**Rings of fractions the hard way**

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Let $R$ be a ring (commutative with unity in what follows) and $D$ a subset of $R$. A standard construction in Commutative Algebra is the localization of $R$ at $D$ or ring of fractions of $R$ with denominators in $D$, defined to be the universal ring to which $R$ maps in such a way that the elements of $D$ map to units. Uniqueness up to isomorphism is clear but one needs to show existence.

The standard construction of this ring is as follows (see e.g. [M]). First, one assumes that $D$ is multiplicatively closed, (that is, finite products of elements of $D$ are also in $D$) since the ring of fractions of $R$ with denominators in $D$ is the same as the ring of fractions of $R$ with denominators in the smallest multiplicatively closed subset of $R$ containing $D$ (which will be denoted by $\bar{D}$). Then one considers the equivalence relation on $R \times D$ given by $(a, b) \sim (c, d)$ if and only if there exists $t \in D$ such that $t(ad - bc) = 0$. The equivalence class of $(a, b)$ is denoted by $a/b$. Addition and multiplication are then defined by the usual rules for fractions $a/b + c/d = (ad + bc)/bd, (a/b)(c/d) = ac/bd$. One is of course left with the task of showing that these operations are well-defined and make the set of equivalence classes into a ring. It would be nice if $R \times D$ were a ring and the equivalence relation were given by an ideal, but this is far from being the case, as the reader can check.

The purpose of this note is to give an alternative construction of the ring of fractions by exhibiting it as a quotient of a suitable ring by an ideal. We will not need the assumption that $D$ is multiplicatively closed.

**Theorem 1.** *Let $R$ be a ring and $D$ a subset of $R$. There exists a ring $Q$ of fractions of $R$ with denominators in $D$. Moreover, if $\phi : R \to Q$ is the the canonical map then $\phi(r)$ is a zero divisor in $Q$ only if $r$ is a zero divisor in $R$ and the kernel of $\phi$ is $\{r \in R | \exists d \in \bar{D}, dr = 0\}$.*

*Proof:* Let $S = R[x_d \mid d \in D]$ (that is, the polynomial ring in as many variables as elements of $D$) and $I = (dx_d - 1 \mid d \in D)$, which is an ideal of $S$. We define $Q$ as $S/I$. Clearly there is a map $\phi : R \to Q$, corresponding to the inclusion $R \subset S$. By construction, the image of $x_d$ in $Q$ is an inverse of $\phi(d)$ so $\phi(D) \subset Q^*$. If $\psi : R \to T$ is a homomorphism, for some ring $T$, with $\psi(D) \subset T^*$ we define a homomorphism $\theta : S \to T$ by letting $\theta = \psi$ on $R$ and $\theta(x_d) = \psi(d)^{-1}$, clearly $I \subset \ker\theta$, so $\theta$ induces a homomorphism $\bar{\theta} : Q \to T$ which satisfies $\bar{\theta} \circ \phi = \psi$, as desired.

To prove the remaining statements, note that the condition that $\phi(r)$ is a zero divisor in $Q$ (or $\phi(r) = 0$) will be expressed by a relation in $S$ which will only involve finitely many variables, so we may assume that $D$ is finite with $n$ elements and proceed by induction on $n$. If $D' = D \cup \{d\}$, then the ring of fractions of $R$ with denominators in $D'$ is the ring of fractions of $Q$ with denominators in $\{d\}$ and this reduces the induction step to the case $n = 1$ which we proceed to do.

Suppose $D = \{d\}$ and write $x$ for $x_d$. If $\phi(r)$ is a zero divisor in $Q$, then there exists $t \in Q, t \neq 0, t\phi(r) = 0$ and therefore there exists polynomials $t(x), s(x) \in R[x]$ with

$$rt(x) = (dx - 1)s(x). \tag{1}$$

Choose $m$ large enough so that $t(x) = \sum_{i=0}^{m+1} t_i x^i$ has degree at most $m+1$ and $s(x) = \sum_{i=0}^{m} s_i x^i$ has degree at most $m$. Equation (1) then gives

$$rt_0 = -s_0,$$

$$rt_i = ds_{i-1} - s_i, i = 1, \ldots, m,$$

$$rt_{m+1} = ds_m.$$

It follows from these equations by induction that

$$d^{j+1} s_{m-j} = r \sum_{i=0}^{j} t_{m+1-i} d^i, j = 1 \ldots, m$$

and therefore

$$r \sum_{i=0}^{m+1} t_{m+1-i} d^i = 0. \tag{2}$$

So, either $r$ is a zero divisor on $R$, or $\sum_{i=0}^{m} t_{m+1-i} d^i = 0$. If the latter holds, we define $u(x) = \sum_{i=0}^{m} u_i x^i$ by $u_0 = -t_0$ and $u_i = du_{i-1} - t_i, i > 0$ and it follows that $t(x) =$

$(dx-1)u(x)$ which implies that $t=0$ in $Q$, contradiction. If $\phi(r)=0$ then we get equation (1) with $t(x)=1$ and we can proceed to obtain equation (2), which then reads $d^{m+1}r=0$ and completes the proof.

We will prove a couple of complements to our main result. First we need a lemma, which states that every element of $Q$ is a fraction.

**Lemma.** *If $R$ is a ring and $D \subset R$ and $Q$ is the ring of fractions of $R$ with denominators in $D$ then for each $q \in Q$, there exists $d \in \bar{D}$ (hence $\phi(d)$ is a unit in $Q$) and $r \in R$ such that $\phi(d)q = \phi(r)$ (i.e. every element of $Q$ is a fraction $\phi(r)\phi(d)^{-1}$).*

*Proof:* The element $q$ of $Q$ corresponds to a polynomial in $S$ and only finitely many variables $x_{d_1}, \ldots, x_{d_n}$ appear in $q$. Multiplying $q$ by $(d_1 \cdots d_n)^N$ for some large $N$ we can replace any appearance of $d_1 x_{d_1}, \ldots, d_n x_{d_n}$ by 1 and obtain an element of $\phi(R)$.

Now we prove the usual fact that integral domains have a field of fractions.

**Theorem 2.** *If $R$ is an integral domain and $D \subset R \setminus \{0\}$ then the ring of fractions of $R$ with denominators in $D$ is an integral domain and if $D = R \setminus \{0\}$ it is a field.*

*Proof:* If $ab=0$ in $Q$, then by the lemma, there exists $d, e \in \bar{D}, r, s \in R$ with $\phi(d)a = \phi(r), \phi(e)b = \phi(s)$. Then $\phi(rs)=0$ and, since $R$ is a domain, we get $rs=0$ from theorem 1. So $r=0$ or $s=0$ and since $\phi(d), \phi(e)$ are units, we get $a=0$ or $b=0$. If $D = R \setminus \{0\}$ and $a \in Q, a \neq 0$, we can get by the lemma a unit $u$ of $Q$ with $ua \in \phi(R)$ and $ua \neq 0$, so $ua$ is a unit in $Q$ and therefore so is $a$.

The following result summarizes the main properties of rings of fractions used in ring theory (see [M])

**Theorem 3.** *If $R$ is a ring and $D \subset R$ and $Q$ is the ring of fractions of $R$ with denominators in $D$ then all ideals of $Q$ are of the form $\phi(I)Q$, where $I$ is an ideal of $R$ and the prime ideals of $Q$ are of the form $\phi(I)Q$, where $I$ is a prime ideal of $R$ disjoint from $D$.*

*Proof:* If $J$ is an ideal of $Q$, then $I = \phi^{-1}(J)$ is an ideal of $R$. Given $a \in J$, there exists $d \in \bar{D}, \phi(d)a \in \phi(R)$ by the lemma, so $\phi(d)a \in \phi(I)$ and since $\phi(d)$ is a unit, $a \in \phi(I)Q$, as desired. If $J$ is prime, then $J$ does not contain any unit of $Q$, thus it meets $\phi(D)$ trivially, so $I$ is disjoint from $D$. Finally, if $I$ is a prime ideal of $R$ disjoint from $D$ then $R/I$ is an integral domain and the image of $D$ in it does not contain zero so its ring of fractions is a domain and it is easily seen to be $Q/\phi(I)Q$ so $\phi(I)Q$ is prime and we are done.

Also, we can easily prove that a derivation on $R$ extends to rings of fractions.

**Theorem 4.** *If $R$ is a ring and $D \subset R$ and $Q$ is the ring of fractions of $R$ with denominators in $D$ and $\delta$ is a derivation on $R$ then $\delta$ extends to $Q$.*

*Proof:* Define $\delta$ on $S$ by $\delta(x_d) = -x_d^2\delta(d), d \in D$ and notice that $\delta(dx_d - 1) = -dx_d^2\delta(d) + x_d\delta(d) = x_d\delta(d)(1 - dx_d) \in I$, so $\delta$ descends to $Q = S/I$, as desired.

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## References.

[M] H. Matsumura, *Commutative Ring Theory*, CUP 1986.

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