Personalized Robo-Advising: Enhancing Investment through Client Interaction

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Abstract

Automated investment managers, or robo-advisors, have emerged as an alternative to traditional financial advisors. Their viability crucially depends on timely communication of information from the clients they serve. We introduce and develop a novel human-machine interaction framework, in which the robo-advisor solves an adaptive mean-variance control problem, with the risk-return tradeoff dynamically updated based on the risk profile communicated by the client. Our model predicts that clients who value a personalized portfolio are more suitable for robo-advising. Clients who place higher emphasis on delegation and clients with a risk profile that changes frequently benefit less from robo-advising.

1 Introduction

Automated investment managers, commonly referred to as robo-advisors, have gained widespread popularity in recent years. The value of assets under management by robo-advisors is the highest in the United States, exceeding $440 billion in 2019 (Backend Benchmarking [2019]). Major robo-advising firms include Vanguard Personal Advisor Services, which manages about $140 billion of assets, Schwab Intelligent Portfolios, with over $40 billion of assets, Wealthfront, with about $20 billion of assets, and Betterment, with about $18 billion of assets under management. The popularity of robo-advisors is also growing in other parts of the world. They manage more than $30 billion in Europe (Statista [2019]) and are rapidly growing in Asia and emerging markets (Burnmark [2017]). According to Abraham et al. [2019], the value of assets under management by robo-advisors is expected to grow at an average annual rate of over 30 percent, reaching an estimated $1.5 trillion by 2023, solely in the United States.

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The first robo-advisors available to the general public were launched in 2008, in the wake of the financial crisis and the ensuing loss of trust in established financial services institutions. To begin with, firms rooted in the technology industry began offering a range of digital financial tools directly to customers, including investment analysis tools, previously only available to financial professionals (FINRA [2016]). Examples of such pioneering robo-advising firms include the now established firms Betterment and Wealthfront. Later on, industry incumbents, such as Vanguard and Charles Schwab, followed suit and began offering their own robo-advising services. This rise of robo-advising since the financial crisis has also undoubtedly been compounded by the seismic shift towards passive investing and exchange-traded funds.\(^1\)

Existing robo-advising systems are based on a one-time interaction with the client. For instance, Vanguard Personal Advisor Services profiles the client based on input received at the outset, which includes financial goals, investment horizon and demographic information. After the investment plan proposed by the robo-advisor is accepted by the client, the robo-advisor autonomously executes trades to reach the desired portfolio allocation (Rossi and Utkus [2019a]). Another popular robo-advising firm, Wealthfront, estimates the client’s subjective risk tolerance by asking whether her objective is to maximize gains, minimize losses, or a mix of the two. The robo-advisor uses the client’s answers to construct a risk parameter which may also depend on additional objective risk indicators, and then solves a mean-variance optimization problem (Lam [2016]). However, with limited or no interaction with the client in the process, the robo-advisor is clearly susceptible to the risk of making decisions based on stale information and, thus, not acting for the client’s best interest.\(^2\)

The distinguishing feature of our framework is that the client and the robo-advisor interact not only at the beginning, but also throughout the investment period. The client wishes to optimally invest her wealth throughout a finite horizon in a discrete market setting. She delegates this task to the robo-advisor, which executes the investment on the client’s behalf, accounting for the evolving nature of her risk profile. In order to effectively tailor the investment advice to the needs of the client, the robo-advisor solicits, on a regular basis, information about the client’s changing risk preferences. At the beginning of the investment process, the client specifies a desired rate of participation, which determines the frequency of her interaction with the robo-advisor. At each interaction time, the client communicates her risk-preferences to the robo-advisor. From that point, and until the subsequent time of interaction, the robo-advisor uses a random walk model to describe the (unobserved) evolution of the client’s risk preferences. The random walk model captures the fact that while the client’s risk preferences may change over time (Guiso et al. [2018]), they are unlikely to


\(^{2}\text{The Monetary Authority of Singapore issued a consultation paper on the regulation of robo-advisor services (MAS [2017]). This includes requirements on the standard of governance and management oversight, and the responsibility of the board and senior management for the monitoring and control of algorithms that process information and generate investment recommendations. See also FINRA [2016].}
exhibit drastic changes over a short time period (Schildberg-Horisch [2018]). The volatility of the random walk quantifies how likely the client’s risk preferences are to change from the ones reported at the most recent interaction with the robo-advisor.

The robo-advisor adopts a multi-period mean-variance optimization criterion. Unlike other risk averse optimization problems considered in the literature, the risk-return tradeoff coefficient is stochastic and consists of two components, one specific to the client and one specific to the robo-advisor. The client-specific component incorporates the most recently communicated risk preferences of the client. This component is updated only when the robo-advisor solicits the client’s input, and its dynamics are described by a finite state Markov chain, whose transition times coincide with the times of interaction between the client and the robo-advisor. The component specific to the robo-advisor, i.e., machine-specific, reflects the robo-advisor’s aversion to uncertainty in the client’s risk preferences. The robo-advisor is less willing to take risk on behalf of the client if the client’s risk preferences are based on stale information, resulting in the risk-return tradeoff coefficient being inflated to a degree determined by the level of distrust in the most recently communicated risk preferences. This is related to the concept of trust in Gennaioli and Vishny [2015], who show that individuals without finance expertise are more willing to take on risk with a financial advisor they trust, with trust based, for example, on personal relationships and good reputation. In their setting, the roles of the client and the financial advisor are reversed compared to our model: the client’s baseline risk aversion in a mean-variance criterion is inflated by a factor representing the client’s level of anxiety from bearing risk with the financial advisor.

We study in detail the underlying optimization problem solved by the robo-advisor. We highlight how both components of the risk-return tradeoff coefficient influence the optimal portfolio strategy, as well as how the Markov chain transition probabilities link the optimal allocations before and after risk preferences are communicated by the client. At each time point, the optimal allocation depends on the future path of the client’s risk preferences. Their distribution depends on the client-specific component of the risk-return tradeoff, which incorporates the most recently communicated risk profile of the client. With frequent interaction, information about the client’s risk preferences arrives gradually, so there is little variation in the client-specific component between consecutive times of interaction. This, in turn, translates into little uncertainty in near-term optimal allocations. However, this allocation uncertainty increases with less frequent interaction, as the random walk model is allowed to evolve “unchecked” over longer periods. Hence, prior to each interaction time, at which the client again communicates her risk preferences, the subsequent optimal allocation (i.e., the optimal allocation at the interaction time) will have a larger variance compared to the case of more frequent updating. Between consecutive times of interaction, the robo-advisor makes investment
decisions based on stale information. Based on its level of uncertainty aversion, it then uses the machinespecific component of the risk-return tradeoff to tilt the optimal portfolio towards a less risky composition. We obtain an explicit expression for the magnitude of this effect, and show that it corresponds to setting the risk-return coefficient to a fixed percentile of the distribution of the client’s unknown risk aversion. In this way, the robo-advisor controls the probability of choosing a portfolio composition that is too risky, given the client’s most recently communicated risk preferences.

Another important modeling element is a measure of client regret, which quantifies the investment implications of the evolving information asymmetry between the client and the robo-advisor. At each point in time, the regret is equal to the expected change in portfolio allocation that would be realized if the robo-advisor knew the client’s actual risk preferences. The regret is determined by the client’s participation rate in the investment process and increases as time elapses since the most recent interaction between the client and the robo-advisor. Moreover, the regret is increasing in the volatility of the random walk that describes the client’s risk preferences; for a given volatility level, the regret is higher if the client communicated a low risk aversion in her latest interaction with the robo-advisor.

We demonstrate how the robo-advisor can calibrate a threshold updating rule, based on expected changes in the client’s risk aversion, in order to maintain a target level of regret for the client. Specifically, the robo-advisor solicits the client’s risk preferences as soon as the expected change in her risk aversion, since the last interaction, breaches a certain threshold. The threshold value depends on the client’s current risk profile and adjusts to changes in her risk profile. For instance, if the client’s risk preferences shift to a level where fluctuations are more likely to occur, then more frequent interaction is required (i.e., a lower threshold). We investigate the benefits of such personalized service, where the mechanism triggering interaction is tailored to the client’s current risk profile. In particular, we compare it to one-size-fits-all updating rules that are tailored to properties of the “average client”. In the latter case, we show that clients with a risk profile that is underrepresented in the robo-advisor’s client body have a higher regret than clients with a common risk profile.

We use our measure of regret to study the suitability of a robo-advisor for a client who trades off low regret with low personal involvement in the investment process. This tradeoff is determined by the client’s frequency of interaction with the robo-advisor. Our model indicates that clients who desire a personalized portfolio strategy (i.e., low regret) are more suitable for robo-advising than clients who prefer a high level of delegation to the robo-advisor (i.e., infrequent interaction). These findings are consistent with empirical

\[\text{3With regret defined in terms of portfolio allocation, this amounts to triggering an update when the expected change in portfolio allocation hits a given threshold. This is analogous to how robo-advisor algorithms trigger portfolio rebalancing as soon as the portfolio composition has drifted sufficiently from the optimal composition (Kaya [2017]), except that here the client’s risk preferences are changing rather than market prices of securities.}\]
evidence provided most recently by Rossi and Utkus [2019b]. We also find that clients with a highly volatile risk profile are less suited for robo-advising. Such clients typically have specific or complex needs, and would require a dedicated human service, rather than robo-advising which is better suited for clients with a conventional risk profile (see, e.g., Phoon and Koh [2018]).

The robo-advising framework proposed herein is an interaction system between a human (the client) and a machine (the robo-advisor). The analysis of this human-machine interaction system presents a novel methodological contribution. Furthermore, the model we propose gives rise to a novel, and rather challenging, adaptive control problem in which the system being controlled always maintains the same dynamics, but the optimality criterion changes at random interaction times between the human and the machine. Specifically, the dynamics of the tradeoff coefficient in the mean-variance criterion adapts to information communicated by the human to the machine. This contrasts with existing literature in adaptive control (see, for instance, Astrom and Wittenmark [1989]), where the optimality criterion always stays the same, and the system dynamics adapt to incoming changes in the environment. To the best of our knowledge, this is the first time that such an adaptive control problem is being considered.

In our framework, the interaction times effectively divide the time interval from initiation and until the terminal date into subperiods, triggering a new optimization problem at each interaction time. This yields a sequence of time-inconsistent problems that are interlinked, because they all share the same terminal date. Notably, at each time, the optimal control depends on the future evolution of the risk-return tradeoff coefficient, until the terminal date, which in turn depends on the system dynamics and the frequency of interaction between the human and the machine.

The rest of this paper is organized as follows. In Section 2, we briefly review related literature. In Section 3, we introduce the main components of our modeling framework. In Section 4, we present the optimal solution to the investment problem. In Section 5, we introduce and study performance metrics for the human-machine interaction framework, and in Section 6 we discuss the calibration of this framework. Section 7 contains concluding remarks and future directions along which the model we propose herein can be extended. Appendix A contains technical results that are used throughout the paper. Appendix B contains the proofs of the main results in Section 4. All remaining proofs are deferred to Appendix C. Appendix D provides a pseudocode for the algorithm to compute the optimal investment strategies in Section 4.
2 Literature Review

The main contribution of our paper is the development of a novel framework that captures the symbiotic nature of the investment process, in which the robo-advisor not only manages a portfolio of behalf of the client but also repeatedly interacts with her to elicit her risk preferences through time.

Methodologically, our work contributes to the literature on time-inconsistent stochastic control (Björk and Murgoci [2013] and Björk et al. [2014]). Other related works include Li and Ng [2010] who solve a multi-period version of the classical Markowitz problem, and Basak and Chabakauri [2010] who solve a continuous-time version of the dynamic mean-variance optimization problem within a potentially incomplete market. A recent study of Dai et al. [2019] develops a dynamic mean-variance framework, in which the investor specifies her target expected return only at inception. The authors obtain explicit formulas for time-consistent policies under stochastic volatility and time-varying Gaussian returns. In all of these works, the risk-return tradeoff is assumed to be constant throughout the investment horizon. By contrast, in our model, the risk-return tradeoff is stochastic, and only observed by the controller (robo-advisor) at random times. Unlike Björk and Murgoci [2013], we consider adaptive control laws rather than the more restrictive feedback laws, and show that the optimal control law for the dynamic mean-variance problem is in fact of feedback form.

Our study also contributes to the growing literature on robo-advising. Noticeable contributions include D’Acunto et al. [2018], who empirically show that the adoption of robo-advising increases portfolio diversification and reduces well-known behavioral biases, such as the disposition effect. Rossi and Utkus [2019a] study the largest US robo-advisor, Vanguard Personal Advisor Services. They show that, for previously self-directed investors that signed up for a hybrid robo-advising service, robo-advising significantly increased the investors’ proportion of wealth invested in low-cost indexed mutual funds, at the expense of individual stocks and active mutual funds. As in D’Acunto et al. [2018], the investors that benefit the most from robo-advisors are those with little investment experience, as well as clients with little mutual fund holdings, or those invested in high-fee active mutual funds.

Rossi and Utkus [2019b] conduct an extensive survey to study the “needs and wants” of individuals when they hire financial advisors. Their results lend support to the theoretical model of Gennaioli and Vishny [2015], indicating that traditionally advised individuals hire financial advisors largely to satisfy various needs other than portfolio return maximization. Namely, they argue that clients choose a traditional financial advisor primarily for the “human touch”, i.e., the ability to interact with a human and receive financial advice from a human, and show that the possibility of establishing a trustworthy relationship with the same advisor increases the client’s perception of value and her overall satisfaction. On the other
hand, robo-advised clients are more interested in the financial performance of their portfolio. These clients are not particularly interested in having access to expert opinion, and do not have a high need for trust. Furthermore, they find that individuals most likely to adopt robo-advising are those interested in acquiring knowledge and improving their investment allocation, while those who wish to completely delegate their investment decisions, and therefore have a need for trust, are more reluctant to consider robo-advising.

3 Modeling Framework

The modeling framework consists of four main components: (i) The robo-advisor, which solves dynamic portfolio optimization problems, incorporating its view of the client’s risk preferences, (ii) the interaction mechanism between the robo-advisor and the client, (iii) the market model for the investment securities, and (iv) the investment criterion.

3.1 Robo-Advisor’s view of Client

The robo-advisor quantifies the client’s risk preferences by a risk aversion parameter. At the start of the investment process, the client communicates to the robo-advisor her initial risk preferences. Most robo-advisors solicit this information by presenting an online questionnaire to the client, asking for information on, e.g., income, education, household status, investment goals, and potential reactions to hypothetical future market events (see, among others, Lam [2016], Ch. 3). The robo-advisor then translates the client’s feedback into a numerical score, herein referred to as the client’s risk aversion parameter. We abstract from the construction of this numerical translation, effectively assuming that the client communicates directly her risk aversion parameter to the robo-advisor.

The client knows her risk aversion parameter at all times, but only communicates it to the robo-advisor at specific updating times. At each updating time, the robo-advisor maps the risk aversion parameter communicated by the client to the nearest value in a finite set of representative risk aversion levels

$$\Gamma := \{\bar{\gamma}_1, \bar{\gamma}_2, \ldots, \bar{\gamma}_K\}, \quad K \geq 1.$$

The discretization above mimics the fact that, in practice, the robo-advisor translates client information into a finite number of risk aversion scores, where each risk score is representative of a category of clients with a similar risk profile (see, e.g., Lam [2016] and Phoon and Koh [2018]). \(^4\) Without loss of generality, we

\(^4\)One of the major robo-advising firms, Wealthfront, constructs a composite Risk Score ranging from 0.5 (most risk averse) to 10.0 (most risk tolerant) in increments of 0.5. Each Risk Score corresponds to one of twenty asset allocations, with target
assume the risk aversion levels in $\Gamma$ to be sorted in increasing order, i.e., $0 < \bar{\gamma}_1 < \bar{\gamma}_2 < \cdots < \bar{\gamma}_K < \infty$.

Between consecutive updating times, the robo-advisor receives no input from the client, and assumes that the (unobserved) evolution of the client’s risk aversion follows a centered random walk with independent and identically distributed increments, whose standard deviation depends on the most recently communicated risk aversion level. For risk aversion level $\gamma \in \Gamma$, we denote this standard deviation by $\sigma_\gamma > 0$, and refer to it as the volatility of the random walk; a higher $\sigma_\gamma$ implies that it is more likely for the client to transition from $\gamma$ to a different risk aversion level. The random walk model captures the fact that while the client’s risk aversion may change over time, it is much more likely to move between neighboring levels than swiftly going from a high risk aversion level to a low risk aversion level, or vice versa. In addition to explaining changes in the client’s risk aversion, the volatility of the random walk accounts for other sources of uncertainty in the communication of risk preferences, such that the fact that the questionnaire used by the robo-advisor to elicit risk preferences cannot provide a complete picture of the client’s risk preferences.

More concretely, at time $n \geq 0$ we denote by $\tau_n \in \{0, 1, \ldots, n\}$ the last updating time prior to and including time $n$, and by $\bar{\gamma}_0 > 0$ the initial risk aversion communicated by the client. We then introduce the random walk process $(\bar{\gamma}_n)_{n \geq 0}$, defined by

$$\bar{\gamma}_n := \gamma_{\tau_{n-1}} + \sigma_{\gamma_{\tau_{n-1}}} (\bar{Z}_{\tau_{n-1}+1} + \cdots + \bar{Z}_n), \quad n > 0, \quad (3.1)$$

where $(\bar{Z}_n)_{n \geq 1}$ is a sequence of i.i.d. $\mathcal{N}(0, 1)$ random variables, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and

$$\gamma_n := \Psi(\bar{\gamma}_{\tau_n}) := \sum_{\gamma \in \Gamma} \gamma 1\{\bar{\gamma}_{\tau_n} \in (\gamma-, \gamma+)\}, \quad n \geq 0, \quad (3.2)$$

where the function $\Psi$ maps its argument to the nearest risk aversion level in $\Gamma$,$^5$ and for each $\gamma \in \Gamma$, the constants $\gamma^\pm$ are defined by

$$\gamma^+ := \begin{cases} \frac{\bar{\gamma}_k + \bar{\gamma}_{k+1}}{2}, & \gamma = \bar{\gamma}_k, \quad 1 \leq k < K, \\ \infty, & \gamma = \bar{\gamma}_K, \end{cases} \quad \gamma^- := \begin{cases} \infty, & \gamma = \bar{\gamma}_1, \\ \frac{\bar{\gamma}_k + \bar{\gamma}_{k-1}}{2}, & \gamma = \bar{\gamma}_K, \quad 1 < k \leq K. \end{cases} \quad (3.3)$$

The process $(\bar{\gamma}_n)_{n \geq 0}$ in (3.1) models the client’s risk aversion between consecutive updating times, and it is not observable by the robo-advisor. Rather, the robo-advisor forms its own view of the client’s risk aversion through the process $(\gamma_n)_{n \geq 0}$, defined in (3.2). In other words, the process $(\gamma_n)_{n \geq 0}$ serves as a proxy of the volatilities ranging from 5.5% per year to 15.0% per year. We refer to https://research.wealthfront.com/whitepapers/investment-methodology/ for additional details.

$^5$The mapping $\Psi$ may project its argument upward or downward. Alternatively, one can define $\Psi$ so that it always projects its argument to a higher risk aversion level, to err on the safe side and decrease the probability of taking more risk than the client specifies.
client’s actual risk aversion. At any time \( n \geq 0 \), it is given by \( \gamma_n = \Psi(\hat{\gamma}_n) \), where \( \hat{\gamma}_n \) is communicated by the client to the robo-advisor at the most recent updating time, \( \tau_n \). Between two consecutive updating times, the process \( (\gamma_n)_{n \geq 0} \) is kept constant at the most recent risk aversion level. This is visually illustrated in Figure 1, that displays a sample path of the client’s risk aversion level. The process \( (\tau_n)_{n \geq 0} \), that determines the times at which the process \( (\gamma_n)_{n \geq 0} \) can take on a new value, is determined by an exogenously specified updating rule, as discussed in the following section.

In summary, the client’s risk aversion is assumed to evolve according to a random walk, that is reset at each updating time to the nearest risk aversion level in \( \Gamma \). Specifically, if \( n > 0 \) is an updating time, then \( \gamma_n = \Psi(\hat{\gamma}_n) \in \Gamma \) is the client’s updated risk aversion level, where \( \hat{\gamma}_n \) is the realization of a random walk starting at the previous updating time \( \tau_{n-1} \), from the corresponding risk aversion level \( \gamma_{\tau_{n-1}} \), with volatility \( \sigma_{\gamma_{\tau_{n-1}}} \). The random walk is then restarted at time \( n \) from the updated risk aversion level \( \gamma_n \), and with volatility \( \sigma_{\gamma_n} \).

**Remark 3.1.** We have not included a drift in the random walk dynamics (3.1), because the process \( (\hat{\gamma}_n)_{n \geq 0} \) captures idiosyncratic shocks to the client’s risk aversion, and should be devoid of a predictable component. Empirical research has identified a clear trend of risk preferences over the life cycle: At a young age, individuals are more willing to take risks than adults, but as they grow older, they become increasingly risk averse, and their risk preferences gradually converge to those of their older counterparts; see, for instance, Levin et al. [2017]. A deterministic risk aversion component (e.g., related to age) can be modeled separately by decomposing the risk aversion process observed by the robo-advisor as \( \gamma_n = \gamma_n^{(1)} + \gamma_n^{(2)} \), where \( (\gamma_n^{(1)})_{n \geq 0} \) is a random process capturing idiosyncratic risk preferences, as described in this section, and \( (\gamma_n^{(2)})_{n \geq 0} \) is a deterministic process.

### 3.2 Interaction between Client and Robo-Advisor

The times of interaction between the client and the robo-advisor are determined by an updating rule, decided at the beginning of the investment horizon, and used throughout the investment horizon. It is specified by a deterministic function

\[
\phi : \Gamma \mapsto \bar{\mathbb{N}} := \{1, 2, \ldots\} \cup \{\infty\},
\]

where \( \phi(\gamma) \) is the time elapsing till the next update of risk preferences, starting at risk aversion level \( \gamma \in \Gamma \),

\[
\phi(\gamma) = \inf\{n > 0 : \tau_n = n | \gamma_0 = \gamma\}.
\]
Examples of such updating rules include the full-information rule, \( \phi \equiv 1 \), where the risk preferences are updated at all times, and the no-information rule, \( \phi \equiv \infty \), where the risk preferences are never updated. All other updating rules lie between those two extreme cases.

For a given updating rule \( \phi \), the corresponding sequence of interaction times between the client and the robo-advisor, \( (T_k^{(\phi)})_{k \geq 0} \), can be iteratively computed as \( T_0^{(\phi)} = 0 \) and

\[
T_k^{(\phi)} = T_k^{(\phi)} + \phi(\tau^{(\phi)}_{T_k^{(\phi)}}), \quad k \geq 0.
\]

(3.4)

Observe that the \( k \)-th updating time, \( T_k^{(\phi)} \), together with the realized risk aversion level, \( \gamma_{T_k^{(\phi)}} \), determines the \( (k + 1) \)-th updating time, but further updating times are random and depend on future realized risk aversion levels. In Lemma A.1-(c) we show that \( (\gamma_k^{(\phi)})_{k \geq 0} \), the sequence of risk aversion levels observed by the robo-advisor, defined by

\[
\gamma_k^{(\phi)} := \gamma_{T_k^{(\phi)}},
\]

(3.5)

is an irreducible and aperiodic Markov chain on \( \Gamma \), with a time-homogeneous transition matrix. In particular, for the special case \( \phi \equiv 1 \), i.e., when the client’s risk preferences are updated at all times, the risk aversion level process \( (\gamma_n)_{n \geq 0} \) is itself a time-homogeneous Markov chain on \( \Gamma \) (see Lemma A.1-(d)).

Notice that the updating rule \( \phi \) determines the updating times at which the client’s risk preferences are communicated. The chosen updating rule therefore determines the uncertainty surrounding the robo-advisor’s investment decisions, arising from the robo-advisor not having access to up-to-date information when allocating the client’s wealth. In Section 5.1 we propose a measure of regret to quantify this effect, and in Section 6.1 we describe how the robo-advisor can calibrate a threshold updating rule to maintain a given target level of regret across risk aversion levels.

**Remark 3.2.** The set of updating rules can be extended to include those characterized by a sequence of deterministic functions, \( (\phi_n)_{n \geq 0} \), such that, for each \( \gamma \in \Gamma \),

\[
\phi_n(\gamma) = \inf \{ n' > n : \tau_{n'} = n' | \tau_n = n, \gamma_n = \gamma \},
\]

is the time until the next update of risk preferences, following an update at time \( n \) that resulted in risk aversion level \( \gamma \). The results in Section 4 still hold for this general case. To keep the analysis herein simple, we work with time-homogeneous updating rules in the sense that \( \phi_n \) is independent of \( n \). That is, for any
Figure 1: The figure displays the client/robo-advisor interaction system. It shows a sample path of the client’s risk aversion level process, \((\gamma_n)_{n \geq 0}\), which is observable by the robo-advisor. The initial risk aversion level is communicated by the client at the beginning of the investment process, \(t = 0\). Subsequent communication times are determined by the updating rule \(\phi\) that is also specified at \(t = 0\). Between consecutive updating times, the risk aversion level viewed by the robo-advisor is constant, and the robo-advisor manages the client’s portfolio without any further input from the client.

\[
\phi_n(\gamma) = \inf\{n' > n : \tau_{n'} = n' | \tau_n = n, \gamma_n = \gamma\} = \inf\{n > 0 : \tau_n = n | \gamma_0 = \gamma\} = \phi_0(\gamma).
\]

This naturally includes the updating rules \(\phi \equiv 1\) and \(\phi \equiv \infty\).

### 3.3 Market Dynamics

The market consists of a risk-free money market account \((B_n)_{n \geq 0}\) and a risky asset \((S_n)_{n \geq 0}\) with dynamics

\[
B_{n+1} = (1 + r)B_n,
\]
\[
S_{n+1} = (1 + \epsilon_{n+1})S_n,
\]

where \(r \geq 0\) is the constant risk-free rate, and \((\epsilon_n)_{n \geq 1}\) is a sequence of i.i.d. random variables, with mean \(\mu > r\) and variance \(\sigma^2 > 0\). This sequence is assumed to be independent of the sequence \((\bar{Z}_n)_{n \geq 1}\), introduced in (3.1), but defined on the same probability space. We denote by \((X_n)_{n \geq 0}\) the wealth process of the client,
allocated between the risky asset and the money market account, with \( \pi_n \) denoting the amount invested in the risky asset. Under a self-financing control law \((\pi_n)_{n \geq 0}\), the wealth dynamics are

\[
X_{n+1}^\pi = (1 + r)X_n^\pi + (\epsilon_{n+1} - r)\pi_n =: RX_n^\pi + Z_{n+1}\pi_n.
\] (3.6)

The random variable \( Z_{n+1} \) denotes the excess return of the risky asset over the risk-free rate, between times \( n \) and \( n + 1 \), and it has mean \( \mu := \mu - r > 0 \) and variance \( \sigma^2 > 0 \).

On the probability space \((\Omega, \mathcal{F}, \mathbb{P})\), supporting the independent sequences of random variables, \((\tilde{Z}_n)_{n \geq 1}\), that drive the risk aversion process in (3.1), and \((Z_n)_{n \geq 1}\), that drive the risky asset process above, we denote by \((F_n)_{n \geq 0}\) the smallest filtration such that

\[
X_{(n)} := (X_k)_{0 \leq k \leq n}, \quad \gamma_{(n)} := (\gamma_k)_{0 \leq k \leq n}, \quad \tau_{(n)} := (\tau_k)_{0 \leq k \leq n},
\]

are measurable with respect to \( F_n \). The control law \((\pi_n)_{n \geq 0}\) in (3.6) is assumed to be adapted to this filtration. Furthermore, for each updating rule \( \phi \), we define the probability measure \( \mathbb{P}^{(\phi)} \) on \((\Omega, \mathcal{F})\), under which the updating times for the client’s risk preferences are given by (3.4), and, for any \( n \geq 0 \), we use the shorthand notation

\[
\mathbb{P}^{(\phi)}(\cdot) := \mathbb{P}^{(\phi)}(\cdot | F_n).
\]

We let \( \mathbb{E}^{(\phi)}_n(\cdot) \) denote the expected value with respect to the probability measure \( \mathbb{P}^{(\phi)}_n \). Note also that the filtration \((F_n)_{n \geq 0}\) models the information that is available to the robo-advisor, and the expected value \( \mathbb{E}^{(\phi)}_n(\cdot) \) thus averages over future paths of both the asset price process \((X_n)_{n \geq 0}\), and the risk aversion process \((\gamma_n)_{n \geq 0}\) observed by the robo-advisor. More specifically, for an \( F_n \)-measurable random variable \( Y \), where \( n' = n + 1 \), the expected value \( \mathbb{E}^{(\phi)}_n[Y] \) averages over future paths of the processes \((X_k)_{n \leq k \leq n'}\) and \((\gamma_k)_{n \leq k \leq n'}\), given the \( \sigma \)-algebra \( F_n \). Note also that under the probability measure \( \mathbb{P}^{(\phi)}_n \), the distribution of \((\gamma_k)_{n \leq k \leq n'}\) is determined by the joint distribution of the future updating times, \((T_k^{(\phi)})_{k \geq 0}\), restricted to the interval \( \{n, n + 1, \ldots, n'\} \), and the corresponding risk aversion levels communicated by the client, \( \gamma_{T_k^{(\phi)}} \).

### 3.4 Investment Criterion

The robo-advisor’s objective is to optimally allocate the client’s wealth using a mean-variance criterion, but also accounting for the stochastic nature of the client’s risk preferences described in Section 3.1. For this purpose, we develop an adaptive extension of such criteria, namely, a dynamic version of the standard
Markowitz [1952] mean-variance problem, that adapts to the client’s changing risk preferences. We proceed to introduce this criterion next, which we refer to as an adaptive mean-variance criterion.

Let $T \geq 1$ be a fixed investment horizon. For each $n \in \{0, 1, \ldots, T - 1\}$ we consider the mean-variance functional

$$J_{n,T}^{(\phi)}(X_n, \gamma(n), \tau(n), \theta, \pi) := \mathbb{E}_n^{(\phi)}[r_{\pi,n,T} - \frac{1}{2} \text{Var}_n^{(\phi)}[r_{\pi,n,T}],$$

where $r_{\pi,n,T}$ is the simple return obtained by following the control law $\pi$ between time $n$ and the terminal date $T$,

$$r_{\pi,n,T} := \frac{X_{T}^{\pi} - X_n}{X_n}.$$  

In the sequel we will also refer to a control law $\pi := (\pi_{n,T})_{0 \leq n < T}$ as a strategy or allocation, and we consider real-valued, $(\mathcal{F}_n)_{n \geq 0}$-adapted control laws of the form

$$\pi_{n,T} = \pi_{n,T}(X_n, \gamma(n), \tau(n), \theta), \quad 0 \leq n < T.$$

The coefficient $\Delta_n > 0$ in (3.7) quantifies the risk-return tradeoff at time $n$, that is determined by the interaction between the client and the robo-advisor. If $\Delta_n$ is chosen to be independent of $n$, then (3.7) reduces to the classical mean-variance criterion. Herein, we propose a much richer structure for $\Delta_n$, in order to incorporate not only the most recently communicated risk aversion level, but also the uncertainty the robo-advisor undertakes in assessing the client’s risk aversion. Specifically, $\Delta_n$ depends on the most recent risk aversion value communicated by the client, $\gamma_n$, and the corresponding time of communication, $\tau_n$. It also depends on a parameter $\theta \geq 0$ that quantifies the level of caution the robo-advisor exhibits due to it being uncertain about the client’s true risk aversion. We choose to define this uncertainty in terms of $\delta_n(\gamma, \tau)$, the conditional standard deviation of the client’s (unknown) risk aversion level at time $n \geq 0$, given that the risk aversion level $\gamma \in \Gamma$ was communicated at the most recent updating time $\tau \leq n$,

$$\delta_n(\gamma, \tau) := \sqrt{\text{Var}[\gamma_n | \gamma_n = \gamma, \tau_n = \tau]}.$$  

This quantity is computable in closed-form, as shown in Lemma A.1-(d). We then consider an additive structure for the risk-return tradeoff coefficient, namely,

$$\Delta_n := \Delta_n(\gamma_n, \tau_n, \theta) := \gamma_n + \theta \delta_n(\gamma_n, \tau_n).$$
The term $\gamma_n$ is the *client-specific* component of the risk-return tradeoff and incorporates the most recently communicated risk aversion level of the client. The term $\theta \delta_n(\gamma_n, \tau_n)$ is specific to the robo-advisor, i.e., it is *machine-specific*, and arises because the robo-advisor does not have knowledge of the client’s risk preferences at all times. At updating times, we have that $\delta_n(\gamma_n, \tau_n) = 0$, and the machine-specific component vanishes. However, between consecutive updating times, $\delta_n(\gamma_n, \tau_n) > 0$ and $\theta$ can, in turn, be viewed as the weight of uncertainty in the robo advisor’s view of the client. A robo-advisor that is very risk averse towards its incomplete knowledge of the client’s risk preferences chooses a high value of $\theta$ (see Section 5.2 for further discussion). This parallels the notion of market price of risk in classical investment theory, but the main difference here is that there is uncertainty about the client’s characteristics rather than about the market dynamics.

At time $n$, the robo-advisor’s objective is to maximize the risk-adjusted return on the client’s wealth, defined by the objective function $J_{n,T}^{(\phi)}$ in (3.7). Through the wealth dynamics (3.6), this function depends on the control law $\pi$ restricted to the time points $\{n, n+1, \ldots, T-1\}$, and the robo-advisor chooses the control $\pi_{n,T}$ given future control decisions until the terminal date, $\pi_{n+1:T} := \{\pi_{n+1,T}, \pi_{n+2,T}, \ldots, \pi_{T-1,T}\}$. Any candidate optimal control law $\pi^*$ is therefore such that for each $n \in \{0, 1, \ldots, T-1\}$,

$$\sup_{\pi \in A_n^*} J_{n,T}^{(\phi)}(X(n), \gamma(n), \tau(n), \theta, \pi) = J_{n,T}^{(\phi)}(X(n), \gamma(n), \tau(n), \theta, \pi^*),$$

where $A_n^* := \{\pi : \pi_{n+1:T} = \pi_{n+1:T}^*\}$ is the set of control laws that coincide with $\pi^*$ after time $n$. If a control law $\pi^*$ satisfying (3.11) exists, we define the corresponding value function at time $n$ as

$$V_{n,T}^{(\phi)}(X(n), \gamma(n), \tau(n), \theta) := J_{n,T}^{(\phi)}(X(n), \gamma(n), \tau(n), \theta, \pi^*).$$

Note that any optimal control naturally depends on the updating rule $\phi$, as such a rule divides the investment horizon into subperiods via the corresponding (random) updating times. Observe that this implies the presence of two temporal scales in our framework. The robo-advisor rebalances the portfolio at times $\{0, 1, \ldots, T-1\}$, while the risk aversion process $(\gamma_k^{(\phi)})_{k \geq 0}$, defined in (3.5), is a Markov chain whose transition times occur at a coarser time scale determined by the updating rule $\phi$. The updating rule also highlights the dynamic feature of the family of mean-variance problems we introduce. Specifically, it is a family of sequentially adaptive problems in the sense that at each updating time, a new problem arises, depending on the realized risk aversion level, but with the same initially determined terminal date.

**Remark 3.3.** At any time $n$, the risk-return coefficient $\Delta_n$ is $\mathcal{F}_n$-measurable, but the optimal strategy at time $n$ depends on the evolution of the client’s risk aversion, throughout the investment horizon. Specifi-
cally, at the initial time, the robo-advisor has a model for the future evolution of the client’s risk aversion, throughout the investment horizon. This model is given by the process \((\gamma_n)_{n \geq 0}\) in (3.2), which depends on the client’s initially communicated risk aversion level, and the updating rule used to determine future times of interaction with the client. When determining the optimal strategy, the robo-advisor averages over future risk aversion paths implied by this model. At time one, if there is not an update, the dynamics of the robo-advisor’s model for the future evolution of the client’s risk aversion remains the same. This continues until the first updating time, at which the robo-advisor updates its model for the future evolution of the client’s risk aversion, throughout the remaining investment horizon, based on the newly communicated risk aversion level. To construct the optimal strategy, the robo-advisor then averages over future risk-aversion paths implied by the updated model, and this process continues until the terminal date is reached.

Finally, we note that maximizing the return on the client’s wealth, as in the objective function (3.7), is equivalent to maximizing the client’s risk-adjusted terminal wealth. Namely, we can define

\[
\tilde{J}_{n,T}^{(\phi)}(X(n), \gamma(n), \tau(n), \theta, \pi) := E_n^\phi[X_T^\pi] - \frac{1}{2} \frac{\Delta n}{X_n} \text{Var}_n^{(\phi)}[X_T^\pi],
\]

and observe that the two objective functions are related by

\[
\tilde{J}_{n,T}^{(\phi)}(X(n), \gamma(n), \tau(n), \theta, \pi) = X_n (1 + J_{n,T}^{(\phi)}(X(n), \gamma(n), \tau(n), \theta, \pi)).
\]

In other words, at any time \(n \in \{0, 1, \ldots, T - 1\}\), maximizing \(J_{n,T}^{(\phi)}\) over \(\pi\) is equivalent to maximizing \(\tilde{J}_{n,T}^{(\phi)}\) over \(\pi\). It then follows that any solution to the optimization problem (3.11) is also a solution of the corresponding optimization problem defined in terms of \(\tilde{J}_{n,T}^{(\phi)}\). Furthermore, from the expression of \(\tilde{J}_{n,T}^{(\phi)}\) it can be easily seen that the risk-return tradeoff in \(J_{n,T}^{(\phi)}\) is decreasing in the current wealth \(X_n\).

**Example 1.** To illustrate the features of our framework, we provide an example with a one year investment horizon consisting of three investment periods, i.e., \(T = 3\). We also consider three representative risk aversion levels given by \(\Gamma = \{\bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3\} = \{2.5, 3.5, 4.5\}\). The optimal amount of wealth allocated to the risky asset is then a strategy \(\pi^* = (\pi_{0,T}^*, \pi_{1,T}^*, \pi_{2,T}^*)\), specifying the allocations made at times \(n = 0, 1, 2\), and the potential interaction times between the client and the robo-advisor are \(n = 1, 2\).

We set the risk aversion volatility to \(\sigma_\gamma = 0.30\), for each risk aversion level \(\gamma \in \Gamma\). As mentioned in Section 3.3, the client’s risk aversion levels at updating times form a Markov chain, and in the special case of risk preferences being updated at all times, the transition matrix \(\Lambda\) (see (A.8)), corresponding to the volatility

\[\text{The optimal allocations corresponding to } \gamma = 3.5 \text{ are close to the classical 60/40 portfolio composition. This is the widely popular passive investing strategy of Jack Bogle, the founder of The Vanguard Group, who is credited with creating the first index fund.} \]
profile specified above, is given by

\[
\Lambda = \begin{pmatrix}
1 - p/2 & p/2 & 0 \\
p/2 & 1 - p & p/2 \\
0 & p/2 & 1 - p/2
\end{pmatrix} = \begin{pmatrix}
0.9 & 0.1 & 0 \\
0.1 & 0.8 & 0.1 \\
0 & 0.1 & 0.9
\end{pmatrix}.
\]

This corresponds to a symmetric random walk with probability \( p = 0.1 \) of jumping to a neighboring level.

The robo-advisor initially communicates with the client at time 0 and then considers three different updating schedules for the client’s risk preferences, corresponding to different rates of client participation in the investment process:

(i) Updates at times \( n = 1 \) and \( n = 2 \), which corresponds to the full-information updating rule, \( \phi \equiv 1 \). In this case, the robo-advisor always knows the client’s risk aversion.

(ii) No further updates after the initial communication, which corresponds to the no-information updating rule, \( \phi \equiv \infty \). In this case, the optimal investment strategy turns out to be the same as in a model where the client’s risk aversion is assumed to be constant.

(iii) The third case lies between the first two cases, and corresponds to updates at every other time point, so \( \phi \equiv 2 \). Hence, the client’s risk preferences are only communicated to the robo-advisor at time \( n = 2 \).

The optimal investment strategy associated with criterion (3.7) naturally depends on which of the three updating schedules for the client’s risk preferences is chosen. In Table 1, we report the optimal proportion of wealth allocated to the risky asset for the two extreme cases, with the risk preferences either updated at both times or never updated.\(^7\) The results show that the optimal proportion at a given time and at a given risk aversion level is not very sensitive to when this risk aversion level was observed (the two numbers within each cell are quite close). However, if the risk aversion level at a given time does not reflect the client’s current risk preferences, there is “regret” associated with the optimal asset allocation. The updating schedule quantifies the tradeoff between having a low regret, which requires frequent updating, and having a low involvement in the investment process, which requires less frequent updating (see Section 5.1 where the concept of “regret” is introduced and this example is continued).

\(^7\)In Section 4, we show that the optimal strategy has the following form: At any time, the amount allocated to the risky asset is proportional to current wealth, with the proportion being a function of both the most recent risk aversion level and the most recent updating time (see (4.3)).
4 Optimal Investment under Client/Robo-Advisor Interaction

In this section, we present the solution to the optimization problem introduced in Section 3.4. It is well known that even if the risk-return tradeoff $\Delta_n$ is constant through time, the family of optimization problems defined by the objective functions in (3.7),

$$\{\sup_{\pi} J_{n,T}^{(\phi)}(X(n), \gamma(n), \tau(n), \theta, \pi)\}_{0 \leq n < T},$$

is time-inconsistent, in the sense that if the control law $\pi^*$ maximizes $J_{n,T}^{(\phi)}(X(n), \gamma(n), \tau(n), \theta, \pi)$, then the restriction of $\pi^*$ to the time points $\{n+1, n+2, \ldots, T-1\}$ may not maximize $J_{n+1,T}^{(\phi)}(X_{n+1}^{\pi^*}, \gamma(n+1), \tau(n+1), \theta, \pi)$, where $X_{n+1}^{\pi^*}$ is equal to $(X_{n}, X_{n+1}^{\pi^*})$, and $X_{n+1}^{\pi^*}$ is obtained by applying the control $\pi_{n,T}^*$ to $X_n$ at time $n$. We refer to Björk and Murgoci [2013] and references therein for a general treatment of time-inconsistent stochastic control in discrete time.

As standard in this literature (see, for instance, Björk and Murgoci [2013] and Björk et al. [2014]), we view the optimization problem in (3.11) as a multi-player game, with one player at each time $n \in \{0, 1, \ldots, T-1\}$ thought of as a future self of the client. Player $n$ then wishes to maximize the objective function $J_{n,T}^{(\phi)}$, but decides only the control $\pi_{n,T}$, while $\pi_{n+1:T}$ are chosen by her future selves. The resulting optimal control strategy, $\pi^*$, is the subgame perfect equilibrium of this dynamic game, and can be computed using backward induction. At time $n = T-1$, the equilibrium control $\pi_{T-1:T}^*$ is obtained by maximizing $J_{T-1,T}^{(\phi)}$ over $\pi_{T-1:T}$, which is a standard optimization problem. For $n < T-1$, the equilibrium control $\pi_{n,T}^*$ is then obtained by letting player $n$ choose $\pi_{n,T}$ to maximize $J_{n,T}^{(\phi)}$, given that player $n'$ will use $\pi_{n',T}^*$, for $n' = n + 1, n + 2, \ldots, T-1$.

The unique structure of our problem, where the client and the robo-advisor interact, implies that the optimal control $\pi^*$ depends also on $\phi$, the updating rule used to solicit the client’s risk preferences, as

<table>
<thead>
<tr>
<th>$\pi_{n,T}^*(\gamma_n, \tau_n, \theta)$</th>
<th>$n = 0$</th>
<th>$n = 1$</th>
<th>$n = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_n = \gamma_1 = 2.5$</td>
<td>$\tau_n = n$</td>
<td>0.789</td>
<td>0.838</td>
</tr>
<tr>
<td></td>
<td>$\tau_n = 0$</td>
<td>0.785</td>
<td>0.836</td>
</tr>
<tr>
<td>$\gamma_n = \gamma_2 = 3.5$</td>
<td>$\tau_n = n$</td>
<td>0.577</td>
<td>0.607</td>
</tr>
<tr>
<td></td>
<td>$\tau_n = 0$</td>
<td>0.579</td>
<td>0.607</td>
</tr>
<tr>
<td>$\gamma_n = \gamma_3 = 4.5$</td>
<td>$\tau_n = n$</td>
<td>0.456</td>
<td>0.476</td>
</tr>
<tr>
<td></td>
<td>$\tau_n = 0$</td>
<td>0.458</td>
<td>0.477</td>
</tr>
</tbody>
</table>

Table 1: For a one year horizon and three investment periods ($T = 3$), the table reports $\pi_{n,T}^*(\gamma_n, \tau_n, \theta)$, the optimal proportion of wealth allocated to the risky asset at time $T$. The unique structure of our problem, where the client and the robo-advisor interact, implies that the optimal control $\pi^*$ depends also on $\phi$, the updating rule used to solicit the client’s risk preferences, as

Electronic copy available at: https://ssrn.com/abstract=3453975
described in Section 3.2.

Next, we first present the solution in the case of a general updating rule $\phi$, and then discuss properties of the solution by focusing on the two extreme cases of full information, $\phi \equiv 1$, and of no information, $\phi \equiv \infty$. As we will show, for a general updating rule $\phi$, the solution exhibits properties of the solutions in those two limiting cases, and in Section 5 we will further analyze how the updating rule $\phi$ determines the client’s regret.

**Proposition 4.1.** The optimization problem (3.11) is solved by the control law

$$
\pi^*_n(T, X_n, \gamma_n, \tau_n, \theta) = \frac{\pi_n(\gamma_n, \tau_n)}{\Delta_n} X_n, \quad 0 \leq n < T.
$$

(4.1)

Above, for each $\gamma_n \in \Gamma$, and each $0 \leq \tau_n \leq n$, we have defined

$$
\mu_n^a(\gamma_n, \tau_n) := \mathbb{E}_n^{(\phi)}[a_n+1(\gamma_{n+1}, \tau_{n+1})], \quad \mu_n^b(\gamma_n, \tau_n) := \mathbb{E}_n^{(\phi)}[b_n+1(\gamma_{n+1}, \tau_{n+1})],
$$

(4.2)

with $a_n(\gamma_n, \tau_n)$ and $b_n(\gamma_n, \tau_n)$ satisfying the backward recursions

$$
an_n(\gamma_n, \tau_n) = \mu_n^{a}(\gamma_n, \tau_n)(R + \bar{\mu} \pi_n(\gamma_n, \tau_n)), \quad 0 \leq n < T,
$$

$$
b_n(\gamma_n, \tau_n) = \mu_n^{b}(\gamma_n, \tau_n)(\sigma^2(\pi_n(\gamma_n, \tau_n))^2 + (R + \bar{\mu} \pi_n(\gamma_n, \tau_n))^2), \quad 0 \leq n < T,
$$

with $a_T(\gamma_T, \tau_T) = b_T(\gamma_T, \tau_T) = 1$, for all $\gamma_T \in \Gamma$ and $0 \leq \tau_T < T$.

In Appendix D, we provide a pseudocode for the backward recursion used to compute the optimal risky asset allocation above. We recall that $R$, $\bar{\mu}$, and $\sigma$, are the market parameters, introduced in Section 3.4, and that $\Delta_n$ is the risk-return tradeoff coefficient, defined in (3.10). We also stress that the optimal solution $\pi^*$ depends on the updating rule $\phi$, but for readability purposes we omit this dependence in the notation. Similarly, the dependence of the $a_n$- and $b_n$-coefficients on $\theta$, $\phi$, and the investment horizon $T$, is not explicitly highlighted. From the optimal allocation formula (4.1), we see that

$$
\pi^*_n(T, X_n, \gamma_n, \tau_n, \theta) = \pi^*_n(T, \gamma_n, \tau_n, \theta) X_n, \quad 0 \leq n < T,
$$

(4.3)

where, with a slight abuse of notation, we use $\pi^*_n(\gamma_n, \tau_n, \theta)$ to denote the optimal proportion of wealth allocated to the risky asset. Hence, the objective function (3.7) exhibits constant relative risk aversion: The optimal portfolio composition at time $n$ is independent of the client’s current wealth, $X_n$.

From (4.1), we also see that the optimal solution is Markovian, as the amount of wealth allocated to
the risky asset at time \( n \) is a function of the current wealth, \( X_n \), the most recently communicated level of the client’s risk aversion, \( \gamma_n \), and the corresponding updating time, \( \tau_n \). However, recalling the discussion at the end of Section 3.3, we stress that the optimal allocation at time \( n \) depends on the future evolution of the client’s risk aversion level, \( (\gamma_k)_{n \leq k < T} \), and, thus, also on the future evolution of the risk-return tradeoff coefficient, \( (\Delta_k)_{n \leq k < T} \). In the recursive formulation, this is seen from the presence of the expected values in (4.2); these values are computed explicitly and further discussed in Remark 4.2 below.

The coefficients \( a_n(\gamma_n, \tau_n) \) and \( b_n(\gamma_n, \tau_n) \) in Proposition 4.1 are the first and second moments of the future value of one dollar invested optimally between time \( n \) and the terminal date \( T \), when the most recent interaction between the client and the robo-advisor occurred at time \( \tau_n \) and the communicated risk aversion level was \( \gamma_n \). That is,

\[
a_n(\gamma_n, \tau_n) = \mathbb{E}_n^{(\phi)}[1 + r_{n,T}^\pi], \quad b_n(\gamma_n, \tau_n) = \mathbb{E}_n^{(\phi)}[(1 + r_{n,T}^\pi)^2],
\]

with the simple return \( r_{n,T}^\pi \) defined in (3.8). It follows that the value function (3.12) is given by

\[
V_{n,T}^{(\phi)}(\gamma_n, \tau_n, \theta) = a_n(\gamma_n, \tau_n) - 1 - \frac{\Delta_n}{2} \left( b_n(\gamma_n, \tau_n) - a_n^2(\gamma_n, \tau_n) \right), \quad 0 \leq n \leq T; \tag{4.4}
\]

which is independent of the current wealth \( X_n \).

**Remark 4.2.** The expected values appearing in the objective function \( J_{n,T}^{\phi} \) in (3.7) are computed by averaging over future paths of the client’s risk aversion (see the discussion at the end of Section 3.3 and Remark 3.3). In the optimal allocation formula (4.1) this averaging takes place through the expected values given in (4.2), which link the optimal allocations at consecutive time points. These expected values admit explicit representations given by

\[
\mu_n^a(\gamma_n, \tau_n) = \sum_{\gamma \in \Gamma} p_{\tau_{n+1}}(\gamma; \gamma_n, \tau_n) a_{n+1}(\gamma, \tau_{n+1}), \quad \mu_n^b(\gamma_n, \tau_n) = \sum_{\gamma \in \Gamma} p_{\tau_{n+1}}(\gamma; \gamma_n, \tau_n) b_{n+1}(\gamma, \tau_{n+1}),
\]

where the transition probabilities \( p_{\tau_{n+1}}(\gamma; \gamma_n, \tau_n) \), for each \( \gamma \in \Gamma \), are given explicitly in Lemma A.1-(a). Observe that, consistently with intuition, the optimal risky asset allocation at time \( n \) is increasing in \( \mu_n^a(\gamma_n, \tau_n) \), the expected future value of one dollar invested in the optimal strategy between times \( n + 1 \) and \( T \), and decreasing in its variance, \( \mu_n^b(\gamma_n, \tau_n) - (\mu_n^a(\gamma_n, \tau_n))^2 \).

There are now two distinct cases. First, if \( \tau_{n+1} = \tau_n < n + 1 \), the client’s risk-preferences are not solicited at time \( n + 1 \), and \( p_{\tau_{n+1}}(\gamma_n; \gamma_n, \tau_n) = 1 \), so \( \mu_n^a(\gamma_n, \tau_n) = a_{n+1}(\gamma_n, \tau_n) \) and \( \mu_n^b(\gamma_n, \tau_n) = b_{n+1}(\gamma_n, \tau_n) \). Second, if \( \tau_{n+1} = n + 1 \), the client’s risk preferences are solicited at time \( n + 1 \), and the probabilities
\{p_{n+1}(\gamma; \gamma_n, \tau_n)\}_{\gamma \in \Gamma} \) link the optimal allocation at time \( n \), corresponding to \( \gamma_n \) and \( \tau_n \), to the optimal allocations at time \( n + 1 \), corresponding to \( \gamma \) and \( n + 1 \), for each \( \gamma \in \Gamma \). In this case, the probability that the realized risk aversion at time \( n + 1 \) differs from \( \gamma_n \) is increasing both in the time since the previous update, \((n + 1) - \tau_n\), and in the volatility of the random walk describing the client’s risk aversion, \( \sigma_{\gamma_n} \) (see Lemma A.1-(a)). These two cases are pictured in Figure 1. The first case corresponds to the constant components of the risk aversion trajectory, where the client and the robo-advisor do not interact. The second case corresponds to the discontinuities observed at the updating times. Prior to each such time, the recursive formulation involves averaging over the distribution of future risk aversion levels, as described above, with one risk aversion level eventually being communicated by the client at the updating time. In other words, prior to each updating time, there is uncertainty in the subsequently communicated risk aversion level, which translates into uncertainty in the optimal allocation at the updating time.

Next, we present the solution to the optimization problem for the two extreme cases \( \phi \equiv 1 \) and \( \phi \equiv \infty \). The former implies that \( \Delta_n = \gamma_n \), so there is no \( \theta \)-dependence, and we denote the optimal proportion of wealth allocated to the risky asset by \( \pi_{n,T}^{*}(\gamma_n, n) \). The latter case implies that \( \Delta_n = \gamma_0 + \theta \delta_n(\gamma_0, 0) \), and we use the notation \( \pi_{n,T}^{*}(\gamma_0, \theta) \). In this case, we also use the superscript \( (\infty) \) in the \( a_n \)- and \( b_n \)-coefficients to distinguish them from the case \( \phi \equiv 1 \).

**Proposition 4.3.**

(a) Assume that \( \phi \equiv 1 \). Then, the optimal proportion of wealth allocated to the risky asset at time \( n \), if the risk-aversion level is \( \gamma_n \in \Gamma \), is given by

\[
\pi_{n,T}^{*}(\gamma_n, n) = \frac{\bar{\mu}/\sigma^2}{\gamma_n} E_n^{(1)} [a_{n+1}(\gamma_{n+1})] - R \gamma_n (E_n^{(1)} [b_{n+1}(\gamma_{n+1})] - (E_n^{(1)} [a_{n+1}(\gamma_{n+1})])^2) + \frac{\bar{\mu}^2}{\sigma^2} E_n^{(1)} [a_{n+1}(\gamma_{n+1})] - (E_n^{(1)} [a_{n+1}(\gamma_{n+1})])^2, 
\]

for \( 0 \leq n < T \), where

\[
a_n(\gamma_n) = E_n^{(1)} [a_{n+1}(\gamma_{n+1})](R + \bar{\mu} \pi_n^{*}(\gamma_n, n)), \quad 0 \leq n < T, \\
b_n(\gamma_n) = E_n^{(1)} [b_{n+1}(\gamma_{n+1})](\sigma^2(\pi_n^{*}(\gamma_n, n))^2 + (R + \bar{\mu} \pi_n^{*}(\gamma_n, n))^2), \quad 0 \leq n < T,
\]

with \( a_T(\gamma_T) = b_T(\gamma_T) = 1 \), for all \( \gamma_T \in \Gamma \).

(b) Assume that \( \phi \equiv \infty \). Then, the optimal proportion of wealth allocated to the risky asset at time \( n \), if the initial risk aversion level is \( \gamma_0 \in \Gamma \), is given by

\[
\pi_{n,T}^{*}(\gamma_0, \theta) = \frac{\bar{\mu}/\sigma^2}{\Delta_n} a_{n+1}^{(\infty)}(\gamma_0) - R \Delta_n (b_{n+1}^{(\infty)}(\gamma_0) - (a_{n+1}^{(\infty)}(\gamma_0))^2) + (\frac{\bar{\mu}}{\sigma})^2 (b_{n+1}^{(\infty)}(\gamma_0) - (a_{n+1}^{(\infty)}(\gamma_0))^2), \quad 0 \leq n < T,
\]
where

\[ a_n^{(\infty)}(\gamma) = a_{n+1}^{(\infty)}(\gamma)(R + \bar{\mu}\pi_n^{*}(\gamma)), \quad 0 \leq n < T, \]
\[ b_n^{(\infty)}(\gamma) = b_{n+1}^{(\infty)}(\gamma)(\sigma^2(\pi_n^{*}(\gamma))^2 + (R + \bar{\mu}\pi_n^{*}(\gamma))^2), \quad 0 \leq n < T, \]

with \( \sigma_T^{(\infty)}(\gamma_0) = b_T^{(\infty)}(\gamma_0) = 1, \) for all \( \gamma_0 \in \Gamma. \)

Recall from Section 3.2 that if \( \phi \equiv 1 \), the risk aversion level process \( (\gamma_n)_{n \geq 0} \) is a Markov chain with transition probability matrix \( \Lambda \) defined in (A.8). The expected values in (4.6) thus become

\[ \mathbb{E}_n^{(1)}[a_{n+1}(\gamma_{n+1})] = \sum_{\gamma \in \Gamma} \Lambda_{\gamma_n, \gamma} a_{n+1}(\gamma), \quad \mathbb{E}_n^{(1)}[b_{n+1}(\gamma_{n+1})] = \sum_{\gamma \in \Gamma} \Lambda_{\gamma_n, \gamma} b_{n+1}(\gamma). \]

In the case \( \phi \equiv \infty \), the entire mass of the distribution at time \( n+1 \) is put on the risk aversion level \( \gamma_n \), i.e.,

\[ \mathbb{E}_n^{(\infty)}[a_{n+1}^{(\infty)}(\gamma_{n+1})] = a_{n+1}^{(\infty)}(\gamma_n), \quad \mathbb{E}_n^{(\infty)}[b_{n+1}^{(\infty)}(\gamma_{n+1})] = b_{n+1}^{(\infty)}(\gamma_n), \]

with \( \gamma_n = \gamma_0 \) (see (4.7)). This holds for all times \( n \); it then follows that, if \( \phi \equiv \infty \), the sequence of optimal allocations, \( (\pi_n^{*}(\gamma_0, \theta))_{0 \leq n < T} \), becomes “open-loop”. In other words, the optimal strategy throughout the investment horizon is determined at the initial time, as it does not depend on future risk aversion levels communicated by the client, simply because there is no communication after the initial time. The value function (4.4) corresponding to \( \phi \equiv 1 \) and \( \phi \equiv \infty \) becomes

\[ V_{n,T}^{(1)}(\gamma_n) = a_n(\gamma_n) - 1 - \frac{\gamma_n}{2} (b_n(\gamma_n) - a_n^2(\gamma_n)), \]
\[ V_{n,T}^{(\infty)}(\gamma_0, \theta) = a_n^{(\infty)}(\gamma_0) - 1 - \frac{\Delta_n}{2} (b_n^{(\infty)}(\gamma_0) - (a_n^{(\infty)}(\gamma_0))^2), \]

respectively, for \( 0 \leq n \leq T. \)

From the two cases discussed in Remark 4.2, we can now see how the optimal solution for a general updating rule \( \phi \) exhibits properties of the optimal solutions in both of the extreme cases analyzed above. Between consecutive updating times, when there is no feedback received from the client, the optimal solution is open-loop, as in the case \( \phi \equiv \infty \), while at updating times the optimal solution depends on the realized risk aversion level, as in the case \( \phi \equiv 1 \) (see Figure 2).

Informally speaking, the updating rule determines the rate of information arrival from the client, with a general updating rule interpolating between the two extreme cases, \( \phi \equiv 1 \) and \( \phi \equiv \infty \). Namely, since the

---

8In fact, the case \( \phi \equiv \infty \) is equivalent to the case \( \phi \equiv 1 \) with constant risk aversion, i.e., \( \sigma_{\gamma} = 0 \) for all \( \gamma \in \Gamma \). In that case \( \Lambda \) is the identity matrix, so all the probability mass at time \( n+1 \) is placed on the time \( n \) risk aversion level.
Figure 2: For investment horizon \( T \), the figure shows how \( \pi^*_n, T(\gamma_n, \tau_n, \theta) \), the optimal proportion of wealth allocated to the risky asset, is determined in a piecewise manner. At the initial time, \( T_0^{(\phi)} = 0 \), the optimal allocation until the first updating time \( T_1^{(\phi)} \) is determined, as \( \{ \pi^*_n, T(\gamma_0, 0, \theta), 0 \leq n < T_1^{(\phi)} \} \) is a deterministic sequence, given the initial risk aversion level \( \gamma_0 \). In general, given the \( k \)-th updating time, \( T_k^{(\phi)} \), and the realized risk aversion level, \( \gamma_{T_k^{(\phi)}} \), the optimal allocations \( \{ \pi^*_n, T(\gamma_{T_k^{(\phi)}}, T_k^{(\phi)}, \theta), T_k^{(\phi)} \leq n < T_{k+1}^{(\phi)} \} \) is a deterministic sequence.

client’s risk aversion evolves like a random walk between consecutive updating times (cf. (3.1)), the variance of the realized risk aversion at an updating time is increasing in the time elapsing since the previous updating time. With frequent updating, there is little variation in the client’s risk aversion between consecutive updating times, as information about the client arrives gradually. With less frequent updating, however, there is a greater uncertainty in the client’s risk aversion level at updating times, as information about the client arrives in larger bursts. In Section 5.1 we quantify the effect of this uncertainty using a measure of regret. In particular, we show that for \( \phi \equiv 1 \) (risk preferences communicated at all times), the regret is zero, while for any other updating rule the regret is nonzero, and it is maximized for \( \phi \equiv \infty \) (risk preferences communicated only once at time zero).

Next, we provide a financial interpretation of the optimal portfolio allocation (4.1). As above, we start with the extreme cases \( \phi \equiv \infty \) and \( \phi \equiv 1 \). For simplicity we assume zero interest rates \( (R = 1) \), with the general case derived in Appendix B. We also set the machine-specific component of the risk-return tradeoff to zero, \( \theta = 0 \), and study the effect of a nonzero \( \theta \) on the optimal investment strategy studied in Section 5.2. We denote by \( Z \) a random variable with the same distribution as \( (Z_n)_{n \geq 1} \) in (3.6), but independent of it. Hence, \( Z' := Z/\sigma \) has mean \( \bar{\mu}/\sigma \) and unit variance.

If \( \phi \equiv \infty \), the optimal allocation at time \( n \), given initial risk aversion \( \gamma_0 \in \Gamma \), can be written as

\[
\pi^*_n, T(\gamma_0; 0) = \pi^*_{T-1, T}(\gamma_0; 0) \frac{1 + V_{n+1}^{(\infty)}(\gamma_0; 0) - \frac{2n}{2} \text{Var}_{n+1}^{(\infty)}[1 + r \pi^*_n, T]}{\text{Var}_{n+1}^{(\infty)}[Z'(1 + r \pi^*_n, T)]}. \tag{4.9}
\]

This shows that the risky asset allocation at time \( n \) is equal to the final period allocation, \( \pi^*_{T-1, T}(\gamma_0; 0) \), multiplied by the value at time \( n + 1 \) of one dollar invested optimally between time \( n + 1 \) and the terminal date \( T \). We then subtract a term that quantifies the uncertainty in this value, and scale everything by a factor that accounts for the current market scenario, using the random variable \( Z' \). This scaling factor
captures the uncertainty in the investment return between times \( n \) and \( n + 1 \). We have

\[
Var_{n+1}^{(\infty)}[Z'(1 + r_{n+1,T}^*)] \geq Var_{n+1}^{(\infty)}[1 + r_{n+1,T}^*].
\]

This interpretation can be readily extended to the case \( \phi \equiv 1 \). For \( \gamma_n \in \Gamma \), we note that \( 1 - \Lambda_{\gamma_n,\gamma_n} \) is the probability that the risk aversion level at time \( n + 1 \) will be different from \( \gamma_n \), and write

\[
\pi_n^*(\gamma_n, n) = \pi_{T-1,T}(\gamma_n, T-1) \frac{1 + \frac{1}{2} Var_{n+1}^{(\infty)}[1 + r_{n+1,T}^*|\gamma_{n+1} = \gamma_n]}{Var_{n+1}^{(\infty)}[Z'(1 + r_{n+1,T}^*)|\gamma_{n+1} = \gamma_n]} + O(1 - \Lambda_{\gamma_n,\gamma_n}). \tag{4.10}
\]

In the above expression, the value function at time \( n + 1 \) is evaluated at the current risk aversion level, \( \gamma_n \). The interpretation in (4.9) extends up to a \( O(\cdot) \) term, which captures the probability that the risk aversion at time \( n + 1 \) will be different from \( \gamma_n \). This probability is larger at a risk aversion level where the transition probability \( 1 - \Lambda_{\gamma_n,\gamma_n} \) is high, i.e., if the uncertainty about the risk aversion level at time \( n + 1 \) is high. In practice, risk preferences are persistent and stable over time (Schildberg-Horisch [2018]), so most of the probability mass at time \( n + 1 \) concentrates on the time \( n \) risk aversion level, \( \gamma_n \).

Finally, for the case of a general updating rule \( \phi \), the optimal allocation (4.1) may be written as

\[
\pi_n^*(\gamma_n, \tau_n; 0) = \pi_{T-1,T}(\gamma_n, \tau_n; 0) \frac{1 + \frac{V_{n+1,T}^{(\phi)}(\gamma_n, \tau_n; 0)}{2} Var_{n+1}^{(\infty)}[1 + r_{n+1,T}^*|\gamma_{n+1} = \gamma_n]}{Var_{n+1}^{(\phi)}[Z'(1 + r_{n+1,T}^*)|\gamma_{n+1} = \gamma_n]} + O(1 - p_{\tau_{n+1}}(\gamma_n; \gamma_n, \tau_n)), \tag{4.11}
\]

where the error term again comes from the fact that there is a nonzero probability that the risk aversion at time \( n + 1 \) will be different from \( \gamma_n \). As in Remark 4.2, two cases now arise. First, between updating times, i.e., if \( \tau_{n+1} = \tau_n < n + 1 \), the error term vanishes as in (4.9). Second, at updating times, i.e., if \( \tau_{n+1} = n + 1 \), the error term is nonzero as in (4.10). This further highlights how the solution for a general updating rule \( \phi \) lies between the solutions in the two extreme cases of \( \phi \equiv \infty \) and \( \phi \equiv 1 \).

5 Performance of the Client/Robo-Advisor Interaction System

In this section, we study the performance of the interaction system formed by the client and the robo-advisor, by considering the investment implications of information asymmetry. The form of the risk-return coefficient in the adaptive mean-variance criterion (see (3.10)) highlights the presence of two effects. First, the robo-advisor may determine the allocation at time \( n \) based on stale information, as the risk aversion level
\(\gamma_n\) is communicated at time \(\tau_n \leq n\). We quantify this \textit{client-specific} effect in Section 5.1, using a proposed measure of regret. We also quantify the extent to which the robo-advisor underestimates the true regret by using a discrete set of risk aversion levels to categorize the client, as opposed to using a continuum of risk aversion levels.\(^9\) Second, if \(\theta > 0\), the robo-advisor is averse with respect to uncertainty in the client’s risk aversion, and thus chooses a less risky portfolio composition. In Section 5.2 we quantify this \textit{machine-specific} effect on the optimal portfolio allocation. In Section 5.3, we use our measure of regret to study the benefits of robo-advising on the basis of the client’s involvement in the investment process.

### 5.1 Client-Specific Effect: Portfolio Regret

The robo-advisor solves the optimal investment problem defined by (3.7) and (3.11), but its investment decisions depend on its uncertainty about the client’s risk aversion. This uncertainty vanishes at the times when the client’s risk preferences are communicated. At each time within an updating interval, defined by two consecutive communication times, the regret is a measure of the difference between the robo-advisor’s optimal asset allocation given the current risk-return tradeoff coefficient, and a benchmark allocation in which the risk-return tradeoff coefficient is obtained via an immediate update of risk preferences. Before formally define this measure of regret, we discuss the rational behind its definition.

In the machine learning literature, regret is traditionally defined as “...the expected decrease in reward gained due to executing the learning algorithm instead of behaving optimally from the very beginning.” (Kaelbling et al. [1996]). In our framework, the reward is quantified using a mean-variance criterion, and cannot be directly used to define regret for the reasons outlined next. First, at a given point in time, an update of the client’s risk preferences changes the mean-variance criterion, so the value functions before and after the update are not directly comparable, and there is no guarantee that the latter is larger than the former. Second, the mean-variance criterion changes through time, as the client’s risk preferences evolve, so the value functions at different time points are not directly comparable.

In our model, we define regret in terms of the control, i.e., the asset allocation, which can naturally be compared before and after an update of risk preferences, as well as at different points in time. Consider a fixed investment horizon \(T\), and an updating rule \(\phi\). The \textit{regret} at time \(n \in \{0, 1, \ldots, T - 1\}\), given that the previous communication of risk preferences took place at time \(\tau \leq n\) and resulted in risk aversion level

\(^9\)While it is theoretically possible to use a continuum of risk aversion levels, it would not be possible to calibrate the model for such a high level of granularity.
γ ∈ Γ, is defined as
\[
\overline{R}_{n,T}^{(\phi)}(\gamma, \tau) := E_n \left[ \frac{|\pi^*_n(T(\gamma_n;0)) - \pi^*_n(T(\tilde{\gamma}_n), n;0)|}{\pi^*_n(T(\tilde{\gamma}_n), n;0)} \bigg| \gamma_n = \gamma, \tau_n = \tau \right].
\] (5.1)

This is the expected relative difference between the robo-advisor’s strategy \(\pi^*_n(T(\gamma_n;0)), i.e., the optimal allocation without updating of risk preferences at time \(n\) (and thus \(\tau_n = \tau\) and \(\gamma_n = \gamma\)), and \(\pi^*_n(T(\tilde{\gamma}_n), n;0)\), i.e., the optimal allocation with updating of risk preferences at time \(n\) (and thus \(\tau_n = \tau\) and \(\gamma_n = \Psi(\tilde{\gamma}_n)\)).

In particular, if \(n\) is an updating time, so \(\tau_n = n\), then \(\overline{R}_{n,T}^{(\phi)}(\gamma, \tau) = 0\). Also observe that we set \(\theta = 0\) to isolate the client-specific component of risk aversion uncertainty, as discussed at the beginning of Section 5.

We then define a worst-case measure of regret as
\[
\overline{R}_T^{(\phi)}(\gamma) := \sup_{0 \leq \tau \leq T-1} \sup_{n \leq \tau < n_T} \overline{R}_{n,T}^{(\phi)}(\gamma, \tau),
\] (5.2)

where \(n_T := (T - \phi(\gamma)) \land T\). For a fixed value of \(\tau\), the quantity \(\sup_{n \leq \tau < n_T} \overline{R}_{n,T}^{(\phi)}(\gamma, \tau)\) is the worst-case regret when starting at risk aversion level \(\gamma\) at time \(\tau\), and following the optimal investment strategy, under the updating rule \(\phi\), until the risk preferences are updated again, or the investment horizon is reached, whichever comes first. Hence, (5.2) provides a worst-case measure of regret both with respect to the starting point of the updating interval, \(\tau\), and with respect to the time point within the updating interval at which the regret is evaluated, \(n\).

In Appendix C, we show that
\[
\overline{R}_T^{(\phi)}(\gamma) \leq \sup_{\tau \leq n < T} \overline{R}_{n,T}^{(\phi)}(\gamma, \tau) \bigg|_{\tau=(T - \phi(\gamma)) \lor 0} + O(1 - p_{\phi(\gamma)}(\gamma; \gamma)),
\] (5.3)

where we have used that \(n_T = T\) for \(\tau = (T - \phi(\gamma)) \lor 0\). That is, the worst-case regret occurs for an updating interval whose right endpoint coincides with the investment horizon \(T\). The error term in (5.3) comes from the fact that the regret is local, i.e., it corresponds to a single updating interval bounded by two consecutive updating times, but the optimal allocations within this interval depend on the risk aversion levels communicated by the client throughout the remaining investment horizon. We also show that
\[
\sup_{\tau \leq n < T} \overline{R}_{n,T}^{(\phi)}(\gamma, \tau) \bigg|_{\tau=(T - \phi(\gamma)) \lor 0} = \overline{R}_{T-1,T}^{(\phi)}(\gamma, \tau) \bigg|_{\tau=(T - \phi(\gamma)) \lor 0} := \overline{R}_T^{(\phi)}(\gamma),
\] (5.4)

which states that within the worst-case updating interval, the maximum regret occurs at the very last time point; because of the random walk assumption (3.1) this is also the point within the updating interval where
Figure 3: For investment horizon $T = 24$ (months), the figure shows the sequence of regret values $\{\hat{R}^{(\phi)}_{n,T}(\gamma_n, \tau_n)\}_{0 \leq n < T}$, defined in (5.1), for $\gamma_n \in \{2.5, 3.5, 4.5\}$, and $\tau_n = \lfloor n/3 \rfloor \times 3$, which corresponds to the updating rule $\phi \equiv 3$ (months). The solid points show the worst-case upper bound $R^{(\phi)}_{T}(\gamma_n)$, defined in (5.4), which is equal to $\hat{R}^{(\phi)}_{n,T}(\gamma_n, \tau_n)$ evaluated at time $n = T - 1$, with $\gamma_n = \gamma$. The set of risk aversion levels is $\Gamma = \{2, 2.5, 3, 3.5, 4, 4.5, 5\}$, and the risk aversion volatility is set to $\sigma = 0.152$, for all $\gamma \in \Gamma$, which corresponds to a 10% probability of leaving the current risk aversion level in a single step (since $\sqrt{4 \times 0.152} \approx 0.30$, this is comparable to the volatility in Example 1 in Section 3.4, where the step size was four months). The market parameters on an annualized basis are set to $r = 0.04$, $\bar{\mu} = 0.08$, and $\sigma = 0.20$.

The variance of the client’s risk aversion is the largest. Both of these properties are illustrated in Figure 3.

Together, (5.3) and (5.4) state that at any time point within any updating interval starting at risk aversion level $\gamma$, the quantity $R^{(\phi)}_{T}(\gamma)$ is an upper bound for the expected relative change in asset allocation that would be observed if there were an update of the client’s risk preferences. Also, observe that $R^{(\phi)}_{T}(\gamma)$ only depends on the updating rule $\phi$ via $\phi(\gamma)$, i.e., the time spent at level $\gamma$ before an update of risk preferences. That is, $R^{(\phi)}_{T}(\gamma) = R^{(\phi')}_{T}(\gamma)$ for any updating rule $\phi'$ such that $\phi'(\gamma) = \phi(\gamma)$.

The upper bound $R^{(\phi)}_{T}(\gamma)$ in (5.4) is a measure of the allocation error suffered by the client with risk aversion level $\gamma \in \Gamma$. Next, we define a corresponding steady-state measure of regret,

$$\hat{R}^{(\phi)}_{T}(\gamma) := \mathbb{E}[R^{(\phi)}_{T}(\gamma)|\gamma \sim \lambda] = \sum_{\gamma \in \Gamma} \lambda(\gamma)R^{(\phi)}_{T}(\gamma), \quad (5.5)$$

which is the regret when starting at a risk aversion level drawn from the stationary distribution of the risk aversion level process $(\gamma_n)_{n \geq 0}$, denoted by $\lambda$. In Lemma A.1-(d) we show that $\lambda$ is uniquely determined.

Remark 5.1. The steady-state regret (5.5) can be interpreted as the average regret through time, for a single client with a long investment horizon (e.g., a young client planning for retirement). Alternatively, at a fixed point in time, the pool of clients of the robo-advisor can be used to construct the stationary distribution
\( \lambda \), as \( \lambda(\gamma) \) represents the proportion of clients having risk aversion level \( \gamma \). The steady-state regret (5.5) can be then interpreted as the cross-sectional average regret of the robo-advisor’s pool of clients.

The following Proposition provides formulas for the regret measure defined in (5.2) and the upper bound defined in (5.4). For each \( \gamma \in \Gamma \) and \( n \geq 0 \), we introduce

\[
\mu_n(\gamma) := E[|\Psi(\tilde{\gamma}_n) - \gamma_0| | \gamma_0 = \gamma, \tau_n = 0], \tag{5.6}
\]

which is equal to the expected absolute change in the client’s risk aversion level, in \( n \) time steps, starting at level \( \gamma \) at time zero, and without any intermediate updates of risk preferences. We also define

\[
\mu^c_n(\gamma) := E[|\tilde{\gamma}_n - \gamma_0| | \gamma_0 = \gamma, \tau_n = 0], \quad n \geq 0. \tag{5.7}
\]

which coincides with the definition of \( \mu_n(\gamma) \), but without applying the function \( \Psi \) to \( \tilde{\gamma}_n \), which projects the random walk \( \tilde{\gamma}_n \) to the nearest risk aversion level in \( \Gamma \). Explicit expression for \( \mu_n(\gamma) \) and \( \mu^c_n(\gamma) \) are given in Lemma A.1-(b).

**Proposition 5.2.** Let \( T \geq 1 \) and \( \phi : \Gamma \rightarrow \tilde{\Gamma} \) be an updating rule. Then, the following assertions hold:

(a) For each \( \gamma \in \Gamma \), the regret measure (5.2) satisfies

\[
\bar{R}_T^{(\phi)}(\gamma) \leq R_T^{(\phi)}(\gamma) + O(1 - p_\phi(\gamma; \gamma)),
\]

where the upper bound is defined in (5.4), and given by (see (5.6))

\[
R_T^{(\phi)}(\gamma) = \frac{\mu(\phi(\gamma) \wedge T) - 1(\gamma)}{\gamma}. \tag{5.8}
\]

(b) For each \( \gamma \in \Gamma \), the value \( R_T^{(\phi)}(\gamma) \) in (5.8) is increasing in \( \phi(\gamma) \) and \( \sigma_\gamma \), and

\[
\lim_{{\phi(\gamma) \wedge T \rightarrow \infty}} R_T^{(\phi)}(\gamma) = \frac{\tilde{\gamma}_K - \tilde{\gamma}_1}{2\gamma}, \quad \lim_{{\sigma_\gamma \rightarrow \infty}} R_T^{(\phi)}(\gamma) = \frac{\tilde{\gamma}_K - \tilde{\gamma}_1}{2\gamma}.
\]

Furthermore, it satisfies

\[
R_T^{(\phi)}(\gamma) \leq R_T^{(\phi)}(\gamma), \tag{5.9}
\]

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Electronic copy available at: https://ssrn.com/abstract=3453975
where $R^{(\phi)}_{T,c}(\gamma)$ is given by (see (5.7))

$$R^{(\phi)}_{T,c}(\gamma) = \frac{\mu^{(\phi)}_{T,c}(\gamma)}{\gamma} = \sqrt{\frac{2}{\pi}} \frac{\sigma_{\gamma}}{\gamma} \sqrt{\phi(\gamma) - 1}. \quad (5.10)$$

In the sequel, we refer to the upper bound $R^{(\phi)}_{T}$ in (5.8) as the \textit{regret of the client}. The Proposition shows that, for $\gamma \in \Gamma$, this quantity is increasing in $\phi(\gamma)$, the time spent at risk aversion level $\gamma$ before an update of risk preferences, and equal to zero if and only if $\phi(\gamma) = 1$. The regret is also increasing in the risk aversion volatility, $\sigma_{\gamma}$. In other words, the longer the time before the client’s risk preferences are communicated, the higher the client’s regret, and this effect is more pronounced at risk aversion levels with a high volatility.

From (5.8) we see that regret is closely related to expected changes in the client’s risk aversion level, which implies that discrepancy in regret between different risk aversion levels depends on the heterogeneity between those levels. The formula for $R^{(\phi)}_{T,c}$ in (5.10) highlights that this heterogeneity can be proxied by the heterogeneity of the proportions $\{\sigma_{\gamma}/\gamma\}_{\gamma \in \Gamma}$, which we refer to as \textit{relative risk aversion volatilities}. Specifically, given a fixed time between updates of risk preferences, i.e., $\phi \equiv c$, for some $c \geq 1$, the regret is higher at risk aversion levels with a high relative volatility. This can also be seen in Figure 3, which shows the client’s regret for three different values of relative risk aversion volatility.

The regret measure $R^{(\phi)}_{T,c}$ depends on the risk aversion levels in the discrete set $\Gamma$. In particular, for a fixed $\gamma \in \Gamma$, $R^{(\phi)}_{T,c}(\gamma)$ increases with the granularity of $\Gamma$. To quantify the effect of this \textit{discretization} on regret, we observe that the difference between $\mu_{T,c}(\gamma)$ and $\mu_{T}(\gamma)$, respectively given in (5.6) and (5.7), is that the latter is based on a continuous range of risk aversion levels, as $\check{\gamma}_{n} \in \mathbb{R}$, while the former is based on the discrete set of risk aversion levels employed by the robo-advisor, as $\Psi(\check{\gamma}_{n}) \in \Gamma$. In turn, the same statement can be made about the corresponding regret measures, $R^{(\phi)}_{T}(\gamma)$ and $R^{(\phi)}_{T,c}(\gamma)$, which are visualized in Figure 4, with the latter being larger, as predicted by the inequality in (5.9). Intuitively, the inequality expresses the fact that even small changes in the underlying random walk model contribute to $R^{(\phi)}_{T,c}$, while such changes are truncated to zero in the discretization used to compute $R^{(\phi)}_{T}$.

The regret measure $R^{(\phi)}_{T,c}$ has the desirable property that it is not affected by the granularity of the set $\Gamma$. For $\gamma \in \Gamma$, we can then define

$$R^{(\phi)}_{T,d}(\gamma) := R^{(\phi)}_{T,c}(\gamma) - R^{(\phi)}_{T}(\gamma) \geq 0,$$

which is the additional regret incurred from using a finite number of risk aversion levels, as opposed to tailoring the risk aversion level to the personal profile of each client. This is a measure of how much the robo-advisor underestimates the regret by using the discrete set $\Gamma$. In particular, $R^{(\phi)}_{T,d}(\gamma)$ is strictly positive.
Figure 4: For investment horizon $T = 24$ (months), the figure shows the regret measures $R_T^{(\phi)}(\gamma)$ and $R_{T, c}^{(\phi)}(\gamma)$, given in Proposition 5.2, for risk aversion level $\gamma \in \{2.5, 3.5, 4.5\}$, and time between updates, $\phi(\gamma)$, ranging from 1 to $T$. The set of risk aversion levels, risk aversion volatilities, and market parameters, are the same as in Figure 3.

unless $\phi(\gamma) = 1$, i.e., if the client’s risk aversion is updated after each time step, in which case both regret measures are zero. Furthermore, $R_{T,d}^{(\phi)}(\gamma)$ decreases with the granularity of risk aversion levels in $\Gamma$, because a higher granularity increases the robo-advisor’s regret measure $R_T^{(\phi)}(\gamma)$ which, in the limit, converges to $R_{T, c}^{(\phi)}(\gamma)$.

**Example 1 (Continued).** We continue Example 1 from Section 3.4 to discuss the regret associated with the robo-advisor’s optimal investment strategy, and its dependence on the uncertainty in the client’s risk aversion determined by the updating rule for the client’s risk preferences.

Consider first the case that no update of the client’s risk preferences takes place, in which case the entire strategy profile is determined at time zero by the client’s initial risk aversion level. That is, if $\tau_n = 0$, and thus $\gamma_n = \gamma_0$, for $n = 1, 2$, then the sequence $(\pi_{0,T}(\gamma_0, 0, \theta), \pi_{1,T}^{*}(\gamma_0, 0, \theta), \pi_{2,T}^{*}(\gamma_0, 0, \theta))$, given in Table 1, is the optimal proportion of wealth allocated to the risky asset. Observe that this is also the optimal strategy of a more “primitive” robo-advisor, namely, one that does not employ a stochastic model for the client’s risk aversion, but rather assumes it to be constant throughout the investment horizon and equal to that communicated at the beginning of the investment horizon.

However, with the client’s risk aversion being stochastic, regret accumulates with time by following the above strategy, with the regret at time $n$ being defined in terms of the relative difference between $\pi_{n,T}^{*}(\gamma_0, 0, \theta)$, and a benchmark allocation that corresponds to an update of risk preferences at time $n$. For instance, if $\gamma_0 = \bar{\gamma}_2$, then at times $n = 1, 2$ the probability of being at one of the other two levels is approximately $np/2$, based on a first order approximation of the transition probabilities corresponding to $\Lambda$. Hence, the worst-case
measure of regret in (5.2), corresponding to the no-information rule \( \phi \equiv \infty \), satisfies

\[
\tilde{R}_T^{(\phi)}(\gamma_2) \approx \left[ \frac{\pi^*_2(T)(\bar{\gamma}_2, 0, \theta) - \pi^*_2(T)(\gamma_1, 0, \theta)}{\pi^*_2(T)(\gamma_1, 0, \theta)} + \frac{\pi^*_2(T)(\bar{\gamma}_2, 0, \theta) - \pi^*_2(T)(\bar{\gamma}_3, 0, \theta)}{\pi^*_2(T)(\bar{\gamma}_3, 0, \theta)} \right] p = 0.114.
\]

For the other extreme case corresponding to the full-information rule \( \phi \equiv 1 \), when the risk preferences are updated at all times, the sequence of optimal allocations is given by \((\pi^*_0(T)(\gamma_0, 0, \theta), \pi^*_1(T)(\gamma_1, 1, \theta), \pi^*_2(T)(\gamma_2, 2, \theta))\), where \( \gamma_n \) is the realized risk aversion level at time \( n \in \{0, 1, 2\} \). In this case, the regret is zero, as the robo-advisor faces no risk of misclassifying the client.

Finally, for the intermediate case where the client’s risk preferences are updated only at time \( n = 2 \), i.e., \( \phi = 2 \), we deduce that the worst-case regret measure satisfies

\[
\tilde{R}_T^{(\phi)}(\gamma_2) \approx \left[ \frac{\pi^*_1(T)(\bar{\gamma}_2, 0, \theta) - \pi^*_1(T)(\gamma_1, 0, \theta)}{\pi^*_1(T)(\gamma_1, 0, \theta)} + \frac{\pi^*_1(T)(\bar{\gamma}_2, 0, \theta) - \pi^*_1(T)(\bar{\gamma}_3, 0, \theta)}{\pi^*_1(T)(\bar{\gamma}_3, 0, \theta)} \right] \frac{p}{2} = 0.055,
\]

which is about half of the regret in the case of zero updates.

### 5.2 Machine-Specific Effect: Portfolio Tilting

We study how the robo-advisor’s aversion towards uncertainty in the client’s risk preferences impacts the optimal portfolio allocation. Recall from (3.10) that at time \( n \), the risk-return tradeoff coefficient is given by

\[
\Delta_n = \gamma_n + \theta \delta_n(\gamma_n, \tau_n) = \begin{cases} 
\gamma_n, & \tau_n = n, \\
\gamma_n + \theta \delta_n(\gamma_n, \tau_n), & \tau_n < n,
\end{cases}
\]

where \( \delta_n(\gamma_n, \tau_n) \) is the conditional standard deviation of the client’s (unknown) risk aversion level, with the most recent risk aversion level, \( \gamma_n \in \Gamma \), having been communicated at time \( \tau_n \leq n \). Then, between updating times, we have \( \delta_n(\gamma_n, \tau_n) > 0 \), and the machine-specific component of the risk-return tradeoff, \( \theta \delta_n(\gamma_n, \tau_n) \), is nonzero. The optimal portfolio is therefore tilted towards a less risky composition, relative to the portfolio corresponding to \( \Delta_n = \gamma_n \). To quantify the magnitude of this change, we define the function

\[
\Delta_n = \gamma_n + \theta \delta_n(\gamma_n, \tau_n) = \begin{cases} 
\gamma_n, & \tau_n = n, \\
\gamma_n + \theta \delta_n(\gamma_n, \tau_n), & \tau_n < n,
\end{cases}
\]

\[
S_{n,T}^{(\phi)}(\gamma, \tau, \theta) := \left| \frac{\pi^*_n(T)(\gamma_n, \tau_n, \theta) - \pi^*_n(T)(\gamma_n, \tau_n; 0)}{\pi^*_n(T)(\gamma_n, \tau_n; 0)} \right|_{\gamma_n=\gamma, \tau_n=\tau},
\]

which is the relative change in the optimal allocation at time \( n \), resulting from using a nonzero value of \( \theta \), given that the previous risk aversion level \( \gamma \in \Gamma \) was communicated at time \( \tau \leq n \).

We then have the following upper bound (see Appendix C for the proof) for \( S_{n,T}^{(\phi)}(\gamma, \tau, \theta) \), which shows explicitly how the shift in allocation vanishes if \( \theta = 0 \), and tends to one as \( \theta \to \infty \). In other words, the
optimal portfolio tends to a risk-free portfolio, as \( \theta \to \infty \).

**Proposition 5.3.** Let \( T \geq 1, \theta \geq 0, \) and \( \phi : \Gamma \mapsto \bar{N} \) be an updating rule. Then, for each \( \gamma \in \Gamma, \) and \( 0 \leq \tau \leq n, \)

\[
\bar{S}^{(\phi)}_{n,T}(\gamma, \tau, \theta) \leq S_n(\gamma, \tau, \theta) + O(\theta(1 - p_{\phi(\gamma)}(\gamma; \gamma))), \quad 0 \leq n < T,
\]

where the upper bound is given by

\[
S_n(\gamma, \tau, \theta) = \frac{\theta \delta_n(\gamma, \tau)}{\gamma + \theta \delta_n(\gamma, \tau)} \in [0, 1),
\]

(5.11)

and satisfies

\[
\lim_{\theta \to 0} S_n(\gamma, \tau, \theta) = 0, \quad \lim_{\theta \to \infty} S_n(\gamma, \tau, \theta) = 1.
\]

Observe that if \( \theta = 0, \) then \( \bar{S}^{(\phi)}_{n,T}(\gamma, \tau, \theta) = 0, \) and that the upper bound \( S^{(\phi)}_n(\gamma, \tau, \theta) \) is independent of the investment horizon \( T. \) Furthermore, since \( \delta_n(\gamma, \tau) = \delta_{n-\tau}(\gamma, 0) \) (see Lemma A.1-(d)), the upper bound only depends on the time since the previous time of communication, \( n - \tau. \) This can be seen more explicitly by using the approximation \( \Psi(\gamma) \approx \gamma^{10} \) to write (see Appendix C)

\[
\theta \delta_n(\gamma, \tau) \approx \theta \sigma_\gamma \sqrt{n - \tau}.
\]

(5.12)

Together with (5.11), this highlights that for a given risk aversion level \( \gamma \in \Gamma, \) the magnitude of the allocation effect is increasing in \( \theta, \) the risk aversion volatility, \( \sigma_\gamma, \) and the time since the previous update, \( n - \tau. \)

We visualize this effect in Figure 5, for \( \theta \in \{0, 0.25, 1\} \)\(^\text{11}\). The left panel corresponds to no updating of the client’s risk preferences, and the blue curve shows the optimal proportion of wealth allocated to the risky asset with \( \theta = 0, \) which is independent of the risk aversion volatility. The red and green curves demonstrate that a larger value of \( \theta \) results in a larger downward shift in allocation and that, for a fixed value of \( \theta, \) a higher risk aversion volatility has the same effect. The black curves show the downward shift in allocation predicted by the upper bound in (5.11) and, furthermore, that the shift is indeed larger compared to the corresponding green and red curves. The right panel corresponds to quarterly updates of the client’s risk preferences, and shows the effect that a nonzero value of \( \theta \) has on the optimal allocation throughout the

\[^{10}\text{The function } \Psi, \text{ defined in (3.2), projects its argument to the nearest risk aversion level in the set } \Gamma, \text{ so this approximation improves with a higher level of granularity in } \Gamma.\]

\[^{11}\text{In Section 6.4 we show how } \theta = 0.25 \text{ and } \theta = 1 \text{ correspond to setting the risk-return tradeoff } \Delta_n \text{ to the right endpoint of a confidence interval for the client’s risk aversion, with significance levels 60% and 85%, respectively. The value } \theta = 0 \text{ corresponds to taking the center of any confidence interval, as the client’s risk aversion evolves like a centered random walk.}\]
Figure 5: For investment horizon $T = 12$ (months), the figure shows the optimal proportion of wealth allocated to the risky asset, $\{\pi^*_n, T(\gamma, \tau_n, \theta)\}_{0 \leq n < T}$, at risk aversion level $\gamma_n = 3.5$, for $\theta \in \{0, 0.25, 1\}$. The left panel sets $\tau_n = 0$, which corresponds to no updating of risk preferences, and considers three different levels of risk aversion volatility, $\sigma_\gamma = c \times 0.152$, where $c \in \{1/\sqrt{2}, 1, \sqrt{2}\}$, for all $\gamma \in \Gamma$. The black curves correspond to $\sigma_\gamma = 0.152$ and show how the allocation corresponding to $\theta = 0$ (blue) shifts downwards based on the upper bound in (5.11). The right panel sets $\tau_n = \lceil n/3 \rceil$, which corresponds to quarterly updates of risk preferences, with risk aversion volatility $\sigma_\gamma = 0.152$. The set of risk aversion levels, and the market parameters, are the same as in Figure 3.

investment horizon, focusing on a single risk aversion volatility. Finally, note that at times of communication, when the robo-advisor observes client’s risk aversion level, the allocations corresponding to a nonzero value of $\theta$ are slightly larger than the one corresponding to $\theta = 0$, in anticipation of the fact that, at future time points, the allocation will be tilted towards a less risky portfolio if $\theta > 0$.

5.3 Client’s Suitability for Robo-Advising

In our model, the client faces a tradeoff between a satisfactory investment performance (i.e., low regret) and the need for trust and peace of mind (i.e., high level of delegation). A client that aims for low regret will communicate frequently with the robo-advisor. On the other hand, a client that desires a higher level of delegation will have a higher regret, because of the increased information asymmetry between the client and the robo-advisor, resulting from the client’s reduced participation in the investment process. Such a client may be better off with a traditional financial advisor, with whom she may establish a more “personal” relationship. These model predictions are consistent with the empirical evidence presented in Rossi and Utkus [2019b], and surveyed in Section 2 herein.

Furthermore, for a fixed level of delegation, the client’s regret is increasing in the volatility of her risk aversion process. In other words, other things being equal, a client with unstable and frequently changing
risk preferences is worse off with the robo-advisor. The risk profile of such a client is likely to be “unusual”,
and the onboarding process used by existing robo-advising systems is unable to provide a complete picture
for clients with complex or specific financial needs. Clients of this type are likely to require tailored services
which go beyond those currently offered by robo-advisors, which are instead better suited to serve clients
with a “typical” risk profile (see, e.g., Phoon and Koh [2018]).

6 Towards the Design of a Personalized Robo-Advisor

In this section we discuss how various components of the robo-advising framework can be calibrated. Section
6.1 shows how the updating rule for the client’s risk preferences can be tailored to the client’s risk aversion
level, in order to maintain a given target level of regret, and Section 6.2 discusses the implications for the
client of the robo-advisor using a less personalized updating rule. Section 6.3 shows how the robo-advisor’s
set of risk aversion levels can be constructed to fix the value of communication for each risk aversion level.
Section 6.4 shows how the machine-specific component of the risk-return tradeoff coefficient can be calibrated
to limit the probability of choosing a too risky portfolio composition for the client.

6.1 Personalizing the Updating Rule

In Section 5.1 we have seen that for a given frequency of interaction between the client and the robo-advisor,
the regret depends on the client’s current risk aversion level, and the corresponding risk aversion volatility.
In this section we show how the robo-advisor can actually calibrate the updating rule for the client’s risk
preferences to maintain a given target level of regret across the various risk aversion levels.

Denote by $\kappa \in \mathbb{R}^+$ the target regret, and assume for notational simplicity that the investment horizon $T$
is such that $R_T^{(\phi)}(\gamma)|_{\phi \equiv T} > \kappa$, for each $\gamma \in \Gamma$, with the regret $R_T^{(\phi)}$ given in (5.8). In other words, without
any updating of the client’s risk preferences, the regret at each risk aversion level exceeds $\kappa$ before the end
of the investment horizon. Then, we define the updating rule $\phi_{\kappa}$ by

$$
\phi_{\kappa}(\gamma) := \sup\{n \geq 1 : R_T^{(\phi)}(\gamma)\big|_{\phi \equiv n} < \kappa\}, \quad \gamma \in \Gamma,
$$

which corresponds to triggering an update of risk preferences right before the regret crosses the level $\kappa$. From
Proposition 5.2 it follows that

$$
\phi_{\kappa}(\gamma) = \inf\{n \geq 1 : \mu_n(\gamma) > b_\kappa(\gamma)\}, \quad \gamma \in \Gamma,
$$

(6.1)
with $\mu_n(\gamma)$ defined in (5.6), and the family of updating thresholds, $\{b_\kappa(\gamma)\}_{\gamma \in \Gamma}$, given by

$$b_\kappa(\gamma) := \gamma \kappa. \tag{6.2}$$

This shows that the updating rule $\phi_\kappa$ is equal to a threshold updating rule that triggers an update of risk preferences as soon as the expected absolute change in the client’s risk aversion, since the previous time of communication, exceeds a certain threshold.\footnote{In fact, from Lemma A.1-(b) it follows that for any updating rule $\phi$, there exists a threshold family $\{b(\phi)(\gamma)\}_{\gamma \in \Gamma}$ such that $\phi(\gamma) = \inf\{n \geq 1 : \mu_n(\gamma) > b(\phi)(\gamma)\}, \ \gamma \in \Gamma$. That is, any updating rule can be framed as a threshold updating rule, based on expected changes in the robo-advisor’s stochastic model for the client’s risk aversion.}

Notice that for a given risk aversion level, $\gamma \in \Gamma$, the threshold $b_\kappa(\gamma)$ does not depend on the risk aversion volatility, $\sigma_{\gamma}$. It then follows from Lemma A.1-(b) that for a given target regret, the time between updates at a given risk aversion level needs to decrease to accommodate a higher risk aversion volatility. However, the threshold is increasing in the risk aversion level itself, $\gamma$, indicating that more frequent updating is required for lower levels of risk aversion, for a fixed level of volatility.\footnote{This property can also be deduced from Figure 3, which shows the regret profile corresponding to different risk aversion states, that all have the same risk aversion volatility.} Intuitively, this is because regret is defined in terms of relative risk aversion changes, and a given absolute risk aversion change results in a larger relative change if the initial risk aversion level is low.

At the beginning of the investment process, the client may specify a desired updating frequency, i.e., $\phi \equiv c$, for some $c \geq 1$, depending on her willingness to participate in the investment process. Given the client’s initial risk aversion level, $\gamma_0 \in \Gamma$, the robo-advisor can then compute the corresponding regret, $\kappa = R^{(\phi)}(\gamma_0)$, and use (6.2) to compute a family of updating thresholds, $\{b_\kappa(\gamma)\}_{\gamma \in \Gamma}$, that allows this client to maintain the same level of regret across risk aversion levels. When a new risk aversion level is communicated by the client, a higher updating frequency may be required to maintain the same investment performance. Alternatively, if the client is unsure or indifferent about her desired updating frequency, the robo-advisor can use the results in this section to calibrate a default updating rule, that is internally consistent, in terms of regret, for all risk aversion levels.

### 6.2 Benefits of Personalization

In the previous section, we have discussed how the robo-advisor can adjust the updating rule to maintain a given level of regret. In this section, we analyze the client’s regret when working with a robo-advisor which provides a less personalized service. Specifically, we analyze the effect on regret of an updating rule that
does not depend on the client’s current risk aversion level.

We consider a robo-advisor that chooses the time between updates of risk preferences to be the same, regardless of the risk aversion level of the client. This frequency might be set arbitrarily by the robo-advisor, without any notion of regret, or it might be calibrated so that the steady-state regret measure $\tilde{R}_{T}^{(\phi)}$, defined in (5.5), is equal to a specific value, without any regard to the heterogeneity of risk aversion levels. In Appendix C, we use the approximation $\Psi(\gamma) \approx \gamma$, as in (5.12), to show that for a given target level of regret $\kappa \in \mathbb{R}^{+}$, we have $\tilde{R}_{T}^{(\phi_{\kappa})} \approx \kappa$ for an updating rule $\tilde{\phi}_{\kappa}$ that satisfies

$$
\tilde{\phi}_{\kappa}(\gamma) \approx \frac{\pi}{2} \kappa^{2} / \left( \sum_{\gamma' \in \Gamma} \lambda(\gamma') \frac{\sigma_{\gamma'}}{\gamma'} \right)^{2}, \quad \gamma \in \Gamma.
$$

(6.3)

Notice that $\tilde{\phi}_{\kappa}(\gamma)$ is independent of the risk aversion level $\gamma$, so the same time between updates is used at all risk aversion levels. This updating rule can be compared to the fixed regret updating rule $\phi_{\kappa}$, defined in (6.1), for which we have, similarly to (6.3), that

$$
\phi_{\kappa}(\gamma) \approx \frac{\pi}{2} \kappa^{2} / \left( \sigma_{\gamma} \gamma \right)^{2}, \quad \gamma \in \Gamma.
$$

(6.4)

To compare the two updating rules above, we recall from Section 5.1 that for $\gamma \in \Gamma$, the proportion $\sigma_{\gamma}/\gamma$ was referred to as the relative risk aversion volatility. From (6.3) we see that in determining the updating rule $\tilde{\phi}_{\kappa}$, the relative risk aversion volatility is weighted by the stationary distribution $\lambda$. In turn, from (6.4) we deduce that as the heterogeneity in risk aversion levels increases, with heterogeneity measured in terms of relative risk aversion volatility, the aggregate updating rule becomes less tailored to individual levels. The effect of heterogeneity can further be witnessed by examining the regret under the aggregate rule $\tilde{\phi}_{\kappa}$, which satisfies

$$
\tilde{R}_{T}^{(\tilde{\phi}_{\kappa})}(\gamma) \approx \kappa \sum_{\gamma' \in \Gamma} \lambda(\gamma') \frac{\sigma_{\gamma}}{(\sigma_{\gamma}/\gamma')}, \quad \gamma \in \Gamma.
$$

(6.5)

This shows that for a given risk aversion level $\gamma \in \Gamma$, the regret is larger (smaller) than $\kappa$ if the relative risk aversion volatility, $\sigma_{\gamma}/\gamma$, is higher (lower) than the weighted average of those volatilities, with the weights determined by the steady-state distribution $\lambda$. We can also see, for instance, that the regret at a risk aversion level with a low weight, perhaps due to a high volatility, can be substantially greater than the target regret $\kappa$. Hence, by using the “one-size-fits-all” updating rule $\tilde{\phi}_{\kappa}$, the robo-advisor is implicitly providing worse service to clients with certain characteristics. That is, lower regret at certain risk aversion levels comes at

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As in the previous section, we assume that the investment horizon $T$ is such that $\tilde{R}_{T}^{(\phi)}|_{\phi=T} > \kappa$. 

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the expense of the regret being higher at other levels.

The analysis above corresponds to a robo-advisor that uses the same updating frequency for all risk aversion levels. Alternatively, one can consider a robo-advisor that employs a threshold updating rule with a \textit{uniform} threshold, i.e., a threshold that does not depend on the risk aversion level. Again, this threshold might be set arbitrarily by the robo-advisor, or it might be calibrated so that the steady-state regret measure in (5.5) is equal to a specific value. In this case, one can similarly show that as the heterogeneity of risk aversion levels increases, the aggregate threshold becomes less tailored to individual levels. This implies, once more, that the robo-advisor is treating clients at different risk aversion levels differently, in the sense that a lower regret at certain levels comes at the expense of a higher regret at other levels.

\subsection*{6.3 Value of Interaction}

In this section, we study the probability that an interaction leads to a change in the robo-advisor’s view of the client’s risk aversion. We analyze how the benefits of an interaction depend on the current risk aversion level of the client.

The threshold family \( \{b_\kappa(\gamma)\}_{\gamma \in \Gamma} \), defined in (6.2), naturally quantifies the tradeoff between having low regret, \( \kappa \), and a low frequency of interactions (i.e., a high threshold). Additionally, it quantifies the tradeoff between low regret, and a low probability of initiating an update when the client’s risk aversion level has not changed (false positive). For a general updating rule \( \phi \), and investment horizon \( T \), the false positive probability is defined as

\[
\alpha_T^{(\phi)}(\gamma) := \sup_{0 \leq n < T} P_n^{(\phi)}(\Psi(\gamma_n + \phi(\gamma)) = \gamma | \gamma_n = \gamma, \tau_n = n) \mathbf{1}_{\{n + \phi(\gamma) < T\}}, \quad \gamma \in \Gamma.
\]

Given that the risk aversion level \( \gamma \) was communicated at time \( n \), this is the probability that the realized risk aversion level at the subsequent updating time (as long as it comes before the terminal date \( T \)) is also equal to \( \gamma \), maximized over all possible values of the communication time \( n \). In Appendix C, we first provide a closed form formula for this probability, and then use the same approximation as in (5.12) to show that, for the fixed regret updating rule \( \phi_\kappa \),

\[
\alpha_T^{(\phi_\kappa)}(\gamma) \approx \Phi\left(\sqrt{\frac{\log \frac{\gamma^+ - \gamma}{\gamma}}{\pi \kappa}}\right) - \Phi\left(\sqrt{\frac{\log \frac{\gamma^- - \gamma}{\gamma}}{\pi \kappa}}\right), \quad \gamma \in \Gamma,
\]

where the constants \( \gamma^\pm \) are defined in (3.3) as the midpoints between neighboring risk aversion levels. This highlights the aforementioned tradeoff between having a low regret, \( \kappa \), that requires frequent updating, and having a low probability of a false positive, \( \alpha_T^{(\phi_\kappa)}(\gamma) \), that requires less frequent updating. Additionally,
we observe that the above expression is independent of the risk aversion volatility, just like the threshold \( b_\kappa(\gamma) \) is. However, it is evident that a uniformly spaced grid of risk aversion levels, i.e. such that \( \gamma^\pm - \gamma \) is independent of \( \gamma \), results in a higher false positive probability for low risk aversion levels.

Informally, this means that the value of an update is not the same for all risk aversion levels. Specifically, for smaller values of \( \gamma \), a higher proportion of updates does not lead to a change in the client’s risk preference classification. To guarantee the same false positive rate across all risk aversion levels, the set \( \Gamma \) needs to be constructed so that the relative distances \( (\gamma^+ - \gamma)/\gamma \) and \( (\gamma - \gamma^-)/\gamma \) are independent of \( \gamma \). That is, a higher granularity is needed for low risk aversion levels compared to high risk aversion levels. The updating rule \( \phi_\kappa \) guarantees, for such a set \( \Gamma \), that both the regret and the false positive probabilities are the same across risk aversion levels.

### 6.4 Calibration of \( \theta \)

We show how the robo-advisor can use the machine-specific component of the risk-return tradeoff coefficient \( \Delta_n \), given in (3.10), to bound the probability of choosing a portfolio for the client that is too risky.

By using the approximation \( \Psi(\bar{\gamma}_n) \approx \bar{\gamma}_n \), as in (5.12), and \( \bar{\gamma}_n \sim \mathcal{N}(\gamma_n, \delta_n(\gamma_n, \tau_n)) \), which follows from (3.2), we have that \( \gamma_n \pm \theta \delta_n(\gamma_n, \tau_n) \) is an approximate 100(1 - \( \alpha \))% confidence interval for \( \Psi(\bar{\gamma}_n) \), the client’s (unknown) risk aversion level at time \( n \), where the significance level is \( \alpha = 2(1 - \Phi(\theta)) \). The risk-return coefficient \( \Delta_n \) is the right endpoint of this confidence interval, i.e., the most risk averse value within the interval. Hence, the probability of the client actually having a higher risk aversion than \( \Delta_n \) is approximately \( \alpha/2 = 1 - \Phi(\theta) \). Equivalently, this is approximately the probability that the optimal portfolio composition under information asymmetry (i.e., with risk-return tradeoff \( \Delta_n \)), is riskier than the optimal portfolio composition under full information (i.e., with \( \Delta_n \) replaced by \( \Psi(\bar{\gamma}_n) \)).

If \( \theta = 0 \), then \( \alpha/2 = 0.5 \), and \( \alpha/2 \) decreases to zero as \( \theta \to \infty \). A robo-advisor wishing to set the probability of choosing a risky portfolio equal to \( \alpha^* \in [0, 0.5] \) can choose \( \theta = \Phi^{-1}(1 - \alpha^*) \). Hence, for a robo-advisor that is more averse against uncertainty in the client’s risk profile (lower \( \alpha^* \)), \( \theta \) will be higher.

### 7 Conclusions and Future Extensions

The past decade has witnessed the emergence of robo-advisors, investment platforms where clients interact directly with an investment algorithm, without the intervention of a human. Recent work has provided empirical evidence on the implications of robo-advising on investment portfolios, and the nature of clients benefit the most from robo-advisors. In the present work, we build a novel modeling framework that is
consistent with those findings.

We present a dynamic investment model between a client and a robo-advisor, where the investment performance criterion automatically adapts to changes in the client’s risk profile. These changes are self-reported by the client, via repeated interaction with the robo-advisor, throughout the investment horizon. The frequency of interaction determines the portfolio regret suffered by the client due to the robo-advisor not having access to up-to-date information. We find that clients placing emphasis on investment performance (i.e., low regret) are more suitable for robo-advising, while clients seeking “peace of mind” through a high level of delegation (i.e., low frequency of interaction), as well as clients with unstable risk preferences, are less suitable for robo-advising.

Our model can be extended in several directions. First, the client’s risk aversion process in Section 3.1 can be enhanced to include a component that captures the overall state of the economy. This is consistent with empirical studies documenting that individuals are willing to take substantially larger risks during periods of economic growth, and are more risk averse during periods of recession (Buccioli and Miniaci [2011], Sahm [2012]). Second, the risk aversion process can be allowed to depend on market returns, with risk aversion going down in market upswings and going up in market downturns, which is also consistent with empirical evidence. This would generate a constraint on the frequency of communication between the client and the robo-advisor. With risk aversion being affected by portfolio performance, such a constraint is needed to prevent market timing (i.e., buying high/selling low) of clients whose risk aversion is highly sensitive to short-term market swings.

The model for the market dynamics in Section 3.3 can be extended to be time-dependent or even having stochastic volatility. It can also naturally be extended to include multiple tradable assets. Furthermore, the portfolio rebalancing times in the investment model of Section 3.4 can be allowed to be random. For instance, rebalancing could be triggered by the portfolio composition having drifted too much from the optimal composition, given the client’s risk aversion level. Robo-advisors generally use such threshold updating rules for rebalancing their portfolios (see Kaya [2017]), and this mirrors how a target regret threshold was used to trigger a communication of the client’s risk preferences (see Section 6.1), with regret defined in terms of the expected change in portfolio composition.

Our framework captures the nature of the investment process between a client and a robo-advisor. However, there is an important tradeoff faced by clients when choosing the type of financial advisor. On the one hand, robo-advisors charge significantly lower fees than traditional “human” financial advisors, which negatively impacts the relative investment performance of the latter. On the other hand, robo-advisors (i.e., “machines”) can be perceived to be less trustworthy than human financial advisors. Indeed, Rossi and
Utkus [2019b] show empirically that algorithmic aversion, i.e., the tendency of clients to more quickly lose confidence in an algorithm than a human after observing them make the same mistake (Dietvorst et al. [2015]), is one of the obstacles for investing through a robo-advisor, even though algorithm aversion is much smaller among younger generations. We leave for future research the study of this tradeoff between a lower fee structure and algorithmic aversion.

A Technical Lemmas

Lemma A.1 contains results related to the client’s risk aversion process, introduced in Section 3.1. Recall that at time \( n \geq 0 \) the client’s stochastic risk aversion is denoted by \( \hat{\gamma}_n \in \mathbb{R} \), while \( \gamma_n \in \Gamma \) is the risk aversion level observed by the robo-advisor (see (3.1)-(3.2)). At updating times we have \( \gamma_n = \Psi(\hat{\gamma}_n) \), but otherwise \( \Psi(\hat{\gamma}_n) \in \Gamma \) is the unobserved risk aversion level that would be observed by the robo-advisor following an update at time \( n \).

Assume now that at time \( n \) the previous communication of risk preferences took place at time \( \tau \leq n \), and resulted in risk aversion level \( \gamma \in \Gamma \). Then, from the robo-advisor’s point of view, the distribution of \( \Psi(\hat{\gamma}_n) \) is characterized by the probabilities

\[
p_n(\gamma'; \gamma, \tau) := \mathbb{P}(\Psi(\hat{\gamma}_n) = \gamma'|\gamma_n = \gamma, \tau_n = \tau), \quad \gamma' \in \Gamma,
\]

which are given in closed form in Lemma A.1-(a), and for \( \tau = 0 \) we let

\[
p_n(\gamma'; \gamma) := p_n(\gamma'; \gamma, 0), \quad \gamma' \in \Gamma.
\]

The expected absolute change in the client’s risk aversion level, since the previous updating time, is then given by

\[
\mu_n(\gamma, \tau) := \mathbb{E}[|\Psi(\hat{\gamma}_n) - \gamma| |\gamma_n = \gamma, \tau_n = \tau],
\]

which is provided in closed form in Lemma A.1-(b), along with \( \mu_n(\gamma) \), defined in (5.6), which is equal to \( \mu_n(\gamma, \tau) \) with \( \tau = 0 \). Similarly to (A.3), we define \( \mu^c_n(\gamma, \tau) \) to be the expected absolute change without projecting the client’s risk aversion to the nearest level in the set \( \Gamma \),

\[
\mu^c_n(\gamma, \tau) := \mathbb{E}[|\hat{\gamma}_n - \gamma| |\gamma_n = \gamma, \tau_n = \tau],
\]

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which is given in Lemma A.1-(c), and so is $\mu_c^c(\gamma)$, defined in (5.7), which is equal to $\mu_c^c(\gamma,\tau)$ with $\tau = 0$. In Lemma A.1-(d) we provide a closed form expression for $\delta_n(\gamma,\tau)$, defined in (3.9), which is equal to the conditional standard deviation of $\Psi(\hat{\gamma}_n)$, as well as the standard deviation

$$\delta_n(\gamma) := \delta_n(\gamma, 0). \quad \text{(A.5)}$$

Finally, parts (e) and (f) of Lemma A.1 show that the sequence of risk aversion levels communicated by the client to the robo-advisor form a time-homogeneous Markov chain on the set of risk aversion levels, $\Gamma$.

**Lemma A.1.**

(a) Let $n \geq 0$, $\tau \leq n$, and $\gamma, \gamma' \in \Gamma$. The probabilities $p_n(\gamma';\gamma,\tau)$ and $p_n(\gamma';\gamma)$, defined in (A.1)-(A.2), satisfy

$$p_n(\gamma';\gamma,\tau) = p_{n-\tau}(\gamma';\gamma) = \begin{cases} 1_{\{\gamma' = \gamma\}}, & \tau = n, \\ \Phi\left(\frac{\gamma' - \gamma}{\sigma_{\gamma'}}\right) - \Phi\left(\frac{\gamma' - \gamma}{\sigma_{\gamma'}}\right), & \tau \leq n, \end{cases} \quad \text{(A.6)}$$

where the constants $(\gamma')^\pm$ are defined in (3.3), and $\Phi$ is the standard normal cumulative distribution function. Furthermore, the probability $p_n(\gamma';\gamma)$ is decreasing in $n \geq 0$, and, for a fixed $n \geq 1$, it is also decreasing in $\sigma_{\gamma}$. Moreover, for $n \geq 1$,

$$\lim_{n \to \infty} p_n(\gamma';\gamma) = \lim_{\sigma_{\gamma'} \to \infty} p_n(\gamma';\gamma) = \begin{cases} 0, & \gamma' \notin \{\tilde{\gamma}_1, \tilde{\gamma}_K\}, \\ 1/2, & \gamma' \in \{\tilde{\gamma}_1, \tilde{\gamma}_K\}. \end{cases}$$

(b) Let $n \geq 0$, $\tau \leq n$, and $\gamma \in \Gamma$. The expected values $\mu_n(\gamma,\tau)$ and $\mu_n(\gamma)$, defined respectively in (A.3) and (5.6), satisfy

$$\mu_n(\gamma,\tau) = \mu_{n-\tau}(\gamma) = \sum_{\gamma' \in \Gamma} p_{n-\tau}(\gamma';\gamma)|\gamma' - \gamma|, \quad \text{(A.7)}$$

where $p_{n-\tau}(\gamma';\gamma)$ is given in (A.6). Furthermore, $\mu_n(\gamma)$ is increasing in $n \geq 0$, and, for a fixed $n \geq 1$, it is also increasing in $\sigma_{\gamma}$. Moreover, for $n \geq 1$,

$$\lim_{n \to \infty} \mu_n(\gamma) = \lim_{\sigma_{\gamma} \to \infty} \mu_n(\gamma) = \frac{\tilde{\gamma}_1 + \tilde{\gamma}_K}{2}.$$

(c) Let $n \geq 0$, $\tau \leq n$, and $\gamma \in \Gamma$. The expected values $\mu_n^c(\gamma,\tau)$ and $\mu_n^c(\gamma)$, defined respectively in (A.4)
and (5.7), satisfy

$$\mu_n^c(\gamma, \tau) = \mu_n^c(\gamma) = \sqrt{\frac{2}{\pi \gamma}} \sqrt{n - \tau}.$$ 

Furthermore, $\mu_n(\gamma)$ is increasing in $n \geq 0$, and $\mu_n(\gamma) \leq \mu_n^c(\gamma)$, where $\mu_n(\gamma)$ is given in (A.7), with the inequality being strict for $n > 0$.

(d) Let $n \geq 0$, $\tau \leq n$, and $\gamma \in \Gamma$. The conditional standard deviations $\delta_n(\gamma, \tau)$ and $\delta_n(\gamma)$, defined in (3.9) and (A.5), respectively, satisfy

$$\delta_n(\gamma, \tau) = \delta_{n-\tau}(\gamma) = \sqrt{\sum_{\gamma' \in \Gamma} p_n(\gamma'; \gamma)(\gamma')^2 - \left( \sum_{\gamma' \in \Gamma} p_n(\gamma'; \gamma) \gamma' \right)^2},$$

where $p_n(\gamma'; \gamma)$ is given in (A.6). Furthermore, $\delta_0(\gamma) = 0$, and

$$\lim_{n \to \infty} \delta_n(\gamma) = \frac{(\bar{\gamma}_K - \bar{\gamma}_1)^2}{4}.$$ 

(e) For any updating rule $\phi$, the process $(\gamma_{\phi(k)})_{k \geq 0}$ is an irreducible and aperiodic Markov chain on $\Gamma$, with respect to the filtration $(G_k)_{k \geq 0}$, where $G_k := F_{T_k}^{(\phi)}$. It has a time-homogeneous transition matrix $\Lambda^{(\phi)}$, given by

$$\Lambda^{(\phi)}_{\gamma', \gamma} = p_{\phi(\gamma')}(\gamma; \gamma'), \quad \gamma, \gamma' \in \Gamma,$$

with $p_{\phi(\gamma')}(\gamma; \gamma')$ given in closed form in (A.6).

(f) If $\phi \equiv 1$, the process $(\gamma_n)_{n \geq 0}$ is an irreducible and aperiodic Markov chain on $\Gamma$, with respect to the filtration $(F_n)_{n \geq 0}$, with a time-homogeneous transition matrix $\Lambda$, given by

$$\Lambda_{\gamma', \gamma} = p_1(\gamma; \gamma'), \quad \gamma, \gamma' \in \Gamma,$$  \hspace{1cm} (A.8)

with $p_1(\gamma; \gamma')$ given in closed form in (A.6). Furthermore, it has a unique stationary distribution, $\lambda := (\lambda(\gamma))_{\gamma \in \Gamma}$, such that for $\gamma, \gamma' \in \Gamma$,

$$\lambda(\gamma) = \lim_{n \to \infty} \Lambda^n_{\gamma', \gamma}.$$
Proof: To show the first equality in (a), we use the (conditional) time-homogeneity of the process \((\tilde{\gamma}_n)_n \geq 0\) to write

\[ p_n(\gamma'; \gamma, \tau) = \mathbb{P}(\Psi(\tilde{\gamma}_n) = \gamma' | \gamma_n = \gamma, \tau_n = \tau), \quad \gamma' \in \Gamma, = \mathbb{P}(\Psi(\tilde{\gamma}_{n-\tau}) = \gamma' | \gamma_{n-\tau} = \gamma, \tau_{n-\tau} = 0) = p_{n-\tau}(\gamma'; \gamma). \]

To show the second equality, we have from the dynamics (3.1),

\[
\begin{align*}
p_n(\gamma'; \gamma) &= \mathbb{P}(\Psi(\tilde{\gamma}_n) = \gamma' | \gamma_0 = \gamma, \tau_n = 0) \\
&= \mathbb{P}(\gamma + \sigma_\gamma (\tilde{Z}_1 + \tilde{Z}_2 + \cdots + \tilde{Z}_n) \in ((\gamma')^-, (\gamma')^+)) \\
&= \Phi\left(\frac{(\gamma')^+ - \gamma}{\sigma_\gamma \sqrt{n}}\right) - \Phi\left(\frac{(\gamma')^- - \gamma}{\sigma_\gamma \sqrt{n}}\right),
\end{align*}
\]

where we used that \(\tilde{Z}_1 + \tilde{Z}_2 + \cdots + Z_n \sim \mathcal{N}(0, n)\). If \(\gamma' = \gamma\), then \(p_n(\gamma; \gamma)\) is decreasing in \(n\) and \(\sigma_\gamma\) as \(\gamma^+ - \gamma > 0\) and \(\gamma^- - \gamma < 0\). Then, the limits follow from the fact that \(0 < (\gamma')^+ - \gamma < \infty\) and \(-\infty < (\gamma')^- - \gamma < 0\), unless \(\gamma' = \tilde{\gamma}_K\) in which case \((\gamma')^+ - \gamma = \infty\), or \(\gamma' = \tilde{\gamma}_1\), in which case \((\gamma')^- - \gamma = -\infty\).

In part (b), the first equality follows by (conditional) time-homogeneity, as the first inequality in (a), and the second equality follows by definition. The limits follow from part (a). To show the monotonicity properties of \(\mu_n(\gamma)\), it is sufficient to notice that, for any \(c > 0\), the probabilities

\[
\begin{align*}
\mathbb{P}(\tilde{\gamma}_n > \gamma_0 + c | \gamma_0 = \gamma, \tau_n = 0) &= \mathbb{P}\left(Z > \frac{c/\sigma_\gamma}{\sqrt{n}}\right), \\
\mathbb{P}(\tilde{\gamma}_n < \gamma_0 - c | \gamma_0 = \gamma, \tau_n = 0) &= \mathbb{P}\left(Z < -\frac{c/\sigma_\gamma}{\sqrt{n}}\right),
\end{align*}
\]

are increasing in \(n \geq 0\) and \(\sigma_\gamma > 0\). In part (c), the first equality follows by (conditional) time-homogeneity, as the first inequality in (a). For the second equality, we again use that \(\tilde{Z}_1 + \tilde{Z}_2 + \cdots + \tilde{Z}_n \sim \mathcal{N}(0, n)\) to write

\[
\mu_n^*(\gamma) = \mathbb{E}[\gamma_0 + \sigma_\gamma (\tilde{Z}_1 + \tilde{Z}_2 + \cdots + \tilde{Z}_n) - \gamma_0] = \sigma_\gamma \sqrt{n} \mathbb{E}[|Z|] = \sqrt{\frac{2}{\pi}} \sigma_\gamma \sqrt{n},
\]

where we also used the absolute moment formula for the Gaussian distribution. It is clear that \(\mu_n^*(\gamma)\) is increasing in \(n\). To show the inequality, we denote by \(f_{\tilde{\gamma}_n}\) the probability density function of \(\tilde{\gamma}_n\), given \(\gamma_0 = \gamma\)
and \( \tau_n = 0 \), and let \( \bar{\gamma}_0 = -\infty \), and \( \bar{\gamma}_K = \bar{\gamma}_{K+1} = \infty \). We then have

\[
\mu_n(\gamma) = \sum_{k=0}^{K} \left[ \int_{\bar{\gamma}_k}^{\bar{\gamma}_{k+1}} f_{\gamma_n}(\gamma') d\gamma' \left| \gamma' - \gamma_0 \right| + \int_{\bar{\gamma}_k}^{\bar{\gamma}_{k+1}} f_{\gamma_n}(\gamma') d\gamma' \left| \gamma_{k+1} - \gamma_0 \right| \right]
\]

\[
\leq \sum_{k=0}^{K} \left[ \int_{\bar{\gamma}_k}^{\bar{\gamma}_{k+1}} f_{\gamma_n}(\gamma') d\gamma' \left| \gamma' - \gamma_0 \right| + \int_{\bar{\gamma}_k}^{\bar{\gamma}_{k+1}} f_{\gamma_n}(\gamma') d\gamma' \left| \gamma_{k+1} - \gamma_0 \right| \right]
\]

\[
= \mu_n(\gamma).
\]

In part (d), the first equality follows by (conditional) time-homogeneity, as the first inequality in (a), and
the second equality follows by definition. The limit follows from part (a). To show (e), recall from (3.4) that
\( T_{k+1}(\phi) = T_k(\phi) + \phi(\gamma_k) \). In turn, using (a) yields

\[
P_{T_k}^{(\phi)}(\gamma_{T_{k+1}} = \gamma | \gamma_{T_k} = \gamma') = P_{T_k}^{(\phi)}(\gamma_{T_k} + \phi(\gamma_k) = \gamma | \gamma_k = \gamma')
\]

\[
= p_{T_k}^{(\phi)}(\gamma; \gamma', T_k^{(\phi)})
\]

Therefore, the transition probabilities are time-homogeneous. We easily deduce that the Markov chain is
irreducible and aperiodic as the random walk \((\bar{\gamma}_n)_{n \geq 0}\) has Gaussian increments. Part (f) follows as a special
43 case of part (e), and by the fact that every irreducible and aperiodic Markov chain with a finite state space
has a unique stationary distribution.

Lemma A.2 yields properties of the optimal portfolio strategy in Section 4. Recall that \( \pi_{n,T}^*(\gamma, \tau, \theta) \)
denotes the optimal proportion of wealth allocated to the risky asset at time \( n \in \{0, 1, \ldots, T - 1\} \), with
\( \gamma \in \Gamma \) representing the most recent risk aversion level, communicated at time \( \tau \in \{0, 1, \ldots, n\} \). If \( \phi \equiv 1 \), the
optimal allocation is denoted by \( \pi_{n,T}^*(\gamma, n) \), and if \( \phi \equiv \infty \), it is denoted by \( \pi_{n,T}^*(\gamma, \theta) \).

Lemma A.2.

(a) The optimal final period allocation (at time \( n = T - 1 \)) is given by

\[
\pi_{T-1,T}^*(\gamma, \tau, \theta) = \frac{1}{\Delta_{T-1}(\gamma; \tau, \theta)} \mu - r \frac{\mu - r}{\sigma^2},
\]

i.e., it is equal to the expected excess return of the risky asset, per unit of variance, with the variance
scaled by the risk-return tradeoff coefficient in (3.10).

(b) The optimal allocation at time \( n \in \{0, 1, \ldots, T - 1\} \) depends only on the time elapsed since the previous
update, \( n - \tau \), and the time until the end of the investment horizon, \( T - n \). That is,

\[
\pi^*_{n,T}(\gamma, \tau, \theta) = \pi^*_{n-\tau,T-\tau}(\gamma, 0, \theta).
\]

Furthermore, if \( \phi \equiv \infty \) and \( \theta = 0 \), then

\[
\pi^*_{n,T}(\gamma; 0) = \pi^*_0, T-n(\gamma; 0),
\]

and, if \( \phi \equiv 1 \), then,

\[
\pi^*_{n,T}(\gamma; n) = \pi^*_0, T-n(\gamma, 0).
\]

(c) Let \( \theta = 0 \). Then, the optimal allocation is increasing between updating times and increasing up to a \( O(\cdot) \) term at updating times. Specifically, for \( n \in \{0, 1, \ldots, T-2\} \),

\[
\pi^*_{n,T}(\gamma; \tau; 0) \leq \pi^*_{n+1,T}(\gamma; \tau_{n+1}; 0) + O(1 - p_{\tau_{n+1}}(\gamma; \gamma, \tau)),
\]

with \( p_{\tau_{n+1}}(\gamma; \gamma, \tau) \) given in Lemma A.1-(a). In particular, if \( \phi \equiv \infty \), then

\[
\pi^*_{n,T}(\gamma; 0) \leq \pi^*_{n+1,T}(\gamma; 0),
\]

while, if \( \phi \equiv 1 \), then

\[
\pi^*_{n,T}(\gamma; n) \leq \pi^*_{n+1,T}(\gamma, n + 1) + O(1 - \Lambda_{\gamma, \gamma}),
\]

with \( \Lambda_{\gamma, \gamma} \) given in (A.8).

(d) Let \( \theta = 0 \). Then, for any \( 0 \leq \tau < T - \phi(\gamma) \), the optimal allocation corresponding to the updating rule \( \phi \) and the optimal allocation corresponding to the no-updating rule, satisfy

\[
\pi^*_{n,T}(\gamma; \tau; 0) = \pi^*_{n,T}(\gamma; 0) + O(1 - p_{\phi(\gamma)}(\gamma; \gamma)), \quad \tau \leq n < \tau + \phi(\gamma).
\]

For \( T - \phi(\gamma) \leq \tau < T \), the two allocations coincide,

\[
\pi^*_{n,T}(\gamma; \tau; 0) = \pi^*_{n,T}(\gamma; 0), \quad \tau \leq n < T.
\]
(c) For \( n = 0, 1, \ldots, T - 1 \), the optimal allocations corresponding to the updating rule \( \phi \equiv \infty \) satisfy

\[
\pi^*_n, T(\gamma; 0) = c_{n, T}(\gamma) \pi^*_{T-1, T}(\gamma; 0),
\]

where \( c_{n, T}(\gamma) \) is increasing in \( \gamma \) and \( c_{T-1, T}(\gamma) = 1 \).

**Proof:** Part (a) follows directly from the optimal allocation formula (4.1) and using that \( a_T = b_T = 1 \). Part (b) follows by showing inductively that \( \pi^*_{T-k, T}(\gamma, \tau, \theta) = \pi^*_{T-k-T-\tau}(\gamma, 0, \theta) \), for \( k = 1, 2, \ldots, T - n \), where the case \( k = 1 \) follows from part (a). Next, we show part (c) for \( \phi \equiv 1 \). In this case, we have

\[
a_{n+1}(\gamma_n) = E^{(1)}_{n+1, \gamma_n}[a_{n+2}](R + \bar{\mu} \pi^*_{n+1, T}(\gamma_n, n + 1)),
\]

\[
b_{n+1}(\gamma_n) = E^{(1)}_{n+1, \gamma_n}[b_{n+2}] (\sigma^2 (\pi^*_{n+1, T}(\gamma_n, n + 1))^2 + (R + \bar{\mu} \pi^*_{n+1, T}(\gamma_n, n + 1))^2)
\]

\[
\geq E^{(1)}_{n+1, \gamma_n}[b_{n+2}] (R + \bar{\mu} \pi^*_{n+1, T}(\gamma_n, n + 1))^2,
\]

where \( E_{n+1, \gamma_n}[a_{n+2}] \) is shorthand notation for \( E_{n+1}[a_{n+2}(\gamma_{n+2})|\gamma_{n+1} = \gamma_n] \), and \( E_{n+1, \gamma_n}[b_{n+2}] \) is shorthand notation for \( E_{n+1}[b_{n+2}(\gamma_{n+2})|\gamma_{n+1} = \gamma_n] \). We then write

\[
\pi^*_{n, T}(\gamma_n, n)
\]

\[
= \frac{\bar{\mu}/\sigma^2}{\gamma_n} a_{n+1}(\gamma_n) - R \gamma_n (b_{n+1}(\gamma_n) - a_{n+1}^2(\gamma_n)) + R_n(\gamma_n)
\]

\[
\leq \frac{\bar{\mu}/\sigma^2}{\gamma_n} E^{(1)}_{n+1, \gamma_n}[a_{n+2}] - R \gamma_n (\bar{\mu} \pi^*_{n+1, T}(\gamma_n, n + 1)) (E^{(1)}_{n+1, \gamma_n}[b_{n+2}] - (E^{(1)}_{n+1, \gamma_n}[a_{n+2}])^2 + R_n(\gamma_n)
\]

\[
\leq \pi^*_{n+1, T}(\gamma_n, n + 1) + R_n(\gamma_n),
\]

where \( R_n(\gamma_n) = O(1 - \Lambda_{\gamma_n, \gamma_n}) \), the first inequality follows from (A.9), and the second inequality follows from the definition of \( \pi^*_{n+1, T}(\gamma_n, n + 1) \), and that \( R + \bar{\mu} \pi^*_{n+1, T}(\gamma_n, n + 1) \geq 1 \). The equality above follows from the definition of \( \pi^*_{n, T}(\gamma_n, n) \), and by using similar steps as those in (B.11). The result for \( \phi = \infty \) is a special case, and the general result is shown in a similar way.

In part (d), we first note that if \( \tau \geq T - \phi(\gamma) \), then \( \pi^*_{n, T}(\gamma, \tau; 0) = \pi^*_{n, T}(\gamma; 0) \), for \( n \geq \tau \). For \( \tau \in \{T - 2\phi(\gamma), T - 2\phi(\gamma) + 1, \ldots, T - \phi(\gamma) - 1 \} \), we can then show recursively that, for \( n = \tau + \phi(\gamma) - 1, \tau +\)
\( \phi(\gamma) - 2, \ldots, \tau + 1, \tau, \)

\[
\pi_{n,T}^*(\gamma, \tau; 0) = \pi_{n,T}^*(\gamma; 0) + O(1 - p_{\phi(\gamma)}(\gamma; \gamma)),
\]

\[
a_n(\gamma, \tau) = a_n(\gamma; \gamma) + O(1 - p_{\phi(\gamma)}(\gamma; \gamma)),
\]

\[
b_n(\gamma, \tau) = b_n(\gamma; \gamma) + O(1 - p_{\phi(\gamma)}(\gamma; \gamma)),
\]

where the \( O(\cdot) \) terms can be justified in the same way as in (B.11). Similarly, one can show the result for \( \tau < T - 2\phi(\gamma) \) and \( n = \tau + \phi(\gamma) - 1, \tau + \phi(\gamma) - 2, \ldots, \tau + 1, \tau \). In part (e), the assertion follows trivially for \( n = T - 1 \). For \( n = T - 2 \), we can explicitly compute \( \pi_{n,T}^*(\gamma; 0) \) and show that the result also holds. Next, assume that \( c_{n,T}(\gamma) \) is increasing in \( \gamma \geq 0 \), for some \( 0 < n < T - 1 \). Then, using the recursive equations for the \( a_n \)- and \( b_n \)-coefficients, we deduce that

\[
c_{n-1,T}(\gamma) = \frac{a_n(\gamma) - R\gamma(b_n(\gamma) - (a_n(\gamma))^2)}{b_n(\gamma) + \left(\frac{\mu}{\gamma}\right)^2 (b_n(\gamma) - (a_n(\gamma))^2)}
\]

\[
= \frac{a_{n+1}(\gamma) - R\gamma(R + \mu p_{\pi_{n,T}^*}(\gamma; 0))(b_{n+1}(\gamma)(\frac{\sigma^2 p_{\pi_{n,T}^*}(\gamma; 0)^2}{(R + \mu p_{\pi_{n,T}^*}(\gamma; 0))^2} + 1) - (a_{n+1}(\gamma))^2)}{R + \mu p_{\pi_{n,T}^*}(\gamma; 0)} b_{n+1}(\gamma)(b_{n+1}(\gamma)(\frac{\sigma^2 p_{\pi_{n,T}^*}(\gamma; 0)^2}{(R + \mu p_{\pi_{n,T}^*}(\gamma; 0))^2} + 1) - (a_{n+1}(\gamma))^2).
\]

We easily deduce that \( c_{n-1,T}(\gamma) \) is increasing in \( \gamma \), by using the assumption on \( c_{n,T}(\gamma) \), and that \( \pi_{n,T}^*(\gamma; 0) \) is decreasing in \( \gamma \), and \( f(x) = x/(1 + x) \) is increasing in \( x \), for \( x \geq 0 \). \( \square \)

**B Proofs of Results in Section 4**

**Proof of Proposition 4.1:** We begin by deriving the HJB system of equations satisfied by any candidate optimal control law for the objective function (3.7). Recall that \( F_n^{(\phi)}(\cdot) \) is shorthand notation for \( F(\phi)(\cdot|F_n) \), and that \( X_{(n)} = (X_k)_{0 \leq k \leq n} \), with analogue identities for \( \gamma_{(n)} \) and \( \tau_{(n)} \). Furthermore, given \( X_{(n)} \), then \( X_{(n+1)} = (X_{(n)}, X_{n+1}) \), where \( X_{n+1} \) is obtained by applying the control \( \pi_n \) to \( X_n \) at time \( n \).

**Proposition B.1.** Assume that an optimal control law \( \pi^* \) for the objective function (3.7) exists. Then the
value function (3.12) satisfies

\[
V^{(\phi)}_{n,T}(X_n, \gamma(n), \tau(n), \theta) = \sup_{x} \left\{ E_n^{(\phi)}[V^{(\phi)}_{n+1,T}(X_{n+1}^\pi, \gamma(n+1), \tau(n+1), \theta)] - \left( E_n^{(\phi)}[f_{n+1,n+1}(X_{n+1}^\pi, \gamma(n+1), \tau(n+1), X_{n+1})] - E_n^{(\phi)}[f_{n+1,n}(X_{n+1}^\pi, \gamma(n+1), \tau(n+1); X_n, \gamma_n, \tau_n, \theta)] \right) \right\}
\]

for \(0 \leq n < T\), with the terminal condition

\[
V^{(\phi)}_{T,T}(X(T), \gamma(T), \tau(T), \theta) = 0.
\]

Herein, the function sequence \((f_{n,k}(X_n, \gamma(n), \tau(n); x', \gamma', \tau', \theta))_{0 \leq n \leq T}\), for any \(x' > 0\), \(\gamma' > 0\), and \(\tau' \geq 0\), and any \(k = 0, 1, \ldots, T\), is determined by the recursion

\[
f_{n,k}(X_n, \gamma(n), \tau(n); x', \gamma', \tau', \theta) = E_n^{(\phi)}[f_{n+1,k}(X_{n+1}^\pi, \gamma(n+1), \tau(n+1); X_{n+1}, \gamma_{n+1}, \tau_{n+1}, \theta)], \quad 0 \leq n < T,
\]

\[
f_{T,k}(X(T), \gamma(T), \tau(T); x', \gamma', \tau', \theta) = \frac{X_T}{x'} - 1 - \frac{\Delta_k(\gamma', \tau', \theta)}{2} \left( \frac{X_T}{x'} \right)^2,
\]

while the function sequence \((g_n(X_n, \gamma(n), \tau(n)))_{0 \leq n \leq T}\) is determined by the recursion

\[
g_n(X_n, \gamma(n), \tau(n)) = E_n^{(\phi)}[g_{n+1}(X_{n+1}^\pi, \gamma(n+1), \tau(n+1))], \quad 0 \leq n < T,
\]

\[
g_{T}(X(T), \gamma(T), \tau(T)) = X_T.
\]

Furthermore, we have, for \(0 \leq n < T\), the probabilistic interpretation,

\[
f_{n,k}(X_n, \gamma(n), \tau(n); x', \gamma', \tau', \theta) = E_n^{(\phi)}\left[ \frac{X_T^\pi}{x'} - 1 - \frac{\Delta_k(\gamma', \tau', \theta)}{2} \left( \frac{X_T^\pi}{x'} \right)^2 \right],
\]

\[
g_n(X_n, \gamma(n), \tau(n)) = E_n^{(\phi)}[X_T^\pi].
\]

**Proof of Proposition B.1:** We begin by deriving the HJB system for the general problem

\[
J_{n,T}(X_n, \gamma(n), \tau(n), \pi, \theta) := E_n[F_n(X_n, \gamma_n, \tau_n, X_T^\pi, \theta)] + G_n(X_n, \gamma_n, \tau_n, E_n[X_T^\pi], \theta), \quad 0 \leq n < T,
\]
where, for simplicity, we drop the superscript \((\phi)\) from the notation. This system is given in Eqs. (B.6)-(B.7), and Proposition B.1 is then a special case with

\[
F_n(x, \gamma, \tau, y, \theta) = \frac{y}{x} - 1 - \frac{\Delta_n(\gamma, \tau, \theta)}{2} \left(\frac{y}{x}\right)^2, \quad G_n(x, \gamma, \tau, y, \theta) = \frac{\Delta_n(\gamma, \tau, \theta)}{2} \left(\frac{y}{x}\right)^2. \tag{B.2}
\]

The proof consists of two parts. First, we derive the recursive equation satisfied by the objective function for any given control law \(\pi\). Then, we derive the system of equations necessarily satisfied by an optimal control law \(\pi^*\).

**Step 1: Recursion for \(J_{n,T}(X(n), \gamma(n), \tau(n), \pi, \theta)\).** For a given control law \(\pi\), we define the functions

\[
f_{n,k}^\pi(x, \gamma, \tau; x', \gamma', \tau', \theta) := E_n[F_k(x', \gamma', \tau', X^\pi_T, \theta)|X(n) = x, \gamma(n) = \gamma, \tau(n) = \tau],
\]

\[
g_n^\pi(x, \gamma, \tau) := E_n[X^\pi_T|X(n) = x, \gamma(n) = \gamma, \tau(n) = \tau],
\]

and write the objective function at time \(n + 1\) as

\[
J_{n+1,T}(X(n+1), \gamma(n+1), \tau(n+1), \pi, \theta) = E_{n+1}[F_{n+1}(X_{n+1}, \gamma_{n+1}, \tau_{n+1}, X^\pi_{T}, \theta)] + G_{n+1}(X_{n+1}, \gamma_{n+1}, \tau_{n+1}, E_{n+1}[X^\pi_{T}], \theta)
\]

\[
= f_{n+1,n+1}^\pi(X(n+1), \gamma(n+1), \tau(n+1); X_{n+1}, \gamma_{n+1}, \tau_{n+1}, \theta)
\]

\[
+ G_{n+1}(X_{n+1}, \gamma_{n+1}, \tau_{n+1}, g_{n+1}^\pi(X_{n+1}, \gamma(n+1), \tau(n+1)), \theta).
\]

Taking expectations with respect to time \(n\) information yields, with \(X^\pi_{(n+1)} = (X(n), X^\pi_{n+1})\),

\[
E_n[J_{n+1,T}(X^\pi_{(n+1)}, \gamma(n+1), \tau(n+1), \pi, \theta)] = E_n[f_{n+1,n+1}^\pi(X^\pi_{(n+1)}, \gamma(n+1), \tau(n+1); X_{n+1}, \gamma_{n+1}, \tau_{n+1}, \theta)]
\]

\[
+ E_n[G_{n+1}(X_{n+1}, \gamma_{n+1}, \tau_{n+1}, g_{n+1}^\pi(X^\pi_{(n+1)}, \gamma(n+1), \tau(n+1)), \theta)].
\]

Adding and subtracting \(J_{n,T}(X(n), \gamma(n), \tau(n), \pi, \theta)\) then gives

\[
E_n[J_{n+1,T}(X^\pi_{(n+1)}, \gamma(n+1), \tau(n+1), \pi, \theta)] = J_{n,T}(X(n), \gamma(n), \tau(n), \pi, \theta)
\]

\[
+ E_n[f_{n+1,n+1}^\pi(X^\pi_{(n+1)}, \gamma(n+1), \tau(n+1); X_{n+1}, \gamma_{n+1}, \tau_{n+1}, \theta)] - E_n[F_n(X_{n}, \gamma_{n}, \tau_{n}, X^\pi_{T}, \theta)]
\]

\[
+ E_n[G_{n+1}(X_{n+1}, \gamma_{n+1}, \tau_{n+1}, g_{n+1}^\pi(X_{n+1}, \gamma(n+1), \tau(n+1))), \theta)] - G_n(X_{n}, \gamma_{n}, \tau_{n}, E_n[X^\pi_{T}], \theta).
\]
By the law of iterated expectations we have

\[ f_{n,k}^\pi(X(n), \gamma(n), \tau(n); x', \gamma', \tau', \theta) = E_n[f_{n+1,k}^\pi(X_{n+1}^\tau, \gamma(n+1), \tau(n+1); x', \gamma', \tau', \theta)], \]

\[ g_{n}^\pi(X(n), \gamma(n), \tau(n)) = E_n[g_{n+1}^\pi(X_{n+1}^\tau, \gamma(n+1), \tau(n+1))], \quad \text{(B.4)} \]

Thus,

\[ J_{n,T}(X_n, \gamma(n), \tau(n), \pi, \theta) = E_n[J_{n+1,T}(X_{n+1}^\tau, \gamma(n+1), \tau(n+1), \pi, \theta)] \]

\[ - \left( E_n[f_{n+1,n+1}^\pi(X_{n+1}^\tau, \gamma(n+1), \tau(n+1); X_{n+1}^\tau, \gamma(n+1), \tau(n+1), \theta)] - E_n[f_{n+1,n}^\pi(X_{n+1}^\tau, \gamma(n+1), \tau(n+1); X_{n+1}^\tau, \gamma(n+1), \tau(n+1), \theta)] \right) \]

\[ - \left( E_n[g_{n+1}^\pi(X_{n+1}^\tau, \gamma(n+1), \tau(n+1); g_{n+1}^\pi(X_{n+1}^\tau, \gamma(n+1), \tau(n+1), \theta)] - G_n(X_n, \gamma(n), \tau(n), E_n[g_{n+1}^\pi(X_{n+1}^\tau, \gamma(n+1), \tau(n+1), \theta)] \right) . \]

**Step 2: Recursion for** \( V_{n,T}(X(n), \gamma(n), \tau(n), \theta) \). Assume that there exists an optimal strategy \( \pi^* \), and consider a strategy \( \pi \) that coincides with \( \pi^* \) after time \( n \), so \( \pi_k(X(k), \gamma(k), \tau(k), \theta) = \pi_k^*(X(k), \gamma(k), \tau(k), \theta) \), for all \( k = n + 1, \ldots, T - 1 \). By definition, we then have

\[ J_{n,T}(X_n, \gamma(n), \tau(n), \pi^*, \theta) = V_{n,T}(X_n, \gamma(n), \tau(n), \theta), \]

\[ J_{n,T}(X_n, \gamma(n), \tau(n), \pi, \theta) \leq V_{n,T}(X_n, \gamma(n), \tau(n), \theta). \]

For the optimal strategy \( \pi^* \), we define

\[ f_{n,k}(x, \gamma, \tau; x', \gamma', \tau', \theta) := f_{n,k}^{\pi^*}(x, \gamma, \tau; x', \gamma', \tau', \theta), \]

\[ g_{n}(x, \gamma, \tau) := g_{n}^{\pi^*}(x, \gamma, \tau). \quad \text{(B.5)} \]

Then, since \( \pi \) and \( \pi^* \) coincide after time \( n \), we have

\[ J_{n+1,T}(X_{n+1}, \gamma(n+1), \tau(n+1), \pi, \theta) = V_{n+1,T}(X_{n+1}, \gamma(n+1), \tau(n+1), \theta), \]

\[ f_{n+1,k}(X_{n+1}, \gamma(n+1), \tau(n+1); x', \gamma', \tau', \theta) = f_{n+1,k}^{\pi^*}(X_{n+1}, \gamma(n+1), \tau(n+1); x', \gamma', \tau', \theta), \]

\[ g_{n+1}(X_{n+1}, \gamma(n+1), \tau(n+1)) = g_{n+1}^{\pi^*}(X_{n+1}, \gamma(n+1), \tau(n+1)). \]
In turn, using the recursion for $J_n,T(X_n),\gamma(n),\tau(n),\pi,\theta$), we may write

$$V_{n,T}(X_n,\gamma(n),\tau(n),\theta) = \sup_{\pi} \left\{ E_n[V_{n+1,T}(X_{n+1}^\pi,\gamma(n+1),\tau(n+1),\theta)] \right\}$$

(B.6)

$$- \left( E_n[f_{n+1.n+1}(X_{n+1}^\pi,\gamma(n+1),\tau(n+1);X_{n+1}^\pi,\gamma(n+1),\tau(n+1),\theta)] - E_n[f_{n+1.n}(X_{n+1}^\pi,\gamma(n+1),\tau(n+1);X_n,\gamma(n),\theta)] \right)$$

$$- \left( E_n[g_{n+1}(X_{n+1}^\pi,\gamma(n+1),\tau(n+1),g_{n+1}(X_{n+1}^\pi,\gamma(n+1),\tau(n+1),\theta)] - G_n(X_n,\gamma(n),\theta) \right) \right\},$$

with terminal condition

$$V_{T,T}(X(T),\gamma(T),\tau(T),\theta) = F_T(X_T,\gamma_T,\tau_T,\theta) + G_T(X_T,\gamma_T,\tau_T,\theta).$$

From (B.3), (B.4), and (B.5), we have that, for any $x' > 0$, $\gamma' > 0$, and $\tau' \geq 0$, and any $k = 0,1,\ldots,T$, the function sequence $(f_{n,k}(X_n,\gamma(n),\tau(n);x',\gamma',\tau',\theta))_{0 \leq n \leq T}$ is determined by the recursion

$$f_{n,k}(X_n,\gamma(n),\tau(n);x',\gamma',\tau',\theta) = E_n[f_{n+1,k}(X_{n+1}^\pi,\gamma(n+1),\tau(n+1);x',\gamma',\tau',\theta)], \quad n = 0,\ldots,T-1,$$

$$f_{T,T}(X(T),\gamma(T),\tau(T);x',\gamma',\tau',\theta) = F_n(x',\gamma',\tau',\theta).$$

Furthermore, the function sequence $(g_{n}(X_n,\gamma(n),\tau(n)))_{0 \leq n \leq T}$ is determined by the recursion

$$g_{n}(X_n,\gamma(n),\tau(n)) = E_n[g_{n+1}(X_{n+1}^\pi,\gamma(n+1),\tau(n+1))], \quad n = 0,\ldots,T-1,$$

$$g_{T}(X(T),\gamma(T),\tau(T)) = X_T.$$

We also have, for $0 \leq n \leq T$, the probabilistic representation

$$f_{n,k}(X_n,\gamma(n),\tau(n);x',\gamma',\tau',\theta) = E_n[F_k(x',\gamma',\tau',X_{T}^\pi,\theta)], \quad g_{n}(X_n,\gamma(n),\tau(n)) = E_n[X_{T}^\pi].$$

We easily conclude.

**Proof of Proposition 4.1:** Assuming the existence of an optimal control law $\pi^*$, the value function at time $n+1$ satisfies

$$V_{n+1,T}(X_{n+1},\gamma(n+1),\tau(n+1),\theta)$$

$$= f_{n+1.n+1}(X_{n+1},\gamma(n+1),\tau(n+1);X_{n+1},\gamma(n+1),\tau(n+1),\theta) + \frac{\Delta_{n+1}}{2} \left( \frac{g_{n+1}^2(X_{n+1},\gamma(n+1),\tau(n+1))}{X_{n+1}} \right)^2.$$
and plugging this into the HJB equation (B.1) gives

\[ V_{n,T}(X_n, \gamma(n), \tau(n), \theta) = \sup_{\pi} \left\{ E_n \left[ f_{n+1,n}(X_{n+1}^\pi, \gamma(n+1), \tau(n+1); X_n, \gamma_n, \tau_n, \theta) \right] + \frac{\Delta_n}{2} \left( \frac{E_n[g_{n+1}(X_{n+1}^\pi, \gamma(n+1), \tau(n+1))]}{X_n} \right)^2 \right\}. \]  

(B.8)

Next, we look for a candidate optimal policy of the form

\[ \pi_n = \pi_n(X_n, \gamma(n), \tau(n), \theta) = \pi_n(X_n, \gamma_n, \tau_n, \theta) = \pi_n(\gamma_n, \tau_n, \theta)X_n. \]

For such a policy, we use the wealth dynamics (3.6) to show that (see Appendix B)

\[ E_n[X_n^\pi] = a_n(\gamma_n, \tau_n)X_n, \quad E_n[(X_n^\pi)^2] = b_n(\gamma_n, \tau_n)X_n^2, \]  

(B.9)

where the \(a_n\)- and \(b_n\)-coefficients are \(\pi\)-dependent and satisfy the recursions

\[ a_n(\gamma_n, \tau_n) = (R + \mu \pi_n(\gamma_n, \tau_n, \theta))E_n[a_{n+1}(\gamma_{n+1}, \tau_{n+1})], \]

\[ b_n(\gamma_n, \tau_n) = (\sigma^2 \pi_n^2(\gamma_n, \tau_n, \theta) + (R + \mu \pi_n(\gamma_n, \tau_n, \theta))^2)E_n[b_{n+1}(\gamma_{n+1}, \tau_{n+1})]. \]  

(B.10)

From (B.2), (B.3), and (B.5), it then follows that for an optimal policy, we have

\[ f_{n,k}(X_n, \gamma(n), \tau(n); x', \gamma', \tau', \theta) = a_n(\gamma_n, \tau_n) \frac{X_n}{x'} - 1 - \frac{\Delta_k(\gamma', \tau', \theta)}{2} b_n(\gamma_n, \tau_n) \left( \frac{X_n}{x'} \right)^2, \]

\[ g_n(X_n, \gamma(n), \tau(n)) = a_n(\gamma_n, \tau_n)X_n. \]

Plugging this into (B.8), using (3.6), and eliminating the arguments from \(a_{n+1}, b_{n+1}, \) and \(\pi_n, \) gives

\[ V_{n,T}(X_n, \gamma(n), \tau(n), \theta) = \sup_{\pi} \left\{ E_n \left[ a_{n+1} \frac{X_{n+1}^\pi}{X_n} - 1 - \frac{\Delta_n}{2} b_{n+1} \left( \frac{X_{n+1}^\pi}{X_n} \right)^2 \right] + \frac{\Delta_n}{2} \left( E_n \left[ a_{n+1} \frac{X_{n+1}^\pi}{X_n} \right] \right)^2 \right\} \]

\[ = \sup_{\pi} \left\{ E_n \left[ a_{n+1}(RX_n + Z_{n+1} \pi_n) - 1 - \frac{\Delta_n}{2} b_{n+1}(RX_n + Z_{n+1} \pi_n)^2 \right] + \frac{\Delta_n}{2} \left( E_n \left[ a_{n+1}(RX_n + Z_{n+1} \pi_n) \right] \right)^2 \right\}. \]

Recalling that \(Z_{n+1}\) has mean \(\bar{\mu}\) and variance \(\sigma^2\) gives

\[ V_{n,T}(X_n, \gamma(n), \tau(n), \theta) = \sup_{\pi} \left\{ E_n[a_{n+1}(RX_n + \bar{\mu} \pi_n) - 1 - \frac{\Delta_n}{2} \pi_n \left( R^2 \pi_n^2 + 2RX_n \bar{\mu} \pi_n + (\bar{\mu}^2 + \sigma^2) \pi_n^2 \right) + \frac{1}{2} \left( E_n[a_{n+1}] \right)^2 (R^2 \pi_n^2 + 2RX_n \bar{\mu} \pi_n + (\bar{\mu}^2 + \sigma^2) \pi_n^2) \right\}. \]  

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This can be rewritten as

\[
V_{n,T}(X_n, \gamma_n, \tau_n, \theta) = \sup_{\pi} \left\{ \mu \left[ \mathbb{E}_n[a_{n+1}] - R\Delta_n(\mathbb{E}_n[b_{n+1}] - (\mathbb{E}_n[a_{n+1}])^2) \right] \pi_n - \frac{1}{2} \frac{\Delta_n}{X_n} \left[ \mathbb{E}_n[b_{n+1}](\sigma^2 + \mu^2) - (\mathbb{E}_n[a_{n+1}])^2 \mu^2 \right] \pi_n^2 - 1 + RX_n \left[ \mathbb{E}_n[a_{n+1}] - \frac{\Delta_n}{2} \mathbb{E}_n[a_{n+1}] R + \frac{\Delta_n}{2} (\mathbb{E}_n[a_{n+1}])^2 R \right] \right\}
\]

and taking the derivative with respect to \( \pi_n \) gives the optimal allocation (4.3). One can then easily check that the HJB equation (B.1) is satisfied by this solution.

\[ \square \]

**Proof of (B.9)-(B.10):** For \( n = T - 1 \), we have by (3.6), that

\[
\mathbb{E}_{T-1} \left[ \frac{X_T}{X_{T-1}} \right] = \mathbb{E}_{T-1}[R + Z_T \pi_{T-1}(\gamma_{T-1}, \tau_{T-1})] = R + \mu \pi_{T-1}(\gamma_{T-1}, \tau_{T-1}) = a_{T-1}(\gamma_{T-1}, \tau_{T-1}).
\]

Next, let \( n \in \{0, 1, \ldots, T - 2\} \) and assume that the result holds for \( n + 1, n + 2, \ldots, T - 1 \). Then,

\[
\mathbb{E}_n \left[ \frac{X_T}{X_n} \right] = \mathbb{E}_n \left[ (R + Z_{n+1} \pi_n(\gamma_n, \tau_n)) \prod_{k=n+1}^{T-1} (R + Z_{k+1} \pi_k(\gamma_k, \tau_k)) \right] = (R + \mu \pi_n(\gamma_n, \tau_n)) \mathbb{E}_n \left[ \prod_{k=n+1}^{T-1} (R + Z_{k+1} \pi_k(\gamma_k, \tau_k)) \right] = (R + \mu \pi_n(\gamma_n, \tau_n)) \mathbb{E}_n[a_{n+1}(\gamma_{n+1}, \tau_{n+1})] =: a_n(\gamma_n, \tau_n).
\]

The result for the \( b_n \)-coefficients can be shown in the same way.

\[ \square \]

**Proof of Proposition 4.3:** Part (a) follows from Proposition 4.1 with \( \tau_n = n \), for all \( n \geq 0 \). Part (b) follows from Proposition 4.1 with \( \tau_n = 0 \), for all \( n \geq 0 \).

\[ \square \]

**Proof of (4.9)–(4.11):** First, consider the case \( \phi \equiv 1 \) and note that, for \( n = 0, 1, \ldots, T - 1 \), there exist constants \( K_a(n, T) \) and \( K_b(n, T) \) such that

\[
1 \leq \sup_{\gamma} a_n(\gamma) < K_a(n, T) < \infty, \quad 1 \leq \sup_{\gamma} b_n(\gamma) < K_b(n, T) < \infty.
\]
Then, we can write
\[
\mathbb{E}_n^{(1)} [a_{n+1}(\gamma_{n+1})] = \Lambda_{\gamma_n, \gamma_n} a_{n+1}(\gamma_n) + \sum_{\gamma \neq \gamma_n} \Lambda_{\gamma_n, \gamma} a_{n+1}(\gamma) =: a_{n+1}(\gamma_n) + R_a(\gamma_n),
\]
\[
\mathbb{E}_n^{(1)} [b_{n+1}(\gamma_{n+1})] = \Lambda_{\gamma_n, \gamma_n} b_{n+1}(\gamma_n) + \sum_{\gamma \neq \gamma_n} \Lambda_{\gamma_n, \gamma} b_{n+1}(\gamma) =: b_{n+1}(\gamma_n) + R_b(\gamma_n),
\]
where
\[
|R_a(\gamma_n)| \leq 2K_a(n, T)(1 - \Lambda_{\gamma_n, \gamma_n}) = O(1 - \Lambda_{\gamma_n, \gamma_n}), \quad |R_b(\gamma_n)| \leq 2K_b(n, T)(1 - \Lambda_{\gamma_n, \gamma_n}) = O(1 - \Lambda_{\gamma_n, \gamma_n}).
\]

Next, recall that the random variable $Z$ is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, independently of $(Z_n)_{n \geq 1}$, but with the same distribution. Thus, $Z' = Z/\sigma$ has mean $\bar{\mu}/\sigma$ and unit variance. In turn,
\[
(\sigma^2 + \bar{\mu}^2) b_{n+1}(\gamma_n) - \mu^2 a_{n+1}(\gamma_n) = \mathbb{E}_n^{(1)} [Z^2 (1 + r_{n+1,T}^*)^2 | \gamma_{n+1} = \gamma_n] + (\mathbb{E}_n^{(1)} [Z (1 + r_{n+1,T}^*) | \gamma_{n+1} = \gamma_n])^2 = \sigma^2 Var_n^{(1)} [Z(1 + r_{n+1,T}^*) | \gamma_{n+1} = \gamma_n] =: \sigma^2 Var_n^{(1)} [Z(1 + r_{n+1,T}^*) | \gamma_{n+1} = \gamma_n] + O(1 - \Lambda_{\gamma_n, \gamma_n}),
\]
and (4.10) follows from Lemma A.1-(a) and (4.8), and by setting $R = 1$.

To justify the second equality above, we work as follows. First, for $n = 0, 1, \ldots, T - 1$, there exists a constant $K(n, T) < \infty$ such that
\[
\sup_{\gamma \in \Gamma} |a_{n+1}(\gamma) - R\gamma (b_{n+1}(\gamma) - a_{n+1}^2(\gamma))| < K(n, T) < \infty.
\]
Second, from the recursive equation (4.6), it follows that $b_{n+1}(\gamma) \geq 1$. Thus, by Jensen’s inequality,
\[
b_{n+1}(\gamma)(\sigma^2 + \mu^2) - a_{n+1}^2(\gamma) \mu^2 \geq b_{n+1}(\gamma) \sigma^2 \geq \sigma^2 > 0, \quad \gamma \in \Gamma.
\]
Identity (4.9) is a special case of (4.10), and (4.11) can be shown in a similar way.
C Proofs of Results in Section 5 and Section 6

Proof of (5.3)-(5.4): Recall from (A.1) that \( p_n(\gamma'; \gamma, \tau) \) is the probability of being at level \( \gamma' \) at time \( n \), after level \( \gamma \) being realized at time \( \tau \), with no intermediate risk aversion updates. Then, we have from Lemma A.2-(d) that

\[
\tilde{R}_{n,T}^{(\phi)}(\gamma, \tau) = \sum_{\gamma' \in \Gamma} p_n(\gamma'; \gamma, \tau) \frac{[\pi_{n,T}(\gamma, \tau; 0) - \pi_{n,T}(\gamma', n; 0)]}{\pi_{n,T}(\gamma', n; 0)}
\]

\[
= \sum_{\gamma' \in \Gamma} p_n(\gamma'; \gamma, \tau) \left( \frac{[\pi_{n,T}(\gamma, \tau; 0) - \pi_{n,T}(\gamma', \tau; 0)]}{\pi_{n,T}(\gamma', \tau; 0)} + O(1 - p_{\phi(\gamma)}(\gamma; \gamma)) \right)
\]

\[
= \sum_{\gamma' \in \Gamma} p_n(\gamma'; \gamma, \tau) \left( \frac{[\pi_{n,T}(\gamma, \tau; 0) - \pi_{n,T}(\gamma', \tau; 0)]}{\pi_{n,T}(\gamma', \tau; 0)} + O(1 - p_{\phi(\gamma)}(\gamma; \gamma)) \right)
\]

\[
= \mathbb{E}_{n}^{(\phi)} \left[ \frac{[\pi_{n,T}(\gamma_n; 0) - \pi_{n,T}(\Psi(\gamma_n); 0)]}{\pi_{n,T}(\Psi(\gamma_n); 0)} \right] \gamma_n = \gamma, \tau_n = \tau + O(1 - p_{\phi(\gamma)}(\gamma; \gamma)).
\]

The second equality can be shown using similar arguments as those used to justify (B.11). For the third equality we used that, for any \( \gamma' \in \Gamma \),

\[
p_n(\gamma'; \gamma, \tau) \leq 1 - p_n(\gamma; \gamma, \tau) = 1 - p_{n-\tau}(\gamma; \gamma) \leq 1 - p_{\phi(\gamma)}(\gamma; \gamma),
\]

which follows from Lemma A.1-(a) and \( n - \tau < \phi(\gamma) \). Next, we claim that for any \( 0 \leq \tau < T \),

\[
\sup_{\tau \leq n < n_r} \tilde{R}_{n,T}^{(\phi)}(\gamma, \tau) = \sup_{\tau \leq n < n_r} \mathbb{E}_{n}^{(\phi)} \left[ \frac{[c_{n,T}(\gamma) - \pi_{n,T}(\gamma_n; 0) - \pi_{n,T}(\Psi(\gamma_n); 0)]}{\pi_{n,T}(\Psi(\gamma_n); 0)} \right] \gamma_n = \gamma, \tau_n = \tau + O(1 - p_{\phi(\gamma)}(\gamma; \gamma))
\]

\[
\leq \sup_{\tau \leq n < n_r} \mathbb{E}_{n}^{(\phi)} \left[ \left| \frac{[\pi_{n,T}(\gamma; 0) - \pi_{n,T}(\Psi(\gamma_n); 0)]}{\pi_{n,T}(\Psi(\gamma_n); 0)} \right| \gamma_n = \gamma, \tau_n = \tau \right] \gamma_n = \gamma, \tau_n = \tau + O(1 - p_{\phi(\gamma)}(\gamma; \gamma))
\]

\[
= \mathbb{E}_{n}^{(\phi)} \left[ \left| \frac{[\pi_{n,T}(\gamma; 0) - \pi_{n,T}(\Psi(\gamma_n); 0)]}{\pi_{n,T}(\Psi(\gamma_n); 0)} \right| \gamma_n = \gamma, \tau_n = \tau \right] \gamma_n = \gamma, \tau_n = \tau + O(1 - p_{\phi(\gamma)}(\gamma; \gamma))
\]

\[
\leq \sup_{\tau \leq n < T} \tilde{R}_{n,T}^{(\phi)}(\gamma, \tau) \right|_{\tau = (T - \phi(\gamma))} + O(1 - p_{\phi(\gamma)}(\gamma; \gamma))
\]

\[
= \tilde{R}_{n,T}^{(\phi)}(\gamma, \tau) \right|_{\tau = (T - \phi(\gamma))} + O(1 - p_{\phi(\gamma)}(\gamma; \gamma)).
\]

Indeed, the first inequality follows from Lemma A.2-(e), because for \( \gamma < \Psi(\gamma_n) \) we have \( c_{n,T}(\gamma) \leq c_{n,T}(\Psi(\gamma_n)) \), which yields that

\[
\pi_{T-1,T}(\Psi(\gamma_n); 0) \leq \frac{c_{n,T}(\gamma)}{c_{n,T}(\Psi(\gamma_n))} \pi_{T-1,T}(\gamma; 0) \leq \pi_{T-1,T}(\gamma; 0),
\]
with the inequalities reversing for \( \gamma > \Psi(\gamma_n) \). The second equality in (C.1) follows from the fact that, conditionally on \( \gamma_n = \gamma \) and \( \tau_n = \tau \), we have \( \tilde{\gamma}_n \sim N(\gamma, (n - \tau)\sigma_n^2) \), and, thus \( \mathbb{P}_n(\tilde{\gamma}_n > c) = \mathbb{E}_n(\tilde{\gamma}_n) > c) \) is increasing in \( n \) for all \( c > 0 \). From this fact, and the definition of \( \mathcal{R}_n(T, \gamma) \), the second inequality and third equality in (C.1) also follow. In turn, (5.3) and (5.4) follow from (C.1), and the fact that the \( O(\cdot) \) term vanishes, and the inequalities become equalities, for \( \tau = (T - \phi(\gamma)) \lor 0 \).

**Proof of Proposition 5.2:** The inequality in part (a) follows from (5.3)-(5.4). To show (5.8), we use that by Lemma A.2-(a),

\[
\mathcal{R}_T^{(\phi)}(\gamma) = \mathbb{E}_n \left[ \frac{\left[ \pi_{n,T}^*(\gamma_n, \tau_n; 0) - \pi_{n,T}^*(\tilde{\gamma}_n, \gamma_n; 0) \right]}{\pi_{n,T}^*(\tilde{\gamma}_n, \gamma_n; 0)} \bigg| \gamma_n = \gamma, \tau_n = \tau \right]_{\gamma = (T - \phi(\gamma)) \lor 0, n = T - 1} = \mathbb{E}_n \left[ \frac{\left[ \Psi(\tilde{\gamma}_n) - \gamma_n \right]}{\gamma_n} \bigg| \gamma_n = \gamma, \tau_n = \tau \right]_{\gamma = (T - \phi(\gamma)) \lor 0, n = T - 1}.
\]

The distribution of \( \tilde{\gamma}_n \) under \( \mathbb{P}_n(\tilde{\gamma}_n) \), given \( \gamma_n = \gamma \) and \( \tau_n = \tau \), is the same as the distribution of \( \tilde{\gamma}_n - \tau \) under \( \mathbb{P} \), given \( \tau_0 = \gamma \). Hence, noting that \( n - \tau = (\phi(\gamma) \land T) - 1 \), for \( \tau = (T - \phi(\gamma)) \lor 0 \) and \( n = T - 1 \), we have

\[
\mathcal{R}_T^{(\phi)}(\gamma) = \mathbb{E}_n \left[ \frac{\left[ \Psi(\tilde{\gamma}_n(\phi(\gamma) \land T) - 1) - \gamma_0 \right]}{\gamma_0} \bigg| \gamma_0 = \gamma \right] = \frac{\mu(\phi(\gamma) \land T) - 1(\gamma)}{\gamma},
\]

where the second equality follows from the definition of \( \mu_n(\cdot) \) in (5.6). In part (b), the monotonicity properties and limits follow from Lemma A.1-(b). The inequality \( \mathcal{R}_T^{(\phi)}(\gamma) \leq \mathcal{R}_{T,c}^{(\phi)}(\gamma) \) and the identity for \( \mathcal{R}_{T,c}^{(\phi)}(\gamma) \), follow from Lemma A.1-(c).

**Proof of Proposition 5.3:** We have

\[
\begin{align*}
\mathcal{S}_{n,T}^{(\phi)}(\gamma, \tau, \theta) &= \frac{\left[ \pi_{n,T}^*(\gamma, \tau, \theta) - \pi_{n,T}^*(\gamma; 0) \right]}{\pi_{n,T}^*(\gamma; 0)} + O(1 - p_{\phi(\gamma)}(\gamma; \gamma)) \\
&= \frac{\pi_{n,T}^*(\gamma + \theta \delta_n(\gamma, \tau; 0) - \pi_{n,T}^*(\gamma; 0)}}{\pi_{n,T}^*(\gamma; 0)} + O(1 - p_{\phi(\gamma)}(\gamma; \gamma)) + O(1 - p_{\phi(\gamma)}(\gamma; \gamma)) \\
&= \frac{\pi_{n,T}^*(\gamma + \theta \delta_n(\gamma, \tau; 0) - \pi_{n,T}^*(\gamma; 0)}}{\pi_{n,T}^*(\gamma; 0)} + O(\theta(1 - p_{\phi(\gamma)}(\gamma; \gamma))) \\
&\leq \frac{\pi_{T-1,T}^*(\gamma + \theta \delta_n(\gamma, \tau; 0) - \pi_{T-1,T}^*(\gamma; 0)}}{\pi_{T-1,T}^*(\gamma; 0)} + O(\theta(1 - p_{\phi(\gamma)}(\gamma; \gamma))) \\
&= \frac{\theta \delta_n(\gamma, \tau)}{\gamma + \theta \delta_n(\gamma, \tau)} + O(\theta(1 - p_{\phi(\gamma)}(\gamma; \gamma))).
\end{align*}
\]

The first equality above follows from Lemma A.1-(d) and similar steps as those used to establish (B.11). For the second equality, first consider \( n \geq \tau \) where \( \tau \) is such that \( \tau + \phi(\gamma) > T - 1 \). Then, we can show
Proof of (6.3)-(6.5): For (6.3), we use the approximation \( \Psi(\bar{\gamma}) \approx \gamma \) and Lemma A.1-(c) to write
\[
\bar{\mathcal{R}}^{(\phi)}_T = \sum_{\gamma \in \Gamma} \lambda(\gamma) R^{(\phi)}_T(\gamma) \approx \sum_{\gamma \in \Gamma} \lambda(\gamma) R^{(\phi)}_{T,c}(\gamma) = \sum_{\gamma \in \Gamma} \lambda(\gamma) \sqrt{\frac{2 \sigma^2}{n} \gamma} \sqrt{\phi(\gamma)}.
\]
We also have that
\[ \bar{\phi}_n := \inf\{ n \geq 1 : \bar{R}_T(\phi)|_{\phi \equiv n} > \kappa \} \approx \kappa^2 \frac{\pi}{2} \left( \sum_{\gamma \in \Gamma} \lambda(\gamma) \frac{\sigma_\gamma}{\gamma} \right)^2, \]
and by combining the above, (6.5) follows.

\[ \square \]

**Proof of (6.6):** By Lemma A.1-(a) we have
\[ \alpha^{(\phi)}(\gamma) = \sup_{0 \leq n < T} \mathbb{P}_n(\Phi^{(\gamma^*-\gamma)}_{\phi_n}(\gamma) = \gamma|\gamma_n = \gamma, \tau_n = n)1_{\{n + \phi(\gamma) < T\}} \]
\[ = \sup_{0 \leq n < T} \mathbb{P}(\Phi^{(\gamma^*-\gamma)}_{\phi_n}(\gamma) = \gamma|\gamma_n = \gamma, \tau_n + \phi(\gamma) = n)1_{\{n + \phi(\gamma) < T\}} \]
\[ = \sup_{0 \leq n < T} p_{\phi(\gamma)}(\gamma; \gamma)1_{\{n + \phi(\gamma) < T\}} \]
\[ = p_{\phi(\gamma)}(\gamma; \gamma)1_{\{\phi(\gamma) < T\}}. \]

Notice, again by Lemma A.1-(a), that \( \alpha^{(\phi)}(\gamma) \) is decreasing in \( \phi(\gamma) \). Also, the probability of a false positive is zero if \( \phi(\gamma) \geq T \), as in this case, there will be no updates prior to the terminal date \( T \). To show (6.6), we first use Lemma A.1-(a) to write
\[ \alpha^{(\phi_n)}(\gamma) = \left[ \Phi\left( \frac{\gamma^+ - \gamma}{\sigma_\gamma \sqrt{\phi_n(\gamma)}} \right) - \Phi\left( \frac{\gamma^- - \gamma}{\sigma_\gamma \sqrt{\phi_n(\gamma)}} \right) \right] 1_{\{\phi_n(\gamma) < T\}}, \]
and we easily conclude using (6.4).

\[ \square \]

**D Pseudocode for Optimal Investment Strategy**

We provide a pseudocode for the backward recursion used to compute the optimal strategy in Proposition 4.1. The optimal strategies in Proposition 4.3 are then special cases. Specifically, for any updating rule \( \phi \), and any \( \theta \geq 0 \), we compute
\[ \pi_{n,k,\tau} := \pi_{n,T}(\bar{\gamma}_k, \tau, \theta), \ a_{n,k,\tau} := a_n(\bar{\gamma}_k, \tau), \ b_{n,k,\tau} := b_n(\bar{\gamma}_k, \tau), \]
for \( 0 \leq n \leq T, \ 1 \leq k \leq K, \) and \( 0 \leq \tau \leq n \). Recall that \( R = 1 + r \), where \( r \geq 0 \) is the risk-free rate, \( \bar{\mu} = \mu - r \) is the excess return of the risky asset and \( \sigma \) its volatility, and the transition probability \( p_n(\gamma'; \gamma) \), for any \( \gamma, \gamma' \in \Gamma \), is given in (A.6). We use \( 1_A \) to be the indicator function of \( A \).
1. Set $a_{T,k,\tau} = 1$, $b_{T,k,\tau} = 1$, and $\pi^*_{T,k,\tau} = 0$, for $k = 1, 2, \ldots, K$ and $\tau = 0, 1, \ldots, T$.

2. For $n = T - 1, T - 2, \ldots, 0$:
   For $k = 1, 2, \ldots, K$:
      For $\tau = n, n - 1, \ldots, 0$:
         \[
         \tau_{n+1} = \mathbf{1}_{\{\tau + \phi(\gamma_k) > n + 1\}} \times \tau + \mathbf{1}_{\{\tau + \phi(\gamma_k) \leq n + 1\}} \times (n + 1),
         \]
         \[
         \mu_a = \sum_{k' = 1}^K p_{n+1-\tau}(\gamma_{k'}; \gamma_k) a_{n+1,k',n+1},
         \]
         \[
         \mu_b = \sum_{k' = 1}^K p_{n+1-\tau}(\gamma_{k'}; \gamma_k) b_{n+1,k',n+1},
         \]
         \[
         \delta = \sum_{k' = 1}^K p_{n-\tau}(\gamma_{k'}; \gamma_k) \gamma_{k'}^2 - (\sum_{k' = 1}^K p_{n-\tau}(\gamma_{k'}; \gamma) \gamma_{k' \gamma_k})^2,
         \]
         \[
         \pi^*_{n,k,\tau} = \frac{\bar{\mu} - R(\gamma_k + \theta \delta) \mu_b - \mu^2}{(\sigma^2 + \mu^2) \mu_b - \mu^2 \mu^2},
         \]
         \[
         a_{n,k,\tau} = \mu_a (R + \bar{\mu} \pi^*_{n,k,\tau}),
         \]
         \[
         b_{n,k,\tau} = \mu_b (\sigma^2 (\pi^*_{n,k,\tau})^2 + (R + \bar{\mu} (\pi^*_{n,k,\tau})^2))^2).
         \]
   End
   End
   End

References


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