Differential operators and interpolation series in power series fields

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Let $k$ be a field and $K$ be the field of formal power series over $k$. That is, the elements of $K$ are of the form $u = \sum_{n=n_0}^{\infty} a_n x^n$, where $a_n \in k$ and $n_0$ is an arbitrary integer. If $a_n \neq 0$ we put $v(u) = n_0$, then $v$ is a valuation on $K$ and $K$ is a local field, i.e., it is complete with respect to this valuation. Let $U$ be an open subset of $K$ and $f : U \rightarrow K$ a function. Besides the usual notion of continuity there is the notion of differentiability for such functions $f$, namely, $f$ is differentiable in $a \in U$ if $\lim_{u \rightarrow a} (f(u) - f(a))/(u - a)$ exists. A natural class of functions to consider is that of differential operators, coming from differentiation with respect to the variable $x$. We can define the Hasse derivations $D^{(r)}, r \geq 0$ by:

$$D^{(r)}(\sum a_n x^n) = \sum \binom{n}{r} a_n x^{n-r}.$$  

**Theorem 1.** The functions $D^{(r)} : K \rightarrow K, r \geq 1$ are $k$-linear, continuous and nowhere differentiable. (Differentiation is not differentiable!)

**Proof:** It is clear that $D^{(r)}$ is $k$-linear and therefore it suffices to check continuity and differentiability at $u = 0$. Plainly $v(D^{(r)}(u)) \geq v(u) - r$, so $D^{(r)}$ is continuous (see also [Go], Prop. 13). Next, $D^{(r)}$ is differentiable at $u = 0$ if and only if $\lim_{u \rightarrow 0} D^{(r)}(u)/u$ exists. However, the sequence $x^n$ converges to 0 but $D^{(r)}(x^n)/x^n = \binom{n}{r} x^{-r}$ does not converge.

Suppose now that $k$ is a finite field with $q$ elements. Then Wagner [W] studied continuous linear functions $f : R \rightarrow K$, where $R = k[[x]]$. He obtained results analogous to classical results of Mahler [M] that gave interpolation series for continuous $p$-adic functions in terms of binomial coefficients. To state Wagner’s result we need to make a few definitions:

$$\Psi_n(u) = \prod_{m \in k[x], \deg m < n} (u - m), n > 0, \Psi_0(u) = u.$$
\[ F_n = (x^{q^n} - x)(x^{q^{n-1}} - x)^q \cdots (x^q - x)^{q-1}, \quad F_0 = 1, \]
\[ L_n = (x^{q^n} - x)(x^{q^{n-1}} - x) \cdots (x^q - x), \quad L_0 = 1. \]

Wagner then proved that every continuous linear function \( f : R \to K \) can be written as \( f = \sum_{n=0}^{\infty} A_n \Psi_n/F_n \), where \( A_n \in K \), \( \lim_{n \to \infty} A_n = 0 \) and moreover \( f \) is differentiable if and only if \( \lim_{n \to \infty} A_n/L_n = 0 \). The \( A_n \) can be obtained as follows. Define:

\[ \Delta_0 f(u) = f(u) \]
\[ \Delta_{n+1} f(u) = \Delta_n f(xu) - x^{q^n} \Delta_n f(u). \]

Wagner then shows that \( A_n = \Delta_n f(1) \). Our next result computes the \( A_n \) for the \( D^{(r)} \).

**Theorem 2.** For all \( u \in R \) we have:

\[ D^{(r)}(u) = \sum_{n=0}^{\infty} A_{nr} \frac{\Psi_n(u)}{F_n}, \]

where \( A_{n1} = (-1)^{n-1} L_{n-1} \) and, for \( r > 1 \),

\[ A_{nr} = (-1)^{n-1} L_{n-1} \sum_{0 < i_1 < \cdots < i_{r-1} < n} \frac{1}{(x - x^{q^{i_1}}) \cdots (x - x^{q^{r-1}})}. \]

**Proof:** We will show that \( \Delta_n D^{(r)} = \sum_{i=0}^{r-1} A_{n,r-i} D^{(i)} \), for \( n \geq 1 \), by induction on \( n \), and the result will follow from Wagner’s results. Clearly, \( \Delta_1 D^{(r)} = D^{(r-1)} \) so the above formula holds for \( n = 1 \). Assume the formula holds for \( n \). From the recursive definition of \( \Delta_{n+1} \) we get that

\[ \Delta_{n+1} D^{(r)} = \sum_{i=0}^{r-1} (A_{n,r-i} (x - x^{q^n}) + A_{n,r-i-1} D^{(i)}) = \sum_{i=0}^{r-1} A_{n+1,r-i} D^{(i)} \]

and this completes the proof.

In particular we get the bizarre formula \( du/dx = \sum_{n=0}^{d} (-1)^{n-1} L_{n-1} \Psi_n(u)/F_n \) for \( u \in k[x], \deg u \leq d \).
Another class of continuous linear functions are \( u \mapsto u \circ b \) for \( b \in xR \). These are differentiable when \( b = x \) or \( v(b) > 1 \). The coefficients of their expansion in Wagner’s basis are given by \((b - x) \cdots (b - x^{q^{n-1}})\). The proof is left to the reader.

Finally we establish the following formula expanding the functions \( u^{q^i} \) in terms of the Hasse derivatives.

**Proposition.** We have

\[
  u^{q^i} = \sum_{r=0}^{\infty} (x^{q^i} - x)^r D^{(r)} u
\]

for \( u \in k[[x]] \).

**Proof:** We begin with the case \( i = 1 \); i.e.,

\[
  u^q = \sum_{r=0}^{\infty} (x^q - x)^r D^{(r)} u.
\]

Note that both sides of this equation are \( k \)-linear, so it suffices to check the formula for \( u = x^m \) and in this case it is straightforward. In this formula for \( u^q \), one can replace \( q \) by \( q^n \), for any \( n \) by extending \( k \) to its extension of degree \( n \). The proposition now follows.

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**References**


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