Entropy Regularization for Mean Field Games with Learning

Xin Guo * Renyuan Xu † Thaleia Zariphopoulou ‡

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Abstract

Entropy regularization has been extensively adopted to improve the efficiency, the stability, and the convergence of algorithms in reinforcement learning. This paper analyzes both quantitatively and qualitatively the impact of entropy regularization for Mean Field Game (MFG) with learning in a finite time horizon. Our study provides a theoretical justification that entropy regularization yields time-dependent policies and, furthermore, helps stabilizing and accelerating convergence to the game equilibrium. In addition, this study leads to a policy-gradient algorithm for exploration in MFG. Under this algorithm, agents are able to learn the optimal exploration scheduling, with stable and fast convergence to the game equilibrium.

1 Introduction

Reinforcement learning (RL) is one of the three basic machine learning paradigms, alongside supervised learning and unsupervised learning. RL is learning via trial and error, through interactions with an environment and possibly with other agents; in RL, an agent takes an action and receives a reinforcement signal in terms of a numerical reward, which encodes the outcome of her action. In order to maximize the accumulated reward over time, the agent learns to select her actions based on her past experiences (exploitation) and/or by making new choices (exploration).

Exploration and exploitation are the essence of RL. Exploration provides opportunities to improve from current sub-optimal solutions to the ultimate global optimal one, yet is time consuming and computationally expensive as over-exploration may impair the convergence to the optimal solution. Meanwhile, pure exploitation, i.e., myopically picking the current solution based solely on past experience, though easy to implement, tends to yield sub-optimal global solutions. Therefore, an appropriate trade-off between exploration-exploitation is crucial for RL algorithms design to improve the learning and the optimization procedure.

Entropy regularization. One common approach to balance the exploration-exploitation in RL is to introduce entropy regularization [1, 18, 20]. In RL setting with more than one agent, there are two major sources of uncertainties: the unknown environment and the actions of the other agents. As such, Shannon entropy and cross-entropy are two natural choices for entropy regularization: the former quantifies the information gain of exploring the environment while the latter measures

*Department of Industrial Engineering and Operations Research, University of California, Berkeley, USA. Email: xinguo@berkeley.edu.

†Mathematical Institute, University of Oxford, Oxford, UK. Email: xur@maths.ox.ac.uk.

‡Departments of Mathematics and IROM, The University of Texas at Austin, Austin, USA, and the Oxford-Man Institute, University of Oxford, Oxford, UK. Email: zariphop@math.utexas.edu.

the benefit from exploring the actions of other agents. This information-theoretic perspective of exploration has been well understood in single-agent RL, see for instance [8, 10, 18, 20, 23].

However, there is virtually no theoretical study on the role of entropy regularization in multi-agent RL (MARL), with the exception of [2]. Indeed, most existing studies are empirical, demonstrating convergence improvement and variance reduction when entropy regularization is added. For instance, [12] showed via empirical analysis that policy features can be learned directly from pure observations of other agents and that the non-stationarity of the environment can be reduced with addition of the cross-entropy; [11] applied the cross-entropy regularization to demonstrate the convergence of fictitious play in a discrete-time model with a finite number of agents while [22] used the cross-entropy loss to train the prediction of other agents' actions via observations of their behavior. The only theoretical work so far can be found in [2] with an infinite horizon setting in which a regularized Q-learning algorithm for stationary discrete-time mean field game was proposed along with its convergence analysis. Still, the problem remains open in the finite time horizon cases, which arises often in many applications.

Optimal exploration scheduling. Another major challenge for both single-agent RL and MARL is exploration efficiency. In practice, there are various heuristic designs of explorations for MARL, including adding random noise in the parameter space [21], the approach of ε-greedy [27], and the method with softmax [14]. However, there is no theoretical validation of these approaches.

Recently, time-invariant Gaussian exploration were applied to single-agent RL ([13], [19], and [24]) and time-dependent "optimal exploration scheduling" was derived for single-agent mean-variance portfolio selection problem in [25]. In these works, the degree of exploration was characterized by the variance of the Gaussian distribution, and the term "optimal exploration scheduling" was coined for the time-dependent variance of the Gaussian distribution.

Exploration schemes are inherently time-dependent, as it is necessary to balance the free exploration at the initial phase and the greedy control policy towards terminal time. Yet, it seems that there is no existing work on analyzing such time-dependent learning policies for MARL, empirical or theoretical.

Our work. In this paper, we propose to study entropy regularization for MARL with a large population, namely, within the framework of the mean field game (MFG). This transition from MARL to MFG with learning is critical to avoid the curse of dimensionality in MARL.

We analyze both quantitatively and qualitatively the impact of entropy regularization in MFG with learning in a finite time horizon. We adopt two different entropies: first the Shannon entropy and then a combination of Shannon entropy and the cross-entropy which we call the *enhanced entropy*.

- We derive explicit Nash equilibrium (NE) solutions (Theorems 2 and 4) for a class of linear-quadratic (LQ) stochastic games. Our study provides a theoretical justification that entropy regularization yields time-dependent policies and, furthermore, helps stabilizing and accelerating convergence to the game equilibrium.

- This theoretical study enables us to design a policy-gradient algorithm for MFG with learning. Under this algorithm, agents are able to learn efficiently the optimal exploration scheduling in an unknown environment and with a large group of competing agents. The convergence to the game equilibrium is stable and fast when appropriate exploration rates are chosen.

Additional related works. Our algorithm is inspired by the recent success of policy-gradient method for single-agent LQ regulators [6], In addition, there is a concurrent work on the global convergence of policy gradient for MFG [26], yet without exploration. We mention here also recent works on two-agent zero-sum LQ games [28] and the LQ mean field control problem with common noise [5].

Organization. The rest of the paper is organized as follows. Section 2 provides the mathematical framework for MFG with learning, Section 3 focuses on analyzing the impact of Shannon entropy and the enhanced entropy in a class of LQ games, and Section 4 proposes a policy-gradient based algorithm and entropy regularization, and provides its numerical performance.

2 Mathematical Formulation

We start with the mathematical formulation of the MFG with learning.

Key ideas. There are several key components for the formulation.

The first component is the *aggregation idea* from the theory of MFG to address the issue of curse of dimensionality in MARL. Specifically, it is to consider N agents, and assume that they are all identical, indistinguishable and interchangeable, and that interactions among them are based on the *macroscopic information*, which is the empirical state distribution and action distribution of the other agents. This allows us to work instead with a representative agent i, her state X_t^i, her policy π_t^i at time $t \in [0,T]$, and her interaction with other agents through the macroscopic information. Since X^i depends on other agents only through the empirical measure, we may then consider both the population state distribution and action distribution if such limits exist when $N \to \infty$. Moreover, the subscript i can be dropped and one can focus on a representative agent in this MFG formulation since all agents are assumed to be identical and indistinguishable.

The second component is how to model learning and exploration via the notion of *randomized policies*, known in the control literature as *relaxed controls* and in game theory as *mixed strategies*, respectively. That is, policies, say π_t, from the representative agent such that $\pi_t \in \mathcal{P}(U)$, with $\mathcal{P}(U)$ a *probability distribution* over an action space U. Mathematically, this means that $\pi_t \in \mathcal{P}(U)$ if and only if

$$\int_U \pi_t(u)du = 1 \quad \text{and} \quad \pi_t(u) \geq 0 \text{ a.e. on } U. \tag{2.1}$$

The third ingredient is the *entropy regularization*, which is adopted to encourage exploration. For this, we will use both the Shannon entropy, and the cross-entropy, denoted by $\mathcal{H}_{SE}$ and $\mathcal{H}_{CE}$, respectively (see (2.5) and (2.6)).

Controlled state process with randomized policies. We incorporate the above components in a finite horizon setting $[0,T]$, $0 < T < \infty$. For this, we denote $\mu := \{\mu_s,\ t \leq s \leq T\}$ to be the flow of population state distribution with $\mu_s \in \mathcal{P}(\mathbb{R})$ and $\alpha := \{\alpha_s,\ t \leq s \leq T\}$ to be the flow of population action distribution with $\alpha_s(\cdot;x) \in \mathcal{P}(U)$, starting from time $t \in [0,T]$. Occasionally, α and μ will also be called the *mean field information*.

Next we define the controlled state process for the representative agent. Given $t \in [0,T]$ and deterministic and exogenous flows, say α and μ with $\mu_t = \nu$, the representative agent adopts a randomized policy $\pi = \{\pi_s \in \mathcal{P}(U), s \in [t,T]\}$ over an admissible policy set $\mathcal{A}$ (to be specified

below). Then, following the paradigm recently proposed in [24], the controlled state process is assumed to follow

$$\begin{aligned} dX_s^\pi &= \left(\int_U b(s, X_s^\pi, \mu_s, \alpha_s, u)\pi_s(u)du\right) ds + \left(\sqrt{\int_U \sigma^2(s, X_s^\pi, \mu_s, \alpha_s, u)\pi_s(u)du}\right) dW_s, \quad (2.2) \\ X_t^\pi &= \xi \sim \nu, \quad \mu_t = \nu, s \in [t, T]. \end{aligned}$$

Here $W = \{W_t\}_{0\le t\le T}$ is a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, \mathbb{P})$, with $\{\mathcal{F}_t\}_{0\le t\le T}$ satisfying the usual conditions; $\nu \in \mathcal{P}^2(U)$ is the distribution of the initial state; ξ is a random variable independent of W and $\mathcal{F}_0$-measurable; and $b, \sigma : [0, T] \times \mathbb{R} \times \mathcal{P}(U) \times \mathcal{P}(U) \times U \hookrightarrow \mathbb{R}$ are, respectively, the drift and volatility of the underlying state process.

Note that the particular form of the state process (2.2) is a consequence of the aggregation of $\widehat{X}_s^u$ over action $u_s \in U$, with

$$d\widehat{X}_s^u = b\left(s, \widehat{X}_s^u, \mu_s, \alpha_s, u_s\right) ds + \sigma\left(s, \widehat{X}_s^u, \mu_s, \alpha_s, u_s\right) dW_s.$$

Such policies $u_s \in U$ are also called *pure strategies* in game theory. Pure strategies and mixed strategies are closely related as discussed in [24]. Indeed, $u = \{u_s, s \in [0, T]\}$ can be regarded as a Dirac distribution $\pi = \{\pi_s(u), s \in [0, T]\}$ where $\pi_s(\cdot) = \delta_{u_s}(\cdot) \in U$. (See [24] for more discussions on this connection).

Game payoff with entropy regularization. The objective of the representative agent is to maximize her payoff function J and solve for

$$V(t \,|\, \mu, \alpha) = \sup_{\pi\in\mathcal{A}} J(t, \pi \,|\, \mu, \alpha), \quad (2.3)$$

where the entropy-regularized payoff is defined as

$$\begin{aligned} J(t, \pi \,|\, \mu, \alpha) = \mathbb{E}\Bigg[\int_t^T \left(\int_U \Big(r(s, X_s, \mu_s, \alpha_s, u)\pi_s(u)du + \lambda_{SE}\mathcal{H}_{SE}(\pi_s) + \lambda_{CE}\mathcal{H}_{CE}(\pi_s, \alpha_s, \mu_s)\Big)\right) ds \\ + g(X_T, \mu_T, \alpha_T)\Big|\mu, \alpha\Bigg]. \quad (2.4) \end{aligned}$$

The Shannon entropy $\mathcal{H}_{SE}$ and cross-entropy $\mathcal{H}_{CE}$ are defined as

$$\begin{aligned} \mathcal{H}_{SE}(\pi_s) &= -\int_U \pi_s(u) \ln \pi_s(u)du, \ \pi_s \in \mathcal{P}(U), \quad (2.5) \\ \mathcal{H}_{CE}(\pi_s, \alpha_s, \mu_s) &= -\int_U \pi_s(u) \int \Big(\ln \alpha_s(u; x)\Big)\mu_s(dx)du, \ \pi_s \in \mathcal{P}(U). \quad (2.6) \end{aligned}$$

In addition, $r : [0, T] \times \mathbb{R} \times \mathcal{P}(U) \times \mathcal{P}(U) \times U \hookrightarrow \mathbb{R}$ and $g : [0, T] \times \mathbb{R} \times \mathcal{P}(U) \times \mathcal{P}(U) \hookrightarrow \mathbb{R}$ are the running reward and terminal reward functions of the representative agent, while $\lambda_{SE} > 0$ is the (temperature) parameter to control the degree of self-exploration and $\lambda_{CE} \ge 0$ is the (temperature) parameter to control the degree of exploration on other agents' actions. From an information-theoretic perspective, $\lambda_{SE}\mathcal{H}_{SE}$ and $\lambda_{CE}\mathcal{H}_{CE}$ quantify the information gain from exploring the unknown environment and the policies chosen by the other agents.

Observable quantities. In a game with learning, the functions b, σ, r and g are *unknown*. The representative agent takes actions while interacting with (the continuum) of the other agents. This interaction takes several rounds.

In each round starting from time 0, she observes $\{\alpha_s\}_{s\le t}$, $\{\mu_s\}_{s\le t}$ and $\{X^\pi_s\}_{s\le t}$ at time $t\in[0,T]$; the reward will not be revealed until time T, the end of each round; at time T, she will observe the *realized* cumulative reward $\widehat{j}\,(0,\pi|\alpha,\mu)$ with

$$\begin{aligned}\widehat{j}\,(0,\pi|\alpha,\mu) \;:=\; & \int_t^T \Big(\int_U \Big(r(s,X^\pi_s,\mu_s,\alpha_s,u)\pi_s(u)du+\lambda_{SE}\mathcal{H}_{SE}(\pi_s)+\lambda_{CE}\mathcal{H}_{CE}(\pi_s,\alpha_s,\mu_s)\Big)\Big)\,ds\\ & +g(X^\pi_T,\mu_T,\alpha_T),\end{aligned}$$

which is associated with the corresponding *single* trajectory $\{X^\pi_s\}_{s\in[0,T]}$ under policy π and the population behavior $\{\alpha_s\}_{s\in[0,T]}$, $\{\mu_s\}_{s\in[0,T]}$ in this round. Note that $\widehat{j}\,(0,\pi|\alpha,\mu)$ is random.

Admissible policies. A policy $\pi\in\mathcal{A}(t,\mu,\alpha)$ is admissible if

(i) for each $s\in[t,T]$, $\pi_s\in\mathcal{P}(U)$ a.s.;

(ii) for each $A\in\mathcal{B}(U)$ with $\mathcal{B}(U)$ being the Borel algebra on U, $\{\int_A\pi_s(u)du, s\in[t,T]\}$ is $\mathcal{F}_t$-progressively measurable;

(iii) the SDE (2.4) admits a unique strong solution $X^\pi:=\{X^\pi_s, s\in[t,T]\}$, with π used;

(iv) the expectation on the right hand side of (2.4) is finite;

(v) there exists a measurable function $\widehat{\pi}:[t,T]\times\mathbb{R}\to\mathcal{P}(U)$ such that

$$\mathbb{P}\Big(\pi_s(du)=\widehat{\pi}_s(du;X^\pi_s),\quad \forall s\in[t,T]\Big)=1.$$

Note that (v) implies the admissible policy is Markovian, i.e., closed-loop policy in feedback form.

Alternative formulation of the MFG with learning. We note that problem (2.3) treats the initial state ξ as a genuine source of randomness, in addition to the stochasticity from the Brownian motion W. Frequently, the following alternative interpretation, with a *deterministic* initial state x is useful for solving analytically the MFG. Specifically, let

$$\begin{aligned}\widetilde{V}(t,x\,|\,\mu,\alpha) :=\;& \sup_{\pi\in\mathcal{A}}\widetilde{J}(t,x\,|\,\pi,\mu,\alpha)\\ :=\;& \mathbb{E}\bigg[\int_t^T\bigg(\int_U(r(X^\pi_s,\mu_s,\alpha_s,u)\pi_s(u)du+\lambda_{SE}\mathcal{H}_{SE}(\pi_s)+\lambda_{CE}\mathcal{H}_{CE}(\pi_s,\alpha_s,\mu_s))\bigg)ds\\ & +g(X^\pi_T,\mu_T,\alpha_T)\Big|X^\pi_t=x,\mu,\alpha\bigg],\end{aligned}\tag{2.7}$$

subject to

$$\begin{aligned}dX^\pi_s \;&=\Big(\int_U b(s,X^\pi_s,\mu_s,\alpha_s,u)\pi_s(u)du\Big)ds+\Big(\sqrt{\int_U\sigma^2(s,X^\pi_s,\mu_s,\alpha_s,u)\pi_s(u)du}\Big)dW_s,\\ X^\pi_t \;&=x,\;\;\mu_t=\nu,\;\;s\in[t,T].\end{aligned}\tag{2.8}$$

Then, it easily follows that

$$\mathbb{E}_{\xi\sim\nu}[\widetilde{V}(t,\xi\,|\,\mu,\alpha)]=V(t\,|\,\mu,\alpha).$$

While conceptually this approach is less general, it is frequently used -as in [16] and therein - to solve the MFG explicitly.

Nash Equilibrium (NE) for MFG with learning. To analyze game (2.4), we adopt the well-known NE criterion.

Definition 1 (NE for MFG)**.** *For game* (2.4) *with an initial state distribution* ν *and state process* (2.8)*, an agent-population profile* $(\pi^*, \mu^*, \alpha^*) := \{(\pi_s^*, \mu_s^*, \alpha_s^*), t \le s \le T\}$ *is called NE if the following conditions hold:*

- **A.** *(Single-agent-side) For the fixed population state-action distribution* (μ^*, α^*) *and any policy* π*,*
$$J(t, \pi \,|\, \mu^*, \alpha^*) \ge J(t, \pi^* \,|\, \mu^*, \alpha^*).$$
- **B.** *(Population-side)* $\pi_s^*(u; x) = \alpha_s^*(u; x)$*, for all* $x \in \mathbb{R}$*. In addition,* $\mathbb{P}_{X_s^*} = \mu_s^*$ *for any* $s \in [t, T]$*, where* X^* *solves* (2.8) *when policy* π^* *is adopted with the initial population state distribution* $\mu_t^* = \nu$*.*

Given an NE (π^*, μ^*, α^*)*,*

$$V(t \,|\, \mu^*, \alpha^*) := J(t, \pi^* \,|\, \mu^*, \alpha^*) = \max_{\pi \in \mathcal{A}} J(t, \pi \,|\, \mu, \alpha^*)$$

is called a game value associated with this NE.

Given (μ^*, α^*), condition **A** captures the optimality of π^* while condition **B** ensures the consistency of the solution so that the state and action flows of the single agent match those of the population. Note that uniqueness of NE for MFG is, in general, rare when mixed strategies are allowed (see, for example, [15]).

3 Shannon Entropy and Enhanced Entropy for MFG with Learning

Given the mathematical formulation for MFG with learning in Section 2, we analyze the information theoretic gain for two types of entropies: Shannon entropy $\mathcal{H}_{SE}$ and enhanced entropy, which is a linear combination of Shannon entropy and cross-entropy $\lambda_{SE}\mathcal{H}_{SE} + \lambda_{CE}\mathcal{H}_{CE}$ with temperature parameters λ_{SE} and λ_{CE}. We study the impact of this entropy regularization within a class of LQ games in a finite time horizon. LQ games are the building blocks of stochastic games and often bring critical insights from their closed-form solutions ([3, 4]). In particular, we will see that the LQ games we analze yield time-dependent optimal policies, with time-dependent Gaussian efficient explorations.

3.1 Game with Shannon Entropy

We start with the case of using only Shannon entropy for exploration, namely

$$\begin{aligned} V_{SE}(t \,|\, \mu) := &\quad \sup_{\pi \in \mathcal{A}} \quad J_{SE}(t, \pi \,|\, \mu) \\ := &\quad \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[\int_t^T \left(\int_{\mathbb{R}} -\frac{Q}{2}(X_s^\pi - m_s)^2 \pi_s(u) du + \lambda_{SE}\mathcal{H}_{SE}(\pi_s)\right) ds \; -\frac{\bar{Q}}{2}(X_T^\pi - m_T)^2 \middle| \, \mu\right], \end{aligned}$$

subject to

$$dX_s^\pi = \left(\int_{\mathbb{R}} (A(m_s - X_s^\pi) + Bu)\pi_s(u) du\right) ds + D\left(\sqrt{\int_{\mathbb{R}} u^2 \pi_s(u) du}\right) dW_s, \quad X_t^\pi = \xi \sim \nu. \tag{MFG-SE}$$

Here $\mu_t = \nu$, and $m_s = \int x\mu_s(dx)$ $(s \in [t,T])$. We assume $A > 0$, $Q > 0$, $\bar{Q} > 0$, and $\lambda_{SE} > 0$. We take the action space to be $U = \mathbb{R}$, and without loss of generality, $B > 0$ and $D > 0$.

We remark that α does not appear in the game formulation (**MFG-SE**). This is because when only Shannon entropy is incorporated, there is no interaction between the policy of the representative agent and the population action distribution $\alpha := \{\alpha_s\}_{s\in[t,T]}$.

There are two types of rewards in this game: the running reward $-\frac{Q}{2}(X_s^\pi - m_s)^2$ that penalizes any deviation from the current average state of the population at time $s \in [t,T]$, and the terminal reward $-\frac{\bar{Q}}{2}(X_T^\pi - m_T)^2$ that penalizes deviation from average state of the population at the terminal time T. There are also two types of interactions: the real time interactions $A(m_s - X_s^\pi)$ and $\frac{Q}{2}(X_s^\pi - m_s)^2$ for $s \in [t,T]$ and the interaction at the terminal time $\frac{\bar{Q}}{2}(X_T^\pi - m_T)^2$.

Next, we present one of the main results herein which provides an explicit NE solution.

Theorem 2 (**MFG-SE**). *Let* $m^* = \mathbb{E}[\xi]$ *and*

$$\widetilde{V}_{SE}(t,x) = -\frac{\eta_t^{SE}}{2}(x-m^*)^2 + \gamma_t^{SE}, \tag{3.1}$$

with

$$\eta_t^{SE} = \bar{Q}\exp\left(-\left(2A + \frac{B^2}{D^2}\right)(T-t)\right) + \frac{Q}{2A + \frac{B^2}{D^2}}\left(1 - \exp\left(-\left(2A + \frac{B^2}{D^2}\right)(T-t)\right)\right) > 0 \tag{3.2}$$

and

$$\gamma_t^{SE} = \frac{\lambda_{SE}}{2}\ln\left(\frac{2\pi\lambda_{SE}}{D^2}\right)(T-t) - \int_t^T \frac{\lambda_{SE}}{2}\ln\left(\eta_z^{SE}\right)dz.$$

Then,

$$V_{SE}^*(t) = \mathbb{E}_{\xi\sim\nu}[\widetilde{V}_{SE}(t,\xi)]$$

is a game value of **(MFG-SE)** *associated with the NE policy*

$$\pi_s^{SE*}(u;x) = \mathcal{N}\left(\frac{B(m^*-x)}{D^2}, \frac{\lambda_{SE}}{D^2\eta_s^{SE}}\right), \quad s \in [t,T]. \tag{3.3}$$

The corresponding controlled state process under (3.3) *is the unique solution of the SDE,*

$$\begin{aligned} dX_s^* &= \left(A + \frac{B^2}{D^2}\right)(m^* - X_s^*)ds + \sqrt{\left(\frac{B(m^*-X_s^*)}{D}\right)^2 + \frac{\lambda_{SE}}{\eta_s^{SE}}}dW_s, \\ X_t^* &= \xi \sim \nu, s \in [t,T]. \end{aligned} \tag{3.4}$$

In addition, the mean state of the population under policy (3.3) *is time-independent, i.e.,*

$$m_s^* = \mathbb{E}[X_s^*] = m^*, \quad s \in [t,T]. \tag{3.5}$$

Remark 3. *Theorem 2 provides important guidance for exploration from an information-theoretic perspective. It suggests that, with Shannon entropy regularization, the associated optimal policy process* $\pi_s^{SE*}(u;X_t^*)$ *implied from* (3.3) *is Gaussian, mean-reverting and with time-dependent variance. This is useful for MARL algorithmic design as the agent can now focus on a much smaller class of policies*

$$\widehat{\pi}_s(u;x) \sim \mathcal{N}\left(\widehat{M}(m_s - x), \widehat{\sigma}_s^2\right), \tag{3.6}$$

with $m_s = \int_{\mathbb{R}} x\mu_s(dx)$, $\widehat{M}$ some scalar, and $\widehat{\sigma}^2 = \{\widehat{\sigma}_s^2\}_{s\in[t,T]}$ a variance exploration process. Meanwhile, she can improve her estimate on $\widehat{M}$ and $\widehat{\sigma}^2$ of the policy while interacting with the system and other agents and observing the outcome at the end of each round of play. Indeed, notice that the controlled state process becomes

$$dX_s^{\widehat{\pi}} = \left(A + B\widehat{M}\right)(m_s - X_s^{\widehat{\pi}})ds + \left(D\sqrt{\widehat{M}^2(m_s - X_s^{\widehat{\pi}})^2 + \widehat{\sigma}_s^2}\right)dW_s, \tag{3.7}$$

with $X_t = \xi$. Thus, the following simple corollary will be useful for MARL (see also more details in Section 4 where this result is used for algorithm design).

Corollary 3.1. *If the representative agent follows policy* (3.6) *under a given mean field information $\mu = \{\mu_s\}_{s\in[t,T]}$, then the payoff is given by*

$$\begin{aligned} J_{SE}(t,\widehat{\pi}\,|\,\mu) &= -\frac{Q}{2}\int_t^T \left(\phi_s^2 - 2m_s\,\widehat{m}_s + m_s^2\right)ds \\ &\quad + \frac{\lambda_{SE}}{2}\int_t^T \log\left(2\pi e\widehat{\sigma}_s^2\right)ds - \frac{\bar{Q}}{2}\left(\phi_T^2 - 2m_T\,\widehat{m}_T + m_T^2\right), \end{aligned} \tag{3.8}$$

where

$$\phi_s^2 = e^{\left(2\widehat{K} + D^2\widehat{M}^2\right)(s-t)}\left(\mathbb{E}[\xi^2] + \int_t^s e^{-D^2\widehat{M}^2(z-t)}d(z)dz\right),$$

with

$$d(s) = -2\mathbb{E}[\xi]\,e^{-\widehat{K}(s-t)}\widehat{K}m_s + \left(\int_t^s e^{-\widehat{K}(z-t)}\widehat{K}m_z dz\right)e^{-\widehat{K}(s-t)}\widehat{K}m_s + e^{-2\widehat{K}(s-t)}D^2\left(\widehat{M}^2m_s^2 - 2\widehat{M}^2m_s\widehat{m}_s + \widehat{\sigma}_s^2\right)$$

$$m_s = \int_{\mathbb{R}}\mu_s(x)dx, \qquad \widehat{m}_s = e^{\widehat{K}(s-t)}\mathbb{E}[\xi] + \int_t^s e^{\widehat{K}(s-z)}\widehat{K}m_z\,dz, \qquad \text{and} \quad \widehat{K} = -\left(A + B\widehat{M}\right).$$

Next, we analyze the game with an additional cross-entropy regularization.

3.2 Game with Enhanced Entropy (Linear Combination of Shannon Entropy and Cross-entropy)

The objective of this game is to find

$$\begin{aligned} V_{EE}(t\,|\,\mu,\alpha) :=& \sup_{\pi\in\mathcal{A}} J_{EE}(t,\pi\,|\,\mu,\alpha) \\ :=& \sup_{\pi\in\mathcal{A}}\mathbb{E}\left[\int_t^T\left(-\frac{Q}{2}(X_s^\pi - m_s)^2 + \lambda_{SE}\mathcal{H}_{SE}(\pi_s) + \lambda_{CE}\mathcal{H}_{CE}(\pi_s,\alpha_s,\mu_s)\right)ds \right. \\ & \left. -\frac{\bar{Q}}{2}(X_T^\pi - m_T)^2\,\middle|\,\alpha,\mu\right], \end{aligned}$$

subject to

$$dX_s = \left(\int_{\mathbb{R}}(A(m_s - X_s^\pi) + Bu)\pi_s(u)du\right)ds + D\sqrt{\int_{\mathbb{R}}u^2\pi_s(u)du}dW_s, \quad X_t^\pi = \xi \sim \nu. \quad \textbf{(MFG-EE)}$$

Here $\mu_t = \nu$, $m_s = \int x\mu_s(dx)$, $s\in[t,T]$, and we assume $\lambda_{SE} > 0$, $\lambda_{CE} \geq 0$, $Q > 0$, $\bar{Q} > 0$, and $A > 0$. Without loss of generality, we further set $B > 0$ and $D > 0$.

Theorem 4 (MFG-EE). *Let* $m^* = \mathbb{E}[\xi]$ *and*

$$\widetilde{V}_{EE}(t,x) = -\frac{\eta_t^{EE}}{2}(x-m^*)^2 + \gamma_t^{EE}, \tag{3.9}$$

with

$$\begin{aligned}\eta_t^{EE} &= \bar{Q}\exp\left(-\left(2A + \frac{B^2}{D^2}\frac{\lambda_{SE}+\lambda_{CE}}{\lambda_{SE}}\right)(T-t)\right) \\ &\quad + \frac{Q}{2A + \frac{B^2}{D^2}\frac{\lambda_{SE}+\lambda_{CE}}{\lambda_{SE}}}\left(1 - \exp\left(-\left(2A + \frac{B^2}{D^2}\frac{\lambda_{SE}+\lambda_{CE}}{\lambda_{SE}}\right)(T-t)\right)\right),\end{aligned} \tag{3.10}$$

where $\eta_t^{EE} > 0$*, for* $t \in [0,T]$*, and*

$$\begin{aligned}\gamma_t^{EE} &= \frac{\lambda_{SE}+\lambda_{CE}}{2}\ln\left(\frac{2\pi(\lambda_{SE}+\lambda_{CE})}{D^2}\right)(T-t) - \int_t^T \frac{\lambda_{SE}+\lambda_{CE}}{2}\ln(\eta_z)dz \\ &\quad + \int_t^T \frac{\lambda_{CE}}{2}\frac{B^2\eta_z(\lambda_{SE}+\lambda_{CE})}{D^2\lambda_{SE}^2}\kappa_z dz,\end{aligned}$$

with

$$\kappa_s^{EE} = e^{(2K+M)(s-t)}\,Var[\xi] + \int_t^s e^{(M+2K)(s-z)}\frac{\lambda_{SE}+\lambda_{CE}}{\eta_z}dz, \tag{3.11}$$

and

$$K = -\left(A + \frac{B^2}{D^2}\right)\frac{\lambda_{SE}+\lambda_{CE}}{\lambda_{SE}}, \quad M = \left(\frac{B}{D}\frac{\lambda_{SE}+\lambda_{CE}}{\lambda_{SE}}\right)^2.$$

Then,

$$V_{EE}^*(t) := \mathbb{E}_{\xi\sim\nu}\left[\widetilde{V}_{EE}(t,\xi\,|\,\mu^*,\alpha^*)\right]$$

is a game value of **(MFG-EE)***, with the associated NE policy*

$$\pi_s^{EE*}(u;x) = \mathcal{N}\left(\frac{\lambda_{SE}+\lambda_{CE}}{\lambda_{SE}}\frac{B(m^*-x)}{D^2}, \frac{\lambda_{SE}+\lambda_{CE}}{D^2\eta_s^{EE}}\right). \tag{3.12}$$

Furthermore, the optimal controlled state process X_s^* *under* (3.12) *is the unique solution of the SDE,*

$$\begin{aligned}dX_s^* &= \left(A + \frac{\lambda_{SE}+\lambda_{CE}}{\lambda_{SE}}\frac{B^2}{D^2}\right)(m^* - X_s^*)ds \\ &\quad + D\sqrt{\left(A + \frac{\lambda_{SE}+\lambda_{CE}}{\lambda_{SE}}\frac{B^2}{D^2}\right)^2(X_s^*-m)^2 + \frac{\lambda_{SE}+\lambda_{CE}}{D^2\eta_s^{EE}}}dW_s, \\ X_t^* &= \xi \sim \nu,\ s \in [t,T].\end{aligned} \tag{3.13}$$

In addition, $\mu_t^* = \mathbb{P}_{X_t^*}$*,* $\alpha_s^*(u;x) = \pi_s^{EE*}(u;x)$*, and the mean state of the population under policy* (3.12) *is time independent, i.e.,*

$$m_s^* = \mathbb{E}[X_s^*] = m^*, \quad s \in [t,T]. \tag{3.14}$$

Before providing detailed proofs and derivation of these NE solutions, a few remarks are in place.

3.3 Discussion.

In both games, with either only Shannon entropy (**MFG-SE**) or with the additional cross-entropy (**MFG-EE**), there are several similarities.

- The form of the optimal policies (3.3) and (3.12) suggests that Gaussian exploration is optimal when entropy regularization is introduced in MFG with learning. This is consistent with recent works of [24, 25] for continuous-time single-agent RL and is also supported by empirical studies of [17] and [21].

- Both means of the optimal policy process $\pi^*_{SE}(u, X^*_t)$ implied from policy (3.3) and the optimal policy process $\pi^*_{EE}(u, X^*_t)$ implied from (3.12) are influenced by the mean field interaction and the current state of the representative agent, while their variances are time-dependent.

  In addition, the strength of mean reversion is quantified by the coefficient $\frac{B}{D^2}$, which indicates that a smaller variance signifies less uncertainty in the game, hence a faster mean reverting policy.

- Equation (3.2) for η^{SE}_s and equation (3.10) for η^{CE}_s suggest that when s is sufficiently small, the term $\frac{Q}{2A+\frac{B^2}{D^2}}\left(1-\exp\left(-\left(2A+\frac{B^2}{D^2}\right)(T-s)\right)\right)$ dominates η^{SE}_s, whereas η^{EE}_s is dominated by $\frac{Q}{2A+\frac{B^2}{D^2}\frac{\lambda_{SE}+\lambda_{CE}}{\lambda_{SE}}}\left(1-\exp\left(-\left(2A+\frac{B^2}{D^2}\frac{\lambda_{SE}+\lambda_{CE}}{\lambda_{SE}}\right)(T-s)\right)\right)$. Thus, when s is small, the cost of exploration is low and the representative agent has more incentive to explore in upcoming times.

  Conversely, when s is sufficiently large and especially when $s \sim T$, η^{SE}_s is dominated by the term $\bar{Q}\exp\left(-\left(2A+\frac{B^2}{D^2}\right)(T-s)\right)$, whereas $\bar{Q}\exp\left(-\left(2A+\frac{B^2}{D^2}\frac{\lambda_{SE}+\lambda_{CE}}{\lambda_{SE}}\right)(T-s)\right)$ dominates η^{EE}_s. Thus, the cost of exploration increases as s approaches T. This implies that the agent is more sensitive to the terminal reward and explore less when the game approaches termination.

- In the very special case $A = Q = 0$, there is no intermediate payoff. Then, the variances of π^*_{SE} and π^*_{EE} decrease when s increases, implying more exploration at the very beginning and less towards the very end.

Despite the above similarities, there is an important difference:

- Shannon entropy and the cross-entropy affect the opitmal policy π^*_{SE} and π^*_{EE} differently. Indeed, the mean of the optimal policy process $\pi^*_{EE}(u, X^*_t)$ depends on the ratio between λ_{CE} and λ_{SE}, while λ_{SE} and λ_{CE} impact the variance of $\pi^*_{EE}(u, X^*_t)$ through both $\frac{\lambda_{CE}}{\lambda_{SE}}$ and $\lambda_{SE}+\lambda_{CE}$. In particular, with the additional cross-entropy, one will explore more and, consequently, the learning procedure would converge faster.

3.4 Derivations and Proofs of Main Results

The derivation of the solutions is based on the classical fixed-point approach for game (2.7). It consists of three steps:

- **Step 1:** Fix a population state-action distribution (μ, α) and an initial state x. Then, solving the MFG is reduced to solving a stochastic control problem with randomized policies (relaxed controls).

- **Step 2:** Let $X_s^{\pi,x}$ be the controlled state process under policy π from the initial state x in Step 1. Update $\alpha'_s(\cdot, y) = \pi_s(\cdot, y)$ for all $y \in \mathbb{R}$ and $s \in [t, T]$. Denote $X_s^{\pi,\xi}$ the controlled state process under π from some random initial state $\xi \sim \nu$. Then, update $\mu'_s = \mathbb{P}_{X_s^{\xi,\pi}}$.
- **Step 3:** Repeat Steps 1 and 2 until (μ', α') converges.

Note that there is no guarantee that the above procedure will yield any MFG solution since **Step 1** may have multiple solutions. Moreover, by the nature of relaxed controls, the potential fixed point(s) would be the fixed point(s) of a set-valued map as described in [15]. Nevertheless, with specific solution structures from **Step 1**, one can build proper verification argument to show that the explicit fixed-point solution is indeed a solution to the MFG problem (2.7).

NE Derivation of (MFG-SE). To ease the exposition, we drop the subscript SE.

Proof of Theorem 2. For a given admissible policy $\pi \in \mathcal{A}$, the forward equation for $p(s, x)$, the density of X_s, is given by,

$$\partial_s p(s,x) = -\partial_x \left(\left(A(m_s - x) + B \int_{\mathbb{R}} u\, \pi_s(u;x) du \right) p(s,x) \right) + \frac{1}{2} \partial_{xx} \left(p(s,x) \int_{\mathbb{R}} D^2 u^2\, \pi_s(u;x) du \right),$$

with initial density $p(t, x) = \nu(x)$. Here, $m_s = \int x p(s, x) dx$, $s \in [t, T]$.

We will first proceed heuristically with an associated HJB equation, derive its solution, and then validate this solution through a verification step.

The HJB equation for the value function $\widetilde{V}(s, x)$ can be written as

$$\begin{aligned} -\partial_s \widetilde{V}(s,x) = \max_{\pi_s \in \mathcal{P}(\mathbb{R})} \Bigg(& \left(A(m_s - x) + B \int_{\mathbb{R}} u \pi(u;s,x) du \right) \widetilde{V}_x(s,x) \\ & - \frac{Q}{2}(m_s - x)^2 - \lambda_{SE} \int_{\mathbb{R}} \pi_s(u;x) \ln \pi_s(u;x) du + \frac{1}{2} \left(\int_{\mathbb{R}} D^2 u^2 \pi_s(u;x) du \right) \partial_{xx} \widetilde{V}(s,x) \Bigg). \end{aligned} \tag{3.15}$$

with terminal condition $\widetilde{V}(T, x) = -\frac{\overline{Q}}{2}(m_T - x)^2$. Recall that $\pi_s(u; x) \in \mathcal{P}(U)$ if and only if (2.1) holds. Solving the constrained maximization problem on the right hand side of (3.15) yields

$$\pi_s^*(u;x) = \frac{\exp\left(\frac{1}{\lambda_{SE}} \left(-\frac{Q}{2}(x - m_s)^2 + \frac{1}{2} D^2 u^2 \partial_{xx} \widetilde{V} + (A(m_s - x) + Bu) \widetilde{V}_x \right) \right)}{\int_{\mathbb{R}} \exp\left(\frac{1}{\lambda_{SE}} \left(-\frac{Q}{2}(x - m_s)^2 + \frac{1}{2} D^2 u^2 \partial_{xx} \widetilde{V} + (A(m_s - x) + Bu) \widetilde{V}_x \right) \right) du}.$$

Thus, the optimal policy is expected to be *Gaussian* with mean $\frac{B \partial_x \widetilde{V}}{-D^2 \partial_{xx} \widetilde{V}}$ and variance $\frac{\lambda_{SE}}{-D^2 \partial_{xx} \widetilde{V}}$, where it is for now assumed (and will be later verified) that $\partial_{xx} \widetilde{V} < 0$, i.e.,

$$\pi_s^*(u;x) = \mathcal{N}\left(\frac{B \partial_x \widetilde{V}}{-D^2 \partial_{xx} \widetilde{V}}, \frac{\lambda_{SE}}{-D^2 \partial_{xx} \widetilde{V}} \right).$$

Therefore,

$$\int_{\mathbb{R}} u \pi_s^*(u;x) du = \frac{B \partial_x \widetilde{V}}{-D^2 \partial_{xx} \widetilde{V}} \quad \text{and} \quad \int_{\mathbb{R}} u^2 \pi_s^*(u;x) du = \left(\frac{B \partial_x \widetilde{V}}{-D^2 \partial_{xx} \widetilde{V}} \right)^2 + \frac{\lambda_{SE}}{-D^2 \partial_{xx} \widetilde{V}}.$$

Next we introduce the ansatz

$$\widetilde{V}(s,x) = -\frac{\eta_s}{2}(m_s - x)^2 + \gamma_s. \tag{3.16}$$

with $\eta_s > 0$. Then, $\partial_x \widetilde{V} = -\eta_s(x - m_s)$ and $\partial_{xx}\widetilde{V} = -\eta_s$, and thus,

$$\int_{\mathbb{R}} u\pi_s^*(u;x)du = \frac{B(m_s - x)}{D^2},$$

and

$$\int_{\mathbb{R}} u^2\pi_s^*(u;x)du = \left(\frac{B(m_s - x)}{D^2}\right)^2 + \frac{\lambda_{SE}}{D^2\eta_s}.$$

Denoting $\kappa = \frac{B}{D^2}$ and plugging in the forward equation for $p(s,x)$ yield

$$\begin{aligned}
\partial_s p(s,x) = \quad & -\partial_x\left[\left((A + B\kappa)(m_s - x)\right)p(s,x)\right] \\
& + \frac{1}{2}D^2\partial_{xx}\left(\left(\kappa^2(m_s - x)^2 + \frac{\lambda_{SE}}{D^2\eta_s}\right)p(s,x)\right), \\
= \quad & (A + B\kappa)\,p(s,x) - \left((A + B\kappa)(m_s - x)\right)\partial_x p(s,x) \\
& + \frac{1}{2}D^2\left(2\kappa^2 p(s,x) + 4\kappa^2(x - m_s)\partial_x p(s,x)\right) \\
& + \frac{1}{2}D^2\left(\left(\kappa^2(m_s - x)^2 + \frac{\lambda_{SE}}{D^2\eta_s}\right)\partial_{xx}p(s,x)\right).
\end{aligned}$$

Multiplying by x and integrating with respect to x show that $dm_s = \left\{\int x\partial_s p(s,dx)\right\}ds = 0$. Therefore, $m_s^* = \mathbb{E}_{\xi\sim\nu}[\xi] =: m^*$. Furthermore, the HJB equation for $s \in [t,T)$ is reduced to

$$\begin{aligned}
-\partial_s\widetilde{V}(s,x) = \quad & \max_{\pi_s\in\mathcal{P}(\mathbb{R})}\left(\left(A(m^* - x) + B\int_{\mathbb{R}} u\pi_s(u;x)du\right)\partial_x\widetilde{V}(s,x)\right. \\
& \left. -\frac{Q}{2}(m^* - x)^2 - \lambda_{SE}\int_{\mathbb{R}}\pi_s(u;x)\ln\pi_s(u;x)du + \frac{1}{2}D^2\left(\int_{\mathbb{R}} u^2\pi_s(u;x)du\right)\partial_{xx}\widetilde{V}(s,x)\right).
\end{aligned} \tag{3.17}$$

with $\widetilde{V}(T,x) = -\frac{\overline{Q}}{2}(m^* - x)^2$. Plugging $\pi_s^*(u;x)$ and using ansatz (3.16), with $m_s = m^*$, into the above HJB give

$$\begin{aligned}
\frac{\dot{\eta}_s}{2}(x - m^*)^2 - \dot{\gamma}_s = \quad & -\left(A(m^* - x) + \frac{B^2(m^* - x)}{D^2}\right)\eta_s(x - m^*) \\
- \quad & \frac{Q}{2}(m^* - x)^2 + \lambda_{SE}\ln\left(\sqrt{\frac{2\pi e\lambda_{SE}}{D^2\eta_s}}\right) \\
- \quad & \frac{1}{2}D^2\left(\left(\frac{B(m^* - x)}{D^2}\right)^2 + \frac{\lambda_{SE}}{D^2\eta_s}\right)\eta_s.
\end{aligned}$$

Direct calculations imply

$$\dot{\eta}_s = \left(2A + \frac{B^2}{D^2}\right)\eta_s - Q \tag{3.18}$$

with $\eta_T = \bar{Q}$, and

$$\dot{\gamma}_s \quad = \quad -\frac{\lambda_{SE}}{2}\ln\left(\frac{2\pi\lambda_{SE}}{D^2}\right) + \frac{\lambda_{SE}}{2}\ln(\eta_s) \tag{3.19}$$

with $\gamma_T = 0$. Then, (3.18) admits the unique solution

$$\eta_s = \bar{Q}\exp\left(-\left(2A + \frac{B^2}{D^2}\right)(T - s)\right) + \frac{Q}{2A + \frac{B^2}{D^2}}\left(1 - \exp\left(-\left(2A + \frac{B^2}{D^2}\right)(T - s)\right)\right),$$

from which it is easy to verify that $\eta_s > 0$ since $A > 0, Q > 0$ and $\bar{Q} > 0$, $s \in [t, T]$. Moreover, (3.19) admits the unique solution

$$\gamma_s = \frac{\lambda_{SE}}{2} \ln\left(\frac{2\pi\lambda_{SE}}{D^2}\right)(T-s) - \int_s^T \frac{\lambda_{SE}}{2} \ln(\eta_z) dz.$$

Consequently, the NE (optimal) policy takes the form

$$\pi_s^*(u; x) = \mathcal{N}\left(\frac{B(m^* - x)}{D^2}, \frac{\lambda_{SE}}{D^2 \eta_s}\right),$$

and the associated optimal state process is the unique solution of the SDE (3.4).

Verification argument. The final step is to verify that m^* is the mean state under policy (3.3) and $V^*(t) := \mathbb{E}_{\xi\sim\nu}[\tilde{V}(\xi, t)] = \mathbb{E}_{\xi\sim\nu}[\frac{\eta_t}{2}(\xi - m^*)^2 + \gamma_t]$ is the corresponding game value.

First, let us fix the mean field information as $m_s = m^*$, $s \in [t, T]$, and also fix the initial state $x \in \mathbb{R}$ and initial time $t \in [0, T]$. Let $\pi \in \mathcal{A}(x)$ and X^π be the associated state process under π solving

$$dX_s^\pi = \left(\int_{\mathbb{R}} (A(m^* - X_s^\pi) + Bu)\pi_s(u)du\right) ds + D\left(\sqrt{\int_{\mathbb{R}} u^2 \pi_s(u)du}\right) dW_s.$$

Denote $\tilde{r}(x, \pi) = -\frac{Q}{2}(x - m^*)^2$, $\tilde{b}(x, \pi) = \int_{\mathbb{R}} (A(m^* - x) + Bu)\pi_s(u)du$, and $\tilde{\sigma}(x, \pi) = D\left(\sqrt{\int_{\mathbb{R}} u^2 \pi(u)du}\right)$. Further, define the stopping time $\tau_n^\pi := \left\{s \geq t : \int_t^T \partial_x \widetilde{V}(t, X_s^\pi)\tilde{\sigma}^2(X_s^\pi, \pi_s)^2 ds \geq n\right\}$, for $n \geq 1$. Then Itô's formula yields

$$\begin{aligned}\widetilde{V}(T \wedge \tau_n^\pi, X_{T\wedge\tau_n^\pi}^\pi) - \widetilde{V}(t, x) &= \int_t^{T\wedge\tau_n^\pi} \left(\frac{1}{2}\partial_{xx}\widetilde{V}(s, X_s^\pi)\tilde{\sigma}^2(X_s^\pi, \pi_s)^2 + \partial_x \widetilde{V}(s, X_s^\pi)\tilde{b}(X_s^\pi, \pi_s)\right) ds \\ &\quad + \int_t^{T\wedge\tau_n^\pi} \partial_x \widetilde{V}(X_s^\pi) dW_s.\end{aligned}$$

Taking expectations, noting that $\widetilde{V}$ solves the HJB equation (3.17), and that π is sub-optimal, we see

$$\begin{aligned}&\mathbb{E}\left[\widetilde{V}(T, X_{T\wedge\tau_n^\pi}^\pi)\right] \\ =\ &\widetilde{V}(t, x) + \mathbb{E}\left[\int_t^{T\wedge\tau_n^\pi} \left(\frac{1}{2}\partial_{xx}\widetilde{V}(s, X_s^\pi)\tilde{\sigma}^2(X_s^\pi, \pi_s)^2 + \partial_x \widetilde{V}(s, X_s^\pi)\tilde{b}(X_s^\pi, \pi_s)\right) ds + \int_t^{T\wedge\tau_n^\pi} \partial_x \widetilde{V}(X_s^\pi) dW_s\right] \\ \leq\ &\widetilde{V}(t, x) - \mathbb{E}\left[\int_t^{T\wedge\tau_n^\pi} \left(\tilde{r}(X_s^\pi, \pi_s) - \lambda \int_{\mathbb{R}} \pi_s(u) \ln \pi_s(u) du\right) ds\right].\end{aligned}$$

Standard calculations yield $\mathbb{E}[\sup_{t\leq s\leq T} |X_s^\pi|^2] \leq N(1 + x^2)e^{NT}$ for some constant $N > 0$, which is independent of n. Sending $n \to \infty$ yields

$$\widetilde{V}(t, x) \geq \mathbb{E}\left[\int_t^T \left(\tilde{r}(X_s^\pi, \pi_s) - \lambda \int_{\mathbb{R}} \pi_s(u) \ln \pi_s(u) du\right) ds - \frac{\bar{Q}}{2}(X_T^\pi - m^*)^2\right],$$

for each $x \in \mathbb{R}$ and $\pi \in \mathcal{A}$. Hence $\widetilde{V}(t, x) \geq V^*(t, x)$, for all $x \in \mathbb{R}$.

On the other hand, the right-hand of (3.15) is maximized at the following policy,

$$\pi_s^*(u;x) = \mathcal{N}\left(\frac{B(m^*-x)}{D^2}, \frac{\lambda_{SE}}{D^2\eta_s}\right).$$

Thus

$$\widetilde{V}(t,x) = \mathbb{E}\left[\int_0^T \left(\tilde{r}(X_s^*,\pi_s) - \lambda\int_{\mathbb{R}} \pi_s^*(u;X_s^*)\ln\pi_s^*(u;X_s^*)du\right)ds - \frac{\bar{Q}}{2}(X_T^*-m^*)^2\right],$$

where X_t^* is the state process under policy process $\pi_s^*(u;X_t^*)$. Hence, $\widetilde{V}(t,x)$ is the optimal value when the mean field information is fixed at $m_s = m^*$, $s\in[t,T]$.

Next, let us show that

$$m^* = \mathbb{E}[X_s^*] \text{ for } s\in[t,T]$$

where X_s^* is the controlled state process under the policy process

$$\pi_s^*(u;X_s^*) = \mathcal{N}\left(\frac{B(m^*-X_s^*)}{D^2}, \frac{\lambda_{SE}}{D^2\eta_s}\right).$$

To this end, denote $K = -\left(A+\frac{B^2}{D^2}\right)$. Then,

$$dX_s^* = (KX_s^* - Km^*)ds + f(s,X_s^*,m^*)dW_s,$$

with

$$f(s,x,m) = \left(\sqrt{\left(\frac{B}{D}(x-m)\right)^2 + \frac{\lambda_{SE}}{\eta_s}}\right).$$

Therefore,

$$e^{-K(s-t)}X_s^* \;=\; \xi + \int_t^s e^{-K(z-t)}\left(-Km^*dz + f(z,X_z^*,m^*)dW_z\right),$$

and $e^{-K(s-t)}\mathbb{E}[X_s^*] = \mathbb{E}[\xi] + \left(e^{-K(s-t)}-1\right)m^*$. Hence, $\mathbb{E}[X_s^*] = m^*$, $s\in[t,T]$.

□

NE Derivation of Game (MFG-EE). To ease the exposition, we drop the subscript EE.

Proof of Theorem 4. For a given Markovian policy $\pi_s(u;x)$, the forward equation for $p(s,x)$, the density of X_s, $s\in[t,T]$ satisfies

$$\partial_s p(s,x) = -\partial_x\left(\left(A(m_s-x) + B\int_{\mathbb{R}} u\pi_s(u;x)du\right)p(s,x)\right) + \frac{1}{2}\partial_{xx}\left(p(s,x)\int_{\mathbb{R}} D^2u^2\pi_s(u;x)du\right),$$

with initial density $p(t,x) = \nu(x)$ and $m_s = \int xp(s,x)dx$.

The HJB equation for the value function $\widetilde{V}(s,x,m)$ can be written as

$$\begin{aligned}-\partial_s\widetilde{V}(s,x) = \quad & \max_{\pi_s\in\mathcal{P}(\mathbb{R})}\left(\left[A(m_s-x) + B\int_{\mathbb{R}} ud\pi_s(u;x)\right]\partial_x\widetilde{V}(s,x) - \frac{Q}{2}(m_s-x)^2\right.\\ & -\lambda_{CE}\int_{\mathbb{R}}\pi_s(u;x)\int\ln\alpha_s(u;x)\mu_s(dx)du - \lambda_{SE}\int_{\mathbb{R}}\pi_s(u;x)\ln\pi_s(u;x)du\\ & \left.+\frac{1}{2}\left(\int_{\mathbb{R}} D^2u^2\pi_s(u;x)du\right)\partial_{xx}\widetilde{V}(s,x)\right),\end{aligned}$$

with $\widetilde{V}(T,x) = -\frac{\bar{Q}}{2}(x-m_T)^2$. Recall that $\pi_s \in \mathcal{P}(U)$ if and only if (2.1) holds. The constrained maximization problem on the right hand side of (3.15) yields

$$\pi_s^*(u;x) = \frac{\exp\left(\frac{1}{\lambda_{SE}}\left(-\frac{Q}{2}(x-m_s)^2 + \frac{1}{2}D^2u^2\partial_{xx}\widetilde{V} + (A(m_s-x)+Bu)\partial_x\widetilde{V} - \lambda_{CE}\int\Big(\ln\alpha_s(u;x)\Big)\mu_s(dx)\Big)\right)}{\int_{\mathbb{R}}\exp\left(\frac{1}{\lambda_{SE}}\left(-\frac{Q}{2}(x-m_s)^2 + \frac{1}{2}D^2u^2\partial_{xx}\widetilde{V} + (A(m_s-x)+Bu)\partial_x\widetilde{V} - \lambda_{CE}\int\Big(\ln\alpha_s(u;x)\Big)\mu_s(dx)\Big)\right)du}.$$

Next, let us introduce the ansatz for the population action distribution for the agent in state y,

$$\alpha_s(u;y) = \mathcal{N}(u \,|\, H_s(y-m_s), L_s) \tag{3.20}$$

with some deterministic processes H_s and L_s, $s \in [t,T]$. Then, $\alpha_s(u;y)$ is Gaussian with mean $H_s(y-m_s)$ and variance $L_s > 0$. We stress that the Gaussian property of $\alpha_s(u;y)$ does not imply the Gaussian property of the aggregated population action distribution $\widetilde{\alpha}(s) = \int \alpha_s(u;y)\mu_s(dy)$.

In turn, $\ln\alpha_s(u;y) = -\frac{1}{2}\ln(2\pi L_s) - \frac{1}{2L_s}(u - H_s(y-m_s))^2$ and

$$\begin{aligned}\int \mu_s(dy)\ln(\alpha_s(u;y)) &= -\frac{1}{2}\ln(2\pi L_s) - \frac{1}{2L_s}\int(u-H_s(y-m_s))^2\mu_t(dy)\\ &= -\frac{1}{2}\ln(2\pi L_s) - \frac{1}{2L_s}(u^2 + H_s^2\mathrm{Var}(\mu_s)),\end{aligned}$$

with $\mathrm{Var}(\mu_s) = \int x^2\mu_s(dx) - (\int x\mu_s(dx))^2$. Therefore, the optimal policy is *Gaussian* with mean $\frac{B\partial_x\widetilde{V}}{-D^2\partial_{xx}\widetilde{V} - \frac{\lambda_{CE}}{L_s}}$ and variance $\frac{\lambda_{SE}}{-D^2\partial_{xx}\widetilde{V} - \frac{\lambda_{CE}}{L_s}}$, i.e.,

$$\pi_s^*(u;x) = \mathcal{N}\left(u \;\middle|\; \frac{B\partial_x\widetilde{V}}{-D^2\partial_{xx}\widetilde{V} - \frac{\lambda_{CE}}{L_s}}, \frac{\lambda_{SE}}{-D^2\partial_{xx}\widetilde{V} - \frac{\lambda_{CE}}{L_s}}\right).$$

Let us for now assume (and verify later) that $-D^2\partial_{xx}\widetilde{V} - \frac{\lambda_{CE}}{L_s} > 0$.

Hence,

$$\int_{\mathbb{R}} u\pi_s^*(u;x)du = \frac{B\partial_x\widetilde{V}}{-D^2\partial_{xx}\widetilde{V} - \frac{\lambda_{CE}}{L_s}},$$

and

$$\int_{\mathbb{R}} u^2\pi_s^*(u;x)du = \left(\frac{B\partial_x\widetilde{V}}{-D^2\partial_{xx}\widetilde{V} - \frac{\lambda_{CE}}{L_s}}\right)^2 + \frac{\lambda_{SE}}{-D^2\partial_{xx}\widetilde{V} - \frac{\lambda_{CE}}{L_s}}.$$

Next, consider the ansatz

$$\widetilde{V}(s,x) = -\frac{\eta_s}{2}(m_s-x)^2 + \gamma_s. \tag{3.21}$$

In turn, $\partial_x\widetilde{V} = -\eta_s(x-m_s)$ and $\partial_{xx}\widetilde{V} = -\eta_s$, together with

$$\int_{\mathbb{R}} u\pi_s^*(u;x)du = \frac{B\eta_s(m_s-x)}{D^2\eta_s - \frac{\lambda_{CE}}{L_s}},$$

and

$$\int_{\mathbb{R}} u^2\pi_s^*(u;x)du = \left(\frac{B\eta_s(m_s-x)}{D^2\eta_s - \frac{\lambda_{CE}}{L_s}}\right)^2 + \frac{\lambda_{SE}}{D^2\eta_s - \frac{\lambda_{CE}}{L_s}}.$$

Denoting $\kappa_s := \frac{B\eta_s}{D^2\eta_s - \frac{\lambda_{CE}}{L_s}}$, $s \in [t,T]$, and plugging in the forward equation for $p(s,x)$, we deduce that

$$\begin{aligned}
\partial_s p(s,x) = \quad & -\partial_x\left((A+B\kappa_s)(m_s-x)p(s,x)\right) \\
& +\frac{1}{2}D^2\partial_{xx}\left(\left(\kappa_t^2(m_s-x)^2 + \frac{\lambda_{SE}}{D^2\eta_s - \frac{\lambda_{CE}}{L_s}}\right)p(s,x)\right) \\
= \quad & (A+B\kappa_s)\,p(t,x) - \left((A+B\kappa_s)(m_s-x)\right)\partial_x p(s,x) \\
& +\frac{1}{2}D^2\left(2\kappa_s^2 p(s,x) + 4\kappa_s^2(x-m_s)\partial_x p(s,x)\right) \\
& +\frac{1}{2}D^2\left(\left(\kappa_s^2(m_s-x)^2 + \frac{\lambda_{SE}}{D^2\eta_s - \frac{\lambda_{CE}}{L_s}}\right)\partial_{xx}p(s,x)\right).
\end{aligned}$$

Multiplying by x and integrating with respect to x give $\partial_s m_s = \int x\partial_s p(s,dx) = 0$ and, thus, $m_s^* = m^* = \mathbb{E}_{\xi\sim\nu}[\xi]$, for $t \le s \le T$. Hence, the HJB equation reduces to

$$\begin{aligned}
-\partial_s\widetilde{V}(s,x) = \quad & \max_{\pi_s\in\mathcal{P}(\mathbb{R})}\Big(\left(A(m^*-x) + B\int_{\mathbb{R}} u d\pi_s(u)\right)\partial_x\widetilde{V}(s,x) - \tfrac{Q}{2}(m^*-x)^2 \\
& -\lambda_{SE}\int_{\mathbb{R}}\pi_s(u)\ln\pi_s(u)du - \lambda_{CE}\int_{\mathbb{R}}\pi_s(u)\int\mu_s(dy)\ln\alpha_s(u;y)dydu \\
& +\tfrac{1}{2}D^2\int_{\mathbb{R}}u^2\pi_s(u)du\partial_{xx}\widetilde{V}(s,x)\Big).
\end{aligned}$$

Plugging $\pi_s^*(u;x)$ and ansatz (3.21) with $m_s = m^*$ into the above HJB, we obtain

$$\begin{aligned}
\frac{\dot\eta_s}{2}(x-m^*)^2 - \dot\gamma_s = \quad & - \left(A(m^*-x) + B\frac{B\eta_s(m^*-x)}{D^2\eta_s - \frac{\lambda_{CE}}{L_s}}\right)\eta_s(x-m^*) + \lambda_{SE}\ln\left(\sqrt{\frac{2\pi e\lambda_{SE}}{D^2\eta_s - \frac{\lambda_{CE}}{L_s}}}\right) \\
& - \frac{1}{2}D^2\left(\left(\frac{(B\eta_s)(m^*-x)}{D^2\eta_s - \frac{\lambda_{CE}}{L_s}}\right)^2 + \frac{\lambda_{SE}}{D^2\eta_s - \frac{\lambda_{CE}}{L_s}}\right)\eta_s - \frac{Q}{2}(m^*-x)^2 \\
& + \frac{\lambda_{CE}}{2}\ln(2\pi L_s) + \frac{\lambda_{CE}}{2L_s}\left(\left(\frac{B\eta_s(m^*-x)}{D^2\eta_s - \frac{\lambda_{CE}}{L_s}}\right)^2 + \frac{\lambda_{SE}}{D^2\eta_s - \frac{\lambda_{CE}}{L_s}}\right) \\
& + \frac{\lambda_{CE}}{2L_s}H_s^2\mathrm{Var}(\mu_s).
\end{aligned}$$

Direct calculations yield

$$\dot\eta_s = 2A\eta_s + \frac{(B\eta_s)^2}{D^2\eta_s - \frac{\lambda_{CE}}{L_s}} - Q, \tag{3.22}$$

$$\dot\gamma_s = \frac{\lambda_{SE}}{2} - \lambda_{SE}\ln\left(\sqrt{\frac{2\pi e\lambda_{SE}}{D^2\eta_s - \frac{\lambda_{CE}}{L_s}}}\right) - \frac{\lambda_{CE}}{2}\ln(2\pi L_s) - \frac{\lambda_{CE}}{2L_s}H_s^2\mathrm{Var}(\mu_s). \tag{3.23}$$

Setting

$$L_s = \frac{\lambda_{SE}}{D^2\eta_s - \frac{\lambda_{CE}}{L_s}} \quad \textit{and} \quad H_s = \frac{-B\eta_s}{D^2\eta_s - \frac{\lambda_{CE}}{L_s}},$$

we deduce

$$H_s = -\frac{B}{D^2}\frac{\lambda_{SE}+\lambda_{CE}}{\lambda_{SE}} \quad \textit{and} \quad L_s = \frac{\lambda_{SE}+\lambda_{CE}}{D^2\eta_s},$$

$$\dot{\eta}_s = 2A\eta_s + \frac{B^2\eta_s}{D^2}\frac{\lambda_{SE}+\lambda_{CE}}{\lambda_{SE}} - Q, \tag{3.24}$$

$$\dot{\gamma}_s = -\frac{\lambda_{SE}+\lambda_{CE}}{2}\ln\left(\frac{2\pi(\lambda_{SE}+\lambda_{CE})}{D^2\eta_s}\right) - \frac{\lambda_{CE}}{2}\frac{B^2\eta_s(\lambda_{SE}+\lambda_{CE})}{D^2\lambda_{SE}^2}\mathrm{Var}(\mu_s). \tag{3.25}$$

Consequently,

$$\pi_s^*(u;x) = \mathcal{N}\left(u \;\middle|\; \frac{B}{D^2}\frac{\lambda_{SE}+\lambda_{CE}}{\lambda_{SE}}(m^*-x), \frac{\lambda_{SE}+\lambda_{CE}}{D^2\eta_s}\right).$$

Denote $-K = \left(A + B\frac{B}{D^2}\right)\frac{\lambda_{SE}+\lambda_{CE}}{\lambda_{SE}}$ and

$$f(s,x,m) = \left(\sqrt{\left(\frac{B}{D}\frac{\lambda_{SE}+\lambda_{CE}}{\lambda_{SE}}(x-m)\right)^2 + \frac{\lambda_{SE}+\lambda_{CE}}{\eta_s}}\right).$$

In turn,

$$dX_s^* = (KX_s^* - Km^*)ds + f(s,X_s^*,m^*)dW_s,$$

and

$$\begin{aligned} d(e^{-K(s-t)}X_s^*) &= -Ke^{-K(s-t)}X_s^*ds + e^{-K(s-t)}dX_s^* \\ &= -Ke^{-K(s-t)}X_s^*ds + e^{-K(s-t)}((KX_s^* - Km^*)ds + f(s,X_s^*,m^*)dW_s) \\ &= e^{-K(s-t)}(-Km^*ds + f(s,X_s^*,m^*)dW_s). \end{aligned}$$

Therefore,

$$e^{-K(s-t)}X_s^* = \xi + \int_t^s e^{-K(z-t)}(-Km^*dz + f(z,X_z^*,m^*)dW_z), \tag{3.26}$$

and

$$e^{-2K(s-t)}\mathrm{Var}[X_s^*] = \mathrm{Var}[\xi] + \mathbb{E}\left[\left(\int_t^s e^{-K(z-t)}f(z,X_z^*,m^*)dW_z\right)^2\right].$$

By Itô's isometry,

$$\begin{aligned} &\mathbb{E}\left[\left(\int_t^s e^{-K(z-t)}f(z,X_z^*,m^*)dW_z\right)^2\right] = \mathbb{E}\left[\int_t^s e^{-2K(z-t)}f^2(z,X_z^*,m^*)dz\right] \\ =\;& \mathbb{E}\left[\int_t^s e^{-2K(z-t)}\left(\left(\frac{B}{D}\frac{\lambda_{SE}+\lambda_{CE}}{\lambda_{SE}}(X_z^*-m^*)\right)^2 + \frac{\lambda_{SE}+\lambda_{CE}}{\eta_z}\right)dz\right] \\ =\;& \int_t^s e^{-2K(z-t)}\left(\left(\frac{B}{D}\frac{\lambda_{SE}+\lambda_{CE}}{\lambda_{SE}}\right)^2\mathrm{Var}[X_z^*] + \frac{\lambda_{SE}+\lambda_{CE}}{\eta_z}\right)dz. \end{aligned}$$

Let

$$y(s) = e^{-2K(s-t)}\mathrm{Var}[X_s^*],$$

$$M = \left(\frac{B}{D}\frac{\lambda_{SE}+\lambda_{CE}}{\lambda_{SE}}\right)^2 \quad \textit{and} \quad b(s) = e^{-2K(s-t)}\frac{\lambda_{SE}+\lambda_{CE}}{\eta_s}.$$

Thus,

$$y(s) = e^{M(s-t)}\left(y(t) + \int_t^s e^{-M(z-t)}b(z)dz\right),$$

and

$$e^{-2K(s-t)}\mathrm{Var}(X_s^*) = e^{M(s-t)}\left[\mathrm{Var}(\xi) + \int_t^s e^{-(M+2K)(z-t)}\frac{\lambda_{SE}+\lambda_{CE}}{\eta_z}dz\right].$$

Therefore,

$$\begin{aligned}\mathrm{Var}[X_s^*] &= e^{(2K+M)(s-t)}\mathrm{Var}[\xi] + e^{2K(s-t)}\int_t^s e^{M(s-z)}b(z)dz \\ &= e^{(2K+M)(s-t)}\mathrm{Var}[\xi] + \int_t^s e^{(M+2K)(s-z)}\frac{\lambda_{SE}+\lambda_{CE}}{\eta_z}dz. \end{aligned} \tag{3.27}$$

Assuming for the moment that $\eta_s > 0$, $s \in [t,T]$, hence $\mathrm{Var}[X_s^*]$ is well-defined. Hence (3.25) reduces to

$$\begin{aligned}\dot{\gamma}_s &= -\frac{\lambda_{SE}+\lambda_{CE}}{2}\ln\left(\frac{2\pi(\lambda_{SE}+\lambda_{CE})}{D^2}\right) \\ &\quad + \frac{\lambda_{SE}+\lambda_{CE}}{2}\ln\eta_s - \frac{\lambda_{CE}}{2}\frac{B^2\eta_s(\lambda_{SE}+\lambda_{CE})}{D^2\lambda_{SE}^2}\kappa_s, \end{aligned} \tag{3.28}$$

with $\gamma_T = 0$, where κ_s, $s \in [t,T]$,

$$\kappa_s := e^{(M+2K)(s-t)}\mathrm{Var}[\xi] + \int_t^s e^{(M+2K)(s-z)}\frac{\lambda_{SE}+\lambda_{CE}}{\eta_z}dz \tag{3.29}$$

with

$$K = -\left(A + B\frac{B}{D^2}\right)\frac{\lambda_{SE}+\lambda_{CE}}{\lambda_{SE}} \quad \text{and} \quad M = D^2\left(\frac{B}{D^2}\frac{\lambda_{SE}+\lambda_{CE}}{\lambda_{SE}}\right)^2.$$

Therefore, equation (3.24) admits the unique solution

$$\begin{aligned}\eta_s &= \bar{Q}\exp\left(-\left(2A + \frac{B^2}{D^2}\frac{\lambda_{SE}+\lambda_{CE}}{\lambda_{SE}}\right)(T-s)\right) \\ &\quad + \frac{Q}{2A + \frac{B^2}{D^2}\frac{\lambda_{SE}+\lambda_{CE}}{\lambda_{SE}}}\left(1 - \exp\left(-\left(2A + \frac{B^2}{D^2}\frac{\lambda_{SE}+\lambda_{CE}}{\lambda_{SE}}\right)(T-s)\right)\right). \end{aligned}$$

We easily deduce that $\eta_s > 0$, $s \in [t,T]$, since $\bar{Q} > 0, Q > 0$ and $A > 0$.

Moreover, (3.28) admits the solution

$$\begin{aligned}\gamma_s &= \frac{\lambda_{SE}+\lambda_{CE}}{2}(T-s)\ln\left(\frac{2\pi(\lambda_{SE}+\lambda_{CE})}{D^2}\right) - \int_s^T \frac{\lambda_{SE}+\lambda_{CE}}{2}\ln(\eta_z)dz \\ &\quad + \int_s^T \frac{\lambda_{CE}}{2}\frac{B^2\eta_z(\lambda_{SE}+\lambda_{CE})}{D^2\lambda_{SE}^2}\kappa_z dz. \end{aligned}$$

The associated optimal policy is given by

$$\pi_s^*(u;x) = \mathcal{N}\left(\frac{\lambda_{SE}+\lambda_{CE}}{\lambda_{SE}}\frac{B(m^*-x)}{D^2}, \frac{\lambda_{SE}+\lambda_{CE}}{D^2\eta_s}\right), \tag{3.30}$$

and that the optimal controlled state process is the unique solution of the SDE (3.13). The verification is similar to the verification of Theorem 2 and is therefore skipped here. □

Proof of Corollary 3.1

Proof. Let $\widehat{K} := -\left(A + B\widehat{M}\right)$ and $f\left(m_s, X_s, \widehat{M}, \widehat{\sigma}_s^2\right) := D\sqrt{\widehat{M}^2(m_s - X_s)^2 + \widehat{\sigma}_s^2}$, $s \in [t, T]$. Then, under policy $\widehat{\pi}$ (3.6),

$$\begin{aligned} dX_s^{\widehat{\pi}} &= \left(\left(A + B\widehat{M}\right) m_s - \left(A + B\widehat{M}\right) X_s^{\widehat{\pi}}\right) ds + f\left(m_s, X_s^{\widehat{\pi}}, \widehat{M}, \widehat{\sigma}_s^2\right) dW_s \\ &= -\widehat{K} m_s ds + K X_s^{\widehat{\pi}} ds + f dW_s. \end{aligned}$$

Using that $d(e^{-\widehat{K}s} X_s^{\widehat{\pi}}) = e^{-\widehat{K}s}\left(-\widehat{K} m_s ds + f\, dW_s\right)$, we have

$$e^{-\widehat{K}(s-t)} X_s^{\widehat{\pi}} = \xi + \int_t^s e^{-\widehat{K}(z-t)}(-\widehat{K} m_z dz + f\, dW_z). \tag{3.31}$$

Hence,

$$\mathbb{E}[X_s^{\widehat{\pi}}] = e^{\widehat{K}(s-t)}\mathbb{E}[\xi] - \int_t^s e^{\widehat{K}(s-z)} \widehat{K} m_z\, dz.$$

Denote this updated mean field information flow as $\widehat{m}_s := \mathbb{E}[X_s^{\widehat{\pi}}]$.

From (3.31) and routine calculations, we have

$$\begin{aligned} e^{-2\widehat{K}(s-t)}\left(X_s^{\widehat{\pi}}\right)^2 &= \xi^2 + 2\xi \int_t^s e^{-\widehat{K}(z-t)}(-\widehat{K} m_z dz + f\, dW_z) + \left(\int_t^s e^{-\widehat{K}(z-t)} \widehat{K} m_z dz\right)^2 \\ &\quad + \left(\int_t^s e^{-\widehat{K}(z-t)} f\, dW_z\right)^2 - 2\int_t^s e^{-\widehat{K}(z-t)} \widehat{K} m_z dz \int_t^s e^{-\widehat{K}(z-t)} f\, dW_z. \end{aligned}$$

Hence,

$$\begin{aligned} &e^{-2\widehat{K}(s-t)}\mathbb{E}\left[\left(X_s^{\widehat{\pi}}\right)^2\right] \\ &= \mathbb{E}[\xi^2] - 2\mathbb{E}[\xi]\int_t^s e^{-\widehat{K}(z-t)} \widehat{K} m_z dz + \left(\int_t^s e^{-\widehat{K}(z-t)} \widehat{K} m_z dz\right)^2 + \mathbb{E}\left(\int_t^s e^{-\widehat{K}(z-t)} f\, dW_z\right)^2 \end{aligned} \tag{3.32}$$

By Itô's isometry,

$$\begin{aligned} &\mathbb{E}\left(\int_t^s e^{-\widehat{K}(z-t)} f\, dW_z\right)^2 = \mathbb{E}\left[\int_t^s e^{-2\widehat{K}(z-t)} f^2\, dz\right] \\ &= \mathbb{E}\left[\int_t^s e^{-2\widehat{K}(z-t)} D^2\left(\widehat{M}^2\left(m_z - X_z^{\widehat{\pi}}\right)^2 + \widehat{\sigma}_z^2\right) dz\right] \\ &= \mathbb{E}\left[\int_t^s e^{-2\widehat{K}(z-t)} D^2 \widehat{M}^2\left(m_z^2 - 2m_z X_z^{\widehat{\pi}} + \left(X_z^{\widehat{\pi}}\right)^2\right) dz + \int_t^s e^{-2K(z-t)} D^2 \widehat{\sigma}_z^2 dz\right] \\ &= \mathbb{E}\left[\int_t^s e^{-2\widehat{K}(z-t)} D^2 \widehat{M}^2\left(X_z^{\widehat{\pi}}\right)^2 dz\right] + \int_t^s e^{-2\widehat{K}(z-t)} D^2\left(\widehat{M}^2 m_z^2 - 2\widehat{M}^2 m_z \widehat{m}_z + \widehat{\sigma}_z^2\right) dz \\ &= D^2\widehat{M}^2 \int_t^s e^{-2\widehat{K}(z-t)}\mathbb{E}\left[\left(X_z^{\widehat{\pi}}\right)^2\right] dz + \int_t^s e^{-2\widehat{K}(z-t)} D^2\left(\widehat{M}^2 m_z^2 - 2\widehat{M}^2 m_z \widehat{m}_z + \widehat{\sigma}_z^2\right) dz \end{aligned} \tag{3.33}$$

Combining (3.32) and (3.33) yields

$$\begin{aligned} e^{-2K(s-t)}\mathbb{E}\left[\left(X_s^{\widehat{\pi}}\right)^2\right] &= \mathbb{E}[\xi^2] - 2\mathbb{E}[\xi]\int_t^s e^{-\widehat{K}(z-t)} \widehat{K} m_z dz + \left(\int_t^s e^{-\widehat{K}(z-t)} \widehat{K} m_z dz\right)^2 \\ &\quad + D^2\widehat{M}^2 \int_t^s e^{-2\widehat{K}(z-t)}\mathbb{E}\left[\left(X_z^{\widehat{\pi}}\right)^2\right] dz + \int_t^s e^{-2\widehat{K}(z-t)} D^2\left(\widehat{M}^2 m_z^2 - 2\widehat{M}^2 m_z \widehat{m}_z + \widehat{\sigma}_z^2\right) dz. \end{aligned}$$

Denoting $y_s = e^{-2\widehat{K}(s-t)}\mathbb{E}\left[\left(X_s^{\widehat{\pi}}\right)^2\right]$, we have

$$y_s - y_t = b_s + D^2\widehat{M}^2\int_t^s y_z dz,$$

where, for $s \in [t,T]$,

$$b_s := -2\mathbb{E}[\xi]\int_t^s e^{-\widehat{K}(z-t)}\widehat{K}m_z dz + \left(\int_t^s e^{-\widehat{K}(z-t)}\widehat{K}m_z dz\right)^2 + \int_t^s e^{-2\widehat{K}(z-t)}D^2\left(\widehat{M}^2 m_z^2 - 2\widehat{M}^2 m_z\widehat{m}_z + \widehat{\sigma}_z^2\right)dz,$$

with $b_t = 0$. Therefore,

$$\int_t^s \dot{y}_z dz = \int_t^s \dot{b}_z dz + D^2\widehat{M}^2\int_t^s y_z dz$$

and

$$y_s = e^{D^2\widehat{M}^2(s-t)}\left(y_t + \int_t^s e^{-D^2\widehat{M}^2(z-t)}\dot{b}(z)dz\right).$$

Finally,

$$\mathbb{E}\left[\left(X_s^{\widehat{\pi}}\right)^2\right] = e^{\left(2\widehat{K}+D^2\widehat{M}^2\right)(s-t)}\left(\mathbb{E}[\xi^2] + \int_t^s e^{-D^2\widehat{M}^2(z-t)}\dot{b}(z)dz\right)$$

with, for $s \in [t,T]$,

$$\begin{aligned}\dot{b}(s) &= -2\mathbb{E}[\xi]\, e^{-\widehat{K}(s-t)}\widehat{K}m_s + \left(\int_t^s e^{-\widehat{K}(z-t)}\widehat{K}m_z dz\right)e^{-\widehat{K}(s-t)}\widehat{K}m_s \\ &\quad + e^{-2\widehat{K}(s-t)}D^2\left(\widehat{M}^2 m_s^2 - 2\widehat{M}^2 m_s\widehat{m}_s + \widehat{\sigma}_s^2\right).\end{aligned}$$

Setting $\phi_s^2 := \mathbb{E}\left[\left(X_s^{\widehat{\pi}}\right)^2\right]$ and $d(s) := \dot{b}(s)$, the rest of the proof follows easily. □

4 Experiment

We now demonstrate how the theoretical results of Theorems 2 and 4 can be used to design algorithms for MFG with learning. The experiment aims to highlight

- how entropy regularization helps to "explore optimally" in a game with learning, and especially in improving the speed of convergence to the NE, and
- how the agent manages to eventually learn the optimal scheduling of the exploration and, in particular, the time-dependent variances (as in (3.3) and (3.12)) over a finite time horizon.

Throughout this section, the experiment is with the inclusion of Shannon entropy only, as the case with the additional cross-entropy may be studied in a similar fashion.

4.1 Set-up

The algorithm design is with discrete time steps $s = 0, 1, 2, \cdots, N$, where $\delta = \frac{T}{N}$ is the step-size. According to Theorem 2 and Corollary 3.1, it suffices to focus on a considerably smaller class of policies of form

$$\widehat{\pi}_s \sim \mathcal{N}\left(\widehat{M}(x_s - m_s), \widehat{\sigma}_s^2\right),$$

which can be fully characterized by the mean state process $m := \{m_s\}_{s=0}^N$ and $\widehat{R} := (\widehat{M}, \widehat{\sigma}^2)$, with $\widehat{\sigma}^2 := \{\widehat{\sigma}_s^2\}_{s=0}^N$. We, then, consider the discrete-time LQ-MFG problem

$$J\left(\widehat{R}, m\right) := \mathbb{E}\left[\sum_{s=0}^{N-1}\Big((x_s - m_s)^2 + \lambda_{SE}\mathcal{H}_{SE}(\pi_s)\Big)\delta - \frac{Q}{2}(X_N - m_N)^2\right], \tag{4.1}$$

where, for $s = 0, 1, \cdots, N-1$,

$$x_{s+1} = x_s + \left(\int_{\mathbb{R}}(A(m_s - X_s) + Bu)\pi_s(u)du\right)\delta + \left(D\sqrt{\int_{\mathbb{R}} u^2\pi_s(u)du}\right)\Delta W_s,\ x_0 = \xi \sim \nu. \tag{4.2}$$

Here, ΔW_s are *IID* $\mathcal{N}(0, \delta)$ random variables and ν is the distribution of the initial state ξ.

4.2 Mean Field Policy Gradient with Exploration

Recall that in the learning setting, the model parameters A, B, D, Q, and $\bar{Q}$ are assumed to be unknown to the agent. She only has access to the *simulated* reward function

$$\widehat{j}\left(\widehat{R}, m\right) := \sum_{s=0}^{N-1}\Big((x_s - m_s)^2 + \lambda_{SE}\mathcal{H}_{SE}(\pi_s)\Big)\delta - \frac{Q}{2}(X_N - m_N)^2,$$

which is associated with a *single* trajectory $\{x_s\}_{s=0}^N$ under the policy characterized by $\widehat{R}$ and the mean state process $m = \{m_s\}_{s=0}^N$. Note, however, that this assumption is weaker than being able to observe $J\left(\widehat{R}, m\right)$ defined in (4.1), as $J\left(\widehat{R}, m\right)$ involves calculating an expectation and requiring observing infinite number of samples.

The algorithm has the following key elements:

- *Information adaptiveness.* In each outer iteration k $k = 1, 2, \cdots, K$, the representative agent can improve her decision (lines 4-11) based on the mean field information m^{k-1} from the previous outer iteration $k-1$. This implies that she has access to a simulator with which she may excercise different policies when other agents keep applying the same policy from the previous outer iteration. This is a standard assumption, see, for example, see [7, 9]. Once the agent stops improving her policy, the mean field information is updated assuming that all agents follow the same improved policy (line 12). In the RL literature, this procedure is sometimes called *self play*.

- *Agent update.* Within each outer iteration k under a fixed mean field information m^{k-1}, the agent will update her estimation of the optimal policy $\widehat{R}$ for I rounds (lines 4-11). Each round corresponds to one gradient descent step (line 10) and requires n samples of the simulated reward function (line 7) associated with the perturbed version of $\widehat{R}^i$ (line 6).

- The gradient term $\nabla J(\widehat{R}^i, m^{k-1})$ in (4.3) is estimated using a *zeroth-order optimization approach* (line 9). That is, the agent only has query access to a sample of the reward function $\widehat{j}(\cdot)$ at input points (R, m), without querying the gradients and higher order derivatives of $\widehat{j}(\cdot)$. Moreover, to avoid the issue of ill-definedness of $\mathbb{E}_{U\sim\mathcal{N}(0,\sigma^2 I)}[J(\widehat{R} + U, m)]$ with a Gaussian smoothing, we choose $\mathbb{S}_r$ by smoothing over the sphere of a ball; hence, step (4.3) in Algorithm 1 is to find, for a given m, a bounded and biased estimate $\widehat{\nabla J(\widehat{R}, m)}$ of $\nabla J(\widehat{R}, m)$.

Algorithm 1 Mean Field Policy Gradient with Exploration

Input: Initial beliefs $m^0 := \{m_s^0\}_{s=0}^N$, distribution of initial policy $\mathcal{D}$, number of trajectories n, smoothing parameter r, learning rate η.
for $k \in \{1, \dots, K\}$ **do**
  Sample initial policy $\widehat{R}^0 \sim \mathcal{D}$.
  for $i \in \{0, \dots, I\}$ **do**
    for $j \in \{1, \dots, n\}$ **do**
      Sample policy $\widehat{R}^{i,j} = \widehat{R}^i + U^{i,j}$ where $U^{i,j}$ is drawn uniformly at random over matrices such that $\|U^{i,j}\|_F = r$.
      Denote $\widehat{j}\left(\widehat{R}^{i,j}, m^{k-1}\right)$ as the single trajectory cost with policy $\widehat{R}^{i,j}$ starting from $x_0^{i,j} \sim \nu$ under fixed mean state m^{k-1}.
    end for
    Obtain the estimate of $\nabla J(\widehat{R}^i, m^{k-1})$:

$$\widehat{\nabla J(\widehat{R}^i, m^{k-1})} = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{r^2} \widehat{j}\left(\widehat{R}^{i,j}, m^{k-1}\right) U^j. \tag{4.3}$$

    Perform policy gradient descent step:

$$\widehat{R}^{i+1} = \widehat{R}^i - \eta \widehat{\nabla J(\widehat{R}^i, m^{k-1})}. \tag{4.4}$$

  end for
  Update mean field information $m^k = \{m_s^k\}_{s=0}^N$ assuming all agents follow policy $\widehat{R}^I$.
end for

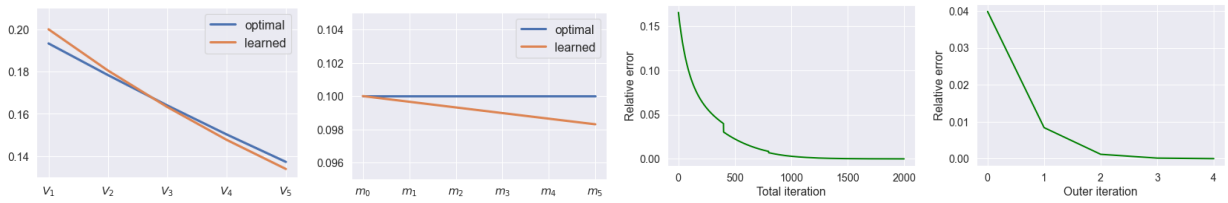
4.3 Results

Model set-up. We take $T = 0.1$, $\Delta = 0.02$, $A = 2.0$, $B = 3.0$, $D = 2.0$, $Q = 3.0$, $\bar{Q} = 2.0$, $\mathbb{E}[\xi] = 0.1$, and $\mathbb{E}[\xi^2] = 1$.

Experiment set-up. We choose $r = 0.01$ and $\eta = 0.05$. Initialization $m_s^0 = 0.0$ $(s = 0, 1, \cdots, N)$, $\widehat{M}^0 \sim \mathcal{N}(0.5, 1)$, and $\widehat{\sigma}_s^0 \sim \mathcal{N}(0.5, 0.1)$ $(s = 0, 1, \cdots, N-1)$, with $K = 10$, $n = 50$, and $I = 400$.

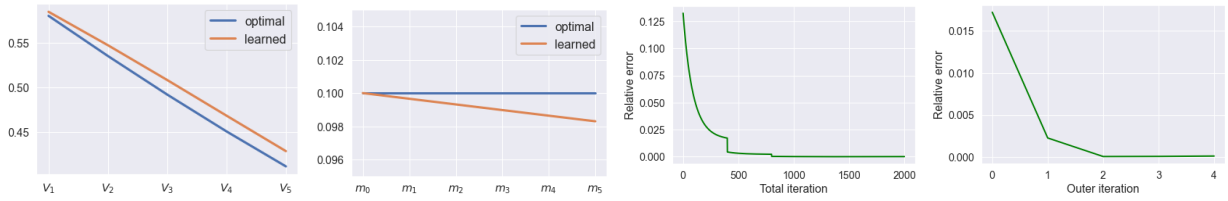
Performance evaluation. Given policy $\widehat{R}$ and mean field information m, define the *relative error* between $(\widehat{R}, m)$ and the mean field solution (R^*, m^*) of problem (4.1)-(4.2) as

$$Err(\widehat{R}, m) := \frac{|J(\widehat{R}, m) - J(R^*, m^*)|}{|J(R^*, m^*)|}. \tag{4.5}$$



(a) $\{V_s\}_{s=0}^N$ (b) $\{m_s\}_{s=0}^N$ (c) Relative error in total iterations. (d) Relative error in outer iterations

Figure 1: Performance of the algorithm when $\lambda_{SE} = 1.0$. (True $M = 0.75$ and learned $\widehat{M} = 0.732$.)



(a) $\{V_s\}_{s=0}^N$ (b) $\{m_s\}_{s=0}^N$ (c) Relative error in total iterations. (d) Relative error in outer iterations

Figure 2: Performance of the algorithm when $\lambda_{SE} = 3.0$, with true $M = 0.75$ and learned $\widehat{M} = 0.736$.

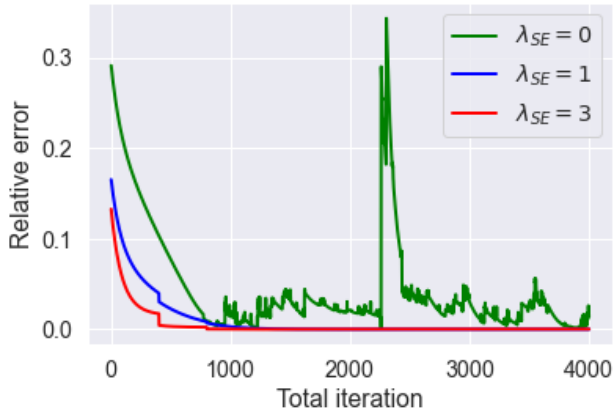


Figure 3: Comparison of relative errors with different λ_{SE}.

Results.

1. *Stability.* As seen from Figure 3, when $\lambda_{SE} = 0$, i.e., when there is no exploration, the algorithms is unstable. Within each outer iteration, errors fluctuate when the representative agent updates her policy under a fixed mean field information. At the end of each outer iteration, there is a sudden jump in error when the population updates their mean field policy. In contrast, the algorithm is stable when exploration is included, i.e., when $\lambda_{SE} > 0$.

2. *Speed of convergence.* As clear from Figures 1b and 2b, Shannon entropy ($\lambda_{SE} > 0$) improves the speed of convergence to the mean field equilibrium. In fact, the algorithm does not

converge without entropy regularization, i.e., when $\lambda_{SE} = 0$; whereas the algorithm converges to the equilibrium solution when $\lambda_{SE} = 1$ and $\lambda_{SE} = 3$. Moreover, the convergence speed is faster with $\lambda_{SE} = 3$ than with $\lambda_{SE} = 1$, with the former converging to the mean field equilibrium within three outer iterations and the latter in five outer iterations.

3. *Accuracy of learned mean field equilibrium.* Figures 1b and 2b show consistency with Theorems 2 and 4. The algorithm is able to learn the mean field information with small errors ($< 5\%$) for both $\lambda_{SE} = 1$ and $\lambda_{SE} = 3$.

4. *Learning optimal scheduling of the exploration policy.* With given parameters, the variance of the Guassian mean field policy (a.k.a., the optimal exploration scheduling) is a decreasing function of time t for both $\lambda_{SE} = 1$ and $\lambda_{SE} = 3$. Figures 1a and 2a suggest that the agent can learn this decreasing function $\{\widehat{\sigma}_s^2\}_{s=0}^T$ with small errors ($< 5\%$).

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