M 328 K 58080 First Midterm

name:

1. Find an integer x simultaneously satifying the following congruences $3x \equiv 5 \pmod 8$ and $x \equiv 3 \pmod{15}$.

If $3x \equiv 5 \pmod 8$, then $x \equiv 9x \equiv 3 \cdot 3x \equiv 3 \cdot 5 \equiv 7 (\bmod 8)$. So we need to solve $x \equiv 7(\bmod 8)$ and $x \equiv 3 \pmod{15}$. We apply the Chinese remainder theorem. First, the Euclidean algorithm gives $2 \cdot 8 + (-1) \cdot 15 = 1$. So, a solution to our problem is $x = 3 \cdot 2 \cdot 8 + 7 \cdot (-1) \cdot 15 = -57$. The general solution is $x \equiv 63 (\bmod 120)$.

2. Let p and q be two distinct prime numbers. How many positive integers divide the integer $p^3 q^{10}$? Justify your answer.

If $a | p^3 q^{10}$ then by the fundamental theorem of arithmetic, the prime factors of a can only be p and q, so $a = p^i q^j$ and if $a | p^3 q^{10}$ we must have $0 \leq i \leq 3$ and $0 \leq j \leq 10$. So we have 4 choices for i and 11 choices for j giving a total of 44 divisors.

3. Let a, b be integers. Show that $(a, b) = (b, 3b - a)$.

Let $d = (a, b)$ and $e = (b, 3b - a)$. So $d | a, d | b$ and we can conclude that $d | (3b - a)$. Since we now know that $d | b$ and $d | (3b - a)$ we conclude that $d | e$. On the other hand, $e | b, e | (3b - a)$ so $e | (3b - (3b - a))$, that is $e | a$, since we now know that $e | a$ and $e | b$ we conclude that $e | d$. Finally, from $d | e$ and $e | d$ and since they are both positive, we get that $e = d$.

4. Let $m > 1$ be an odd integer. Prove that every integer is congruent modulo m to an element of the set $\{2n + 3 | n = 1, 2, \ldots, m\}$.

Given an integer a, we can solve the congruence $2x \equiv a - 3 (\bmod m)$, since $(2, m) = 1$ as m is odd. If we take a solution x which is the least positive residue modulo m, we get $0 \leq x \leq m - 1$. Now put $n = x$ if $x \neq 0$ and $n = m$ if $x = 0$, so in both cases $n \equiv x (\bmod m)$ so $2n + 3 \equiv 2x + 3 \equiv a - 3 + 3 = a (\bmod m)$ and, by construction, $1 \leq n \leq m$ as we wanted.