Differentials of the Second Kind in Characteristic p

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The theory of differentials on a Riemann surface X is well-developed. Using the basic tools of integration, we can easily take the residue of a differential ω on X. In particular, if ω has zero residue at every point of X, we call ω a differential of the second kind. If f is a rational function on X, then df is a differential on X and in fact is a differential of the second kind. Therefore we can consider the factor space of the differentials of the second kind modulo the exact differentials. If instead of being a Riemann surface, X is a curve defined over an algebraically closed field k of characteristic zero, the same theory can be developed for the differentials of the second kind modulo exact differentials. When X is defined over a field of characteristic $p > 0$, then this factor space is noticeably smaller in dimension. It is our goal to construct a vector space, isomorphic to this quotient in characteristic zero, that retains many of the properties of the differentials of the second kind modulo exact differentials in characteristic zero but for curves defined over an algebraically closed field of any characteristic.

We begin with some preliminaries. Our objects of study will revolve around curves X defined over an algebraically closed ground field k. These curves are assumed to be smooth, projective, and irreducible. We will make repeated use of the Serre Duality theorem, so we state it here. We denote by $\mathcal{O}$ the sheaf of regular functions on X, Ω the sheaf of differentials (or meromorphic differentials) on X, and Ω_0 the sheaf of differentials of the first kind (or holomorphic differentials) on X.

Serre Duality. *Let X be a curve. Then there is an isomorphism*

$$H^0(X, \Omega_0) \cong H^1(X, \mathcal{O})^*;$$

moreover, this duality induces an isomorphism between the cohomology group $H^1(X, \Omega_0)$ *and the ground field* k*.*

The explicit isomorphism between $H^1(X, \Omega_0)$ and k will be used later, so we will show here how it is given by the trace map $tr : H^1(X, \Omega_0) \to k$ defined as follows. Let $P \in X$ be a point and $(\psi_{ij}) \in H^1(X, \Omega_0)$. The same cocycle, viewed in the first cohomology group with coefficients in the sheaf of differentials $H^1(X, \Omega)$, splits as $\psi_{ij} = \psi_i - \psi_j$ since $H^1(X, \Omega) = 0$. Thus we define

$$tr(\psi_{ij}) = \sum_{P \in X} res_P(\psi_i)$$

where for each P, we choose a U_i containing P and consider the residue of the corresponding ψ_i. This is independent of which open set U_i and corresponding differential is chosen since if $P \in U_i \cap U_j$,

$$res_P(\psi_i) = res_P(\psi_{ij} + \psi_j) = res_P(\psi_{ij}) + res_P(\psi_j) = res_P(\psi_j)$$

since $res_P(\psi_{ij}) = 0$ because it is holomorphic at P.

Note that since $H^0(X, \Omega_0)$ and $H^1(X, \mathcal{O})$ are dual and $H^1(X, \mathcal{O})$ has finite dimension as a k-vector space, then so does $H^0(X, \Omega_0)$ and these dimensions are equal.

1 Curves in Characteristic 0

Let X be a curve defined over an algebraically closed field k of characteristic zero. For this section, we view X with the "complex" topology. Note that there is a short exact sequence of sheaves on X

$$0 \to k \to \mathcal{O} \xrightarrow{d} \Omega_0 \to 0$$

since locally every holomorphic differential ω has a primitive that is a regular function on X. This short exact sequence of sheaves induces an exact sequence of cohomology groups

$$0 \to H^0(X, k) \to H^0(X, \mathcal{O}) \to H^0(X, \Omega_0) \to H^1(X, k) \to H^1(X, \mathcal{O}) \to H^1(X, \Omega_0).$$

Now $H^0(X,k) \cong k(X) = k$ and $H^0(X,\mathcal{O}) \cong \mathcal{O}(X)$ which is the vector space of regular functions on X. But the only functions that are regular on all of X are the constants, so $H^0(X,\mathcal{O}) \cong k$ as well. Hence our sequence becomes

$$0 \to H^0(X,\Omega_0) \to H^1(X,k) \to H^1(X,\mathcal{O}) \to H^1(X,\Omega_0).$$

¿From Serre Duality, $\dim_k H^0(X,\Omega_0) = \dim_k H^1(X,\mathcal{O}) = g$ where g is the genus of the curve X. To show that our sequence reduces to the short exact sequence

$$0 \to H^0(X,\Omega_0) \to H^1(X,k) \to H^1(X,\mathcal{O}) \to 0,$$

we will show that $\dim\ H^1(X,k) = 2g$, as a k-vector space.

First, let $\mathcal{M}$ denote the sheaf of rational functions on X, and let $\mathcal{D}$ be the sheaf of differentials of the second kind, that is differentials that have zero residue at every point. Then there is a short exact sequence

$$0 \to k \to \mathcal{M} \xrightarrow{d} \mathcal{D} \to 0$$

of sheaves on the curve X. Once again this induces an exact sequence of cohomology groups

$$0 \to H^0(X,k) \to H^0(X,\mathcal{M}) \to H^0(X,\mathcal{D}) \to H^1(X,k) \to \\ H^1(X,\mathcal{M}) \to H^1(X,\mathcal{D}).$$

But $H^1(X,\mathcal{M}) = 0$ which gives that

$$H^0(X,\mathcal{M}) \to H^0(X,\mathcal{D}) \to H^1(X,k) \to 0,$$

if we isolate the end of the new sequence. Then clearly there is an isomorphism

$$H^1(X,k) \cong \mathcal{D}/d\mathcal{M}$$

which is the vector space of differentials of the second kind on X modulo the exact differentials on X. This quotient, as a k-vector space, has dimension $2g$ by a standard argument in algebraic geometry [3]. Thus we have proven both the exactness of the sequence

$$0 \to H^0(X,\Omega_0) \to H^1(X,k) \to H^1(X,\mathcal{O}) \to 0,$$

and given the cohomology group $H^1(X,k)$ an interpretation in terms of differential forms on the curve X [2].

Both the exactness of the sequence and the interpretation of $H^1(X,k)$ given above depend heavily on the assumption that k has characteristic zero. In a paper by Rosenlicht [4], it is demonstrated that if the ground field k has characteristic $p > 0$, then the vector space of differentials of the second kind modulo exact differentials has dimension over k no greater than g; even the definition given above for differentials of the second kind is not appropriate for fields of positive characteristic. It is our goal to find a suitable generalization of $H^1(X,k)$ that will retain many of the properties of this cohomology group (and hence differentials of the second kind), but for arbitrary characteristic.

2 Exactness

We will first construct a short exact sequence

$$0 \to H^0(X,\Omega_0) \to \mathbb{H}(X) \to H^1(X,\mathcal{O}) \to 0$$

where X is a curve defined over an algebraically closed field of constants k of any characteristic. We begin by carefully defining the term $\mathbb{H}(X)$ and proving the exactness of the above sequence.

2.1 Definition of $\mathbb{H}(X)$

Let $\mathfrak{U} = (U_i)_{i\in I}$ be an open cover of X. Then $\mathbb{H}(X,\mathfrak{U})$ is the set

$$\{(\omega_i, f_{ij}) : f_{ij} \in \mathcal{O}(U_{ij}),\, \omega_i \in \Omega_0(U_i), \text{ and } \omega_i - \omega_j = df_{ij}\}/\sim$$

where each (f_{ij}) is taken modulo coboundaries in $H^1(X,\mathcal{O})$ and each ω_i is taken modulo the exact differentials df_i. Thus an element of $\mathbb{H}(X,\mathfrak{U})$ is a family of elements of the form (ω_i, f_{ij}) where if we restrict our attention to just the set (f_{ij}), we have an element in the cohomology group $H^1(X,\mathcal{O})$. On the other hand, looking at just the family (ω_i), one has a set of differentials without poles modulo exact differentials defined on some open set with the relation on the intersection of open sets that $\omega_i - \omega_j = df_{ij}$. In particular, the hypercocycle $(\omega_i, f_{ij}) \sim 0$ if the family (f_{ij}), viewed as a cocycle in $H^1(X,\mathcal{O})$, splits as $f_{ij} = f_i - f_j$ and $\omega_i = df_i$. To move to a definition of $\mathbb{H}(X)$ that is independent of an open cover, we shall use a direct limit over all open covers.

This approach requires the two lemmas. Before these lemmas can be stated and proven, some notation is necessary.

Let $\mathfrak{U} = (U_i)_{i\in I}$ and $\mathfrak{V} = (V_k)_{k\in K}$ be two open covers of X. Then $\mathfrak{V}$ is finer than $\mathfrak{U}$, denoted $\mathfrak{V} < \mathfrak{U}$, if every V_k is contained in some U_i. This gives the existence of a map $\tau : K \to I$ between the indexing sets such that $V_k \subset U_{\tau(k)}$ for all $k \in K$. Using this map τ, it is possible to define a new map

$$t^{\mathfrak{U}}_{\mathfrak{V}} : \mathbb{H}(X, \mathfrak{U}) \to \mathbb{H}(X, \mathfrak{V})$$

as follows. Let $(\omega_i, f_{ij}) \in \mathbb{H}(X, \mathfrak{U})$; then $t^{\mathfrak{U}}_{\mathfrak{V}}((\omega_i, f_{ij})) = (\psi_k, g_{kl})$ where $g_{kl} = f_{\tau(k),\tau(l)}$ restricted to $V_k \cap V_l$ for all k, l and $\psi_k = \omega_{\tau(k)}$ restricted to V_k for all k. This map takes coboundaries into coboundaries and exact differentials into exact differentials, so it is well-defined and a homomorphism.

Lemma 1. *For $\mathfrak{V} < \mathfrak{U}$, the mapping*

$$t^{\mathfrak{U}}_{\mathfrak{V}} : \mathbb{H}(X, \mathfrak{U}) \to \mathbb{H}(X, \mathfrak{V})$$

is independent of the choice of refining map $\tau : K \to I$.

Proof. Let $\tilde{\tau} : K \to I$ is another refining map such that $V_k \subset U_{\tilde{\tau}(k)}$ for every $k \in K$. Let $(\omega_i, f_{ij}) \in \mathbb{H}(X, \mathfrak{U})$ and define

$$g_{kl} = f_{\tau(k),\tau(l)} \mid V_k \cap V_l \quad \text{and} \quad \tilde{g}_{kl} = f_{\tilde{\tau}(k),\tilde{\tau}(l)} \mid V_k \cap V_l$$
$$\psi_k = \omega_{\tau(k)} \mid V_k \quad \text{and} \quad \tilde{\psi}_k = \omega_{\tilde{\tau}(k)} \mid V_k.$$

To show that the map $t^{\mathfrak{U}}_{\mathfrak{V}}$ is independent of the refining map, we need to show that $(g_{kl}) - (\tilde{g}_{kl})$ splits and that $\psi_k - \tilde{\psi}_k = dg_k$ for some g_i. To show that $(g_{kl}) - (\tilde{g}_{kl})$ splits, first note that $V_k \subset U_{\tau(k)} \cap U_{\tilde{\tau}(k)}$, so one can define $h_k = f_{\tau(k),\tilde{\tau}(k)}$ restricted to V_k. Thus on $V_k \cap V_l$,

$$\begin{aligned} g_{kl} - \tilde{g}_{kl} &= f_{\tau(k),\tau(l)} - f_{\tilde{\tau}(k),\tilde{\tau}(l)} \\ &= f_{\tau(k),\tau(l)} + f_{\tau(l),\tilde{\tau}(k)} - f_{\tau(l),\tilde{\tau}(k)} - f_{\tilde{\tau}(k),\tilde{\tau}(l)} \\ &= f_{\tau(k),\tilde{\tau}(k)} - f_{\tau(l),\tilde{\tau}(l)} = h_k - h_l. \end{aligned}$$

On V_k, it is also true that

$$\psi_k - \tilde{\psi}_k = \omega_{\tau(k)} - \omega_{\tilde{\tau}(k)} = df_{\tau(k),\tilde{\tau}(k)}$$

by the relations defining $\mathbb{H}(X, \mathfrak{V})$. This gives that $\psi_k - \tilde{\psi}_k = dh_k$, proving the lemma. □

Lemma 2. *For $\mathfrak{V} < \mathfrak{U}$, the mapping*

$$t_{\mathfrak{V}}^{\mathfrak{U}} : \mathbb{H}(X, \mathfrak{U}) \to \mathbb{H}(X, \mathfrak{V})$$

is injective.

Proof. We have to show that if the image of (ω_i, f_{ij}) is equivalent to zero, then so is (ω_i, f_{ij}). First, this means that if the image of the (f_{ij}) component splits relative to $\mathfrak{V}$, then it splits relative to $\mathfrak{U}$.

Suppose $f_{\tau(k),\tau(l)} = g_k - g_l$ on $V_k \cap V_l$. Then on $U_i \cap V_k \cap V_l$,

$$g_k - g_l = f_{\tau(k),\tau(l)} = f_{i,\tau(l)} - f_{i,\tau(k)}$$

so that $f_{i,\tau(k)} + g_k = f_{i,\tau(l)} + g_l$. This gives an element

$$h_i = f_{i,\tau(k)} + g_k \text{ on } U_i \cap V_k.$$

Then on the intersection $U_i \cap U_j \cap V_k$, one has

$$f_{ij} = f_{i,\tau(k)} + f_{\tau(k),j} = f_{i,\tau(k)} + g_k - f_{j,\tau(k)} - g_k = h_i - h_j.$$

Thus (f_{ij}) splits relative to the covering $\mathfrak{U}$.

Thus we have proven that $(\omega_i, f_{ij}) \sim (\omega_i, 0)$ in $\mathbb{H}(X, \mathfrak{U})$, and we know that $(\omega_i, f_{ij}) \sim 0$ in $\mathbb{H}(X, \mathfrak{V})$. In particular, we also have that $(\omega_i, 0) \sim 0$ in $\mathbb{H}(X, \mathfrak{V})$. From the definition of $\mathbb{H}(X, \mathfrak{V})$, the hypercocycle $(\omega_i, 0)$ gives an element $\omega \in H^0(X, \Omega_0)$ by $\omega \mid_{V_i} = \omega_i$; the relation $\omega_i - \omega_j = df_{ij} = 0$ on $V_i \cap V_j$ shows that ω is well-defined on the intersection of two open sets in the cover. Since $(\omega_i, 0) \sim 0$, there exist functions $f_i \in \mathcal{O}(V_i)$ such that $\omega_i = df_i$ and $f_i - f_j = f_{ij} = 0$ on $V_i \cap V_j$. But then $(f_i) \in H^0(X, \mathcal{O})$, proving that there exists a regular function f on X such that $f_i = f \mid_{V_i}$; such a regular function on X is an element of k, so $\omega_i = df_i = 0$ proving that $\omega = 0$ and hence that $(\omega_i, f_{ij}) \sim 0$ in $\mathbb{H}(X, \mathfrak{U})$ and that

$$t_{\mathfrak{V}}^{\mathfrak{U}} : \mathbb{H}(X, \mathfrak{U}) \to \mathbb{H}(X, \mathfrak{V})$$

is injective. □

We are now ready to define $\mathbb{H}(X)$. Given open covers $\mathfrak{W} < \mathfrak{V} < \mathfrak{U}$, we have

$$t_{\mathfrak{W}}^{\mathfrak{V}} \circ t_{\mathfrak{V}}^{\mathfrak{U}} = t_{\mathfrak{W}}^{\mathfrak{U}}.$$

Define an equivalence relation $\sim$ by two equivalence classes $\xi \in \mathbb{H}(X, \mathfrak{U})$ and $\eta \in \mathbb{H}(X, \mathfrak{U}')$ are equivalent, $\xi \sim \eta$, if there exists some common refinement $\mathfrak{V}$ with $\mathfrak{V} < \mathfrak{U}$ and $\mathfrak{V} < \mathfrak{U}'$ such that

$$t^{\mathfrak{U}}_{\mathfrak{V}}(\xi) = t^{\mathfrak{U}'}_{\mathfrak{V}}(\eta).$$

The set of equivalence classes taken over all open covers is the direct limit

$$\varinjlim \mathbb{H}(X, \mathfrak{U})$$

which we will now show can be computed in terms of a single open cover.

Proposition. *Let $\mathfrak{U} = (U_i)_{i \in I}$ be an open cover of X such that $\mathbb{H}(U_i) = 0$ for every $i \in I$. Then $\mathbb{H}(X, \mathfrak{U}) \cong \mathbb{H}(X)$.*

Proof. We will show that given $\mathfrak{V} < \mathfrak{U}$, the map $t^{\mathfrak{U}}_{\mathfrak{V}}$ is an isomorphism. We have already established that this map is injective, so we need only prove surjectivity. Let $\tau : K \to I$ be a refining map with $V_k \subset U_{\tau(k)}$ for all $k \in K$. Then given a hypercocycle $(\omega_k, f_{kl}) \in \mathbb{H}(X, \mathfrak{V})$, we must show there exists a hypercocycle $(\psi_i, F_{ij}) \in \mathbb{H}(X, \mathfrak{U})$ such that $(\omega_k, f_{kl}) - (\psi_{\tau(k)}, F_{\tau(k),\tau(l)})$ is zero in $\mathbb{H}(X, \mathfrak{V})$.

The family $(U_i \cap V_k)_{k \in K}$ covers U_i and $U_i \cap \mathfrak{V}$ will denote this covering. By hypothesis, $\mathbb{H}(U_i, U_i \cap \mathfrak{V}) = 0$ so there exist $g_{ik} \in \mathcal{O}(U_i \cap V_k)$ such that

$$f_{kl} = g_{ik} - g_{il} \text{ on } U_i \cap V_k \cap V_l.$$

On the intersection $U_i \cap U_j \cap V_k \cap V_l$, the relation

$$g_{jk} - g_{ik} = g_{jl} - g_{il}$$

holds, giving rise to elements $F_{ij} \in \mathcal{O}(U_i \cap U_j)$ such that

$$F_{ij} = g_{jk} - g_{ik} \text{ on } U_i \cap U_j \cap V_k.$$

Now (F_{ij}) satisfies the cocycle condition; let $h_k = g_{\tau(k),k}$ on V_k. Then on $V_k \cap V_l$ one has

$$\begin{aligned} F_{\tau(k),\tau(l)} - f_{kl} &= (g_{\tau(l),k} - g_{\tau(k,k)}) - (g_{\tau(l),k} - g_{\tau(l),l}) \\ &= g_{\tau(l),l} - g_{\tau(k),k} = h_l - h_k. \end{aligned}$$

Now on $U_i \cap V_k \cap V_l$,

$$\omega_k - \omega_l = df_{kl} = dg_{ik} - dg_{il}$$

and thus

$$\omega_k - dg_{ik} = \omega_l - dg_{il}$$

giving an element of $\Omega_0(U_i)$ by $\psi_i = \omega_k - dg_{ik}$ on $U_i \cap V_k$. Then it is clear that $\omega_k - \psi_i = dg_{ik}$ restricted to V_k. The only thing left to check is that $\psi_i - \psi_j = dF_{ij}$. On $U_i \cap U_j \cap V_k$,

$$\psi_i - \psi_j = (\omega_k - dg_{ik}) - (\omega_k - dg_{jk}) = dg_{jk} - dg_{ik} = dF_{ij};$$

since k was arbitrary, this holds on all of $U_i \cap U_j$. □

The set $\mathbb{H}(X)$ forms a k-vector space. Addition is performed componentwise, using a common refinement of open covers if necessary. Multiplication by scalars is given by $\lambda(\omega_i, f_{ij}) = (\lambda\omega_i, \lambda f_{ij})$.

2.2 The Exact Sequence

Using the Leray-type result of the previous proposition, we can compute $\mathbb{H}(X)$ using a single covering of X. Now we are ready to prove the exactness of the sequence

$$0 \to H^0(X, \Omega_0) \to \mathbb{H}(X) \to H^1(X, \mathcal{O}) \to 0$$

where the map $H^0(X, \Omega_0) \to \mathbb{H}(X)$ is given by

$$\omega \mapsto (\omega_i, 0) \text{ where } \omega_i = \omega \mid U_i$$

and the map $\mathbb{H}(X) \to H^1(X, \mathcal{O})$ is given by

$$(\omega_i, f_{ij}) \mapsto (f_{ij}).$$

Theorem 1. *The sequence*

$$0 \to H^0(X, \Omega_0) \to \mathbb{H}(X) \to H^1(X, \mathcal{O}) \to 0$$

is exact.

Proof. First one has to check that the map $H^0(X, \Omega_0) \to \mathbb{H}(X)$ is actually a map into $\mathbb{H}(X)$. On $U_i \cap U_j$,

$$0 = \omega_i - \omega_j = df_{ij}$$

so this map does give an element of $\mathbb{H}(X)$. Now we must check that this map is injective; assume that $\omega \in \Omega_0(X)$ maps to zero in $\mathbb{H}(X)$. Then there exist $f_i \in \mathcal{O}_i$ such that

$$\omega = df_i \text{ on } U_i, \text{ and}$$
$$f_i - f_j = f_{ij} = 0 \text{ on } U_{ij}.$$

This means that the f_i give a global element f in $\mathcal{O}(X)$, but then $f \in k$ and $\omega = df = 0$. It is clear that the kernel of the map $\mathbb{H}(X) \to H^1(X, \mathcal{O})$ is the image of $H^0(X, \Omega_0) \to \mathbb{H}(X)$, so to prove that the above sequence is exact, it only remains to be shown that $\mathbb{H}(X) \to H^1(X, \mathcal{O})$ is surjective. Let $(f_{ij}) \in H^1(X, \mathcal{O})$ be given. Then if we can find $\omega_i \in \Omega_0(U_i)$ such that $\omega_i - \omega_j = df_{ij}$, then the hypercocycle (ω_i, f_{ij}) would map to (f_{ij}). Thus we must show that (df_{ij}) splits in $H^1(X, \Omega_0)$. Using the isomorphism between $H^1(X, \Omega_0)$ and k given by Serre Duality, (df_{ij}) splits if and only if $tr(df_{ij}) = 0$. Now (f_{ij}) splits when viewed as a cocycle in $H^1(X, \mathcal{M})$, the first cohomology group with coefficients in the sheaf of meromorphic functions. Thus we can write $f_{ij} = f_i - f_j$ on $U_i \cap U_j$. Thus it is possible to split (df_{ij}) as $df_{ij} = df_i - df_j$ on $U_i \cap U_j$. Therefore

$$tr(df_{ij}) = \sum_{P \in X} res_P(df_i) = 0$$

since exact differentials have no residue. □

3 Differentials of the Second Kind

Now that we have proven that the sequence

$$0 \to H^0(X, \Omega_0) \to \mathbb{H}(X) \to H^1(X, \mathcal{O}) \to 0$$

is exact, we would like to investigate any similarities these groups have to the corresponding cohomology groups in the exact sequence

$$0 \to H^0(X, \Omega_0) \to H^1(X, k) \to H^1(X, \mathcal{O}) \to 0$$

for fields k of characteristic zero.

3.1 An Isomorphism Theorem

Assume that k has characteristic zero. We previously demonstrated an isomorphism between the vector space of differentials of the second kind modulo exact differentials and the cohomology group $H^1(X, k)$. Thus there is an isomorphism between $\mathbb{H}(X)$ and this vector space of differentials when $char(k) = 0$. We now explicitly construct this isomorphism.

Theorem 2. *Let k be an algebraically closed field of characteristic zero, and X a curve defined over k. Then $\mathbb{H}(X)$ is isomorphic, as a k-vector space, to the vector space of differentials of the second kind modulo exact differentials.*

Proof. First, define a map from $\mathbb{H}(X)$ to the vector space of differentials of the second kind as follows. Let $(\omega_i, f_{ij}) \in \mathbb{H}(X)$ be a hypercocycle. Then by construction,

$$\omega_i - \omega_j = df_{ij} = df_i - df_j \text{ on } U_i \cap U_j$$

where we take $f_{ij} = f_i - f_j$ in the cohomology group $H^1(X, \mathcal{M})$; then f_i and f_j are rational functions defined over U_i and U_j, respectively. Thus there is the relation $\omega_i - df_i = \omega_j - df_j$ on the intersection $U_i \cap U_j$. Define the image of the hypercocycle (ω_i, f_{ij}) in the vector space of differentials of the second kind to be the differential ω where

$$\omega|_{U_i} = \omega_i - df_i.$$

Assume that the hypercocycle (ω_i, f_{ij}) maps to zero in the vector space of differentials of the second kind. Then $\omega_i - df_i = 0$, so it is immediate that $\omega_i = df_i$ and thus the original hypercocycle can be written $(0, f_{ij})$. Proving that the cocycle $(f_{ij}) \in H^1(X, \mathcal{O})$ splits is sufficient to show that the map from $\mathbb{H}(X)$ to the vector space of differentials of the second kind is injective. Since $H^1(X, \mathcal{M}) = 0$, $f_{ij} = f_i - f_j$ where the f_i are rational functions on U_i. Assume that f_i has a pole at the point $P \in U_i$. Then, since k has characteristic zero, df_i also has a pole at P. But $df_i = \omega_i$ which is by hypothesis without poles on U_i. Thus f_i is regular on U_i, and $(f_{ij}) = 0$ in $H^1(X, \mathcal{O})$. Now assume that there exists some hypercocycle (ω_i, f_{ij}) that maps to an exact differential dg in the vector space of differentials of the second kind. Then on U_i, it follows from the definition of $\mathbb{H}(X)$ that $\omega_i - df_i = dg$. Hence $\omega_i = dg + df_i = d(g + f_i)$ and the hypercocycle (ω_i, f_{ij}) is equivalent to $(0, f_{ij})$. If the cocycle $(f_{ij}) \in H^1(X, \mathcal{O})$ splits, then the only hypercocycle that maps to an exact differential is the zero hypercocycle and hence the image of the constructed map lies within the vector space of differentials of the second kind modulo exact differentials. We know that on U_i, $\omega_i - df_i = dg$. Thus $\omega_i = d(g + f_i)$; since ω_i has no poles in U_i, $d(g + f_i)$ also has no poles in U_i. By the same argument as above, $g + f_i$ is regular on U_i. Therefore the image of the cochain $(g + f_i)$ under the coboundary map is

$$(g + f_i) - (g + f_j) = f_i - f_j = f_{ij}$$

and $(f_{ij}) \in H^1(X, \mathcal{O})$ splits.

It is trivial to show that this is a k-vector space homomorphism, so to prove the isomorphism between $\mathbb{H}(X)$ and the differentials of the second kind modulo exact differentials, we need only construct a two-sided inverse. Let ω be a differential of the second kind on X. Let $\{P_1, P_2, \ldots, P_n\}$ be the set of points at which ω has poles, which we know is a finite set. By the proposition, we can assume that our open cover $(U_i)_{i \in I}$ is fine enough so that no pole lies in the intersection of two open sets. If U_k is an open set such that ω has no poles on U_k, then set $\omega_k = \omega$. If U_i is an open set containing the point P_i, then we can write

$$\omega = \omega_i - df_i$$

where ω_i is a differential without poles on U_i; it is possible to write ω in this way by an equivalent definition of differentials of the second kind that will be discussed later. Then the map from the vector space of differentials of the second kind to $\mathbb{H}(X)$ is defined by

$$\omega \mapsto (\omega_i, f_{ij})$$

where ω_i is the differential above in the representation of ω on U_i, and $f_{ij} = f_i - f_j$ where f_i is some rational function such that $\omega = \omega_i - df_i$. First, we prove that this map is well defined. Assume that, on U_i, $\omega = \omega_i - df_i$ and $\omega = \psi_i - dg_i$ are two ways to write ω, and hence ω gets mapped to the hypercocycles (ω_i, f_{ij}) and (ψ_i, g_{ij}). To show the map above is well-defined, we must prove that $(\omega_i, f_{ij}) - (\psi_i, g_{ij}) \sim 0$ in $\mathbb{H}(X)$. First, note the relations

$$\omega + df_i = \omega_i \qquad \text{and} \qquad \omega + dg_i = \psi_i$$

on the open set U_i. Thus on U_i,

$$\omega_i - \psi_i = (\omega + df_i) - (\omega + dg_i) = df_i - dg_i = d(f_i - g_i).$$

Since $\omega_i - \psi_i$ has no poles on U_i, we can conclude as above that $f_i - g_i$ has no poles on U_i. Using the same calculations gives

$$f_{ij} - g_{ij} = (f_i - f_j) - (g_i - g_j) = (f_i - g_i) - (f_j - g_j)$$

so $(f_{ij} - g_{ij})$ splits in $H^1(X, \mathcal{O})$. This shows that $(\omega_i, f_{ij}) - (\psi_i, g_{ij})$ is equivalent to the zero hypercocycle in $\mathbb{H}(X)$ and so the map from differentials of the second kind to $\mathbb{H}(X)$ is well-defined.

Denote by τ the map from $\mathbb{H}(X)$ to the differentials of the second kind modulo exact differentials, and σ the map from differentials of the second kind to $\mathbb{H}(X)$; we will prove that $\tau \circ \sigma = \sigma \circ \tau = 1$. Let ω be an element of the vector space of differentials of the second kind modulo exact differentials, with $\omega = \omega_i - df_i$ on the open set U_i. Then $\sigma(\omega) = (\omega_i, f_{ij})$ and $\tau((\omega_i, f_{ij})) = \psi$ where

$$\psi|_{U_i} = \omega_i - df_i,$$

so ψ is visibly equal to ω and $\tau \circ \sigma = 1$. Now let $(\omega_i, f_{ij}) \in \mathbb{H}(X)$; then $\tau((\omega_i, f_{ij})) = \omega$ where

$$\omega|_{U_i} = \omega_i - df_i$$

writing $f_{ij} = f_i - f_j$ with f_i, f_j rational functions. Then clearly we can take $\sigma(\omega)$ to be the hypercocycle (ω_i, f_{ij}), proving $\sigma \circ \tau = 1$. □

It would be desirable to be able to recover as much of the previous theorem as possible when the characteristic of the ground field k is $p > 0$. To make the discussion clear, we introduce another definition of differentials of the second kind that, in characteristic 0, is equivalent to the usual definition of a differential having zero residue at every point; the definition we present now is the same one used by Rosenlicht [4]. A differential of the second kind is a differential ω such that at every point $P \in X$, ω can be written, on some neighborhood of P, as the sum of a differential without poles and an exact differential. In characteristic zero, if t is a local parameter at P, then a differential of the first kind on some neighborhood of P can be written $\omega = y\ dt$ where y is a power series

$$y = \sum_{i=0}^{\infty} c_i t^i;$$

an exact differential can be written $\psi = dz$, so

$$\psi = d\Big(\sum_{j=-m}^{\infty} c_j t^j\Big) = \Big(\sum_{j=-m}^{\infty} j c_j t^{j-1}\Big) dt.$$

It is then easy to see that in characteristic zero a differential with zero residue at every point can be written as $\omega + \psi$ as above, and a differential $\omega + \psi$ clearly has no residue at every $P \in X$.

Now let ω be a differential of the second kind on X, a curve defined over the algebraically closed ground field k of positive characteristic. Then on the open set U_i, we can write $\omega = \omega_i - df_i$ for some rational function f_i. We would like to define the same map σ as above sending the differential ω to the hypercocycle (ω_i, f_{ij}). First we check that this map is well-defined as before; let $\omega = \omega_i - df_i = \psi_i - dg_i$ on U_i. Then the requirement that σ be well-defined is that the hypercocycle $(\omega_i, f_{ij}) - (\psi_i, g_{ij})$ is equivalent to the zero hypercocycle in $\mathbb{H}(X)$. Once again, we find that

$$\omega_i - \psi_i = (\omega + df_i) - (\omega + dg_i) = df_i - dg_i = d(f_i - g_i)$$

on U_i. To conclude that $(\omega_i, f_{ij}) - (\psi_i, g_{ij}) \sim 0$, it is necessary that

(1) $(f_{ij} - g_{ij})$ splits in $H^1(X, \mathcal{O})$, that is $f_{ij} - g_{ij} = (f_i - g_i) - (f_j - g_j)$ where $f_i - g_i \in \mathcal{O}(U_i)$, and
(2) $\omega_i - \psi_i = d(f_i - g_i)$ where $f_i - g_i$ is the regular function on U_i given in (1).

Thus it is required that $f_i - g_i$ is a regular function on U_i; unfortunately, we cannot conclude this from the construction given when k has positive characteristic. The condition that $d(f_i - g_i)$ is a differential without poles on U_i merely implies that the derivative of the polar part of $f_i - g_i$ is zero, not that it has no polar part as in characteristic zero. Conversely, requiring that $f_i - g_i$ be regular on U_i is also a sufficient condition to ensure that $(\omega_i, f_{ij}) - (\psi_i, g_{ij}) \sim 0$; thus if we want to prove that the differentials of the second kind modulo exact differentials are isomorphically embedded in $\mathbb{H}(X)$, a different map will be necessary.

To construct the correct map, first choose an open cover of X with the following properties. For every pole of ω - say at the point P_i - there exists a unique open set U_i containing P_i with the property that U_i contains no other point of X at which ω has a pole. We also have an open set U_0 in the cover with the property that ω has no poles in U_0 and it contains a point $Q \in U_0$ that is in no other open set of the cover. Now let $\omega = \omega_i - df_i$ on U_i; we will now explicitly construct the functions f_i to ensure that the map σ is well-defined. We know that on U_i,

$$df_i = \sum_{i=-m}^{\infty} ic_i t^{i-1} dt$$

where $m \in \mathbb{Z}$ and the characteristic of k does not divide i in $\mathbb{Z}$ for all i. Producing a function having polar part

$$\sum_{\substack{i \in \mathbb{Z}^- \\ char(k) \nmid i}} c_i t^i$$

and any other poles outside of U_i would prove that the map σ as defined above would be well-defined as any two functions f_i and g_i meeting this criteria would clearly have as their difference a regular function on U_i.

To produce such a function, we need to use the Riemann-Roch Theorem, stated here without proof. First we introduce the following notation. Let D be a divisor on a curve X and let $L(D)$ denote the vector space of functions f, along with the zero function, that have the property that the divisor of f is greater than or equal to $-D$. These are the functions, then, such that $(f) \geq -D$ or $(f) + D \geq 0$. Let $l(D)$ denote the dimension of this vector space over k.

Theorem of Riemann-Roch. *There exists a divisor K and an integer g such that*

$$l(D) = deg(D) + 1 - g + l(K - D)$$

for all divisors D on X.

It is easy to show that if the degree of some divisor D is negative, then $l(D) = 0$; this also gives that if $deg(D) > 2g - 2$, then $l(K - D) = 0$ since it can be shown that $deg(K) = 2g - 2$.

We make use of the divisors $D = j \cdot P_i + 2g \cdot Q$ where $0 < j \leq m$. This divisor clearly has degree larger than $2g - 2$ for all j, so by Riemann-Roch

$$l(D) = deg(D) + 1 - g = g + j + 1.$$

Since all of these vector spaces have different dimensions depending on j, we know that there exist functions with polar part

$$\sum_{i=-j}^{0} a_i t^i$$

on U_i and $a_{-j} \neq 0$. Let f_{-m+1} be a function with a pole of order $-m+1$ at P_i and all its other poles at Q as given by the Riemann-Roch theorem. Multplying by a suitable constant gives f_{-m+1} the same coefficient as the desired function for the t^{-m+1} term. Now we repeat the process, choosing a function f_{-m+2} using Riemann-Roch and then multplying by the correct constant so that when added to f_{-m+1}, the resulting function has the correct coefficients for the t^{-m+1} and t^{-m+2} terms. This produces a function which we denote f_i as above. It is clear that if another function g_i was produced in this fashion, $f_i - g_i$ would be regular on U_i since they have the same polar parts on that open set, hence the map σ from differentials of the second kind modulo exact differentials to $\mathbb{H}(X)$ is well-defined. Now we have proved

Theorem 3. *The map σ is well-defined, with left inverse τ as shown above. Thus σ isomorphically embeds the differentials of the second kind modulo exact differentials in $\mathbb{H}(X)$. Therefore the map τ from $\mathbb{H}(X)$ to the vector space of differentials of the second kind modulo exact differentials is surjective, and if k has characteristic zero both of these maps are isomorphisms of k-vector spaces.*

3.2 Self-Duality of $\mathbb{H}(X)$

By the Serre Duality theorem, we obtain the duality of the first and third terms in the short exact sequence

$$0 \to H^0(X, \Omega_0) \to H^1(X, k) \to H^1(X, \mathcal{O}) \to 0$$

when k has characteristic zero. It is also true that the middle term $H^1(X, k)$ is self-dual, although this duality is easiest to prove using the isomorphic vector space of differentials of the second kind modulo exact differentials on X. Let ω and η be two such differentials. Let $P \in X$ be a point; then at P, the differntial η can be written $\eta = dg_P$. Then we define the pairing $\langle , \rangle$ by

$$\langle \omega, \eta \rangle = \sum_{P \in X} res_P(g_P \omega).$$

A proof that $\langle , \rangle$ is a non-degenerate pairing can be found in Chevalley [1]. This proves that $\mathbb{H}(X)$ is self-dual when k has characteristic zero and provides the motivation for the next theorem. But first, let $x = (\omega_i, f_{ij})$ and $y = (\eta_i, g_{ij})$ be elements of $\mathbb{H}(X)$. Then define

$$\langle x, y \rangle = \sum_{P \in X} res_P(g_i\omega_i - f_i\eta_i + f_i dg_i)$$

where $P \in U_i$ and $f_{ij} = f_i - f_j$ with the f_i rational functions on U_i.

Theorem 4. *$\mathbb{H}(X)$ is self-dual under the pairing $\langle , \rangle$ when X is defined over any algebraically closed field k.*

Proof. First we must prove that this pairing is well-defined. If $P \in U_i \cap U_j$, then to show that this pairing is well-defined, we need to show that $\langle x, y \rangle$ gives the same result regardless of whether we use the functions and differentials given on U_i or given on U_j. Thus

$$\begin{aligned}\langle x, y \rangle &= \sum_{P \in X} res_P(g_i\omega_i - f_i\eta_i + f_i dg_i) \\ &= \sum_{P \in X} res_P((g_{ij} + g_j)(\omega_j + df_{ij}) - (f_{ij} + f_j)(\eta_j \\ &\qquad + dg_{ij}) + (f_{ij} + f_j)(dg_{ij} + dg_j))\end{aligned}$$

using the relations

$$\begin{array}{ccc} f_{ij} = f_i - f_j & \text{and} & \omega_i - \omega_j = df_{ij} \\ g_{ij} = g_i - g_j & \text{and} & \eta_i - \eta_j = dg_{ij}. \end{array}$$

Since the residue is an additive homomorphism, after multiplying these terms out, we can separate the resultant residue into

$$\sum_{P \in X} res_P(g_j\omega_j - f_j\eta_j + f_j dg_j) + \sum_{P \in X} res_P(\psi)$$

where

$$\psi = g_{ij}\omega_j + g_{ij}df_{ij} + g_j df_{ij} - f_{ij}\eta_j - f_{ij}dg_{ij} - f_j dg_{ij} + f_{ij}dg_{ij} + f_j dg_{ij} + f_{ij}dg_i.$$

Now $g_{ij}\omega_i - f_{ij}\eta_j$ has no poles on $U_i \cap U_j$, and thus has no residue. After cancellation, we are left showing that

$$g_{ij}df_{ij} + g_j df_{ij} + f_{ij}dg_i$$

has zero residue. Since $g_{ij} = g_i - g_j$, this differential is equal to $g_i df_{ij} + f_{ij} dg_i$. Noting that

$$g_i df_{ij} + f_{ij} dg_i = d(g_i f_{ij})$$

and that exact differentials have no residue, we have shown that

$$\sum_{P \in X} res_P(\psi) = 0$$

and the pairing $\langle , \rangle$ is well-defined.

Let $\omega \in H^0(X, \Omega_0)$, so that its image in $\mathbb{H}(X)$ is the hypercocycle $(\omega_i, 0)$ where $\omega_i = \omega|_{U_i}$. Let $\eta \in H^0(X, \Omega_0)$ be another differential with image $(\eta_i, 0)$ in the same fashion. Then

$$\langle \omega, \eta \rangle = \sum_{P \in X} res_P(0) = 0;$$

since we have proven that the sequence

$$0 \to H^0(X, \Omega_0) \to \mathbb{H}(X) \to H^1(X, \mathcal{O}) \to 0$$

is exact, the above calculation proves that if we identify $\omega \in H^0(X, \Omega_0)$ with its image in $\mathbb{H}(X)$, then $\langle \omega, x \rangle$ depends only on the image of the hypercocycle x in $H^1(X, \mathcal{O})$. Let $\overline{x}$ denote the image of $x \in \mathbb{H}(X)$ in $H^1(X, \mathcal{O})$. Now let $x \in \mathbb{H}(X)$ be a hypercocycle such that $\langle x, y \rangle = 0$ for all $y \in \mathbb{H}(X)$. Let y be the image in $\mathbb{H}(X)$ of a differential $H^0(X, \Omega_0)$. Then $\langle x, y \rangle$ depends only on $\overline{x}$, and in fact

$$\langle \overline{x}, y \rangle = \sum_{P \in X} res_P(-f_i \psi_i)$$

if $x = (\omega_i, f_{ij})$ and $y = (\psi_i, 0)$. Recall the function on $H^0(X, \Omega_0) \times H^1(X, \mathcal{O})$ given by Serre Duality. Given (ψ_i) and (f_{ij}), first write $f_{ij} = f_i - f_j$ as rational functions. Now $(f_{ij}\psi_i)$ is an element of $H^1(X, \Omega_0)$, so we can apply the map

$$tr : H^1(X, \Omega_0) \to k$$

previously defined to the cocycle $(f_{ij}\psi_i)$ Clearly we can split this cocycle as $f_{ij}\psi_i = f_i\psi_i - f_j\psi_i$. Thus

$$tr(f_{ij}\psi_i) = \sum_{P \in X} res_P(f_i\psi_i).$$

This is, up to sign, the pairing $\langle \overline{x}, y \rangle$, so $\langle \overline{x}, y \rangle = 0$ for all $y \in H^0(X, \Omega_0)$ if and only if (f_{ij}) splits in $H^1(X, \mathcal{O})$. Therefore $x \sim (\omega_i, 0)$ in $\mathbb{H}(X)$. Then $\langle x, y \rangle = \sum_{P \in X} res_P(g_i \omega_i)$ where $y = (\psi_i, g_{ij})$. Applying Serre Duality with the fixed differential ω, $\omega|_{U_i} = \omega_i$, we know that $\langle x, y \rangle = 0$ for all y if and only if $\omega = 0$. Thus the only hypercocycle that pairs with every other hypercocycle to yield zero is the zero hypercocycle. Since we are dealing with two vector spaces - both $\mathbb{H}(X)$ actually - of equal dimension, this proves that the pairing $\langle , \rangle$ makes $\mathbb{H}(X)$ self-dual. $\square$

Chevalley also proves that his pairing on differentials of the second kind modulo exact differentials is anti-symmetric, so $\langle \omega, \eta \rangle = -\langle \eta, \omega \rangle$. We will prove that the pairing given above on $\mathbb{H}(X)$ is also anti-symmetric. By the definition of the pairing,

$$\langle x, y \rangle = \sum_{P \in X} res_P(g_i \omega_i - f_i \eta_i + f_i dg_i) = -\sum_{P \in X} res_P(f_i \eta_i - g_i \omega_i - f_i dg_i).$$

Since $d(f_i g_i) = f_i dg_i + g_i df_i$,

$$\langle x, y \rangle = -\sum_{P \in X} res_P(f_i \eta_i - g_i \omega_i - d(f_i g_i) + g_i df_i)$$

which is equal to $-\langle y, x \rangle$ since the residue is an additive map and exact differentials have no residue.

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