \zeta = 0\} = \emptyset.$$

And if $\overline{\zeta}_2 \neq 0$ then $a(\xi) \cdot \zeta = 0$ if and only if $A'(\xi) = -\overline{\zeta}_1 / \overline{\zeta}_2$. Since A' is one to one on I, this could occur at most for one value of ξ. This shows that a verifies the non-degeneracy condition (20) on I since for any $\zeta \neq 0$:

$$\mathcal{L}\{\xi \mid a(\xi) \cdot \zeta = 0\} = 0.$$

So, from Proposition 5.3, f^τ is suitable on $]-r_\alpha, r_\alpha[\times I$. To sum up, we have shown that for any $I \in \mathcal{A}_1 \cup \mathcal{A}_2$, the function f^τ is suitable for almost every $y_1 \in]-r_\alpha, r_\alpha[\times I$.

(iii) Since $\mathcal{A}_1 \cup \mathcal{A}_2$ is countable, we have that for almost every $y_1 \in]-r_\alpha, r_\alpha[$, f^τ is suitable almost everywhere in $\{y_1\} \times I$, for **any** $I \in \mathcal{A}_1 \cup \mathcal{A}_2$. Let us fix such a y_1. From Lemma 2.1, $f^\tau(y_1, \cdot)$ is a BV function, so it is continuous almost everywhere. Let us consider a continuity point ξ. If $a'(\xi) \neq 0$, then there exists $I \in \mathcal{A}_2$ such that $\xi \in I$. So from (ii), f^τ is suitable at this point (y_1, ξ). The same conclusion holds if $\xi \in I$ with $I \in \mathcal{A}_1$. The last situation corresponds to a $\xi \in]-L, L[$ verifying $a'(\xi) = 0$ but for which there exists a sequence ξ_n converging to ξ and verifying $a'(\xi_n) \neq 0$ for every n. But for all those ξ_n, $f^\tau(y_1, \cdot)$ is a χ-function function on a neighborhood of ξ_n so f^τ is suitable at (y_1, ξ_n). Since $f^\tau(y_1, \cdot)$ is continuous at ξ, $a(\cdot) \cdot \nu(y_1) f^\tau(y_1, \cdot)$ is also continuous at this point. The limit $a(\xi) \cdot \nu(y_1) f^\tau(y_1, \xi)$ can only be $\mathrm{sign}(\xi) a(\xi) \cdot \nu(y_1)$ or 0. This implies that $f^\tau(y_1, \xi)$ is suitable at (y_1, ξ). We have shown the property for almost every $(y_1, \xi) \in]-r_\alpha, r_\alpha[\times]-L, L[$.

Theorem 2.2 follows from Lemma 4.1. □

7 Appendix

Proof of Lemma 5.1: For every integer N, we denote

$$M_\delta^N(y_1) = \frac{1}{\delta} \tilde{m}\left( ]0, N\delta[\times (y_1 +]-N\delta, N\delta[) \times I \right).$$

Since M_δ^N is nonnegative, the L^1 norm of M_δ^N is :

$$\begin{aligned}
&\int_{-r_\alpha}^{r_\alpha} M_\delta^N(y_1)\, dy_1 \\
&= \int_{-r_\alpha}^{r_\alpha} \frac{1}{\delta} \int_0^{N\delta} \int_{-N\delta}^{N\delta} \int_I \tilde{m}(s, y_1 + z_1, \xi)\, d\xi\, dz_1\, ds\, dy_1 \\
&\leq \frac{1}{\delta} \int_{-N\delta}^{N\delta} \int_0^{N\delta} \int_I \int_{-r_\alpha - N\delta}^{r_\alpha + N\delta} \tilde{m}(s, y_1, \xi)\, dy_1\, d\xi\, ds\, dz_1.
\end{aligned}$$

We denote abusively $\tilde{m}(ds, dz_1, d\xi) = \tilde{m}(s, z_1, \xi)\, ds\, dz_1\, d\xi$ in this computation as if it was a function. This calculation is still correct since we just use the Fubini Theorem and a linear change of variable which are valid for measures. The last inequality can be written as:

$$\begin{aligned}&\int_{-r_\alpha}^{r_\alpha} M_\delta^N(y_1)\,dy_1\\&\leq \frac{1}{\delta}\int_{-N\delta}^{N\delta} \tilde{m}(]0,N\delta[\times]-r_\alpha-N\delta, r_\alpha+N\delta[\times I)\,dz_1\\&\leq N\tilde{m}(]0,N\delta[\times]-r_\alpha-N\delta, r_\alpha+N\delta[\times I).\end{aligned}$$

By monotone convergence, since $\underset{\delta>0}{\cap}]0,N\delta[=\emptyset$, this converges to 0 when δ converges to 0. Finally the L^1 norm of M_δ^N converges to 0 so there exists a subsequence δ_n and a set $\mathcal{E}_N \subset]-r_\alpha, r_\alpha[$ with $\mathcal{L}(]-r_\alpha, r_\alpha[\backslash\mathcal{E}_N)=0$ such that for every $y_1\in\mathcal{E}_N$ $M_{\delta_n}^N(y_1)$ converges to 0 when δ_n goes to 0. By diagonal extraction, we can choose δ_n such that for every integer N and every $y_1\in\mathcal{E}_N$, $M_{\delta_n}^N(y_1)$ converges to 0. This sequence δ_n with subset $\mathcal{E}=\cap_N\mathcal{E}_N$ verifies the required condition. □

Proof of Lemma 5.2: For every integer N we denote:

$$F_{\delta_n}^N(y_1)=\int_I\int_{-N}^{N}|f^\tau(y_1,\xi)-f^\tau(y_1+\delta_n\underline{y}_1,\xi)|\,d\underline{y}_1\,d\xi.$$

Since $f^\tau\in L^\infty(]-r_\alpha,r_\alpha[\times I)$, the L^1 norm of this function goes to zero as n tends to ∞ so there exists a subsequence still denoted δ_n and a subset $\mathcal{E}'_N\subset\mathcal{E}$ with $\mathcal{L}(]-r_\alpha,r_\alpha[\backslash\mathcal{E}'_N)=0$ such that for every $y_1\in\mathcal{E}'_N$, $F_{\delta_n}^N(y_1)$ converges to 0 when n tends to infinity. By diagonal extraction we can find a subsequence such that this holds true for every N. Then this subsequence and $\mathcal{E}'=\cap_N\mathcal{E}'_N$ fulfill the required condition for the first limit. We consider in the same way the term with $\tilde{a}^0$ noticing that

$$\tilde{a}^0(y_1,\xi)-\tilde{a}^0(y_1+\delta_n\underline{y}_1,\xi)=a^1(\xi).[\nabla\gamma_\alpha(y_1)-\nabla\gamma_\alpha(y_1+\delta_n\underline{y}_1)]$$

with $\nabla\gamma_\alpha\in L^\infty(]-r_\alpha,r_\alpha[)$. □

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