Witt vectors and topological cyclic homology

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$$\begin{array}{ccc} \textbf{Arithmetic geometry} & \longleftrightarrow & \textbf{Homotopy theory} \\ \text{Witt vectors} & & \text{Tate cohomology of } C_n \\ \text{de Rham-Witt complex} & & \text{THH/TR/TC} \end{array}$$

$$
\begin{tikzcd}
 & \begin{array}{c}\text{combinatorics}\\ \text{of cyclic words}\end{array} \arrow[dl] \arrow[dr] & \\
\begin{array}{c}\textbf{Arithmetic geometry}\\ \text{Witt vectors}\\ \text{de Rham-Witt complex}\end{array} \arrow[rr, leftrightarrow] & & \begin{array}{c}\textbf{Homotopy theory}\\ \text{Tate cohomology of } C_n\\ \text{THH/TR/TC}\end{array}
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combinatorics
of cyclic words

Arithmetic geometry
Witt vectors
de Rham-Witt complex

Homotopy theory
Tate cohomology of C_n
THH/TR/TC

Recurring theme

$$\left(\sum X_i\right)^n \quad \text{versus} \quad \sum X_i^n$$

Philosophy: Arithmetic over the sphere

FinSet

Philosophy: Arithmetic over the sphere

$$\begin{tikzcd} \mathrm{FinSet} \arrow[d, "\text{decategorify}"'] \\ \mathbb{N} \end{tikzcd}$$

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$$\begin{tikzcd}
\mathsf{FinSet} \arrow[d, "\text{decategorify}"'] & \\
\mathbb{N} \arrow[r, "\text{group complete}"'] & \mathbb{Z}
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Philosophy: Arithmetic over the sphere

- $\mathrm{FinSet}^{\simeq}$ is a **groupoid:** a 1-truncated **space**

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- $(B\Sigma, \amalg, \times)$ is a "commutative semiring space", just as good as the commutative semiring $(\mathbb{N}, +, \times)$

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- $(B\Sigma, \amalg, \times)$ is a "commutative semiring space", just as good as the commutative semiring $(\mathbb{N}, +, \times)$
- can form the group completion of $B\Sigma$ **inside spaces, without decategorifying.** This gives a **commutative ring spectrum.**

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Slogan

$\mathbb{N}$ knows about **arithmetic.**

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Programme

Revisit arithmetic arguments using combinatorial thinking and techniques of (equivariant) stable homotopy theory.

Frobenius I

$$(x+y)^p \equiv x^p + y^p \bmod p$$

Proof (numerical).

Look at denominator of $\binom{p}{k} = \frac{p!}{k!(p-k)!}$ $\square$

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Corollary (Frobenius)

$$R \xrightarrow{\varphi} R/p$$

Frobenius II

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Proof (combinatorial).

Let $S = \{x, y\}$, $S^{\times p} \circlearrowleft C_p$. Then every orbit except for x^p and y^p is free. $\square$

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Corollary (non-commutative Frobenius)

$$R \xrightarrow{\varphi} R/([R,R]+(p))$$

Tate cohomology

G finite group, M a G-module

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Definition (norm map)

$$N(m) = \sum_{g \in G} g \cdot m$$

defines

$$H_0(G, M) = M_G \xrightarrow{N} M^G = H^0(G, M)$$

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$$\hat{H}^n(G, M) = \begin{cases} H_{-(n+1)}(G, M) & n \leq -2 \\ \ker N & n = -1 \\ \operatorname{coker} N & n = 0 \\ H^n(G, M) & n \geq 1 \end{cases}$$

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Slogan

Tate cohomology kills free orbits; sees only *interesting* fixed points.

Tate cohomology

Example

Tate cohomology of finite cyclic groups is 2-periodic:

$$\hat{H}^*(C_n, M) = \begin{cases} \text{coker } N & * \text{ even} \\ \ker N & * \text{ odd} \end{cases}$$

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$$\begin{aligned}\hat{H}^*(C_n, M) &= \begin{cases} \operatorname{coker} N & * \text{ even} \\ \ker N & * \text{ odd} \end{cases} \\ &= \begin{cases} M/nM & * \text{ even} \\ M[n] & * \text{ odd} \end{cases} \quad \text{if the action on } M \text{ is trivial}\end{aligned}$$

Frobenius III

Theorem (Nikolaus-Scholze)

The functor $T_p(X) = (X^{\otimes p})^{tC_p}$ *is exact:*

$$T_p(X \oplus Y) = T_p(X) \oplus T_p(Y)$$

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$$M = \mathbb{Z} \otimes \{x, y\}, \;\; M^{\otimes p} \circlearrowleft C_p$$

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What does it mean?

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$$\begin{aligned} \mathrm{tr}\left((X+Y)^4\right) = \mathrm{tr}(X^4) + 4\,\mathrm{tr}(X^3Y) + 4\,\mathrm{tr}(X^2Y^2) + 4\,\mathrm{tr}(XY^3) + \mathrm{tr}(Y^4) \\ + 2\,\mathrm{tr}\left((XY)^2\right) \end{aligned}$$

Traces and Tate cohomology

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Remark

If λ_i eigenvals of X, then $\operatorname{tr}(X)^n = (\sum \lambda_i)^n$ and $\operatorname{tr}(X^n) = \sum \lambda_i^n$

Cyclotomic trace

cyclotomic trace: “trace of monodromy”

$$
\begin{tikzcd}
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Made precise by Ayala–Mazel-Gee–Rozenblyum.

Witt vectors

Fix a prime p (omitted from notation). Witt vectors are a functor

$$W\colon \mathsf{CRing} \longrightarrow \mathsf{CRing}$$
$$W_n\colon \mathsf{CRing} \longrightarrow \mathsf{CRing}$$
$$W(A) = \varprojlim_n W_n(A)$$

For example,

$$W_n(\mathbb{F}_p) = \mathbb{Z}/p^n\mathbb{Z}, \quad W(\mathbb{F}_p) = \mathbb{Z}_p$$
$$W(\mathbb{F}_{p^n}) = W(\mathbb{F}_p[\zeta_{p^n-1}]) = \mathbb{Z}_p[\zeta_{p^n-1}]$$

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Excellent blog post on Witt vectors (and much more):
`https://ayoucis.wordpress.com/2017/02/20/the-fontaine-winterberger-theorem-going-full-tilt/`

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Dishonesty box

generalizes positional (digit) notation:

$$\text{think of } (a_0, a_1, a_2, \dots) \in W(A) \quad \text{as} \quad \dots a_2 a_1 a_0 = \sum a_i p^i$$

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How to 'carry' in an arbitrary ring?

Carrying

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for $p=2$, can define

$$W_2(R) = R \times R$$
$$(a_0,a_1)+(b_0,b_1) = (a_0+b_0, a_1+b_1 - a_0 b_0)$$

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in general,

$$(a_0,a_1)+(b_0,b_1)=\left(a_0+b_0, a_1+b_1-\frac{1}{p}\left[(a_0+b_0)^p-a_0^p-b_0^p\right]\right)$$

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e.g. $p = 5$, calculate $17 + 7 = 24$ in $\mathbb{Z}/25$:

$$\begin{array}{r} 32 \\ +12 \\ \hline 44 \end{array} \text{ using ordinary base 5 but } \begin{array}{r} {\scriptstyle 3} \\ 22 \\ +02 \\ \hline 04 \end{array} \text{ using Witt vectors}$$

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Keyword: Teichmüller representatives
$\tau(2 \in \mathbb{F}_5) = 7 \in \mathbb{Z}/25$, $\tau(4 \in \mathbb{F}_5) = 24 \in \mathbb{Z}/25$

de Rham-Witt complex

$X/\mathbb{F}_p$ smooth scheme. The **de Rham-Witt complex** $W\Omega_X^*$ is a lift of Ω_X^* to $\mathbb{Z}_p$; $W\Omega_X^0 = W(\mathcal{O}_X)$. Also truncated versions $W_n\Omega_X^*$.

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Example

$X = \mathbb{G}_m = \mathrm{Spec}(\mathbb{F}_p[T^{\pm 1}])$, then

$$p^k T^{1/p^k} \in W\Omega^0_X, \quad T^{1/p^k}\,\mathrm{dlog}\,T \in W\Omega^1_X$$

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Remark

If $\mathfrak{X}$ is a smooth lift of X to $\mathbb{Z}_p$ and has a lift of Frobenius $\mathfrak{X} \xrightarrow{\varphi} \mathfrak{X}$, then there is a quasi-isomorphism $\Omega^*_{\mathfrak{X}/\mathbb{Z}_p} \xrightarrow{\sim} W\Omega^*_X$.

Topological Restriction homology

Classically:

$$\mathrm{TR}^n = \mathrm{THH}^{C_{p^{n-1}}}, \quad \mathrm{TR} = \varprojlim_n \mathrm{TR}^n$$

Nikolaus-Scholze:

$$\begin{tikzcd}
\mathrm{TR} \arrow[r] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & \cdots \arrow[r] & \mathrm{TR}^3 \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & \mathrm{TR}^2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{THH} \arrow[d, "\varphi"] \\
 & & & & \mathrm{THH}^{hC_p} \arrow[r, "\mathrm{can}"'] \arrow[d, "\varphi"] & \mathrm{THH}^{tC_p} \\
\vdots & & \mathrm{THH}^{hC_{p^2}} \arrow[rr, "\mathrm{can}"'] & & (\mathrm{THH}^{tC_p})^{h(C_{p^2}/C_p)} &
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Witt vectors and TR

For a commutative ring A,

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This suggests one possible definition of **non-commutative Witt vectors:** for a non-commutative ring A, let

$$W_n(A) = \pi_0 \operatorname{TR}^n(A), \quad W(A) = \pi_0 \operatorname{TR}(A).$$

Witt vectors and TR

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Can give a **purely algebraic** formula for this (Hesselholt, Nikolaus). **No longer a ring,** e.g. $W_1(A) = A/[A, A]$.

dRW complex and TR

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$$\pi_* \operatorname{TR}^n(A) = W_n\Omega^*_A \otimes S\{\sigma\}, \quad |\sigma| = 2$$
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Conjectural algebraic formula for this (Kaledin, S???).

Hochschild-Witt complex

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WIP: prove this actually computes TR.