Maximal ideals of $\mathbf{Z}[x]$.

The maximal ideals of $\mathbf{Z}[x]$ are of the form $(p, f(x))$ where p is a prime number and $f(x)$ is a polynomial in $\mathbf{Z}[x]$ which is irreducible modulo p. To prove this let M be a maximal ideal of $\mathbf{Z}[x]$. Assume first that $M \cap \mathbf{Z} \neq (0)$. As $\mathbf{Z}/(M \cap \mathbf{Z})$ injects into $\mathbf{Z}[x]/M$, we conclude that $\mathbf{Z}/(M \cap \mathbf{Z})$ is a domain, so $M \cap \mathbf{Z}$ is a prime ideal of $\mathbf{Z}$ so $M \cap \mathbf{Z} = (p)$, where p is a prime number. Let M' be the image of M in $\mathbf{Z}/(p)[x]$ and you may check that $\mathbf{Z}[x]/M$ is isomorphic to $\mathbf{Z}/(p)[x]/M'$. So $M' = (f_0(x))$ where $f_0(x) \in \mathbf{Z}/(p)[x]$ is irreducible. Take now $f(x) \in \mathbf{Z}[x]$ reducing to $f_0(x)$ modulo p and it should be clear that $M = (p, f(x))$ as described above.

We still need to show that $M \cap \mathbf{Z} \neq (0)$. Let's assume the contrary and get a contradiction. Let M_1 be the ideal of $\mathbf{Q}[x]$ generated by M. It is a proper ideal of $\mathbf{Q}[x]$ so $M_1 = (f(x)), \deg f(x) > 0$ and we may assume without loss of generality that $f(x)$ is a polynomial in $\mathbf{Z}[x]$ of content 1. We show that $M = (f(x))$. Indeed, if $h \in M$, then h is a multiple of f by an element of $\mathbf{Q}[x]$ and using Gauss lemma we get that this element of $\mathbf{Q}[x]$ is actually in $\mathbf{Z}[x]$.

To finish the proof we show that $\mathbf{Z}[x]/(f(x))$ is not a field if $\deg f(x) > 0$. For this purpose, choose $a \in \mathbf{Z}, f(a) \neq 0, \pm 1$ and a prime p dividing $f(a)$. Let $\phi : \mathbf{Z}[x] \to \mathbf{Z}/(p)$ be the unique homomorphism with $\phi(x) = a \bmod p$. Then ϕ factors through $\mathbf{Z}[x]/(f(x))$ since $\phi(f(x)) = 0$. Now, $\mathbf{Z}[x]/(f(x))$ is infinite, so $\bar{\phi} : \mathbf{Z}[x]/(f(x)) \to \mathbf{Z}/(p)$ is not bijective. If we show that $\bar{\phi}$ is not the zero map, then $\ker \bar{\phi}$ will be a non-trivial ideal of $\mathbf{Z}[x]/(f(x))$ and it won't be a field. If $\bar{\phi}$ is the zero map, then $\bar{\phi}(1) = 0$, i.e., there exists polynomials $u, v \in \mathbf{Z}[x]$ with $1 = u(x)f(x) + pv(x)$. Putting $x = a$ we get a contradiction since $u(a)f(a) + pv(a)$ is divisible by p as well as being equal to 1.

Valuation domains

Let V be a valuation domain, F its field of fractions and W a valuation domain with $V \subset W \subset F$ and maximal ideal M. We want to show that $W = \{a/s \mid a \in V, s \in V - M\}$. Let $U = \{a/s \mid a \in V, s \in V - M\}$. It is easy to check that U is a domain and $V \subset U \subset W$. So U is a valuation domain and if P is its maximal ideal, $M \subset P$. As $F - W = \{x \in F^* \mid x^{-1} \notin M\}$ and similarly for U, to show that $W = U$ it is enough to show that $M = P$. If $x = a/s \in P$, then $a \in M$ for otherwise $x^{-1} = s/a \in U$. As, $s^{-1} \in W$ we conclude that $x = as^{-1} \in M$, since M is an ideal, so $P \subset M$ thus $P = M$ and we are done.