Splitting of Log Gromov-Witten Invariants with Toric Gluing Strata

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3CinG Workshop

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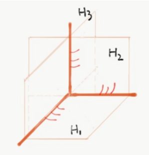
A Degeneration Example[1]

- Smooth cubic surface in $\mathbb{P}^3$ defined by $f_3 + x_1x_2x_3 = 0$.
- Consider $Y' \to \mathbb{A}^1$ defined by $t \cdot f_3 + x_1x_2x_3 = 0$.
- Blow up the 9 ordinary double points. The central fiber $X_0 = H_1 \cup H_2 \cup H_3$ normal crossing.
- Divisorial log structure from X_0.
- Y is log smooth over $\mathbb{A}^1$.



Logarithmic Gromov-Witten invariants are deformation invariant.

Types of the map that contribute to the Gromov-Witten count on the central fiber are identified by the rigid tropical curves.

Question: How to relate the Log Gromov-Witten invariants on H_1, H_2 and H_3 to the Log Gromov-Witten invariants on X_0? Thus, the Gromov-Witten invariants on the general fiber.

[1] [Abramovich-Chen-Gross-Siebert,2017], Decomposition of Degenerate Gromov-Witten Invariants

A Brief History

- Relative Gromov-Witten Invariants and expanded degeneration
  [Li-Ruan, 2001], [Li, 2001, 2002], [Ionel-Parker, 2003,2004]

- Logarithmic expansions for smooth divisor
  [Kim, 2010]

- Orbifold expansions for smooth divisor
  [Abramovich-Fantechi, 2011]

- Logarithmic expansions in the normal crossing setting
  [Ranganathan, 2019]

- Punctured Log Gromov-Witten Invariants
  [Abramovich-Chen-Gross-Siebert, 2020]

Punctured Invariants

Consider a stable map to $X_0 = H_1 \cup H_2 \cup H_3$, with the curve having four components. C_1, C_2, C_3 are mapped to H_1, H_2, H_3 and C_4 is contracted to a point. There is a marked point for each of C_1 and C_2.

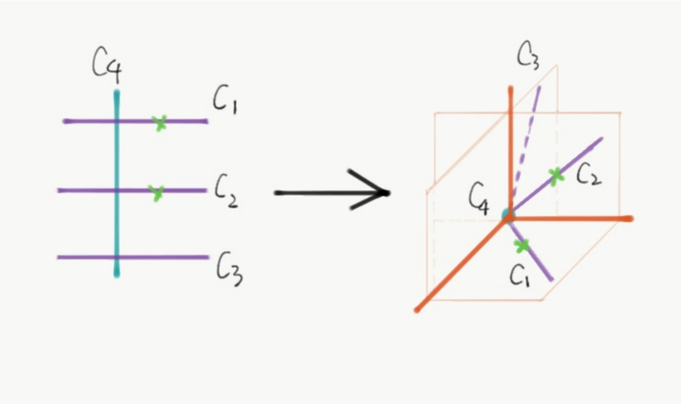
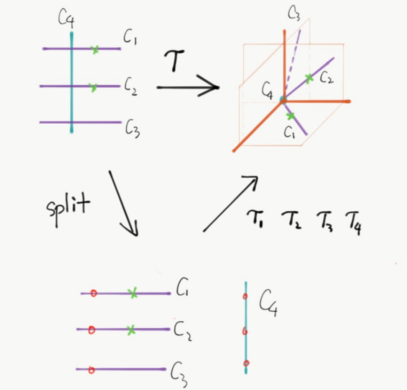


Figure: A stable map to X_0

Punctured Invariants



- Splitting along the nodes, we obtain punctured curves. Each node become a punctured point for the component that contains it, with contact orders the opposite direction.
- Use type $\boldsymbol{\tau}$ to pack the discrete data of the map, including the dual graph of the curve with genus, contact order of each puncutres and nodal point, the image strata and the image class.
- We obtain smaller types $\boldsymbol{\tau}_1, ..., \boldsymbol{\tau}_r$ after splitting.

Punctured Invariants

- A punctured map marked by a type $\boldsymbol{\tau}$ lies in the closure of the open locus of maps with type $\boldsymbol{\tau}$.

Theorem (Abramovich-Chen-Gross-Siebert)

The moduli space $\mathscr{M}(X, \boldsymbol{\tau})$ of basic stable puncutred log maps marked by $\boldsymbol{\tau}$ is a proper Deligne-Mumford stack.

There is a natural finite, representable splitting morphism

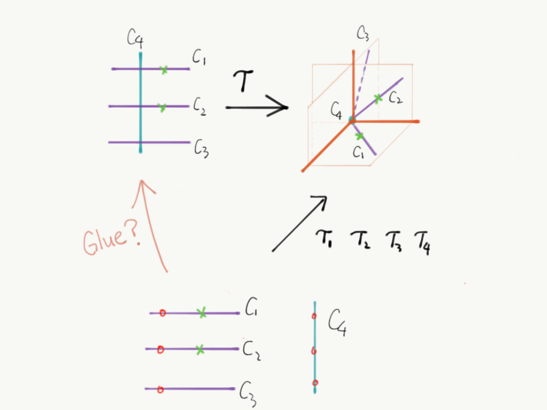
$$\delta : \mathscr{M}(X, \boldsymbol{\tau}) \to \prod_{i=1}^{r} \mathscr{M}(X, \boldsymbol{\tau}_i).$$

Punctured Invariants

How about the other direction?

In this example, we are gluing along a log point $(\star, \mathbb{N}^3)$.

But it is already very nontrivial.



Gluing Process

Logarithmic gluing is more than schematic gluing.

Example: Glue along a point

- $$\begin{tikzcd} A \arrow[r] \arrow[d] & (\star, \mathbb{N}) \arrow[d, "f"] \\ (\star, \mathbb{N}) \arrow[r, "f"] & (\star, \mathbb{N}) \end{tikzcd}$$
  with f sends $1 \mapsto 2$ on monoid $\mathbb{N} \to \mathbb{N}$
- Fine, saturated log fiber product $A = (\star, \mathbb{N}) \sqcup (\star, \mathbb{N})$.
- One point is mapped $t \to (t, t)$. The other one is mapped $t \to (t, -t)$.

- $$\begin{tikzcd} A' \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d, "f'"] \\ \mathbb{A}^1 \arrow[r, "f'"] & \mathbb{A}^1 \end{tikzcd}$$ with $f' : t \mapsto t^2$
- Schematic fiber product is $\operatorname{Spec} \Bbbk[x, y]/(x^2 = y^2)$.
- The intersection point is a double point!
- Log fiber product normalizes the double point.

Gluing Process

On the other hand, logarithmic gluing is closely related to gluing the tropical spaces.

Lemma (Molcho)

The fine, saturated logarithmic fiber product of toric varieties $X \times_Y Z$ is a disjoint union of m toric varieties. Each of them is isomorphic to the toric variety associated to the fan $\Sigma(X) \times_{\Sigma(Y)} \Sigma(Z)$.

- In the case we are interested in, the multiplicity m is a lattice index.
- Virtual gluing of punctured invariants admits a toric local model!

Virtual Gluing

- The Artin fan $\mathcal{X}$ of X is an algebraic object connecting the geometric map with the tropical picture.
- $\mathrm{Hom}(C, \mathcal{X}) \to \mathrm{Hom}(\Sigma(C), \Sigma(X))$ is a bijection.
- The moduli space of maps $\mathfrak{M}(\mathcal{X}, \boldsymbol{\tau})$ is idealized log smooth.
- In order to glue schematically, we need to enhance $\mathfrak{M}(\mathcal{X}, \boldsymbol{\tau})$ with the scheme structure

$$\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}) := \mathfrak{M}(\mathcal{X}, \boldsymbol{\tau}) \times_{\underline{\mathcal{X}}} \underline{X}.$$

Theorem (Abramovich-Chen-Gross-Siebert)

There is a Cartesian diagram

$$\begin{array}{ccc} \mathscr{M}(X, \boldsymbol{\tau}) & \xrightarrow{\delta} & \prod_{i=1}^{r} \mathscr{M}(X, \boldsymbol{\tau}_i) \\ \downarrow{\hat{\varepsilon}} & & \downarrow{\varepsilon} \\ \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}) & \xrightarrow{\delta'} & \prod_{i=1}^{r} \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}_i), \end{array}$$

where the horizontal maps are proper, representable and the vertical maps admit perfect obstruction theories. For $\alpha \in A_*(\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}))$

$$\varepsilon^! \delta'_*(\alpha) = \delta_* \hat{\varepsilon}^!(\alpha).$$

Gluing Formalism

In order to glue logarithmically, a further enhancement of $\mathfrak{M}^{\text{ev}}(\mathcal{X}, \boldsymbol{\tau})$ is needed to allow a logarithmic evaluation map to X.

Theorem (Abramovich-Chen-Gross-Siebert)

There is a Cartesian diagram in the category of fine, saturated log stacks

$$\begin{tikzcd} \widetilde{\mathfrak{M}'}^{\text{ev}}(\mathcal{X}, \boldsymbol{\tau}) \arrow[r, "\delta^{\text{ev}}"] \arrow[d, "\text{ev}_E"] & \prod_{i=1}^{r} \widetilde{\mathfrak{M}'}^{\text{ev}}(\mathcal{X}, \boldsymbol{\tau}_i) \arrow[d, "\text{ev}_{\mathbf{L}}"] \\ \prod_E X \arrow[r, "\Delta"] & \prod_E X \times X, \end{tikzcd}$$

where horizontal maps are proper, representable.

Theorem (Abramovich-Chen-Gross-Siebert)

There is an isomorphism of the underlying stack $\underline{\widetilde{\mathfrak{M}'}}^{\text{ev}}(\mathcal{X}, \boldsymbol{\tau})_{\text{red}} \to \underline{\mathfrak{M}}^{\text{ev}}(\mathcal{X}, \boldsymbol{\tau})$.

Local Model of Gluing Formalism

Lemma (W.)

The evaluation maps $\mathrm{ev}_E : \widetilde{\mathfrak{M}'}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}) \to X$ *is idealized log smooth.*

Idealized Log smooth $\approx$ Locally a closed immersion of toric strata + toric morphism

Toric Morphism: Tropical Evaluation Maps

- The basic/minimal log structure is the universal log structure of the type. Dually, the basic cone $\overline{\tau}$ is the universal family of the tropical maps to $\Sigma(X)$ of type $\boldsymbol{\tau}$.
- The evaluation cone $\widetilde{\tau}$ is the universal family of tropical maps with a marking on each evaluation edge and leg.
- Evaluate at markings provide the tropical evaluation map $\mathrm{ev} : \widetilde{\tau} \to \Sigma(X)$.

Closed Immersion of Toric Strata: An ideal structure comes from $\boldsymbol{\tau}$-marking, puncturing and the evaluation marking.

Local Model of Gluing Formalism

Example $X_0 = H_1 \cup H_2 \cup H_3$

- Tropical space $\Sigma(X_0) = \mathbb{R}^3_{\geq 0}$.
- Basic cone $\overline{\tau}$ for $\boldsymbol{\tau}$ is $\mathbb{R}_{\geq 0}$, the ray generated by the image of central vertex v_4 in $\Sigma(X)$.
- Evaluation cone $\widetilde{\tau}$ for $\boldsymbol{\tau}$ is $\{(v_4, l_1, l_2, l_3) \in R^4_{\geq 0} | v_4 + l_i \cdot u_i \in \Sigma(X_0)\}$.
- The tropical evaluation map $\Sigma(\mathrm{ev}_i) : (v_4, l_1, l_2, l_3) \mapsto v_4 + l_i \cdot u_i$.

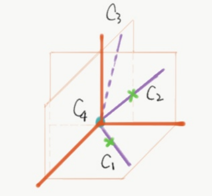


Figure: $C \to X_0$

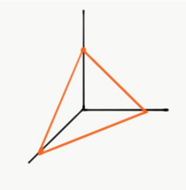


Figure: $\Sigma(X_0)$

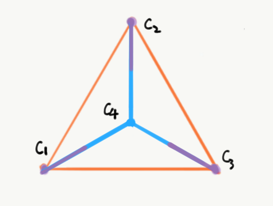


Figure: Tropical Map

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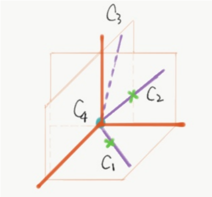

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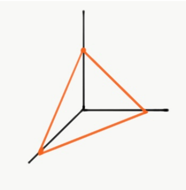

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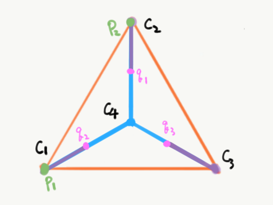


Figure: Tropical map with evaluation marking

Fulton-Sturmfels Formula

- Let X be a toric variety with fan (Σ_X, N_X) with $\delta : X \to X \times X$.
- A vector in $N_X \times N_X$ defines a $\mathbb{G}_m$-action on $X \times X$ from the one parameter group $v : \mathbb{G}_m \to T_X \times T_X$.
- Geometrically, if we take the limit $t \to 0$, we get a torus invariant subscheme that is rationally equivalent to $\delta(X)$.
- Combinatorially, if we displace $\delta_N(N_X)$ by vector v, we can read off the cones $\boldsymbol{\rho}$ that are the components of the subscheme directly.

Lemma(Fulton-Sturmfels)

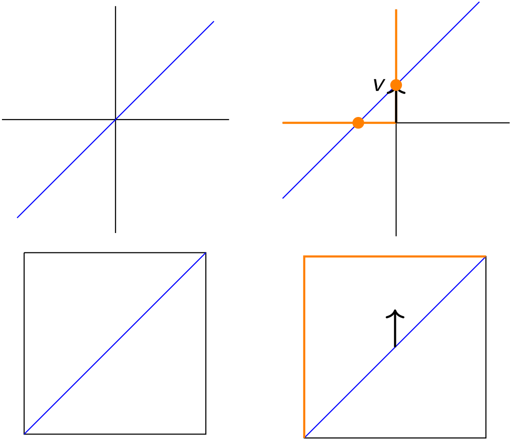
In the Chow group $A_*(X \times X)$,

$$\delta_*[X] = \sum_{\boldsymbol{\rho}} m_{\boldsymbol{\rho}} \cdot [V_X(\rho_1)] \times [V_X(\rho_2)],$$

with $m_{\boldsymbol{\rho}} = [N_X : \mathrm{coker}(\delta_N)(N_{\rho_1} \times N_{\rho_2})]$.

Fulton-Sturmfels Formula

Example: $\delta : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$



$$v : \mathbb{G}_m \to T_X \times T_X$$
$$t \mapsto (1, t)$$

$$\delta_*[\mathbb{P}^1] = [\infty \times \mathbb{P}^1] + [\mathbb{P}^1 \times 0]$$

A displacement vector defines a log structure compatible rational deformation of $\delta(X)$ in $X \times X$.

Splitting Formula of Punctured Invariants

Assumption: Assume the gluing strata X_E is isomorphic to a toric strata of a toric variety Z.

$$\begin{array}{ccc}
\widetilde{\mathfrak{M}'}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}) & \xrightarrow{\delta^{\mathrm{ev}}} & \prod_{i=1}^{r} \widetilde{\mathfrak{M}'}^{\mathrm{ev}}(\mathcal{X}, \boldsymbol{\tau}_i) \\
\downarrow \mathrm{ev}_E & & \downarrow \mathrm{ev}_{\mathbf{L}} \\
Z & \xrightarrow{\quad\Delta\quad} & Z \times Z,
\end{array}$$

- A generic displacement vector $v \in N_Z \times N_Z$.
- Types $\boldsymbol{\rho} = (\boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_r)$ go over the limit strata under the $\mathbb{G}_m$-action associated to v.

Theorem (W.)

There is a virtual pushforward

$$\delta_*[\mathscr{M}(X, \boldsymbol{\tau})]^{\mathrm{virt}} = \sum_{\boldsymbol{\rho}} \frac{m_{\boldsymbol{\rho}}}{|\operatorname{Aut} \boldsymbol{\rho}|} \prod_{i=1}^{r} j_{\boldsymbol{\rho}_i *}[\mathscr{M}(X, \boldsymbol{\rho}_i)]^{\mathrm{virt}},$$

with $m_{\boldsymbol{\rho}} = [N_Z : \operatorname{coker}(\delta_N) \circ \operatorname{ev}(N_{\widetilde{\rho}})]$.

Splitting Formula of Punctured Invariants

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$$

- A generic displacement vector $v\in N_Z\times N_Z$.
- Types $\boldsymbol{\rho}=(\boldsymbol{\rho}_1,\ldots,\boldsymbol{\rho}_r)$ go over the limit strata under the $\mathbb{G}_m$-action associated to v.

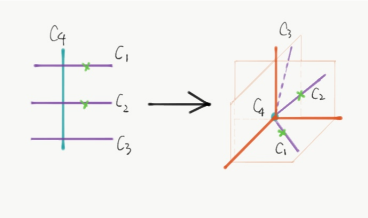
Corollary (Numerical Splitting Formula)

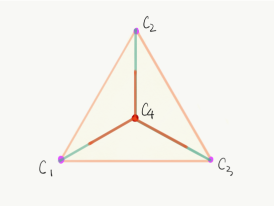
Let $\beta\in H^*(\underline{X}^n)$. Suppose there is a Künneth decomposition $\beta=\prod_{i=1}^{r}\beta_i$. Then

$$\int_{[\mathscr{M}(X,\boldsymbol{\tau})]^{\mathrm{virt}}}\mathrm{ev}_{\boldsymbol{\tau}}^*(\beta)=\sum_{\boldsymbol{\rho}}\frac{m_{\boldsymbol{\rho}}}{|\operatorname{Aut}\boldsymbol{\rho}|}\prod_{i=1}^{r}\int_{[\mathscr{M}(X,\boldsymbol{\rho}_i)]^{\mathrm{virt}}}\mathrm{ev}_{\boldsymbol{\rho}_i}^*(\beta_i),$$

with $\mathrm{ev}_{\boldsymbol{\omega}}:\underline{\mathscr{M}}(X,\boldsymbol{\omega})\to\underline{X}^{n_{\boldsymbol{\omega}}}$ for any type $\boldsymbol{\omega}$.

The Degeneration Example





Displacement along one edge already gives us the types with the expected dimension.

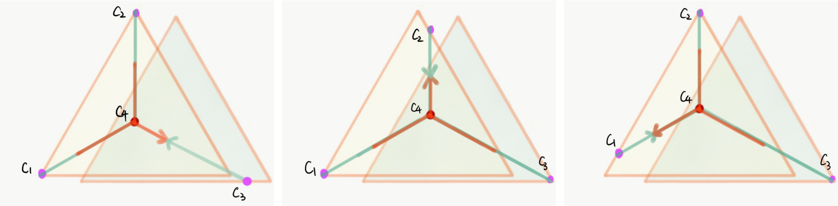
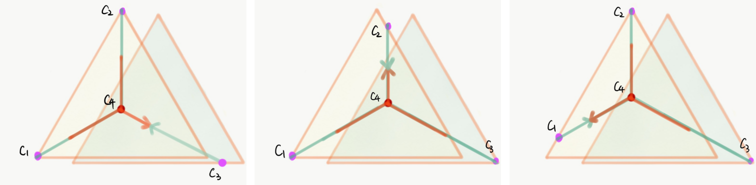


Figure: Three Types after Displacement

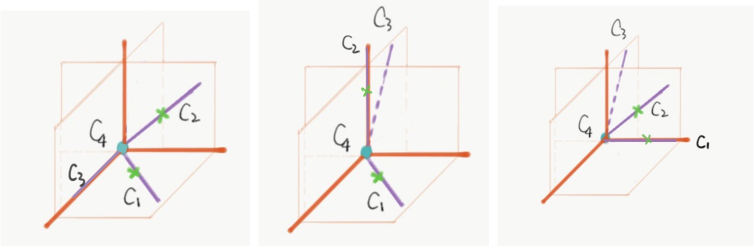
The Degeneration Example

Three cases correspond to the types where one of C_1, C_2, C_3 degenerate to the strict transform of intersection of two planes.

Tropical Types:



Geometric Maps:



- If we require the marked points to pass through two general points, then only the first type contributes to the Gromov-Witten count.

Application

- Mikhalkin/Nishinou-Siebert Correspondence Theorem

Question: Let X be a toric variety and $Z_i \subseteq X$ be the closure of a subtorus of the maximal torus in X. How many n-marked rational curves are there in a toric variety X with the i-th marked points lies in Z_i?

Translation: Consider a type $\boldsymbol{\tau}$ with a graph G containing one genus 0 vertex and n contact order 0 legs. What is the degree of $[\mathscr{M}(X, \boldsymbol{\tau}, \sigma)]^{\mathrm{virt}}$, with $\mathscr{M}(X, \boldsymbol{\tau}, \sigma)$ defined by the fiber diagram

$$\begin{array}{ccc} \mathscr{M}(X, \boldsymbol{\tau}, \sigma) & \longrightarrow & \mathscr{M}(X, \boldsymbol{\tau}) \\ \downarrow & & \downarrow \mathrm{ev} \\ \prod_{i=1}^{n} Z_i & \longrightarrow & X^n ? \end{array}$$

Step 1: A generic displacement vector $v = (P_1, \ldots, P_n) \in N_X^n$ picks a finite collection of tropical types. (The rigid tropical curves!)

Step 2: By splitting along all of the edges for each type, the multiplicity gives the count contributed by the type.

What if the gluing strata is not toric?

Displacement vector v associated to a toric morphism $f : X \to X \times X$ provides a degeneration of $f(X)$ in X. In other words, there is a family of maps over $\mathbb{A}^1$

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "\widetilde{f}"] \arrow[dr, "\pi"'] & X \times X \times \mathbb{A}^1 \arrow[d] \\ & \mathbb{A}^1 \end{tikzcd}$$

with $\widetilde{f}|_{\pi^{-1}(1)} \cong f$.

Assume a generic degeneration exists for the gluing strata, then we are happy!

$$\begin{tikzcd} \mathfrak{M} \arrow[r] \arrow[d, "\mathrm{ev}"] & M \times \mathbb{A}^1 \arrow[d] \\ \mathfrak{X} \arrow[r, "\widetilde{f}"] \arrow[dr, "\pi"'] & X \times X \times \mathbb{A}^1 \arrow[d] \\ & \mathbb{A}^1 \end{tikzcd}$$

Example: Toric stratas, Gluing strata with constant log structure...

Next question: What are the conditions for the generic degeneration to exist in general?

Thank you!