

# Witt vectors and topological cyclic homology

Yuri Sulyma

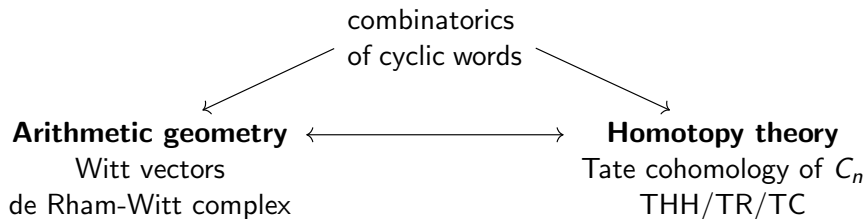
University of Texas at Austin

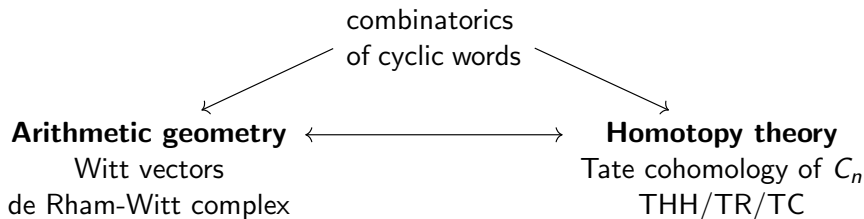
July 11, 2018

**Arithmetic geometry**  
Witt vectors  
de Rham-Witt complex



**Homotopy theory**  
Tate cohomology of  $C_n$   
THH/TR/TC





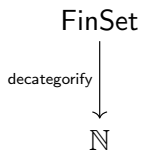
### Recurring theme

$$\left(\sum X_i\right)^n \quad \text{versus} \quad \sum X_i^n$$

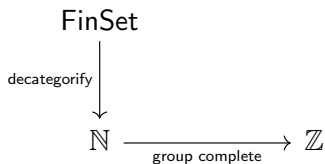
# Philosophy: Arithmetic over the sphere

FinSet

# Philosophy: Arithmetic over the sphere



# Philosophy: Arithmetic over the sphere



## Philosophy: Arithmetic over the sphere

- $\text{FinSet}^{\simeq}$  is a **groupoid**: a 1-truncated **space**

$$\text{FinSet}^{\simeq} = \coprod_{n \geq 0} B\Sigma_n =: B\Sigma$$



## Philosophy: Arithmetic over the sphere

- $\text{FinSet}^{\simeq}$  is a **groupoid**: a 1-truncated **space**

$$\text{FinSet}^{\simeq} = \coprod_{n \geq 0} B\Sigma_n =: B\Sigma$$

- $(B\Sigma, \Pi, \times)$  is a “commutative semiring space”, just as good as the commutative semiring  $(\mathbb{N}, +, \times)$

## Philosophy: Arithmetic over the sphere

- $\text{FinSet}^{\simeq}$  is a **groupoid**: a 1-truncated **space**

$$\text{FinSet}^{\simeq} = \coprod_{n \geq 0} B\Sigma_n =: B\Sigma$$

- $(B\Sigma, \Pi, \times)$  is a “commutative semiring space”, just as good as the commutative semiring  $(\mathbb{N}, +, \times)$
- can form the group completion of  $B\Sigma$  **inside spaces, without decategorifying**. This gives a **commutative ring spectrum**.

# Philosophy: Arithmetic over the sphere

$$\begin{array}{ccc} B\Sigma & & \\ \downarrow \text{deategorify} & & \\ \mathbb{N} & \xrightarrow{\text{group complete}} & \mathbb{Z} \end{array}$$

## Philosophy: Arithmetic over the sphere

$$\begin{array}{ccc} B\Sigma & \xrightarrow{\text{group complete}} & \mathbb{S} \\ \text{deategorify} \downarrow & & \downarrow \text{deategorify} \\ \mathbb{N} & \xrightarrow{\text{group complete}} & \mathbb{Z} \end{array}$$

## Philosophy: Arithmetic over the sphere

$$\begin{array}{ccc} B\Sigma & \xrightarrow{\text{group complete}} & \mathbb{S} \\ \text{deategorify} \downarrow & & \downarrow \text{deategorify} \\ \mathbb{N} & \xrightarrow{\text{group complete}} & \mathbb{Z} \end{array}$$

### Slogan

$\mathbb{N}$  knows about **arithmetic**.

$B\Sigma$  knows about **combinatorics**.

# Philosophy: Arithmetic over the sphere

$$\begin{array}{ccc} B\Sigma & \xrightarrow{\text{group complete}} & \mathbb{S} \\ \text{deategorify} \downarrow & & \downarrow \text{deategorify} \\ \mathbb{N} & \xrightarrow{\text{group complete}} & \mathbb{Z} \end{array}$$

## Slogan

$\mathbb{N}$  knows about **arithmetic**.

$B\Sigma$  knows about **combinatorics**.

## Programme

Revisit arithmetic arguments using combinatorial thinking and techniques of (equivariant) stable homotopy theory.

# Frobenius I

$$(x + y)^p \equiv x^p + y^p \pmod{p}$$

Proof (numerical).

Look at denominator of  $\binom{p}{k} = \frac{p!}{k!(p-k)!}$

# Frobenius I

$$(x + y)^p \equiv x^p + y^p \pmod{p}$$

Proof (numerical).

Look at denominator of  $\binom{p}{k} = \frac{p!}{k!(p-k)!}$  □

Corollary (Frobenius)

$$R \xrightarrow{\varphi} R/p$$



## Frobenius II

$$(x + y)^p \equiv x^p + y^p \pmod{p}$$

Proof (combinatorial).

Let  $S = \{x, y\}$ ,  $S^{\times p} \circlearrowleft C_p$ . Then every orbit except for  $x^p$  and  $y^p$  is free. □

## Frobenius II

$$(x + y)^p \equiv x^p + y^p \pmod{p}$$

Proof (combinatorial).

Let  $S = \{x, y\}$ ,  $S^{\times p} \circlearrowleft C_p$ . Then every orbit except for  $x^p$  and  $y^p$  is free. This reasoning works even if  $x$  and  $y$  don't commute.  $\square$

## Frobenius II

$$(x + y)^p \equiv x^p + y^p \pmod{p}$$

Proof (combinatorial).

Let  $S = \{x, y\}$ ,  $S^{\times p} \circlearrowleft C_p$ . Then every orbit except for  $x^p$  and  $y^p$  is free. This reasoning works even if  $x$  and  $y$  don't commute.  $\square$

Corollary (non-commutative Frobenius)

$$R \xrightarrow{\varphi} R/([R, R] + (p))$$

# Tate cohomology

$G$  finite group,  $M$  a  $G$ -module

# Tate cohomology

$G$  finite group,  $M$  a  $G$ -module

Definition (norm map)

$$N(m) = \sum_{g \in G} g \cdot m$$

defines

$$H_0(G, M) = M_G \xrightarrow{N} M^G = H^0(G, M)$$

# Tate cohomology

$G$  finite group,  $M$  a  $G$ -module

Definition (norm map)

$$N(m) = \sum_{g \in G} g \cdot m$$

defines

$$H_0(G, M) = M_G \xrightarrow{N} M^G = H^0(G, M)$$

Definition (Tate cohomology)

$$M_{hG} \xrightarrow{N} M^{hG}$$

# Tate cohomology

$G$  finite group,  $M$  a  $G$ -module

## Definition (norm map)

$$N(m) = \sum_{g \in G} g \cdot m$$

defines

$$H_0(G, M) = M_G \xrightarrow{N} M^G = H^0(G, M)$$

## Definition (Tate cohomology)

$$M_{hG} \xrightarrow{N} M^{hG} \longrightarrow M^{tG}$$

# Tate cohomology

$G$  finite group,  $M$  a  $G$ -module

## Definition (norm map)

$$N(m) = \sum_{g \in G} g \cdot m$$

defines

$$H_0(G, M) = M_G \xrightarrow{N} M^G = H^0(G, M)$$

## Definition (Tate cohomology)

$$M_{hG} \xrightarrow{N} M^{hG} \longrightarrow M^{tG}$$

$$\hat{H}^*(G, M) = \pi_{-*} M^{tG}$$



# Tate cohomology

$$H_0(G, M) = M_G \xrightarrow{N} M^G = H^0(G, M)$$

$$\hat{H}^n(G, M) = \begin{cases} H_{-(n+1)}(G, M) & n \leq -2 \\ \ker N & n = -1 \\ \operatorname{coker} N & n = 0 \\ H^n(G, M) & n \geq 1 \end{cases}$$

# Tate cohomology

$$H_0(G, M) = M_G \xrightarrow{N} M^G = H^0(G, M)$$

$$\hat{H}^n(G, M) = \begin{cases} H_{-(n+1)}(G, M) & n \leq -2 \\ \ker N & n = -1 \\ \operatorname{coker} N & n = 0 \\ H^n(G, M) & n \geq 1 \end{cases}$$

## Slogan

Tate cohomology kills free orbits; sees only *interesting* fixed points.

# Tate cohomology

## Example

Tate cohomology of finite cyclic groups is 2-periodic:

$$\hat{H}^*(C_n, M) = \begin{cases} \text{coker } N & * \text{ even} \\ \text{ker } N & * \text{ odd} \end{cases}$$

# Tate cohomology

## Example

Tate cohomology of finite cyclic groups is 2-periodic:

$$\hat{H}^*(C_n, M) = \begin{cases} \text{coker } N & * \text{ even} \\ \text{ker } N & * \text{ odd} \end{cases}$$

$$= \begin{cases} M/nM & * \text{ even} \\ M[n] & * \text{ odd} \end{cases} \quad \text{if the action on } M \text{ is trivial}$$

## Frobenius III

### Theorem (Nikolaus-Scholze)

The functor  $T_p(X) = (X^{\otimes p})^{tC_p}$  is exact:

$$T_p(X \oplus Y) = T_p(X) \oplus T_p(Y)$$

## Frobenius III

### Theorem (Nikolaus-Scholze)

The functor  $T_p(X) = (X^{\otimes p})^{tC_p}$  is exact:

$$T_p(X \oplus Y) = T_p(X) \oplus T_p(Y)$$

### Theorem (Kaledin)

$M$  abelian group

$$\hat{H}^0(C_p, M^{\otimes p}) \cong M/p$$

## Frobenius III

### Theorem (Nikolaus-Scholze)

The functor  $T_p(X) = (X^{\otimes p})^{tC_p}$  is exact:

$$T_p(X \oplus Y) = T_p(X) \oplus T_p(Y)$$

### Theorem (Kaledin)

$M$  abelian group

$$\hat{H}^0(C_p, M^{\otimes p}) \cong M/p$$

$$M = \mathbb{Z} \otimes \{x, y\}, \quad M^{\otimes p} \circlearrowleft C_p$$

$$\hat{H}^0(C_p, M^{\otimes p}) = \mathbb{Z}/p \otimes \{x^p, y^p\} \cong M/p$$

What about  $C_{p^n}$ ?



What about  $C_{p^n}$ ?

$$M = \mathbb{Z} \otimes \{x, y\}, \quad M^{\otimes 4} \circlearrowleft C_4$$

What about  $C_{p^n}$ ?

$$M = \mathbb{Z} \otimes \{x, y\}, \quad M^{\otimes 4} \circlearrowleft C_4$$

$$H^0(C_4, M^{\otimes 4}) = \{x^4, y^4, (xy)^2 + (yx)^2, \\
 x^3y + x^2yx + xyx^2 + yx^3, \\
 x^2y^2 + xy^2x + y^2x^2 + yx^2y, \\
 xy^3 + y^3x + y^2xy + yxy^2\}$$

What about  $C_{p^n}$ ?

$$M = \mathbb{Z} \otimes \{x, y\}, \quad M^{\otimes 4} \circlearrowleft C_4$$

$$H^0(C_4, M^{\otimes 4}) = \{x^4, y^4, (xy)^2 + (yx)^2, \\
 x^3y + x^2yx + xyx^2 + yx^3, \\
 x^2y^2 + xy^2x + y^2x^2 + yx^2y, \\
 xy^3 + y^3x + y^2xy + yxy^2\}$$

$$\hat{H}^0(C_4, M^{\otimes 4}) = (\mathbb{Z}/4 \otimes \{x^4, y^4\}) \\
 \oplus (\mathbb{Z}/2 \otimes \{(xy)^2 + (yx)^2\})$$

What about  $C_{p^n}$ ?

$$M = \mathbb{Z} \otimes \{x, y\}, \quad M^{\otimes 4} \circlearrowleft C_4$$

$$\begin{aligned} H^0(C_4, M^{\otimes 4}) = & \{x^4, y^4, (xy)^2 + (yx)^2, \\ & x^3y + x^2yx + xyx^2 + yx^3, \\ & x^2y^2 + xy^2x + y^2x^2 + yx^2y, \\ & xy^3 + y^3x + y^2xy + yxy^2\} \end{aligned}$$

$$\begin{aligned} \hat{H}^0(C_4, M^{\otimes 4}) = & (\mathbb{Z}/4 \otimes \{x^4, y^4\}) \\ & \oplus (\mathbb{Z}/2 \otimes \{(xy)^2 + (yx)^2\}) \end{aligned}$$

What does it mean?

# Matrix trace

$$\mathrm{tr}(AB) = \mathrm{tr}(BA)$$

# Matrix trace

$$\mathrm{tr}(AB) = \mathrm{tr}(BA)$$

$$\mathrm{tr}(ABC) = \mathrm{tr}(BCA) = \mathrm{tr}(CAB)$$



$$\mathrm{tr}(ACB) = \mathrm{tr}(CBA) = \mathrm{tr}(BAC)$$

# Matrix trace

$$\mathrm{tr}(AB) = \mathrm{tr}(BA)$$

$$\mathrm{tr}(ABC) = \mathrm{tr}(BCA) = \mathrm{tr}(CAB)$$



$$\mathrm{tr}(ACB) = \mathrm{tr}(CBA) = \mathrm{tr}(BAC)$$

$$\mathrm{tr}\left((X+Y)^4\right) = \mathrm{tr}(X^4) + 4 \mathrm{tr}(X^3 Y) + 4 \mathrm{tr}(X^2 Y^2) + 4 \mathrm{tr}(XY^3) + \mathrm{tr}(Y^4) + 2 \mathrm{tr}\left((XY)^2\right)$$

## Traces and Tate cohomology

$$\hat{H}^0(C_4, M^{\otimes 4}) = (\mathbb{Z}/4 \otimes \{x^4, y^4\}) \oplus (\mathbb{Z}/2 \otimes \{(xy)^2 + (yx)^2\})$$

$$\begin{aligned} \text{tr} \left( (X+Y)^4 \right) &= \text{tr}(X^4) + 4 \text{tr}(X^3 Y) + 4 \text{tr}(X^2 Y^2) + 4 \text{tr}(XY^3) + \text{tr}(Y^4) \\ &\quad + 2 \text{tr}((XY)^2) \end{aligned}$$



## Traces and Tate cohomology

$$\hat{H}^0(C_4, M^{\otimes 4}) = (\mathbb{Z}/4 \otimes \{x^4, y^4\}) \oplus (\mathbb{Z}/2 \otimes \{(xy)^2 + (yx)^2\})$$

$$\begin{aligned} \text{tr} \left( (X+Y)^4 \right) &= \text{tr}(X^4) + 4 \text{tr}(X^3 Y) + 4 \text{tr}(X^2 Y^2) + 4 \text{tr}(XY^3) + \text{tr}(Y^4) \\ &\quad + 2 \text{tr}((XY)^2) \end{aligned}$$

### Slogan

$\hat{H}^0(C_{p^n}, M^{\otimes p^n})$  records expansion of  $\text{tr} \left( (X + Y)^{p^n} \right) \pmod{p^n}$

## Traces and Tate cohomology

$$\hat{H}^0(C_4, M^{\otimes 4}) = (\mathbb{Z}/4 \otimes \{x^4, y^4\}) \oplus (\mathbb{Z}/2 \otimes \{(xy)^2 + (yx)^2\})$$

$$\text{tr} \left( (X+Y)^4 \right) = \text{tr}(X^4) + 4 \text{tr}(X^3 Y) + 4 \text{tr}(X^2 Y^2) + 4 \text{tr}(XY^3) + \text{tr}(Y^4) \\ + 2 \text{tr}((XY)^2)$$

### Slogan

$\hat{H}^0(C_{p^n}, M^{\otimes p^n})$  records expansion of  $\text{tr} \left( (X + Y)^{p^n} \right) \pmod{p^n}$

### Remark

If  $\lambda_i$  eigenvals of  $X$ , then  $\text{tr}(X)^n = \left( \sum \lambda_i \right)^n$  and  $\text{tr}(X^n) = \sum \lambda_i^n$

# Cyclotomic trace

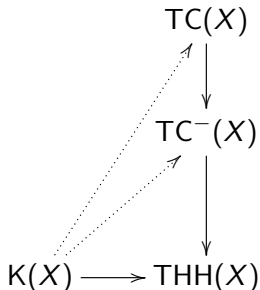
cyclotomic trace: “trace of monodromy”

$$\begin{array}{ccc} & \text{TC}(X) & \\ & \downarrow & \\ & \text{TC}^-(X) & \\ & \downarrow & \\ K(X) & \longrightarrow & \text{THH}(X) \end{array}$$

The diagram illustrates the cyclotomic trace map. It shows a commutative triangle with vertices  $K(X)$ ,  $\text{TC}(X)$ , and  $\text{THH}(X)$ . A solid arrow points from  $K(X)$  to  $\text{THH}(X)$ . A solid arrow points from  $\text{TC}(X)$  to  $\text{TC}^-(X)$ , and another solid arrow points from  $\text{TC}^-(X)$  to  $\text{THH}(X)$ . Dotted arrows represent the cyclotomic trace maps: one from  $K(X)$  to  $\text{TC}(X)$  and another from  $K(X)$  to  $\text{TC}^-(X)$ .

# Cyclotomic trace

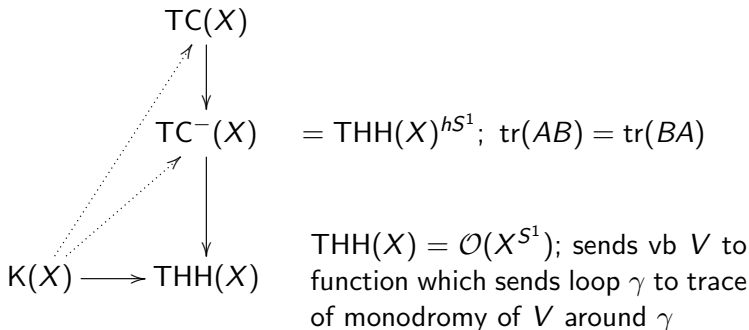
cyclotomic trace: “trace of monodromy”



$THH(X) = \mathcal{O}(X^{S^1})$ ; sends  $\text{vb } V$  to function which sends loop  $\gamma$  to trace of monodromy of  $V$  around  $\gamma$

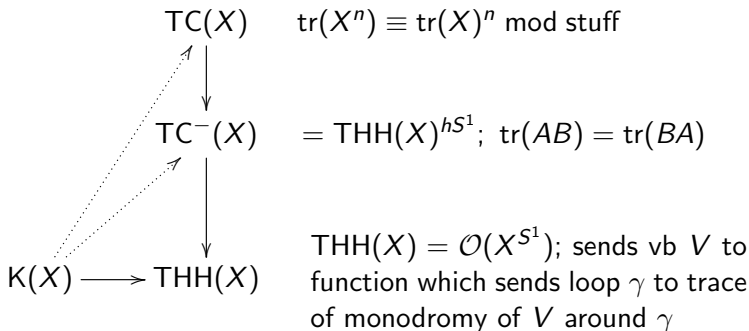
# Cyclotomic trace

cyclotomic trace: “trace of monodromy”



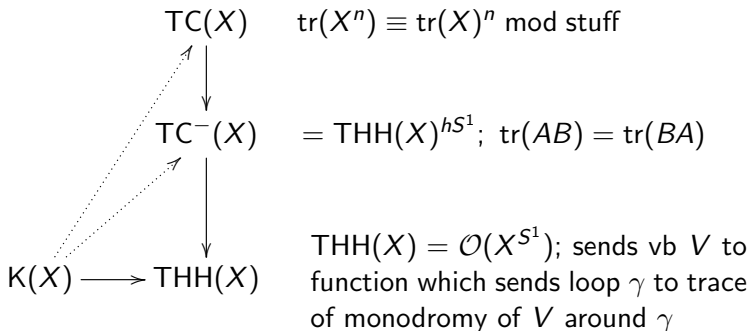
# Cyclotomic trace

cyclotomic trace: “trace of monodromy”



# Cyclotomic trace

cyclotomic trace: “trace of monodromy”



Made precise by Ayala–Mazel–Gee–Rozenblyum.

# Witt vectors

Fix a prime  $p$  (omitted from notation). Witt vectors are a functor

$$W: \text{CRing} \longrightarrow \text{CRing}$$

$$W_n: \text{CRing} \longrightarrow \text{CRing}$$

$$W(A) = \varprojlim_n W_n(A)$$

For example,

$$W_n(\mathbb{F}_p) = \mathbb{Z}/p^n\mathbb{Z}, \quad W(\mathbb{F}_p) = \mathbb{Z}_p$$

$$W(\mathbb{F}_{p^n}) = W(\mathbb{F}_p[\zeta_{p^n-1}]) = \mathbb{Z}_p[\zeta_{p^n-1}]$$



# Witt vectors

Fix a prime  $p$  (omitted from notation). Witt vectors are a functor

$$W: \text{CRing} \longrightarrow \text{CRing}$$

$$W_n: \text{CRing} \longrightarrow \text{CRing}$$

$$W(A) = \varprojlim_n W_n(A)$$

For example,

$$W_n(\mathbb{F}_p) = \mathbb{Z}/p^n\mathbb{Z}, \quad W(\mathbb{F}_p) = \mathbb{Z}_p$$

$$W(\mathbb{F}_{p^n}) = W(\mathbb{F}_p[\zeta_{p^n-1}]) = \mathbb{Z}_p[\zeta_{p^n-1}]$$

Excellent blog post on Witt vectors (and much more):

<https://ayoucis.wordpress.com/2017/02/20/the-fontaine-winterberger-theorem-going-full-tilt/>

# Witt vectors

- " $W(A) = A[[p]]$ "

# Witt vectors

- " $W(A) = A[[p]]$ "
- $W(A) = \prod_{k=0}^{\infty} A$  as a set, exotic addition law

# Witt vectors

- " $W(A) = A[[p]]$ "
- $W(A) = \prod_{k=0}^{\infty} A$  as a set, exotic addition law

## Dishonesty box

generalizes positional (digit) notation:

think of  $(a_0, a_1, a_2, \dots) \in W(A)$  as  $\dots a_2 a_1 a_0 = \sum a_i p^i$

# Witt vectors

- " $W(A) = A[[p]]$ "
- $W(A) = \prod_{k=0}^{\infty} A$  as a set, exotic addition law

## Dishonesty box

generalizes positional (digit) notation:

think of  $(a_0, a_1, a_2, \dots) \in W(A)$  as  $\dots a_2 a_1 a_0 = \sum a_i p^i$

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A & \longrightarrow & W_2(A) & \longrightarrow & A & \longrightarrow & 0 \\
 & & \text{'tens'} & & & & \text{'ones'} & & 
 \end{array}$$

# Witt vectors

- " $W(A) = A[[p]]$ "
- $W(A) = \prod_{k=0}^{\infty} A$  as a set, exotic addition law

## Dishonesty box

generalizes positional (digit) notation:

think of  $(a_0, a_1, a_2, \dots) \in W(A)$  as  $\dots a_2 a_1 a_0 = \sum a_i p^i$

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A & \longrightarrow & W_2(A) & \longrightarrow & A & \longrightarrow & 0 \\
 & & \text{'tens'} & & & & \text{'ones'} & & 
 \end{array}$$

How to 'carry' in an arbitrary ring?

# Carrying

$$\left. \begin{array}{ll} c(0, 0) = 0 & c(1, 0) = 0 \\ c(0, 1) = 0 & c(1, 1) = 1 \end{array} \right\} \implies c(a, b) = ab$$

# Carrying

$$\left. \begin{array}{l} c(0, 0) = 0 \quad c(1, 0) = 0 \\ c(0, 1) = 0 \quad c(1, 1) = 1 \end{array} \right\} \implies c(a, b) = ab$$

for  $p = 2$ , can define

$$W_2(R) = R \times R$$
$$(a_0, a_1) + (b_0, b_1) = (a_0 + b_0, a_1 + b_1 - a_0 b_0)$$



# Carrying

$$\left. \begin{array}{l} c(0, 0) = 0 \quad c(1, 0) = 0 \\ c(0, 1) = 0 \quad c(1, 1) = 1 \end{array} \right\} \implies c(a, b) = ab$$

for  $p = 2$ , can define

$$W_2(R) = R \times R$$
$$(a_0, a_1) + (b_0, b_1) = (a_0 + b_0, a_1 + b_1 - a_0 b_0)$$

in general,

$$(a_0, a_1) + (b_0, b_1) = (a_0 + b_0, a_1 + b_1 - \frac{1}{p} [(a_0 + b_0)^p - a_0^p - b_0^p])$$

# Lies

**WARNING!** for  $p \neq 2$  this does not recover usual digit notation.

# Lies

**WARNING!** for  $p \neq 2$  this does not recover usual digit notation.  
 e.g.  $p = 5$ , calculate  $17 + 7 = 24$  in  $\mathbb{Z}/25$ :

$$\begin{array}{r} 32 \\ +12 \\ \hline 44 \end{array} \text{ using ordinary base 5 but} \quad \begin{array}{r} \phantom{3}22 \\ +02 \\ \hline 04 \end{array} \text{ using Witt vectors}$$

# Lies

**WARNING!** for  $p \neq 2$  this does not recover usual digit notation.  
 e.g.  $p = 5$ , calculate  $17 + 7 = 24$  in  $\mathbb{Z}/25$ :

$$\begin{array}{r} 32 \\ +12 \\ \hline 44 \end{array} \text{ using ordinary base 5 but } \begin{array}{r} \phantom{3}22 \\ +02 \\ \hline 04 \end{array} \text{ using Witt vectors}$$

Keyword: **Teichmüller representatives**

$$\tau(2 \in \mathbb{F}_5) = 7 \in \mathbb{Z}/25, \tau(4 \in \mathbb{F}_5) = 24 \in \mathbb{Z}/25$$

# de Rham-Witt complex

$X/\mathbb{F}_p$  smooth scheme. The **de Rham-Witt complex**  $W\Omega_X^*$  is a lift of  $\Omega_X^*$  to  $\mathbb{Z}_p$ ;  $W\Omega_X^0 = W(\mathcal{O}_X)$ . Also truncated versions  $W_n\Omega_X^*$ .

# de Rham-Witt complex

$X/\mathbb{F}_p$  smooth scheme. The **de Rham-Witt complex**  $W\Omega_X^*$  is a lift of  $\Omega_X^*$  to  $\mathbb{Z}_p$ ;  $W\Omega_X^0 = W(\mathcal{O}_X)$ . Also truncated versions  $W_n\Omega_X^*$ .

## Example

$X = \mathbb{G}_m = \text{Spec}(\mathbb{F}_p[T^{\pm 1}])$ , then

$$p^k T^{1/p^k} \in W\Omega_X^0, \quad T^{1/p^k} d\log T \in W\Omega_X^1$$

# de Rham-Witt complex

$X/\mathbb{F}_p$  smooth scheme. The **de Rham-Witt complex**  $W\Omega_X^*$  is a lift of  $\Omega_X^*$  to  $\mathbb{Z}_p$ ;  $W\Omega_X^0 = W(\mathcal{O}_X)$ . Also truncated versions  $W_n\Omega_X^*$ .

## Example

$X = \mathbb{G}_m = \text{Spec}(\mathbb{F}_p[T^{\pm 1}])$ , then

$$p^k T^{1/p^k} \in W\Omega_X^0, \quad T^{1/p^k} d\log T \in W\Omega_X^1$$

## Remark

If  $\mathfrak{X}$  is a smooth lift of  $X$  to  $\mathbb{Z}_p$  and has a lift of Frobenius  $\mathfrak{X} \xrightarrow{\varphi} \mathfrak{X}$ , then there is a quasi-isomorphism  $\Omega_{\mathfrak{X}/\mathbb{Z}_p}^* \xrightarrow{\sim} W\Omega_X^*$ .

# Topological Restriction homology

Classically:

$$\mathrm{TR}^n = \mathrm{THH}^{C_{p^{n-1}}}, \quad \mathrm{TR} = \varprojlim_n \mathrm{TR}^n$$

Nikolaus-Scholze:

$$\begin{array}{ccccccc}
 \mathrm{TR} & \longrightarrow & \dots & \longrightarrow & \mathrm{TR}^3 & \longrightarrow & \mathrm{TR}^2 & \longrightarrow & \mathrm{THH} \\
 \downarrow \lrcorner & & & & \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow \varphi \\
 & & & & & & \mathrm{THH}^{hC_p} & \xrightarrow{\mathrm{can}} & \mathrm{THH}^{tC_p} \\
 & & & & & & \downarrow \varphi & & \\
 & & & & & & & & \\
 & & & & & & \mathrm{THH}^{hC_{p^2}} & \xrightarrow{\mathrm{can}} & (\mathrm{THH}^{tC_p})^{h(C_{p^2}/C_p)}
 \end{array}$$



# Witt vectors and TR

For a commutative ring  $A$ ,

$$W_n(A) = \pi_0 \mathrm{TR}^n(A)$$

$$W(A) = \pi_0 \mathrm{TR}(A)$$

# Witt vectors and TR

For a commutative ring  $A$ ,

$$W_n(A) = \pi_0 \mathrm{TR}^n(A)$$

$$W(A) = \pi_0 \mathrm{TR}(A)$$

This suggests **one possible** definition of **non-commutative Witt vectors**: for a non-commutative ring  $A$ , let

$$W_n(A) = \pi_0 \mathrm{TR}^n(A), \quad W(A) = \pi_0 \mathrm{TR}(A).$$

# Witt vectors and TR

For a commutative ring  $A$ ,

$$W_n(A) = \pi_0 \mathrm{TR}^n(A)$$

$$W(A) = \pi_0 \mathrm{TR}(A)$$

This suggests **one possible** definition of **non-commutative Witt vectors**: for a non-commutative ring  $A$ , let

$$W_n(A) = \pi_0 \mathrm{TR}^n(A), \quad W(A) = \pi_0 \mathrm{TR}(A).$$

Can give a **purely algebraic** formula for this (Hesselholt, Nikolaus). **No longer a ring**, e.g.  $W_1(A) = A/[A, A]$ .

# dRW complex and TR

For a commutative ring  $A$ ,

$$\pi_* \mathrm{TR}^n(A) = W_n \Omega_A^* \otimes S\{\sigma\}, \quad |\sigma| = 2$$

$$\pi_* \mathrm{TR}(A) = W \Omega_A^*$$

# dRW complex and TR

For a commutative ring  $A$ ,

$$\pi_* \mathrm{TR}^n(A) = W_n \Omega_A^* \otimes S\{\sigma\}, \quad |\sigma| = 2$$

$$\pi_* \mathrm{TR}(A) = W \Omega_A^*$$

This suggests **one possible** definition of a **non-commutative de Rham-Witt complex**: for a non-commutative ring  $A$ , let

$$W \Omega_X^* = \pi_* \mathrm{TR}(A).$$

# dRW complex and TR

For a commutative ring  $A$ ,

$$\pi_* \mathrm{TR}^n(A) = W_n \Omega_A^* \otimes S\{\sigma\}, \quad |\sigma| = 2$$

$$\pi_* \mathrm{TR}(A) = W \Omega_A^*$$

This suggests **one possible** definition of a **non-commutative de Rham-Witt complex**: for a non-commutative ring  $A$ , let

$$W \Omega_X^* = \pi_* \mathrm{TR}(A).$$

**Conjectural** algebraic formula for this (Kaledin, S???)

# Hochschild-Witt complex

**Very roughly,** Kaledin's construction is as follows:

# Hochschild-Witt complex

**Very roughly**, Kaledin's construction is as follows:

- pick a lift  $\mathfrak{A}$  of  $A$  to  $\mathbb{Z}_p$



# Hochschild-Witt complex

**Very roughly**, Kaledin's construction is as follows:

- pick a lift  $\mathfrak{A}$  of  $A$  to  $\mathbb{Z}_p$
- consider the cyclic bar construction computing  $\mathrm{HH}(\mathfrak{A}/\mathbb{Z}_p)$

# Hochschild-Witt complex

**Very roughly**, Kaledin's construction is as follows:

- pick a lift  $\mathfrak{A}$  of  $A$  to  $\mathbb{Z}_p$
- consider the cyclic bar construction computing  $\mathrm{HH}(\mathfrak{A}/\mathbb{Z}_p)$
- apply the construction  $\hat{H}^0(C_{p^n}, -^{\otimes p^n})$  to this whole complex

# Hochschild-Witt complex

**Very roughly**, Kaledin's construction is as follows:

- pick a lift  $\mathfrak{A}$  of  $A$  to  $\mathbb{Z}_p$
- consider the cyclic bar construction computing  $\mathrm{HH}(\mathfrak{A}/\mathbb{Z}_p)$
- apply the construction  $\hat{H}^0(C_{p^n}, -\otimes p^n)$  to this whole complex
- the geometric realization of this new complex is  $W_n\Omega_A^*$   
 ( $W_n\mathrm{HH}_*(A)$  in Kaledin's notation)
- $W\Omega_A^* = \varprojlim W_n\Omega_A^*$

# Hochschild-Witt complex

**Very roughly**, Kaledin's construction is as follows:

- pick a lift  $\mathfrak{A}$  of  $A$  to  $\mathbb{Z}_p$
- consider the cyclic bar construction computing  $\mathrm{HH}(\mathfrak{A}/\mathbb{Z}_p)$
- apply the construction  $\hat{H}^0(C_{p^n}, -\otimes p^n)$  to this whole complex
- the geometric realization of this new complex is  $W_n\Omega_A^*$   
( $W_n\mathrm{HH}_*(A)$  in Kaledin's notation)
- $W\Omega_A^* = \varprojlim W_n\Omega_A^*$

WIP: prove this actually computes TR.