Portfolio decisions in incomplete markets

Thaleia Zariphopoulou
The University of Texas at Austin
Models with undiversifiable risks

Market environment

Traded assets: \( S_t = \left( S_t^1, \ldots, S_t^m \right) \) stocks

\( B_t \) riskless bond maturing at \( T \)

Stochastic factors: \( Y_t = \left( Y_t^1, \ldots, Y_t^n \right) \)

Probabilistic model

\((S, Y)\) is an \((m + n)\)—dimensional semimartingale

on the probability space \((\Omega, F, (F_t), \mathbb{P})\)
Investment plans: \( \boldsymbol{\alpha}_t = \left( \alpha^1_t, \ldots, \alpha^m_t \right) \)

Wealth process: \( X_t \)

\[
X_t = \sum_{i=1}^m \int_0^t \alpha^i_s dS^i_s - \int_0^t dC_s + \sum_{j=1}^n \int_0^t Y^j_s \, ds
\]

Objectives

Value function

\[
V(x, S, Y, t) = \sup_{\mathcal{A}} \mathbb{E} \left( \int_t^T U_1(C_s) \, ds + U_2(X_T) \mid \mathcal{F}_t \right)
\]

or

\[
V(x, S, Y, t) = \sup_{\mathcal{A} \times \mathcal{T}_{[t,T]}} \mathbb{E} \left( \int_t^T U_1(C_s) \, ds + U_2(X_{\tau}) \mid \mathcal{F}_t \right)
\]
Optimal policies

$\alpha^*_s$ feedback functionals involving partial derivatives of the value function

Sensitivity and robustness

- investment horizon
- Sharpe ratios
- correlation
- risk aversion
Undiversifiable risks

• **Stochastic Sharpe ratio**
  
  \[ dS_s = \mu(Y_s, s) S_s ds + \sigma(Y_s, s) S_s dW^1_s \]
  \[ dY_s = b(Y_s, s) ds + a(Y_s, s) dW_s \]
  \[ \lambda(Y_s, s) = \frac{\mu(Y_s, s)}{\sigma(Y_s, s)} \]

• **Stochastic volatility**
  
  \[ dS_s = \mu(S_s, s) S_s ds + \sigma(S_s, Y_s, s) S_s dW^1_s \]
  \[ dY_s = b(Y_s, s) ds + a_1(S_s, Y_s, s) dW_s + a_2(S_s, Y_s, s) d\overline{W}_s \]
\textbf{Labor income}

\[ dX_s = rX_s ds + \mu(S_s, s)\alpha_s ds - C_s ds + Y_s ds + \sigma(S_s, s)\alpha_s dW_s^1 \]

\[ Y_T \quad \text{lump-sum at } T \]

\textbf{Derivatives on nontraded assets}

\[ C = C(S_s, Y_s) \]

\[ S_s = (S_s^1, ..., S_s^m) \]

\[ Y_s = (Y_s^1, ..., Y_s^n) \]
Solution approach

Optimal portfolio choice \rightleftharpoons \text{Derivative valuation}

Objectives

• Model independence
• Minimal assumptions
• Sensitivity
• Robustness
• Computational tractability
Canonical example

Nested complete market model

**Portfolio choice**

\[ dS_s = \mu (S_s, s) S_s ds + \sigma (S_s, s) S_s dW_s^1 \]
\[ dX_s = \mu (S_s, s) \alpha_s ds + \sigma (S_s, s) \alpha_s dW_s^1 \]

\[ V (x, S, t) = \sup_{\alpha} E \left( \int_t^T U_1(C_s) ds + U_2(X_T) \mid X_t = x, S_t = S \right) \]

**Claim valuation**

\[ C = C(S_T) \]
\[ \nu_t(C) = \nu(S, t) = E_{\mathbb{P}^*}(C(S_T) \mid S_t = S) \]
Extended case

\[ V^C(x, S, t) = \sup_{\alpha} \mathbb{E} \left( \int_t^T U_1(C_s) \, ds + U_2(X_T - C(S_T)) \mid X_t = x, S_t = S \right) \]

\[ V^C(x, S, t) = V^0(x - E_{\mathbb{P}^*}(C(S_T)), S, t) \]
\[ = V^0(x - \nu_t(C), S, t) \]

\[ \alpha_s^* = \alpha_s^{0,*} + \sigma(S_s, s) \left( \frac{\partial \nu(S_s, s)}{\partial S} \right) \]

\[ C_s^* \sim \nabla_x V \]

Level sets of dynamic utilities (Tiu and Z.)
Objectives

Nested complete market models

adjusted utility payoffs
adjusted wealth levels

Models with undiversifiable risks

Need to construct the analogue of ‘prices’ in incomplete markets
Structural result

Under minimal model assumptions:

• Value functions are invariant on ‘indifference price’ surfaces

• Optimal policies consist of two parts, one corresponding to a ‘complete market’ type investment rule and a second corresponding to the risk monitoring policy of an inherent pseudoclaim priced by indifference.
Optimal investment behavior and derivative valuation in incomplete markets

- The case of power (CRRA) and exponential (CARA) preferences
- Isomorphic behavior of the two classes of value functions
- Identification of supporting claims
- Indifference pricing of supporting claims
- Hedging strategies versus optimal investment policies
Optimal investments under CRRA preferences

Market environment

\[ dS_s = M(Y_s, s) S_s \, ds + \Sigma(Y_s, s) S_s \, dW_s^1 \]
\[ dY_s = B(Y_s, s) \, ds + A(Y_s, s) \, dW_s \]

riskless bond of zero interest rate

Preferences

\[ U(x) = \frac{x^\alpha}{\alpha} \quad (\alpha < 0) \]
Value function

\[ V(x, y, t) = \sup_{\pi} E \left( \frac{X_T^\alpha}{\alpha} \middle| X_t = x, Y_t = y \right) \]

State controlled wealth process

\[ dX_s = M(Y_s, s)\pi_s ds + \sum(Y_s, s)\pi_s dW_s^1 \]

\[ X_t = x, \quad x \geq 0 \]

Objective

Characterize the optimal investment process \( \pi^*_s \)

Feedback controls \( \pi^*_s = \pi^*(X^*_s, Y_s, s) \)

(Wachter, Campell and Viciera, Liu, ... )
The Hamilton-Jacobi-Bellman equation

\[ V_t + \max_\pi \left( \frac{1}{2} \sum_{y,t} (y_t)^2 \pi^2 V_{xx} + \pi \left( R \sum_{y,t} A(y_t) V_{xy} + M(y_t) V_x \right) \right) \]

\[ + \frac{1}{2} A^2(y,t) V_{yy} + B(y,t) V_y = 0 \]

\[ V(x,y,T) = \frac{x^\alpha}{\alpha} ; \quad (x,y,t) \in D = R^+ \times R \times [0,T] \]
Optimal policies

\[ \pi_s^* = \pi^*(X_s^*, Y_s, s) \]

\[ = - \left( \frac{M(Y_s, s)}{\Sigma^2(Y_s, s)} \right) \frac{V_x(X_s^*, Y_s, s)}{V_{xx}(X_s^*, Y_s, s)} \left( R \frac{A(Y_s, s)}{\Sigma(Y_s, s)} \right) \frac{V_{xy}(X_s^*, Y_s, s)}{V_{xx}(X_s^*, Y_s, s)} \]

\[ dX_s^* = M(Y_s, s) \pi_s^* ds + \Sigma(Y_s, s) \pi_s^* dW_s^1 \]
• **Normalized HJB Equation** (Krylov, Lions)

Non-compact set of admissible controls

\[
\max_{\pi} \left( \frac{1}{1 + \pi^2} \left( V_t + \max_{\pi} \left( \frac{1}{2} \Sigma^2(y, t) \pi^2 V_{xx} + \pi (RA(y, t) \Sigma(y, t) V_{xy} \\
+ M(y, t)V_x) \right) + \frac{1}{2} A^2(y, t)V_{yy} + B(y, t)V_y \right) \right) = 0
\]

\[U(x, y, T) = \frac{x^\alpha}{\alpha}\]

\[V\] is the unique constrained viscosity solution of the normalized HJB equation.

• \(V\) is a constrained viscosity solution of the original HJB equation (Duffie-Z.)

• \(V\) is unique in the appropriate class (Ishii-Lions, Duffie-Z., Katsoulakis, Touzi, Z.)
Solution

\[ V(x, y, t) = \frac{x^\alpha}{\alpha} v(y, t)^\varepsilon \quad \varepsilon = \frac{1 - \alpha}{1 - \alpha + R^2 \alpha} \]

\[ v_t + \frac{1}{2} A^2(y, t) v_{yy} + \left( B(y, t) + R \frac{\alpha}{1 - \alpha} L(y, t) A(y, t) \right) v_y \]

\[ + \frac{1}{2 \varepsilon (1 - \alpha)} L^2(y, t) v = 0 \]

\[ L(y, t) = \frac{M(y, t)}{\Sigma(y, t)} \]

\[ \pi^*(x, y, t) = \frac{1}{1 - \alpha} \frac{M(y, t)}{\Sigma^2(y, t)} x + R \frac{\varepsilon}{1 - \alpha \Sigma(y, t)} \frac{A(y, t)}{v(y, t)} v_y(y, t) x \]
An isomorphism between optimal investments under CRRA and CARA preferences

Market environment

\[ dS_s = \mu(Y_s, s) \, ds + \sigma(Y_s, s) \, S_s \, d\tilde{W}_s^1 \]
\[ dY_s = b(Y_s, s) \, ds + a(Y_s, s) \, d\tilde{W}_s \]
\[ \rho \text{ correlation of } (\tilde{W}_1, \tilde{W}) \]

Preferences

\[ \tilde{U}(x) = -e^{-\gamma x}, \quad \gamma > 0 \]

Value function

\[ \tilde{V}(x, y, t) = \sup_{\pi} E \left( - e^{-\gamma X_T} \mid X_t = x, \ Y_t = y \right) \]
Main result

\[ \rho^2 = \frac{|\alpha|}{1 + |\alpha|} R^2, \quad \lambda^2 = \frac{|\alpha|}{1 + |\alpha|} L^2 \]

\[ b(\cdot, t) - \rho \lambda(\cdot, t) a(\cdot, t) \equiv B(\cdot, t) - R \frac{|\alpha|}{1 + |\alpha|} L(\cdot, t) a(\cdot, t) \]

\[ A(\cdot, t) \equiv a(\cdot, t) \]

\[ \frac{V(x, y, t)}{U(x)} = \frac{\tilde{V}(x, y, t)}{\tilde{U}(x)} \]
Idea of the proof

CRRA preferences : \[ V(x, y, t) = \frac{x^\alpha}{\alpha} v(y, t)^{1-\alpha + R^2 \alpha} \]

CARA preferences : \[ \tilde{V}(x, y, t) = -e^{-\gamma x} \tilde{v}(y, t)^{1-\rho^2} \]

\[ 1 - \frac{\alpha}{1 - \alpha + R^2 \alpha} = \frac{1}{1 - \rho^2} \]

\[ v, \tilde{v} \text{ solve the same linear pde's} \]
Insights

• Invariance of value functions of two classes of preferences

\[
\frac{V(x, y, s)}{V(x, y, t)} = \frac{\tilde{V}(x, y, s)}{\tilde{V}(x, y, t)}
\]

• Sufficient to understand the optimal investment behavior under CARA preferences

• Use of indifference pricing theory

• Pricing measures and valuation algorithms
Optimal investments under CRRA preferences

Optimal investments under CARA preferences

Hedging strategies of supporting claims that are priced by indifference
The CARA expected utility model

- \( \tilde{U}(x) = -e^{-\gamma x} \)

- \( \tilde{V}(x, y, t) = -e^{-\gamma x} \tilde{v}(y, t)\delta \); \( \delta = \frac{1}{1 - \rho^2} \)

\[
\tilde{v}_t + \frac{1}{2}a^2(y, t)\tilde{v}_{yy} + (b(y, t) - \rho \lambda(y, t)a(y, t))\tilde{v}_y = \frac{1}{2}(1 - \rho^2)\lambda^2(y, t)\tilde{v}
\]

\( \lambda(y, t) = \frac{\mu(y, t)}{\sigma(y, t)} \)

- \( \pi^*(x, y, t) = \frac{\mu(y, t)}{\gamma\sigma^2(y, t)} + \rho \frac{a(y, t)}{\sigma(y, t)} \frac{\delta}{\gamma} \frac{v_y(y, t)}{v(y, t)} \)
Probabilistic representation

- $\tilde{V}(x, y, t) = -e^{-\gamma x} \left( E_{\mathbb{Q}} \left( e^{-\int_t^T \frac{1}{2}(1-\rho^2)\lambda^2(Y_s, s) ds \mid Y_t = y} \right) \right)^{\frac{1}{1-\rho^2}}$

- $\mathbb{Q}$ minimal relative entropy measure

- $dY_s = (b(Y_s, s) - \rho \lambda(Y_s, s) a(Y_s, s)) ds + a(Y_s, s) dW^\mathbb{Q}$

- Path-dependent elements
Indifference prices of path-dependent claims

• Path-dependent claim

\[ C(Y) = C(Y_s; t \leq s \leq T) = \int_t^T c_1(Y_s, s) \, ds + c_2(Y_T) \]

\[ dS_s = \mu(Y_s, s) S_s \, ds + \sigma(Y_s, s) S_s \, dW_s^1 \]

\[ dY_s = b(Y_s, s) \, ds + a(Y_s, s) \, dW_s \]

• Buyer’s indifference price

\[ \tilde{V}^C(x, y, t) = \sup_{\pi} E \left( -e^{-\gamma(X_T + C(Y_s; t \leq s \leq T))} \bigg| X_t = x, \ Y_t = y \right) \]

\[ \tilde{V}^0(x, y, t) = \tilde{V}^C(x - h_t(C), y, t) \]
• Non-linear pricing functional

\[ \mathcal{E}_Q(G(Y)/Y_t = y) = -\frac{1}{\gamma(1 - \rho^2)} \ln E_Q\left( \exp(-\gamma(1 - \rho^2)G(Y)) \right| Y_t = y \]

\[ G(Y) = G(Y_s; t \leq s \leq T) \]

**Indifference price**

\[ h_t = \mathcal{E}_Q\left( C(Y_s; t \leq s \leq T) + \int_t^T 1 \frac{\lambda^2(Y_s, s)}{\gamma} ds \right| Y_t = y \]

\[ -\mathcal{E}_Q\left( \int_t^T 1 \frac{\lambda^2(Y_s, s)}{\gamma} ds \right| Y_t = y \)
Indifference price quasilinear pde

\[
\begin{align*}
    h_t + \frac{1}{2}a^2(y, t)h_{yy} + \left( b(y, t) - \rho \lambda(y, t)a(y, t) + a^2(y, t) \frac{v_y(y, t)}{v(y, t)} \right)h_y \\
    + \frac{1}{2}\gamma(1 - \rho^2)a^2(y, t)h_y^2 = \frac{1}{2\gamma} \lambda^2(y, t) + c_1(y, t) \\
    h(y, T) = c_2(y)
\end{align*}
\]
Optimal investments for CARA preferences

Supporting path-dependent claim

\[ \Lambda(Y_s; t \leq s \leq T) = \int_t^T -\frac{1}{2\gamma} \lambda^2(Y_s, s) \, ds \]

Indifference price

\[ h_s(\Lambda) = h(Y_s, s) \]

\[ h(y, t) = \mathcal{E}_Q \left( \int_t^T -\frac{1}{2\gamma} \lambda^2(Y_s, s) \, \bigg| \, Y_t = y \right) \]
**Indifference price quasilinear pde**

\[ h_t(\Lambda(Y_s; t \leq s \leq T)) = h(Y_t, t) \]

\[ h_t + \frac{1}{2} a^2(y, t) h_{yy} + (b(y, t) - \rho \lambda(y, t) a(y, t)) h_y \\ + \frac{1}{2} \gamma (1 - \rho^2) a^2(y, t) h_y^2 = \frac{1}{2\gamma} \lambda^2(y, t) \]

\[ h(y, T) = 0 \]
Optimal investments for CARA preferences

\[ \pi^* = \pi^M_s + H_s \]

\[ \pi^M_s = \pi^M(Y_s, s) = \frac{1}{\gamma} \frac{\mu(Y_s, s)}{\sigma^2(Y_s, s)} \]

\[ H_s = H(Y_s, s) = \rho \frac{a(Y_s, s)}{\sigma(Y_s, s)} h_y(Y_s, s) \]

\( H_s \) is the **hedging strategy** of

\[ \Lambda(Y_s; t \leq s \leq T) = - \int_t^T \frac{1}{2\gamma} \lambda^2(Y_s, s) \, ds \]
Hedging strategies for $\Lambda(Y_s; t \leq s \leq T)$

$$
\Lambda(Y_s; t \leq s \leq T) = h(Y_t, t) + \int_t^T \frac{a(Y_s, s)}{\sigma(Y_s, s)} h_y(Y_s, s) \frac{dS_s}{S_s} \\
- \int_t^T \frac{1}{2} \gamma(1 - \rho^2) a^2(Y_s, s) h_y^2(Y_s, s) \, ds + \\
+ \int_t^T \sqrt{1 - \rho^2} a(Y_s, s) h_y(Y_s, s) \, dW_s^\perp
$$
Sensitivity and robustness of optimal policies

- Sign of the unhedged demand component
- Monotonicity with respect to the level of $Y$
- Behavior in terms of the time horizon
- Behavior in terms of the correlation
- Behavior in terms of the Sharpe ratio of traded asset
Technical tools

• The unhedged demand of component $h_y$ solves a viscous Burger’s equation

• Maximum principle for quasilinear and reaction-diffusion pde

• Probabilistic representation for emerging pseudoclaims related to “indifference greeks”
Sign of $h_y$

$$f = h_y$$

$$f_t + \frac{1}{2} a^2(y, t) f_{yy} + (b(y, t) - \rho \lambda(y, t)a(y, t)) f_y$$

$$+ (b_y(y, t) - \rho \lambda_y(y, t)a_y(y, t)) f = (1 - \rho^2) \lambda(y, t) \lambda_y(y, t)$$

$$f(y, T) = 0$$

$$\lambda \uparrow (\downarrow) \implies h_y < 0 \ (> 0)$$

Portfolio component:

$$H(y, t) = \rho \frac{a(y, t)}{\sigma(y, t)} h_y(y, t)$$
• Dependence on the trading horizon

\[ g = f_t = (h_y)_t \]

\[ g_t + \frac{1}{2} a^2 g_{yy} + (aa_y + a^2 f + (b - \rho \lambda a))g_y \]

\[ + (a^2 f_y + 2aa_y f + b_y - \rho \lambda y a - \rho \lambda a_y)g = 0 \]

\[ g(y, T) = (1 - \rho^2)\lambda \lambda_y \]

\[ \lambda \uparrow (\downarrow) \implies h_y \downarrow (\uparrow) \text{ w.r.t } (T - t) \]
The extended model

\[ dS_s = \mu(S_s, s) S_s \, ds + \sigma(Y_s, s) S_s \, dW^1_s \]
\[ dY_s = b(Y_s, s) \, ds + a(Y_s, s) \, dW_s \]

Supporting pseudo-claim

\[ \Lambda((S_s, Y_s) ; t \leq s \leq T) = - \int_t^T \frac{1}{2\gamma} \frac{\mu^2(S_s, s)}{\sigma^2(Y_s, s)} \, ds \]

Explicit solutions do not (should not!) exist.
Optimal investment policies

\[ \pi^*_s = \pi^M_s + H_s \]

\[ \pi^M_s = \pi^M(Y_s, s) = \frac{1}{\gamma} \frac{\mu(Y_s, s)}{\sigma^2(Y_s, s)} \]

\[ H_s = H(Y_s, s) = \rho \frac{a(Y_s, s)}{\sigma(Y_s, s)} h_y(Y_s, s) \]

\( H_s \) is the **hedging strategy** of

\[ \Lambda(Y_s; t \leq s \leq T) = - \int_t^T \frac{1}{2\gamma} \frac{\mu^2(S_s, s)}{\sigma^2(Y_s, s)} ds \]
Indifference price representation

The indifference price process is given by

\[ h_s(\Lambda) = H(S_s, Y_s, s) \]

where \( H(S, y, t) \) is the unique viscosity solution of the quasilinear price equation

\[
H_t + \mathcal{L}(H(S, y, t)) + \frac{1}{2} \gamma(1 - \rho^2)a^2(y, t)H_y^2 = \frac{1}{2} \frac{\mu^2(S, t)}{\sigma^2(y, t)}
\]

\[ H(S, y, T) = 0 \]

The operator \( \mathcal{L}(.) \) is given by

\[
\mathcal{L}(.) = \frac{1}{2} \sigma^2(y, t) \frac{\partial^2}{\partial S^2} + \rho a(y, t) \sigma(y, t) \frac{\partial^2}{\partial S \partial y} + \frac{1}{2} a^2(y, t) \frac{\partial^2}{\partial y^2} + \left( b(y, t) - \rho \frac{\mu(S, t)}{\sigma(y, t)} a(y, t) \right) \frac{\partial}{\partial y}
\]
Remarks

i) $H$ may be directly related to a quadratic stochastic control problem

ii) $H$ may be related to a BSDE

But, there is no direct Feynman-Kac type representation of $H$ available up to now!
Valuation algorithm

- **Pricing measure** $\mathbb{Q}$: a martingale measure satisfying

\[
\mathbb{Q}(Y_{t+dt} \mid \mathcal{F}_t \vee \mathcal{F}_t^{S,t+dt}) = \mathbb{P}(Y_{t+dt} \mid \mathcal{F}_t \vee \mathcal{F}_t^{S,t+dt})
\]

The indifference price $h_t(S_s, Y_s, s)$ is given by an iterative pricing algorithm

\[
h_t(S_s, Y_s, s) = E_{\mathbb{Q}}\left( F^{-1} \left( E_{\mathbb{Q}}\left( F \left( \int_s^{s+ds} - \frac{1}{2\gamma} \lambda^2(S_u, Y_u, u) \, du \\ + h_{s+ds}(S_{s+ds}, Y_{s+ds}, s + ds) \right) \bigg| \mathcal{F}_s \vee \mathcal{F}_{s+ds} \bigg| \mathcal{F}_s \right) \right)
\]

\[
h_T(S_T, Y_T, T) = 0
\]
A novel probabilistic representation of solutions to quasilinear PDEs arises

The solution \( H(S, y, t) \) of the quasilinear price equation

\[
H_t + \mathcal{L}(H(S, y, t)) + \frac{1}{2} \gamma (1 - \rho^2) a^2(y, t) H_y^2 = \frac{1}{2\gamma} \frac{\mu^2(S, t)}{\sigma^2(y, t)}
\]

with

\[
\mathcal{L}(.) = \frac{1}{2} \sigma^2(y, t) \frac{\psi^2}{\psi S^2} + \rho a(y, t) \sigma(y, t) \frac{\psi^2}{\psi S \psi y} + \frac{1}{2} a^2(y, t) \frac{\psi^2}{\psi y^2}
\]

\[
+ \left( b(y, t) - \rho \frac{\mu(S, t)}{\sigma(y, t)} a(y, t) \right) \frac{\psi}{\psi y}
\]

is given by

\[
H(S, y, t) = \lim_{ds \to 0, s \to t} h_t(S_{s+ds}, Y_{s+ds}, s + ds)
\]

**Proof:** Based on convergence results of supermartingales, behavior of nonadditive measures and robustness properties of viscosity solutions.
Future directions

• Optimal investment behavior under homothetic preferences in non-Markovian models

• Stochastic labor income

• Models of intermediate consumption

  Identification and pricing of supporting pseudo claims