Hadamard's Inequality

Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_N$ be (column) vectors in $\mathbb{R}^N$ and let

$$A = \begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_N \end{pmatrix}$$

be the corresponding $N \times N$ real matrix. Then Hadamard's inequality asserts that

$$|\det A| \le \prod_{N=1}^{N} \|\mathbf{a}_n\|, \tag{1}$$

where $\| \ \|$ is the Euclidean norm on vectors in $\mathbb{R}^N$. Perhaps the simplest proof of this inequality is as follows.

By the Gram-Schmidt process we can establish the existence of an orthonormal basis $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_N$ for $\mathbb{R}^N$ such that

$$\operatorname{span}_{\mathbb{R}} \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} = \operatorname{span}_{\mathbb{R}} \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\} \tag{2}$$

for each $n = 1, 2, \ldots, N$. Write

$$B = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \ldots & \mathbf{b}_N \end{pmatrix}$$

for the corresponding $N \times N$ real, orthogonal matrix. By orthonormality each vector $\boldsymbol{\xi}$ in $\mathbb{R}^N$ has an expansion as

$$\boldsymbol{\xi} = \sum_{n=1}^{N} \langle \boldsymbol{\xi}, \mathbf{b}_n \rangle \mathbf{b}_n,$$

and then

$$\|\boldsymbol{\xi}\|^2 = \sum_{n=1}^{N} |\langle \boldsymbol{\xi}, \mathbf{b}_n \rangle|^2.$$

However, (2) implies that each vector $\mathbf{a}_m$ has a shorter expansion of the form

$$\mathbf{a}_m = \sum_{n=1}^{m} \langle \mathbf{a}_m, \mathbf{b}_n \rangle \mathbf{b}_n. \tag{3}$$

Alternatively, let $C = (c_{kl})$ be the $N \times N$, upper triangular matrix defined by

$$c_{kl} = \langle \mathbf{a}_l, \mathbf{b}_k \rangle \quad \text{if} \quad 1 \le k \le l,$$

and

$$c_{kl} = 0 \quad \text{if} \quad l < k \le N.$$

Then it is easy to check that (3) can be restated as

$$A = BC.$$

Again using the fact that B has orthonormal columns and the fact that C is upper triangular, we get

$$\begin{aligned}\big(\det A\big)^2 &= \det A^T A = \det C^T B^T B C \\ &= \det C^T C = \big(\det C\big)^2 \\ &= \prod_{n=1}^{N} |\langle \mathbf{a}_n, \mathbf{b}_n \rangle|^2 \\ &\leq \prod_{n=1}^{N} \Big\{ \sum_{m=1}^{n} |\langle \mathbf{a}_m, \mathbf{b}_n \rangle|^2 \Big\} \\ &= \prod_{n=1}^{N} \|\mathbf{a}_n\|^2,\end{aligned}$$

which gives (1). This argument also shows that there is equality in Hadamard's inequality if and only if

$$|\langle \mathbf{a}_n, \mathbf{b}_n \rangle|^2 = \sum_{m=1}^{n} |\langle \mathbf{a}_m, \mathbf{b}_n \rangle|^2$$

for each n. That is, if and only if

$$\mathbf{a}_n = \langle \mathbf{a}_n, \mathbf{b}_n \rangle \mathbf{b}_n$$

for each n, and this is equivalent to the vectors $\mathbf{a}_n$ being pairwise orthogonal.