Group actions on boundaries of convex divisible domains

Theodore Weisman
University of Texas - Austin

November 2, 2019

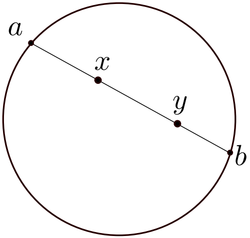
Definition

Closed convex projective manifold: quotient $M = \Omega/\Gamma$, where:

- $\Omega \subset \mathbb{R}P^d$ is open and properly convex ("convex domain")
- $\Gamma \subset \mathrm{PGL}(d+1, \mathbb{R})$ is discrete and torsionfree
- The quotient Ω/Γ is compact

We say Γ *divides* Ω.
Koszul $\implies$ deformations of Γ in $\mathrm{PGL}(d+1)$ are still discrete, and give a family of real projective structures on M.

$\mathbb{H}^2 \subset \mathbb{R}P^2$



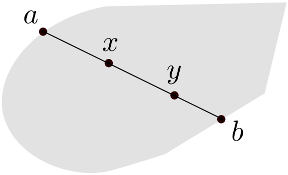
$$d_{\mathbb{H}}(x, y) = \frac{1}{2} \log[a, x; y, b]$$

Question: If a subgroup $\Gamma \subset \mathrm{PGL}(d+1)$ preserves a convex domain $\Omega \subset \mathbb{R}P^d$, when does it divide Ω?

Theorem (Benoist, 2004)

Let Γ divide Ω. Then Ω is strictly convex $\iff$ Γ is Gromov-hyperbolic.

Define: *Hilbert metric* on Ω, so that Γ acts by isometries.

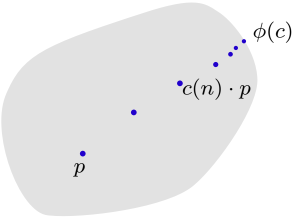


$$d_\Omega(x, y) = \log[a, x; y, b]$$

Ω divisible and strictly convex $\implies$ d_Ω is a hyperbolic metric space.

When a Gromov-hyperbolic group Γ divides Ω, get *equivariant homeomorphism* $\phi : \partial\Gamma \to \partial\Omega$.

$c \in \partial\Gamma$
$c : \mathbb{N} \to \Gamma$

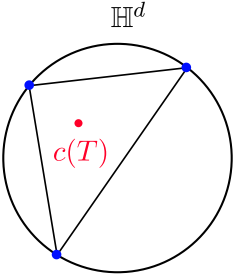


Theorem (Benoist)

Let Ω be a strictly convex domain with C^1 boundary, and $\Gamma \subset \mathrm{PGL}(d+1)$ preserve Ω.
If there exists an equivariant homeomorphism $\partial\Gamma \to \partial\Omega$, then Γ divides Ω.

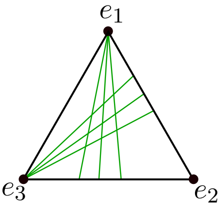
Any Gromov hyperbolic Γ acts properly discontinuously and cocompactly on *triples*:

$$\partial\Gamma^{(3)} = \{(x, y, z) \in \partial\Gamma^3 : x, y, z \text{ distinct}\}$$



In $\mathbb{H}^d$ case, there's an equivariant projection map $c : \partial\Omega^{(3)} \to \Omega$.

When does Γ divide Ω when Ω is not strictly convex?



$$\{\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 : \lambda_i > 0\}$$

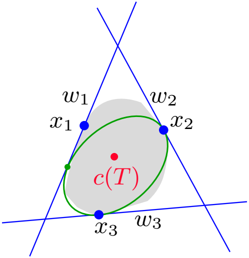
$$\mathbb{Z}^2 \simeq \left\{ \begin{pmatrix} 2^a & & \\ & 2^b & \\ & & 1 \end{pmatrix} : a, b \in \mathbb{Z} \right\}$$

Γ does not in general act properly discontinuously on triples in $\partial\Omega$!

Definition

$\partial_{\mathcal{F}}\Omega = \{(x, w) : x \in \partial\Omega, w \text{ supporting hyperplane of } \Omega \text{ at } x\}$

Consider *transverse triples* $T = \{(x_i, w_i)\}_{i=1,2,3}$ in $\partial_{\mathcal{F}}\Omega$:



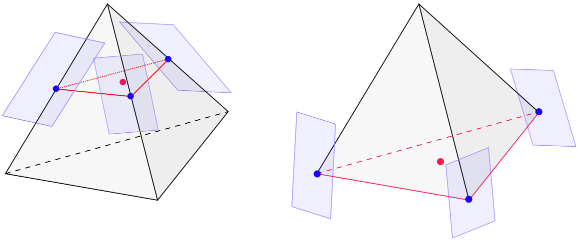
Projective invariants of T:

- Triple ratio: pick lifts for $x_i \in \mathbb{R}P^d, w_j \in (\mathbb{R}P^d)^*$.

$$\rho(T) = \frac{w_1(x_2)w_2(x_3)w_3(x_1)}{w_1(x_3)w_2(x_1)w_3(x_2)}$$

- Center of T: $c(T)$ = unique point in span of x_i's left invariant by projective transformations permuting the flags of T.

Triples in $\partial_{\mathcal{F}}\Omega$:



Theorem (W)

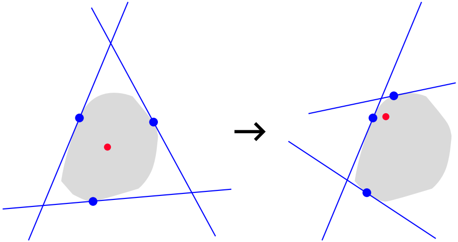
Let $\Gamma \subset \mathrm{PGL}(d+1)$ *preserve a convex domain* Ω. *Then* Γ *divides* Ω *if and only if for all sufficiently large* $k \in \mathbb{R}^+$, Γ *acts properly discontinuously and cocompactly on the space of triples* T *in* Γ *satisfying* $1/k < \rho(T) < k$ *and* $c(T) \in \Omega$.

Proof sketch

Show that the map

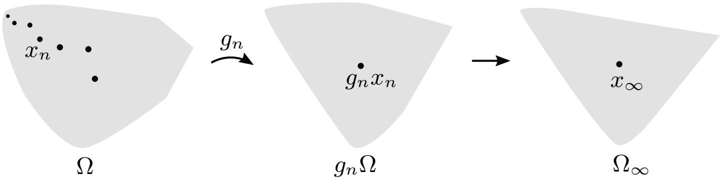
$$c : \{\text{triangles with triple ratio } \leq k\} \to \Omega$$

is *proper* and *quasisurjective* (with respect to Hilbert metric). Properness: a degenerating sequence of transverse triples in $\partial\Omega$ has center converging to $\partial\Omega$.



Proof sketch (continued)

Suppose not quasisurjective: there's $x_n \in \Omega$ far away from the center of every triangle with triple ratio k.
We can find group elements $g_n \in \mathrm{PGL}(d+1)$ such that $g_n x_n \to x_\infty$ and $g_n \Omega \to \Omega_\infty$.
(Benzecri compactness theorem)



Must be a transverse triple in $\partial\Omega_\infty$, whose center is *finitely far* from $x_\infty \implies$ contradiction.

Further questions

- Does a similar criterion imply a notion of *convex cocompactness* in convex projective domains?
- What abstract conditions on the group Γ does this action on the boundary imply?