Surfaces in $\mathbf{P}^3$ over finite fields

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Abstract: For a smooth surface X in $\mathbf{P}^3$ of degree d, defined over a finite field $\mathbf{F}_q$ with q elements, q prime, we prove that X has at most $d(d+q-1)(d+2q-2)/6+d(11d-24)(q+1)$ points with coordinates in $\mathbf{F}_q$.

0. Introduction

In this paper we will obtain an upper bound on the number of rational points on a surface X in $\mathbf{P}^3$ of degree d over a finite field $\mathbf{F}_q$, which in a certain range of d and q improves on all other known bounds.

The bound will be obtained by counting the number of points P of X in an algebraic closure of $\mathbf{F}_q$ whose image under the Frobenius map lies in one of the asymptotic lines to X at P. Recall that the asymptotic lines to a surface X in $\mathbf{P}^3$ are the lines that touch X at a point with multiplicity at least three. Usually, given a general point P of X, there are two asymptotic lines tangent to X at P. We will make this statement more precise later when we investigate the geometry of asymptotic lines in detail. This is, of course, a very classical topic in algebraic geometry but has only been investigated in characteristic zero. We will obtain some new geometric results in positive characteristic and raise a few questions.

This bound is the analogue for surfaces of theorem 0.1 of [SV], which is about plane curves. In [SV] a general result is proved for morphisms of curves to projective spaces of arbitrary dimension with many consequences and one would hope to similarly extend the results here. This seems to be considerably harder but we hope to return to it in a later paper.

We will also compare our bound with the other known bounds for surfaces over finite fields and discuss some examples.

1. Geometry

In this section we fix an algebraically closed field k of characteristic $p > 0$. Let X be a smooth surface of degree $d > 1$ over k defined by the equation $f = 0$, where f is a homogeneous polynomial of degree d in $k[x_0, x_1, x_2, x_3]$. We will study various geomtric properties of X.

We begin with the dual variety of X. Let $\mathbf{P}^{3^*}$ be the dual projective space and $\gamma : X \to \mathbf{P}^{3^*}, P \mapsto T_P X$ be the Gauss map. As usual, $T_P X$ denotes the tangent plane to X at P, given by the equation $\sum_{i=0}^{3} f_{x_i}(P)\, x_i = 0$, where, here and elsewhere, f_{x_i} denotes the partial derivative $\partial f / \partial x_i$. So γ is given by $P \mapsto (f_{x_0}(P) : f_{x_1}(P) : f_{x_2}(P) : f_{x_3}(P))$. We denote $\gamma(X)$ by X^*, which is called the dual variety of X. If p does not divide $d(d-1)$, γ is birational and $\deg X^* = d(d-1)^2$ (See, e.g. [HK]).

The points of ramification of the Gauss map are known as parabolic points and the points where the derivative of the Gauss map is zero are known as planar points. Every line through a planar point in the tangent plane at the point touches the surface with multiplicity at least three. On a non-planar point, the general line through the point in the tangent plane at the point touches the surface with multiplicity two and by a local argument it easy to see that, in such a point, there are either two or one lines touching the surface with multiplicity at least three; these are the asymptotic lines. The parabolic points are exactly those for which there is exactly one such line. We call the points for which there are exactly two such lines, ordinary points. To obtain the asymptotic lines through $P \in X$ one considers the first and second polars to X at a point $Q = (y_0 : y_1 : y_2 : y_3)$, that is $\sum f_{x_i} y_i, \sum f_{x_i x_j} y_i y_j$. The line $\overline{PQ}$ is an asymptotic line to X at P if and only if the first and second polars to X at Q vanish at P.

If the Gauss map is inseparable, all points are parabolic (or planar). Otherwise the set of parabolic points form a curve, the parabolic curve with equation $H = \det(f_{x_i x_j}) = 0$ (the polynomial H is the Hessian of f) and is therefore a curve of degree $4d(d-2)$. Of course, the parabolic curve may not be reduced. For instance, planar points are easily seen to be singularities of the parabolic curve and, therefore, if the surface has a one-dimensional set of planar points, these will form a multiple component of the parabolic curve.

We will call a surface ordinary if the Gauss map is birational and, therefore, there is an open set of ordinary points. Let $\mathbf{G}$ be the Grassmannian of lines in $\mathbf{P}^3$ and define $Z' = \{(P, L) \in X \times \mathbf{G} \mid (X \cdot L)_P \geq 3\}$, in other words the set of pairs (P, L) where L is an asymptotic line at P. If X has a one-dimensional set of planar points, then Z' has a component which is a $\mathbf{P}^1$-bundle over this set, since every line in the tangent plane of a planar point is asymptotic. We let Z_0 be this component of Z' and define Z to be the union of the remaining components of Z'. We call Z the asymptotic double cover of X and, indeed if X is ordinary Z is a double cover of X. (The asymptotic double cover was introduced in [McS] for surfaces without planar points). We note that Z does not have to be irreducible. We say that the surface X is non-split if its asymptotic double cover is irreducible and split otherwise.

A point $P \in X$ is called flecnodal and a line L a flecnodal line through P if $(X \cdot L)_P \geq 4$. The name flecnodal is the traditional one ([Sa], pg.277) and evokes the fact that, if P is an ordinary flecnodal point, the curve $X \cap T_P X$ has a node at P with one branch being inflectional at P, with the line L being the tangent to said branch. Modern authors ([K], [McS], [V]) refer to asymptotic flex points instead of flecnodal points, because the flecnodal points are the inflection points of the asymptotic curves. Since the asymptotic curves are not algebraic and may not exist in positive characteristic we prefer to use the term flecnodal. It may happen, in positive characteristic, that all points on X are flecnodal and we conjecture that this is only possible when p divides $d(d-1)(d-2)$. We will prove some partial results towards this conjecture below. It is an old result of Salmon (see [Sa], pg. 278) that the flecnodal points of X are cut out by a homogeneous polynomial of degree $11d - 24$ and the proof is valid in arbitrary characteristic (see also [K], [McS], [V], 12.8). Thus, if not all points are flecnodal, the set of flecnodal points form a curve, the flecnodal curve, of degree $d(11d - 24)$.

Define a rational map $\phi : Z' \to \mathbf{P}^2$ as follows. Fix a plane $H \subset \mathbf{P}^3$ and if $(P, L) \in Z', L \not\subset H$, define $\phi(P, L) = L \cap H$. We claim that $\deg \phi = d(d-1)(d-2)$. Indeed, for a general Q in H, to find the inverse image of Q we need to find the points P which annihilate the equation f of X and also the the first and second polars to X at Q, these are equations of degree respectively d,$d-1$ and $d-2$ and for all Q the system will have solutions, so ϕ is surjective. A surjective map between varieties of the same dimension is generically finite so the system has finitely many solutions for general Q. It follows that $\deg\phi = d(d-1)(d-2)$. We will use ϕ extensively below.

We can also define another map $\phi' : Z' \to \mathbf{P}^2$ as follows. Take a generic point P_0 of $\mathbf{P}^3$ and consider the $\mathbf{P}^2$ of lines through P_0. Given an element $(P, L) \in Z'$ define $\phi'(P, L)$ as the plane spanned by L and P_0, provided P_0 is not in L. To compute the degree of ϕ' take a plane H through P_0. The points in Z' with image H are pairs (P, L) with $L \subset H$, as it is easy to see. To count these points, notice that the condition that $L \subset H$ and $(X \cdot L)_P \geq 3$ is the same as L having triple contact with $X \cap H$ at P, thus we are counting the number of inflections of the plane curve $X \cap H$ of degree d and they number $3d(d-2)$, therefore $\deg\phi' = 3d(d-2)$. The computation of these degrees is presumably the meaning of eqs. 3 and 4 on pg. 236 of [Sc]. We will not use ϕ' in what follows.

Theorem 1. *Not all points are flecnodal on a non-split smooth surface of degree d if p does not divide $d(d-1)(d-2)$.*

Proof: Note that the hypotheses imply that d is at least three and p is at least five. Let P be a point on X and choose affine coordinates x, y, z such that $P = (0, 0, 0)$ and $T_P X = \{z = 0\}$. Near P we can describe X by $z = u(x, y)$ for some power series $u(x, y)$. At a point $P_0 = (x_0, y_0, z_0)$ near P we can consider lines through P_0 given by $L : y - y_0 = \lambda(x - x_0), z - z_0 = \mu(x - x_0)$. Such a line is in $T_{P_0}X$ if and only if $\mu = u_x(P_0) + u_y(P_0)\lambda$ and is asymptotic if and only if, in addition, $u_{xx}(P_0) + 2u_{xy}(P_0)\lambda + u_{yy}(P_0)\lambda^2 = 0$. Also, the line L intersects the plane at infinity in the point with (affine) coordinates (λ, μ) and this gives an analytic expression for the map ϕ.

We will first show that ϕ is ramified everywhere on any two-dimensional component of Z_0 for any surface. Let C be a one-dimensional component of the locus of planar points, P a general point of C and t a local parameter for C at P and choose coordinates (x, y, z)

as above. Thus, $u_{xx} = u_{xy} = u_{yy} = 0$ along C, (t, λ) serve as local coordinates for Z_0 at P and ϕ is given by $(t, \lambda) \mapsto (\lambda, u_x + u_y\lambda)$, where the partial derivatives are computed at $(x(t), y(t), z(t))$. The derivative of ϕ in these coordinates is given by

$$\begin{pmatrix} \partial\lambda/\partial t & \partial\lambda/\partial\lambda \\ \partial(u_x + u_y\lambda)/\partial t & \partial(u_x + u_y\lambda)/\partial\lambda \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & u_y \end{pmatrix}$$

which has zero determinant, as claimed.

Now we will show that, for an arbitrary surface, the map ϕ is ramified at a point (P, L) in Z with P a smooth and ordinary point of X if L is a flecnodal line at P. We again choose coordinates x, y, z as above. Since P is assumed ordinary, there are two asymptotic lines through P on the plane $z = 0$ and we can make a linear change on the coordinates x, y so that these lines are $x = 0$ and $y = 0$. We further assume that the flecnodal line L is given by $y = 0$. In terms of u this means that $u = xy + ax^2y + bxy^2 + cy^3 + \cdots$. Therefore

$$u_{xx} = 2ay + \cdots$$

$$u_{xy} = 1 + 2ax + 2by + \cdots$$

$$u_{yy} = 2bx + 6cy + \cdots.$$

It follows that, if λ and μ as above give the asymptotic line at points near P which are close to L, then λ and μ vanish at P and from the above they have local expansions

$$\lambda = -ay + \cdots$$

$$\mu = y + \cdots.$$

Thus, The derivative of ϕ in the local coordinates x, y is given by

$$\begin{pmatrix} \lambda_x & \lambda_y \\ \mu_x & \mu_y \end{pmatrix}$$

and, at $P = (0, 0)$, this gives

$$\begin{pmatrix} 0 & -a \\ 0 & 1 \end{pmatrix}$$

and this shows not only that ϕ is ramified at (P, L) but also that the kernel of the derivative of ϕ is L regarded as a line in $T_P Z$ which is isomorphic to $T_P X$ under the projection $Z \to X$.

It follows that, if all points of X are flecnodal and Z is irreducible, then ϕ is ramified everywhere, so it is inseparable, since it was proved finite above. It follows that p divides the degree of ϕ, namely $d(d-1)(d-2)$, contradicting $1 < d < p$.

Remark: The conclusion of the theorem still holds, with essentially the same proof, if X is defined over some non-algebraically closed subfield of k and Z is assumed to be merely irreducible over this subfield. This will be useful in the sequel.

Proposition 1. *If X is a split smooth surface of degree d with $2 < d < p$, then not all points are flecnodal.*

Proof: We proceed as in the proof of the previous theorem. If all points of X are flecnodal then it follows that ϕ is inseparable on a component Z_1, say, of Z. Since Z is a double cover of X, we must have Z_1 birational to X and we identify (an open set of) Z_1 to (an open set of) X in what follows. If all points of X are parabolic then p divides $d(d-1)$ which is impossible for $d < p$. So the generic point of X is ordinary and we obtain, from the proof of theorem 1, that the derivative of ϕ has rank one at the the generic point. A section (on an open set) of the kernel of the derivative of ϕ gives us a vector field δ, which induces a derivation, also denoted by δ on the function field of Z_1. Therefore, ϕ factors through an inseparable map $Z_1 \to Y$, for some surface Y and this map is induced by the derivation δ. As the derivative of ϕ has rank one at the the generic point, the function field of Y is not contained in the p-th powers of the functions on Z_1, so we can take a function w on Z_1 (coming from a function on Y) which is not a p-th power but satisfies $\delta w = 0$. The level curves $w = c, c \in k$ of w are tangent to the vector field corresponding to δ. On the other hand, this vector field is contained in the kernel of the derivative of ϕ, by construction. Thus, this vector field is in the direction of the flecnodal lines. This implies that the tangents to the level curves of w are flecnodal lines.

We now prove a lemma. For the definition of the order sequence $\varepsilon_0, \varepsilon_1, \ldots$ of a curve see [SV] section 1.

Lemma 1. *Assume $p > 3$. If a curve C on a smooth surface X is such that the tangent line to C at a generic point is a flecnodal line to X at the same point, then either C is contained in the parabolic locus of X or the second order of $C \subset \mathbf{P}^3$ satisfies $\varepsilon_2 > 2$.*

Proof: We can choose a general point on C and affine coordinates such that the point is given by $(0,0,0)$ and, around this point, X is given by $z = u(x,y)$ for some power series u and C is given by $y = g(x), z = u(x, g(x))$ for some power series g. The condition that the tangent line to C at a generic point is a flecnodal line to X at the same point translates into the following conditions, where $\lambda = g'$:

$$u_{xx} + 2u_{xy}\lambda + u_{yy}\lambda^2 = 0$$
$$u_{xxx} + 3u_{xxy}\lambda + 3u_{xyy}\lambda^2 + u_{yyy}\lambda^3 = 0.$$

Differentiating the first equation gives

$$u_{xxx} + 3u_{xxy}\lambda + 3u_{xyy}\lambda^2 + u_{yyy}\lambda^3 + 2(u_{xy} + u_{yy}\lambda)\lambda' = 0.$$

In view of the second equation, this simplifies to $(u_{xy} + u_{yy}\lambda)\lambda' = 0$. So, either $u_{xy}+u_{yy}\lambda = 0$, which means that λ is a double root of the equation $u_{xx}+2u_{xy}\lambda+u_{yy}\lambda^2 = 0$ and thus C is contained in the parabolic locus of X, or we have $g'' = \lambda' = 0$. If the second option holds, we also have $u(x, g(x))'' = u_{xx} + 2u_{xy}\lambda + u_{yy}\lambda^2 + u_y\lambda' = 0$ proving that $\varepsilon_2 > 2$.

Before returning to the proof of the proposition we remark that the above argument works in characteristic zero and, under this condition, the fact that $\varepsilon_2 > 2$ implies that C is a line. From this it follows the classical result that a surface in characteristic zero all whose points are flecnodal is ruled.

Going back to the proof of the proposition, we may assume that not all level curves of w are parabolic, for otherwise all points of X would be parabolic and this is impossible for $d < p$. We apply the lemma to a generic level curve C of w. Then, from corollary 1.9

of [SV] we deduce that $\varepsilon_2 \geq p$. We proceed to show that the intersection multiplicity of X and $T_P C$ at a general point of C is at least p.

Using the notation of the lemma, we can adjust the coordinates such that $g(x) = x^{\varepsilon_2} + \cdots$ and $u(x, g(x)) = x^{\varepsilon_3} + \cdots$ and this immediately implies that $\partial^j u/\partial x^j(0,0) = 0, j < \varepsilon_2$ which shows that the intersection multiplicity of X and the line $y = z = 0$ is at least ε_2, as was to be shown.

Since we assumed that $d < p$ and proved that the intersection multiplicity of X and $T_P C$ is at least p, we conclude that $T_P C$ is contained in X for all P in C. If $T_P C$ varies with P we conclude that X is ruled, which is impossible since X is smoooth of degree at least three. If $T_P C$ is independent of P then C is a line and, as C was a generic level curve of a non-contant function w, again we obtain that X is ruled, which is impossible. This completes the proof.

Corollary 1. *If X is a smooth surface in $\mathbf{P}^3$ of degree d with $2 < d < p$, then X contains at most $d(11d - 24)$ lines.*

Proof: It is clear that a line contained in X is automatically a subset of the flecnodal locus. Under the hypotheses of the corollary, the flecnodal locus is a curve of degree $d(11d - 24)$. The result follows.

2. The main result

Theorem 2. *Let X be a smooth surface in $\mathbf{P}^3$ of degree d, defined over $\mathbf{F}_q$ with q prime and assume that $2 < d < q$. Let m be the number of lines contained in X. Then $\#X(\mathbf{F}_q) \leq d(d+q-1)(d+2q-2)/6 + m(q+1)$. In particular, $\#X(\mathbf{F}_q) \leq d(d+q-1)(d+2q-2)/6 + d(11d-24)(q+1)$.*

Proof: We will count the number of points P of X in an algebraic closure of $\mathbf{F}_q$ whose image under the Frobenius map lies in one of the asymptotic lines to X at P. This set of points consists of the points $(x_0 : x_1 : x_2 : x_3)$ which satisfy $f = \sum f_{x_i} x_i^q = \sum f_{x_i x_j} x_i^q x_j^q = 0$. Consider the scheme S given by the above equations. Clearly $S \subset X$. By theorem 1 and proposition 1, since we assumed $2 < d < q$, not all points of X are flecnodal. We will show that S is not the whole of X and that, outside of the lines contained in X, the rational points of X are isolated points of S with multiplicity at least 6, with equality outside of the flecnodal curve.

Since S has degree $d(d+q-1)(d+2q-2)$ and a line has at most $q+1$ rational points, the result will follow.

Consider then a rational point of X. We choose affine coordinates such that the point is the origin $(0,0,0)$, that the tangent plane is $z = 0$ and that the affine equation for X is $g(x,y,z) = 0$. We write $g = z + g_2 + g_3 + \cdots$, where g_i is homogeneous of degree i. Since $q \geq 4$ we get that, up to third order terms, the equations cutting down S around $(0,0,0)$ are g,

$$(x^q - x)g_x + (y^q - y)g_y + (z^q - z)g_z = -(xg_x + yg_y + zg_z) + \cdots = -z - 2g_2 - 3g_3 + \cdots$$

and

$$-2(d-1)(xg_x + yg_y + zg_z) + x^2 g_{xx} + y^2 g_{yy} + z^2 g_{zz} + 2xyg_{xy} + 2xzg_{xz} + 2yzg_{yz} + \cdots =$$
$$-2(d-1)z - 2(2d-3)g_2 - 3(2d-4)g_3 + \cdots,$$

which is locally like the intersection of $g_2 = 0$ and $g_3 = 0$ on $z = 0$ and therefore intersect at the origin with multiplicity at least six. If the point is non-flecnodal, $g_2 = 0$ and $g_3 = 0$ on $z = 0$ have no common factor and therefore intersect at the origin with multiplicity exactly six.

We will now show that, if a rational point belongs to a curve C inside S then this curve must be a line contained in X. Suppose that the point is the origin of a coordinate system and that locally, X is given by $z = u(x,y)$ for some power series u and C is given by $y = g(x), z = u(x, g(x))$ for some power series g. Now, S is cut out by the equations $z = u(x,y)$, $u^q - u = u_x(x^q - x) + u_y(y^q - y)$ and $u_{xx}(x^q - x)^2 + 2u_{xy}(x^q - x)(y^q - y) + u_{yy}(y^q - y)^2 = 0$. If we assume that C is contained in S then, differentiating the second equation with respect to x, we get $u_{xx}(x^q - x) + u_{xy}((x^q - x)g' + (y^q - y)) + u_{yy}(y^q - y)g' = 0$, solving for g'

and using the third equation yieds $g' = (y^q - y)/(x^q - x)$. Recalling that $y = g(x)$ along C this last equation gives $g'' = 0$. It also follows then that $u(x, g(x))'' = 0$. Now we can conclude that C is a line exactly as in the end of the proof of proposition 1.

Remark: It is possible to improve the bound in the theorem somewhat using excess intersection formulae.

3. Other bounds, consequences and examples

In the case of smooth surfaces in $\mathbf{P}^3$ of degree d, Deligne's theorem [D] tells us that, since we have $b_1 = b_3 = 0$ and $b_2 = (d^3 - 4d^2 + 6d - 2)$ for the Betti numbers, $\#X(\mathbf{F}_q) \leq q^2+1+(d^3-4d^2+6d-2)q$. An elementary bound, easy to prove is $\#X(\mathbf{F}_q) \leq d(q^2+q+1)$. This can be improved slightly:

Proposition 2. *Let X be a smooth surface $\mathbf{P}^3$ of degree d, then $\#X(\mathbf{F}_q) \leq (d-1)(q+1)^2 + q + 1$. If, moreover X does not contain a line defined over $\mathbf{F}_q$, then $\#X(\mathbf{F}_q) \leq (d-1)q^2 + (d-2)(q+1) + 1$.*

Proof: If $X(\mathbf{F}_q)$ is empty, there is nothing to prove. Otherwise, let $P \in X(\mathbf{F}_q)$ and consider lines through P. Assume first that there is no line through P defined over $\mathbf{F}_q$ contained in X. Of the q^2+q+1 lines through P defined over $\mathbf{F}_q$, those not contained in T_PX meet X in at most $d-1$ points besides P and those contained in T_PX meet X in at most $d-2$ points besides P. The inequalities follow in this case. If X contains a line L, consider instead the $q+1$ planes defined over $\mathbf{F}_q$ containing L. Each such plane meets X in a plane curve of degree $d-1$ besides L. This curve has at most $(d-1)(q+1)$ points defined over $\mathbf{F}_q$ (by an easier version of the same argument). Adding up the contribution of all planes together with the $q+1$ points of L gives the bound.

Lachaud and Tsfasman have some asymptotic results ([LT],[T]) based on explicit formulae which do not compare with our results in the range that our bounds are interesting.

Of course, in our setup $\#X(\mathbf{F}_q) \leq \#\mathbf{P}^3(\mathbf{F}_q) = q^3 + q^2 + q + 1$ and this cannot be improved when $d > q$ (see below for an example). It can be improved however for $d \leq q$ as the previous proposition shows. Our bound only beats these two elementary bounds for $d < 0.06q$ approximately. Our bound is better than Deligne's if $d > \sqrt{q/3}$, approximately.

A consequence of our bound is also that, if the number of points of $X(\mathbf{F}_q)$ is sufficiently large then X must contain lines. One way in which the number of points of $X(\mathbf{F}_q)$ is large is if many of the eigenvalues of the Frobenius acting on $H^2(X, \mathbf{Q}_l)$ are equal to q. This is in agreement with the Tate conjecture for surfaces over finite fields. This conjecture states that the rank of the Néron-Severi group of a surface $X/\mathbf{F}_q$ coincides with the multiplicity of q as an eigenvalue of the Frobenius acting on $H^2(X, \mathbf{Q}_l)$.

Our first example is the surface $x_1^q x_3 - x_1 x_3^q + x_0^q x_2 - x_0 x_2^q$ (see [R]), it is smooth of degree $q+1$ and has q^3+q^2+q+1 points over $\mathbf{F}_q$, i.e., it passes through every rational point in $\mathbf{P}^3$. That shows that one only expects to improve on the bound q^3+q^2+q+1 for $d \leq q$. Also this surface consists entirely of planar points.

Consider the Fermat surfaces $x_0^d + x_1^d = x_2^d + x_3^d$, over $\mathbf{F}_q$ with $d|(q-1)$. Then the points with $x_0^d = x_2^d, x_1^d = x_3^d$ give $(q-1)d^2$ points with $x_0 = 1$ say. Another $(q-1)d^2$ points come from $x_0^d = x_3^d, x_1^d = x_2^d$ and a further $(q-1)d^2$ points come from $x_0^d + x_1^d = x_2^d + x_3^d = 0$ if $(q-1)/d$ is even. Some of these points have been counted twice but the end result is about $2(q-1)d^2$ or $3(q-1)d^2$ according to whether $(q-1)/d$ is odd or even. Note that all these points are on lines contained in the surface. Sometimes there are more points. For a numerical example take $d = 48$ and $q = 1009$. Our bound is 41879040, the bound from proposition 2 is 47945710 and Deligne's bound is a whopping 103595040. The actual number of points is a mere 6303792. Note that the Hessian of a Fermat surface is $d(d-1)(x_0x_1x_2x_3)^{d-2}$ which is a square when d is even, so the surface is split. These surfaces also provide examples of pathological geometric behaviour. For instance, for $d = p+1$ all points are planar and when $d = p+2$ the generic point is ordinary and flecnodal.

The Schur quartic has 64 lines and is given by $x_0^4 + x_0 x_1^3 = x_2^4 + x_2 x_3^3$ and 64 is the maximum number of lines on a smooth quartic surface in characteristic zero ([Se]). It might be possible to adapt Segre's proof to characteristic $p \geq 5$ using the results of section

1. The Hessian is $81x_0^4x_2^4$ is a square so the surface is split.

The sextic $x_0x_1(x_0^4 - x_1^4) = x_2x_3(x_2^4 - x_3^4)$ contains 180 lines but it is not split. (This example is from [E]).

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