Personalized Robo-Advising: an Interactive Investment Process

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Abstract
Automated investment managers, or robo-advisors, have emerged as an alternative to traditional human advisors. Their viability crucially depends on the frequency of interaction with the clients they serve. We develop a novel client/robo-advisor interaction framework, in which the robo-advisor solves a multi-period mean-variance optimization problem where the risk-return trade-off adapts to the risk profile communicated by the client. In determining the communication schedule, the client faces a trade-off between a more personalized portfolio strategy and an increased level of delegation to the robo-advisor. Our model predicts that a client that values having a more personalized portfolio prefers the robo-advisor over a human-advisor, unless the latter is highly sophisticated or the client’s risk profile changes frequently. A client that places more emphasis on delegation favors the human-advisor, consistently with empirical findings.

1 Introduction
Automated investment managers, commonly referred to as robo-advisors, have gained widespread popularity in recent years. The value of assets under management by robo-advisors is the highest in the United States, exceeding $400 billion in 2018 (Abraham et al. [2019]). Major robo-advising firms include Vanguard Personal Advisor Services, which manages about $112 billion of assets, Schwab Intelligent Portfolios, managing nearly $33 billion of assets, Betterment, with about $14 billion of assets, and Wealthfront with about $10 billion of assets under management. The popularity of robo-advisors is also growing in other parts of the world. They manage more than €100 million in Europe (Burnmark [2017]), and are rapidly growing in Asia (Forbes [2017]) and emerging markets. According to Abraham et al. [2019], the value of assets under management is expected to grow at an average annual rate of over 30 percent, reaching an estimated $1.5 trillion by 2023, solely in the United States.

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The first robo-advisors available to the general public were launched in 2008, in the wake of the financial crisis and the ensuing loss of trust in established financial services institutions. To begin with, firms rooted in the technology industry began offering a range of digital financial tools directly to customers, including investment analysis tools, previously only available to financial professionals (FINRA [2016]). Since then, the emergence of robo-advising firms has caused a significant disruption to the investment management landscape, by appealing to a new tech-savvy generation of wealth. The rise of robo-advisors has also been compounded by the seismic shift towards passive investing and exchange-traded funds, since the financial crisis (Deloitte [2014], Kaya [2017]).

Existing robo-advising systems are based on a one-time interaction with the client. For instance, Vanguard Personal Advisor Services profiles the client based on input received at the outset, which includes financial goals, investment horizon and demographic information. After the investment plan proposed by the robo-advisor is accepted by the client, the robo-advisor autonomously executes trades to reach the desired portfolio allocation (Rossi and Utkus [2019a]). Another popular robo-advising firm, Wealthfront, estimates the client’s subjective risk tolerance by asking once whether her objective is to maximize gains, minimize losses, or a mix of the two. The robo-advisor uses the client’s answers to construct a risk parameter which may also depend on additional objective risk indicators, and then solves a mean-variance optimization problem (Lam [2016]). With limited or no human intervention in the process, however, the robo-advisor is clearly susceptible to the risk of making decisions based on stale information and thus not acting in the client’s best interest.

The distinguishing feature of our framework is that the client and the robo-advisor interact not only at the beginning, but also throughout the investment period. The client wishes to optimally invest her wealth throughout a discrete, finite period horizon. She delegates this task to the robo-advisor, which executes the investment process on the client’s behalf, accounting for the time varying nature of her risk profile. In order to effectively tailor the investment advice to the needs of the client, the robo-advisor solicits, on a regular basis, information about the client’s changing risk preferences. At the beginning of the investment process the client specifies a desired rate of participation, which determines the frequency of interactions between the client and the robo-advisor. At each interaction time, the risk-preferences are communicated by the client to the robo-advisor. From that point, and until the subsequent time of interaction, the robo-advisor uses a random walk model to describe the (unobserved) evolution of the client’s risk preferences. The random walk model captures the fact that while the client’s risk preferences may change over time (Guiso et al.

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2The Monetary Authority of Singapore (MAS) issued a consultation paper on the regulation of robo-advisor services (MAS [2017]). MAS set out minimum requirements on the standard of governance and management oversight of digital advisors, including the responsibility of the Board and Senior Management for the monitoring and control of algorithms that process questions and generate recommendations. See also FINRA [2016].
[2018]), they are unlikely to exhibit drastic changes over a short time period (Schildberg-Horisch [2018]). The volatility of the random walk quantifies how likely the client’s risk preferences are to change from the ones reported at the latest interaction with the robo-advisor.

The robo-advisor adopts a multi-period mean-variance optimization criterion. Unlike other risk averse optimization problems considered in the literature, the risk-return trade-off coefficient is stochastic and consists of two components, one specific to the client and one specific to the robo-advisor. The client-specific component incorporates the most recently communicated risk preferences of the client, and is updated only when the robo-advisor solicits the client’s input. The dynamics of these client-specific risk preferences are described by a finite state Markov chain whose transition times coincide with the times of interaction. The contribution of the robo-advisor to the risk-return trade-off reflects its aversion against uncertainty in the client’s risk preferences. The robo-advisor is less willing to take risk on behalf of the client if the client’s risk preferences are based on stale information, resulting in the risk-return trade-off coefficient being inflated to a degree determined by the level of distrust in the most recently communicated risk preferences. This is related to the concept of trust in Gennaioli and Vishny [2015], who show that individuals without finance expertise are more willing to take on risk with a financial advisor they trust, with trust based on, e.g., personal relationships and persuasive advertising. In their setting, the roles of the client and the financial advisor are reversed compared to our model: the client’s baseline risk aversion in their mean-variance utility function is inflated by a factor representing the level of anxiety suffered by the client from bearing risk with the financial advisor.

We present a recursive solution to the optimization problem. We highlight how both components of the risk-return trade-off coefficient influence the optimal portfolio strategy, and how the Markov chain transition probabilities link the optimal allocations before and after risk preferences are communicated by the client. With frequent interaction, information about the client’s risk preferences arrives gradually, so there is little variation in the client’s risk preferences between consecutive times of interaction, and therefore little uncertainty in near-term optimal allocations. The allocation uncertainty increases with a less frequent interaction schedule. This is because the optimal allocation depends on future realizations of the client’s risk preferences, and the robo-advisor becomes more uncertain about them as the random walk is allowed to evolve over longer periods. Hence, prior to each interaction time, the subsequent optimal allocation will have a larger variance compared to the case of more frequent updating.

At times of no interaction, the robo-advisor makes investment decisions based on stale information, and as a result it tilts the optimal portfolio towards a less risky composition. We obtain an explicit expression for the magnitude of this effect, and show that it corresponds to setting the risk-return coefficient to a fixed
percentile of the distribution of the client’s unknown risk aversion, thus limiting the probability of choosing a portfolio composition that is too risky.

We introduce a measure of regret that quantifies the investment implications of the information asymmetry between the client and the robo-advisor. At each point in time, the regret is equal to the change in portfolio allocation that would be realized if the robo-advisor knew the client’s actual risk preferences. The regret is increasing in the time elapsing since the last interaction. Moreover, the regret is increasing in the volatility of the random walk that describes the client’s risk preferences, and for a given volatility level, the regret is higher if the risk preferences last communicated by the client correspond to low risk aversion. The client’s regret is determined by her participation rate in the investment process, and we show that this uniquely determines a threshold parameter such that the robo-advisor solicits the client’s risk preferences as soon as the expected change in the client’s risk aversion since the last interaction time breaches this threshold. Furthermore, we demonstrate how the threshold needs to be adjusted as the client’s risk preferences change, in order to maintain the same level of regret. For instance, if the client’s risk preferences shift to a level where fluctuations are more likely to occur, then more frequent interaction is required. We investigate the benefits of such a personalized service, where the mechanism triggering interaction is tailored to the client’s current risk profile, over one-size-fits-all updating rules that are tailored to properties of the “average client”. We show that clients with a risk profile that is underrepresented in the robo-advisor’s client body have a higher regret than clients with a common risk profile.

We compare our robo-advisor framework with that of a stylized traditional human advisor. On the one hand, human advisors are expected to know the client’s risk attitude better than robo-advisors. Empirical evidence provided by Rossi and Utkus [2019b] indicates that clients indeed choose a traditional financial advisor primarily to interact with a human. In their paper, they show that the possibility of establishing a personal long-term relationship with the same advisor increases the client’s perception of value and her overall satisfaction. In our stylized model of human advising, we capture this situation by assuming that the human advisor knows the client’s risk preferences at all times. On the other hand, robo-advisors are expected to be more technologically sophisticated, and employ algorithms that adapt the investment decisions to the client’s changing risk profile. We quantify the human advisor’s skill level in terms of the maximum investment horizon for which he is able to optimally solve the multi-period investment problem. Using the same measure of regret as for the robo-advisor, we show that the client’s regret associated with the human advisor’s investment decisions increases with his short-sightedness. A human advisor who can only look one period ahead repeatedly solves a single-period optimization problem, and maximizes the client’s regret. By contrast, with a human advisor who solves the problem optimally, just like the robo-advisor, the client has
zero regret. We compare the client’s regret with the robo-advisor, which results from information asymmetry, i.e., the client’s risk preferences not being observable by the robo-advisor, to the client’s regret with the human advisor, who is fully aware of the client’s risk preferences at all times but solves the investment problem suboptimally.

Our analysis shows that unless the human-advisor is near the level of sophistication of the robo-advisor, or if the client’s risk preferences are very volatile, then a moderate time between updates is sufficient for the client to favor the robo-advisor. For instance, most robo-advisors encourage their clients to revisit their risk preferences at least quarterly, which is sufficient for them to have a lower regret with the robo-advisor. This suggests that clients who want little involvement in the investment process (i.e., don’t want frequent interaction with the robo-advisor) are less likely to choose the robo-advisor. These results are consistent with the empirical evidence presented in Rossi and Utkus [2019b], where individuals who put emphasis on delegating financial decisions (i.e., don’t want much involvement and prefer to trust the financial advisor) are less likely to use the services of a robo-advisor. On the other hands, clients who care about investment performance (i.e., want low regret) are more likely to choose the robo-advisor, again consistent with the empirical results in Rossi and Utkus [2019b].

The analysis of the proposed robo-advising framework presents a novel methodological contribution. To the best of our knowledge, our paper is the first to solve an adaptive control problem in which the system being controlled always maintains the same dynamics, but the investment criterion changes by adapting the risk-return trade-off coefficient at random interaction times. This contrasts with existing literature in adaptive control (see, for instance, Astrom and Wittenmark [1989]), where the optimization criterion always stays the same, and the system dynamics adapt to changes in the environment. In our framework, the interaction times effectively split the investment horizon into subperiods, triggering a new mean-variance optimization problem at each communication time. This yields a sequence of time-inconsistent problems that are interlinked, because they all share the same target horizon. At each time, the optimal control strategy and value functions depend on future realizations of the risk-return coefficient, which in turn depends on the volatility of risk preferences and the frequency of interaction.

The rest of this paper is organized as follows. In Section 2, we briefly review related literature. In Section 3, we introduce the main components of our modeling framework. In Section 4, we present the optimal solution to the investment problem. In Section 5, we study performance-metrics of the human-machine interaction system. In Section 6, we compare the performance of the robo-advisor and the human-advisor. In Section 7, we discuss the calibration of the human-machine interaction system. Section 8 contains concluding remarks and future directions along which the model can be extended. Appendix A collects technical results.
that are used throughout the paper. Appendix B contains the proofs of the main results in Section 4. All remaining proofs are deferred to Appendix C. Appendix D provides pseudocode to compute the optimal investment strategy in Section 4.

2 Literature Review

The main contribution of our paper is to develop a framework that captures the symbiotic nature of the investment process, in which the robo-advisor repeatedly interacts with the client to elicit risk preferences.

Our work contributes to the literature on time-inconsistent stochastic control (Björk and Murgoci [2013] and Björk et al. [2014]). Other related works include Li and Ng [2010] who solve a multi-period version of the classical Markowitz problem, and Basak and Chabakauri [2010] who solve a continuous-time version of the dynamic mean-variance optimization problem within a potentially incomplete market. A recent study of Dai et al. [2019] develops a dynamic mean-variance framework, in which the investor specifies her target expected return only at inception. They obtain explicit formulas for the time-consistent policies in a mean-variance framework with stochastic volatility and time-varying Gaussian returns. In all these papers, the risk-return trade-off is assumed to be constant throughout the investment horizon. By contrast, the risk-return trade-off is stochastic in our model, and only observed by the controller (robo-advisor) at random times. Unlike Björk and Murgoci [2013], we consider adapted control laws rather than the more restrictive feedback laws, and show that the optimal control law for the dynamic mean-variance problem is in fact of feedback form.

Our study also contributes to the growing literature on robo-advising. Noticeable contributions include D’Acunto et al. [2018], who empirically show that the adoption of robo-advising increases portfolio diversification and reduces well-known behavioral biases such as the disposition effect. They also show that these effects are more pronounced for undiversified investors, suggesting that technology levels the playing ground for less savvy investors. Rossi and Utkus [2019a] study the largest US robo-advisor, Vanguard Personal Advisor Services. Upon adoption, they show that it significantly increases the proportion of clients’ wealth invested in low-cost indexed mutual funds, at the expense of individual stock holding and active mutual funds. As in D’Acunto et al. [2018], the clients that benefit the most from adoption are those with little investment experience, as well as clients with little mutual fund holdings and those invested in high-fee active mutual funds.

Rossi and Utkus [2019b] conduct a survey to study the wants and needs of individuals when they hire financial advisors. Their results lend support to the theoretical model of Gennaioli and Vishny [2015], indicating that traditionally-advised individuals hire financial advisors largely to satisfy needs other than
portfolio return maximization. Robo-advised clients, on the other hand, are more interested in the financial performance of their portfolio. These clients are not particularly interested in having access to expert opinions and do not have a high need for trust. Furthermore, they find that individuals most likely to adopt robo-advising are those interested in acquiring knowledge and improving their investment allocation, while those that wish to completely delegate their investment decisions, and therefore have a need for trust, are more reluctant to consider robo-advising. These pros and cons are reflected in the comparison of the robo-advisor and the human-advisor in our model. Clients that place a high emphasis on delegation, or low personal involvement, potentially at the expense of a high portfolio regret, are less likely to choose the robo-advisor over the human-advisor, while clients that are willing to have a high participation rate in the investment process, in order to keep portfolio regret low, are more likely to choose the robo-advisor.

3 Model

The modeling framework consists of four main components: (i) The robo-advisor, which solves the dynamic portfolio optimization problem based on its view of the client’s risk preferences, (ii) the interaction mechanism between the robo-advisor and the client, (iii) the market model for the investment securities, and (iv) the investment criterion.

3.1 Robo-Advisor’s View of Client

The robo-advisor summarizes the client’s risk preferences by a risk aversion parameter. At the start of the investment process, the client communicates to the robo-advisor her initial risk preferences. Most robo-advisors solicit this information by presenting an online questionnaire to the client, asking for information on, e.g., income, education, household status, investment goals, and potential reactions to hypothetical future market events (see Lam [2016], Ch. 3). The robo-advisor then translates the client’s feedback into a numerical risk score, herein referred to as the client’s risk aversion parameter. We abstract away from the construction of this mapping, which is equivalent to assuming that the client communicates directly her risk aversion parameter to the robo-advisor.

The client knows her risk aversion parameter at all times, but only communicates it to the robo-advisor at specific updating times. At each updating time, the robo-advisor maps the risk aversion parameter communicated by the client to the nearest value in a finite set of risk aversion levels

\[ \Gamma := \{\bar{\gamma}_1, \bar{\gamma}_2, \ldots, \bar{\gamma}_K\}, \]
where $K \geq 1$. Without loss of generality, we assume that the levels are sorted in increasing order of risk aversion, i.e., $0 < \bar{\gamma}_1 < \bar{\gamma}_2 < \cdots < \bar{\gamma}_K < \infty$.

Between consecutive updating times, the robo-advisor receives no input from the client, and assumes that the evolution of the client’s risk aversion follows a random walk with volatility that depends on the risk aversion level. We denote by $\sigma_\gamma > 0$ the volatility of a random walk starting in risk aversion level $\gamma \in \Gamma$; a higher $\sigma_\gamma$ implies that it is more likely for the client to transition from $\gamma$ to a different risk aversion level. The random walk model captures the fact that while the client’s risk aversion may change over time, it is much more likely to move between neighboring levels than swiftly going from a high risk aversion level to a low risk aversion level or vice versa.  

More concretely, at time $n \geq 0$ we denote by $\tau_n \in \{0, 1, \ldots, n\}$ the last updating time prior to and including time $n$, and by $\gamma_0 \in \Gamma$ the client’s initial risk aversion level. We then define the random walk process $(\bar{\gamma}_n)_{n \geq 0}$ by

$$\bar{\gamma}_n := \gamma_{\tau_n} + 1 \times (\bar{Z}_{\tau_n-1} + \cdots + \bar{Z}_n), \quad n > 0,$$

where $(\bar{Z}_n)_{n \geq 1}$ is a sequence of i.i.d. $\mathcal{N}(0, 1)$ random variables, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. At time $n$, the client’s risk aversion level, as viewed by the robo-advisor, is then given by

$$\gamma_n := \Psi(\bar{\gamma}_{\tau_n}) := \sum_{\gamma \in \Gamma} \gamma \times 1\{\bar{\gamma}_{\tau_n} \in (\gamma-, \gamma+)\},$$

where the function $\Psi$ maps its argument to the nearest risk aversion level in $\Gamma$. Above, $1_A$ denotes the indicator of an event $A$, and for each $\gamma \in \Gamma$, the constants $\gamma^{\pm}$ are defined by

$$\gamma^+ := \begin{cases} \frac{\bar{\gamma}_k + \bar{\gamma}_{k+1}}{2}, & \gamma = \bar{\gamma}_k, 1 \leq k < K, \\ \infty, & \gamma = \bar{\gamma}_K, \end{cases}, \quad \gamma^- := \begin{cases} \infty, & \gamma = \bar{\gamma}_1, \\ \frac{\bar{\gamma}_k + \bar{\gamma}_{k-1}}{2}, & \gamma = \bar{\gamma}_K, 1 < k \leq K. \end{cases}$$

The robo-advisor only observes the client’s risk aversion level process $(\gamma_n)_{n \geq 0}$, which is constant between consecutive updating times, i.e., for any $n \geq 0$ we have $\gamma_n = \gamma_{\tau_n}$. This is visually illustrated in Figure 1, that displays a sample path of the client’s risk aversion level. The process $(\tau_n)_{n \geq 0}$, that pins down the process $(\gamma_n)_{n \geq 0}$, is determined by an exogenously specified updating rule, as discussed in the following section.

In summary, the client’s risk aversion is assumed to evolve according to a random walk, that is reset at

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3 These volatilities can be calibrated to historically observed transitions between risk aversion levels (see Section 7.5).

4 As argued in Schildberg-Horisch [2018], individual risk preferences are moderately stable over time, and exogenous preference shocks are rare events. Moreover, temporary changes in self-control, stress, and emotions typically induce small changes in risk preferences.
each updating time to the nearest risk aversion level in $\Gamma$. That is, if $n > 0$ is an updating time ($\tau_n = n$), then $\gamma_n = \Psi(\tilde{\gamma}_n) \in \Gamma$ is the client’s updated risk aversion level, where $\tilde{\gamma}_n$ is the realization of a random walk starting at the previous updating time $\tau_{n-1}$, from the corresponding risk aversion level $\gamma_{\tau_{n-1}}$, with volatility $\sigma_{\gamma_{n-1}}$. The random walk is then restarted at time $n$ from the updated risk aversion level $\gamma_n$, and with volatility $\sigma_{\gamma_n}$.

Remark 3.1. We have not included a drift in the random walk dynamics (3.1), because the process $(\tilde{\gamma}_n)_{n \geq 0}$ captures idiosyncratic shocks to the client’s risk aversion, and should be devoid of a predictable component.

Empirical research has identified a clear trend of risk preferences over the life cycle: At a young age, individuals are more willing to take risks than adults, but as they grow older, they become less willing to take risks, and their risk preferences gradually converge to those of their older counterparts; see, for instance, Levin et al. [2017]. A deterministic risk aversion component (e.g., related to age) can be modeled separately by decomposing the risk aversion process observed by the robo-advisor as $\gamma_n = \gamma_n^{(1)} + \gamma_n^{(2)}$, where $(\gamma_n^{(1)})_{n \geq 0}$ is a random process capturing idiosyncratic risk preferences, as described in this section, and $(\gamma_n^{(2)})_{n \geq 0}$ is a deterministic process.

3.2 Interaction Between Client and Robo-Advisor

The times of interaction between the client and the robo-advisor are determined by an updating rule, specified at the beginning of the investment period by a deterministic function

$$\phi : \Gamma \mapsto \bar{\mathbb{N}} := \{1, 2, \ldots \} \cup \{\infty\},$$

where $\phi(\gamma)$ is the time elapsing till the next update of risk preferences, starting in risk aversion level $\gamma \in \Gamma$, $\phi(\gamma) = \inf\{n > 0 : \tau_n = n | \gamma_0 = \gamma\}$.

Examples of such updating rules include the full-information rule, where $\phi \equiv 1$ and the risk preferences are updated at all times, and the no-information rule, where $\phi \equiv \infty$ and the risk preferences are never updated. All other updating rules lie between those two extreme cases.

For a given updating rule $\phi$, the corresponding sequence of interaction times between the client and the robo-advisor, $(T_k^{(\phi)})_{k \geq 0}$, can be iteratively computed as $T_0^{(\phi)} = 0$ and

$$T_{k+1}^{(\phi)} = T_k^{(\phi)} + \phi(\gamma_{T_k^{(\phi)}}), \quad k \geq 0. \quad (3.4)$$
Observe that the realized risk aversion level at the \( k \)-th updating time determines the \((k + 1)\)-th updating time, but further updating times are random and depend on future realized risk aversion levels. In Lemma A.1-(c) we show that \( (\gamma_k^{(\phi)})_{k \geq 0} \), the sequence of risk aversion levels observed by the robo-advisor,

\[
\gamma_k^{(\phi)} := \gamma_{T_k^{(\phi)}}, \tag{3.5}
\]

is an irreducible and aperiodic Markov chain on \( \Gamma \), with a time-homogeneous transition matrix \( \Lambda^{(\phi)} \). In particular, for the special case \( \phi \equiv 1 \), i.e., when the client’s risk preferences are updated at all times, the risk aversion level process \( (\gamma_n)_{n \geq 0} \) is itself a time-homogeneous Markov chain with a transition matrix \( \Lambda \) (see Lemma A.1-(d)).

In the sequel, we will devote special attention to the threshold updating rule, that triggers an update of risk preferences as soon as the expected absolute deviation of the client’s risk aversion from the previously communicated value, computed using the random walk risk aversion model discussed earlier, exceeds a threshold. To specify the threshold updating rule, we define for each \( \gamma \in \Gamma \) and \( n \geq 0 \),

\[
\mu_n(\gamma) := \mathbb{E}[|\Psi(\hat{\gamma}_n) - \gamma_0| | \gamma_0 = \gamma, \tau_n = 0], \tag{3.6}
\]

which is the expected absolute change in the client’s risk aversion level, in \( n \) time steps, starting in level \( \gamma \) at time zero, and without any intermediate updates of risk preferences. An explicit expression for \( \mu_n(\gamma) \) is given in Lemma A.1-(b). For a threshold \( b := (b_\gamma)_{\gamma \in \Gamma} \) whose value depends on the risk aversion level,

\[
b_\gamma \in \bar{\mathbb{R}}^+ := [0, \infty) \cup \{\infty\}, \quad \gamma \in \Gamma,
\]

the corresponding threshold updating rule, \( \phi_b \), is then given by

\[
\phi_b(\gamma) := \inf\{n \geq 1 : \mu_n(\gamma) > b_\gamma\}, \quad \gamma \in \Gamma. \tag{3.7}
\]

Finally, notice that by determining the updating times at which the client’s risk preferences are communicated, the updating rule \( \phi \) affects the distribution of the risk aversion level process \( (\gamma_n)_{n \geq 0} \) in (3.2). It thus affects the optimal investment strategy for the portfolio optimization problem described in Section 3.4. Hence, the chosen updating rule determines the regret arising from the robo-advisor not having access to up-to-date information when allocating the client’s wealth (see Section 5.1 for the definition of regret in this setting). The regret is increasing in the time spent in a given level without an update of risk preferences, and while the client may at the outset specify an updating rule with equally spaced updating times, in Section
Figure 1: The figure displays the client/robo-advisor interaction system. It shows a sample path of the client’s risk aversion level process \((\gamma_n)_{n \geq 0}\) that is observable by the robo-advisor. The initial risk aversion level is communicated by the client at the beginning of the investment process. Subsequent communication times are determined by the updating rule \(\phi\) that is also specified at the beginning of the investment process. Between consecutive updating times the risk aversion level is constant, and the robo-advisor manages the client’s portfolio without any further input from the client.

7.1 we describe how the updating rule can be calibrated by the robo-advisor to maintain a given target level of regret across risk aversion levels.

**Remark 3.2.** The set of updating rules can be extended to include those characterized by a sequence of deterministic functions, \((\phi_n)_{n \geq 0}\), such that for each \(\gamma \in \Gamma\),

\[
\phi_n(\gamma) = \inf\{n' > n : \tau_{n'} = n' | \tau_n = n, \gamma_n = \gamma\},
\]

is the time until the next update of risk preferences, following an update at time \(n\) resulting in risk aversion level \(\gamma\). The results in Section 4 continue to hold in this case, but all specific updating rules considered herein are time-homogeneous in the sense that \(\phi_n\) is independent of \(n\). That is, for any \(n \geq 0\) they satisfy

\[
\phi_n(\gamma) = \inf\{n' > n : \tau_{n'} = n' | \tau_n = n, \gamma_n = \gamma\} = \inf\{n > 0 : \tau_n = n | \gamma_0 = \gamma\} = \phi_0(\gamma).
\]

This includes the updating rules \(\phi \equiv 1\) and \(\phi \equiv \infty\), and the thresholding rule \(\phi_b\).
3.3 Market Dynamics

The market consists of a risk-free money market account \((B_n)_{n \geq 0}\) and a risky asset \((S_n)_{n \geq 0}\) with dynamics

\[
B_{n+1} = (1 + r)B_n, \\
S_{n+1} = (1 + \epsilon_{n+1})S_n,
\]

where \(r \geq 0\) is the constant risk-free rate, and \((\epsilon_n)_{n \geq 1}\) is a sequence of i.i.d. random variables, with mean \(\mu > r\) and variance \(\sigma^2 > 0\). They are assumed to be independent of the sequence \((\tilde{Z}_n)_{n \geq 1}\) introduced in (3.1), but defined on the same probability space. We denote by \((X_n)_{n \geq 0}\) the self-financing wealth process of the client. At time \(n \geq 0\), the wealth \(X_n\) is allocated between the risky asset and the money market account, with \(\pi_n\) denoting the amount invested in the risky asset. The wealth dynamics corresponding to a control law \((\pi_n)_{n \geq 0}\) is then given by

\[
X^\pi_{n+1} = (1 + r)X^n + (\epsilon_{n+1} - r)\pi_n =: RX^n_n + Z_{n+1}\pi_n. \tag{3.8}
\]

The random variable \(Z_{n+1}\) denotes the excess return of the risky asset over the risk-free rate, between times \(n\) and \(n+1\), and it has mean \(\bar{\mu} := \mu - r > 0\) and variance \(\sigma^2 > 0\).

On the probability space \((\Omega, \mathcal{F}, P)\), carrying the independent sequences of random variables, \((\tilde{Z}_n)_{n \geq 1}\), that drives the risk aversion process in (3.1), and \((Z_n)_{n \geq 1}\), that drives the risky asset process above, we denote by \((\mathcal{F}_n)_{n \geq 0}\) the smallest filtration such that

\[
X_{(n)} := (X_k)_{0 \leq k \leq n}, \quad \gamma_{(n)} := (\gamma_k)_{0 \leq k \leq n}, \quad \tau_{(n)} := (\tau_k)_{0 \leq k \leq n},
\]

are measurable with respect to \(\mathcal{F}_n\). For any updating rule \(\phi\) we define the probability measure \(P^{(\phi)}\), under which the updating times for the client’s risk preferences are given by (3.4), and for any \(n \geq 0\) we use the shorthand notation

\[
P_{n}^{(\phi)}(\cdot) := P^{(\phi)}(\cdot | \mathcal{F}_n).
\]

3.4 Investment Criterion

The robo-advisor’s objective is to optimally allocate the client’s wealth, accounting for the dynamic nature of the client’s risk preferences described in Section 3.1. To this purpose, we develop an adaptive mean-variance
criterion, namely, a dynamic version of the standard Markowitz [1952] mean-variance problem, that adapts to the client’s changing risk preferences.

Specifically, for a fixed investment horizon \( T \geq 1 \) and a control law \( \pi := (\pi_{n,T})_{0 \leq n < T} \) such that \( \pi_{n,T} \in \mathbb{R} \) for \( 0 \leq n < T \), we consider for each \( n \in \{0, 1, \ldots, T - 1\} \) the mean-variance function

\[
J^{(\phi)}_{n,T}(X(n), \gamma(n), \tau(n), \pi; \theta) := \mathbb{E}_n^{(\phi)} [r_{n,T}^\pi] - \frac{\Delta_n}{2} \times \text{Var}_n^{(\phi)}[r_{n,T}^\pi],
\]

(3.9)

where \( r_{n,T}^\pi \) is the simple return obtained by following the control law \( \pi \) between time \( n \) and the terminal date \( T \),

\[
r_{n,T}^\pi := \frac{X_T^\pi - X_n}{X_n}.
\]

(3.10)

At time \( n \), the objective is to maximize the return on the client’s wealth, which depends on the control law \( \pi \) restricted to the time points \( \{n, n + 1, \ldots, T - 1\} \). In the sequel we will also refer to a control law \( \pi \) as a strategy or allocation, and we consider adapted real-valued control laws of the form

\[
\pi_{n,T} = \pi_{n,T}(X(n), \gamma(n), \tau(n); \theta).
\]

(3.11)

We define the optimal value function corresponding to the problem above as

\[
V^{(\phi)}_{n,T}(X(n), \gamma(n), \tau(n); \theta) := \sup_{\pi} J^{(\phi)}_{n,T}(X(n), \gamma(n), \tau(n), \pi; \theta).
\]

(3.12)

The solution to (3.9) depends on the updating rule \( \phi \), that splits the investment horizon into subperiods via the corresponding (random) updating times. This highlights the dynamic feature of the mean-variance problem. It is also an adaptive or sequential problem in the sense that at each updating time, a new problem arises, depending on the realized risk aversion value, but with the same initially determined client-specific investment horizon.

**Remark 3.3.** Observe the presence of two temporal scales in our framework. The robo-advisor rebalances the portfolio at times \( \{0, 1, \ldots, T - 1\} \), while the risk aversion process \( (\gamma_k^{(\phi)})_{k \geq 0} \), defined in (3.5), is a Markov chain whose transition times occur at a coarser time scale determined by the updating rule \( \phi \).

The coefficient \( \Delta_n > 0 \) in (3.9) quantifies the risk-return trade-off resulting from the interaction between the client and the robo-advisor. If \( \Delta_n \) is independent of \( n \), then we recover the classical mean-variance criterion. In our model, the risk-return trade-off depends on the most recent risk aversion value communicated.
by the client, $\gamma_n$, and the corresponding time of communication, $\tau_n$. It also depends on a parameter $\theta \geq 0$ that quantifies the level of caution the robo-advisor exhibits due to it being uncertain about the client’s true risk aversion. We consider an additive structure for the risk-return trade-off coefficient, i.e.,

$$
\Delta_n := \Delta_n(\gamma_n, \tau_n; \theta) := \gamma_n + \theta \times \delta_n(\gamma_n, \tau_n).
$$

(3.13)

The term $\gamma_n$ is the \textit{client-specific} component of the risk-return trade-off, and it incorporates the most recently communicated risk aversion level of the client. The term $\theta \times \delta_n(\gamma_n, \tau_n)$ is specific to the robo-advisor, i.e., \textit{machine-specific}, and it arises due to the information asymmetry between the client and the robo-advisor; $\delta_n(\gamma, \tau)$ is the conditional standard deviation of the client’s (unknown) risk aversion level at time $n \geq 0$, given that $\gamma \in \Gamma$ is the last communicated risk aversion level at time $\tau \leq n$,

$$
\delta_n(\gamma, \tau) := \sqrt{\text{Var}[\Psi(\gamma_n) | \gamma_n = \gamma, \tau_n = \tau]};
$$

(3.14)

and is computable in closed-form as shown in Lemma A.1-(d). At updating times, $\delta_n(\gamma_n, \tau_n) = 0$, and the machine-specific component vanishes. However, between consecutive updating times, $\delta_n(\gamma_n, \tau_n) > 0$, and $\theta$ can be viewed as the \textit{weight of uncertainty in the robo advisor’s view of the client}. A robo-advisor that is very risk averse towards incomplete knowledge of the client’s risk preferences chooses a high value of $\theta$ (see Section 5.2 for further discussion). This parallels the notion of market price of risk in classical investment theory, but the main difference here is that there is uncertainty about the client’s characteristics rather than the market outcome.

The machine specific component discussed above is related to the concept of trust in Gennaioli and Vishny [2015]. They show that clients without finance expertise are more willing to take on risk with a financial advisor they trust, with trust based on, e.g., personal relationships and persuasive advertising. The client’s baseline risk aversion in a mean-variance utility function is then inflated by a factor representing the level of anxiety suffered by the client from bearing risk with the financial advisor. In our framework, the roles of the client and the financial advisor are reversed. The robo-advisor is less willing to take on risk on behalf of the client if the risk aversion $\gamma_n$ is based on stale information, resulting in the risk-return trade-off coefficient $\Delta_n$ being inflated by the level of distrust in the value $\gamma_n$.

Finally, observe that maximizing the return on the client’s wealth, as in the objective function (3.9), is equivalent to maximizing the client’s risk-adjusted terminal wealth. Namely,

$$
\tilde{j}_{n,T}^{(\phi)}(X_n, \gamma(n), \tau(n), \pi; \theta) := X_n \times (1 + j_{n,T}^{(\phi)}(X_n, \gamma(n), \tau(n), \pi; \theta)) = \mathbb{E}_n^{(\phi)}[X_T] - \frac{1}{2} \frac{\Delta_n}{X_n} \times \text{Var}_n^{(\phi)}[X_T],
$$

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and maximizing \( J_n^{(\phi)} \) at time \( n \) is equivalent to maximizing \( J_n^{(\phi)} \). This also shows that the risk-return trade-off is implicitly decreasing in the current wealth \( X_n \), and the optimal solution presented in Section 4 indeed exhibits decreasing absolute risk aversion - the amount invested in the risky asset is increasing in the client’s current wealth. Furthermore, the optimal solution exhibits a constant relative risk aversion - the optimal fraction of wealth invested in the risky asset is independent of the client’s current wealth.

Example 1. To illustrate the features of the framework introduced in this section, we provide an example with a one year investment horizon and three investment periods, i.e., \( T = 3 \). We consider three risk aversion levels given by \( \Gamma = \{ \bar{\gamma}_1, \bar{\gamma}_2, \bar{\gamma}_3 \} = \{2.5, 3.5, 4.5\} \).\(^5\) The optimal amount of wealth allocated to the risky asset is then a strategy \( \pi^* = (\pi^*_0, \pi^*_1, \pi^*_2) \), specifying the allocations made at times \( n = 0, 1, 2 \), and the potential interaction times between the client and the robo-advisor are \( n = 1, 2 \).

We set the risk aversion volatility to \( \sigma_\gamma = 0.30 \), for each risk aversion level \( \gamma \in \Gamma \). As mentioned in Section 3.3, the client’s risk aversion levels at updating times form a Markov chain, and in the special case of risk preferences being updated at all times, the transition matrix \( \Lambda \) in (A.8), corresponding to the volatility profile specified above, is given by

\[
\Lambda = \begin{pmatrix}
1 - p/2 & p/2 & 0 \\
p/2 & 1 - p & p/2 \\
0 & p/2 & 1 - p/2
\end{pmatrix} = \begin{pmatrix}
0.9 & 0.1 & 0 \\
0.1 & 0.8 & 0.1 \\
0 & 0.1 & 0.9
\end{pmatrix}.
\]

This corresponds to a symmetric random walk with probability \( p = 0.1 \) of jumping to a neighboring level.

The robo-advisor uses the threshold strategy \( \phi_b \), defined in (3.7), to determine when to solicit the client’s risk preferences: At each time, the robo-advisor computes the absolute change in the client’s risk aversion since the last time risk preferences were communicated, and prompts the client for an update if this value exceeds a threshold.

If the initial communication of risk preferences resulted in risk aversion level \( \gamma_0 \in \Gamma \), then the expected change in \( n \geq 0 \) time steps can be approximated as \( \mu_n(\gamma_0) \approx \sqrt{2/\pi} \sigma_\gamma \sqrt{n} \) (see Lemma A.1-(c)), using the approximation \( \Psi(\bar{\gamma}_n) \approx \bar{\gamma}_n \) in (3.6). Given a threshold \( b \geq 0 \) (assumed to be independent of the level \( \gamma_0 \)), we then consider three possibilities:

1. For \( 0 \leq b < \mu_1(\gamma_0) \) there will be an update at times \( n = 1 \) and \( n = 2 \), i.e., the robo-advisor always knows the client’s risk aversion. This corresponds to the full-information updating rule \( \phi_b \equiv 1 \).

\(^5\)The optimal allocations corresponding to \( \gamma = 3.5 \) are close to the classical 60/40 portfolio composition. This is the widely popular passive investing strategy of Jack Bogle, the founder of The Vanguard Group, who is credited with creating the first index fund.
(ii) For $b \geq \mu_2(\gamma_0)$ there will be no further updates, which results in the same optimal strategy as in a model with constant risk aversion. This corresponds to the no-information updating rule $\phi_b \equiv \infty$.

(iii) The third case lies between the first two cases, with $\mu_1(\gamma_0) \leq b < \mu_2(\gamma_0)$, and the client’s risk preferences only updated at time $n = 2$.

The optimal investment strategy associated with the criterion (3.9) depends on which of the three updating schedules for the client’s risk preferences is chosen. In Table 1, we report the optimal proportion of wealth allocated to the risky asset for the two extreme cases, with the risk preferences either updated at both times or never updated.\(^6\) The results show that the optimal proportion at a given time and in a given risk aversion level is not very sensitive to when this risk aversion level was observed (the two numbers within each cell are quite close). However, if the risk aversion level does not reflect the client’s current risk aversion, there is “regret” associated with the asset allocation, and the updating threshold $b$ quantifies the trade-off between having a low regret, which requires frequent updating, and having a low involvement in the investment process, which requires less frequent updating (see Section 5.1 where this example is continued).

<table>
<thead>
<tr>
<th>$\gamma_n$</th>
<th>$\tau_n = n$</th>
<th>$\tau_n = 0$</th>
<th>$\tau_n = n$</th>
<th>$\tau_n = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_1 = 2.5$</td>
<td>0.789</td>
<td>0.785</td>
<td>0.838</td>
<td>0.836</td>
</tr>
<tr>
<td>$\gamma_2 = 3.5$</td>
<td>0.577</td>
<td>0.579</td>
<td>0.607</td>
<td>0.607</td>
</tr>
<tr>
<td>$\gamma_3 = 4.5$</td>
<td>0.456</td>
<td>0.458</td>
<td>0.476</td>
<td>0.477</td>
</tr>
</tbody>
</table>

Table 1: For a one-year investment horizon and three subperiods ($T = 3$), the table reports $\pi_n^*(\gamma_n, \tau_n; \theta)$, the optimal proportion of wealth allocated to the risky asset at time $n \in \{0, 1, 2\}$, with the most recent risk aversion level being $\gamma_n \in \Gamma$, revealed at time $\tau_n \leq n$. The market parameters on an annualized basis are given by $r = 0.04$, $\bar{\mu} = 0.08$, and $\sigma = 0.20$. We set $\theta = 0$.

4 Optimal Portfolio Allocation Under Client/Robo-Advisor Interaction

In this section, we present the solution to the optimization problem (3.9). To the best of our knowledge, the criterion (3.9) has never been studied in the case where the risk-return trade-off coefficient follows a Markov chain. Hence, the solution of this problem presents a contribution in its own right. Herein we also consider

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\(^6\)Section 4 shows that the optimal strategy has this form: At any time, the number of dollars allocated to the risky asset is proportional to current wealth, with the proportion being a function of the most recent risk aversion level, and the most recent updating time (see (4.1)-(4.2)).
adapted control laws of the form (3.11), rather than feedback control laws of the form $\pi_{n,T}(X_{n},\gamma_{n},\tau_{n};\theta)$, as in Björk and Murgoci [2013], and show that the optimal adapted control for (3.9) is indeed of feedback form.

It is well known that even if the risk-return trade-off $\Delta_n$ in (3.9) is constant through time, then the family of optimization problems

$$\{\sup_{\pi} J_{n,T}^{(\phi)}(X_n(\gamma_{n}),\tau_{(n)};\pi;\theta)\}_{0\leq n<T},$$

is time-inconsistent, in the sense that if $\pi^*$ is optimal for $J_{n,T}^{(\phi)}(X_n(\gamma_{n}),\tau_{(n)};\pi;\theta)$, then the restriction of $\pi^*$ to the time points $\{n+1,n+2,\ldots,T-1\}$ may not be optimal for $J_{n+1,T}^{(\phi)}(X_{n+1}^{\pi^*}(\gamma_{n+1}),\tau_{(n+1)};\pi;\theta)$, where $X_{n+1}^{\pi^*}$ is equal to $(X_n,X_{n+1}^{\pi^*})$, and $X_{n+1}^{\pi^*}$ obtained by applying the control $\pi_{n,T}^{\pi^*}$ to $X_n$ at time $n$. We refer to Björk and Murgoci [2013] and references therein for a general treatment of time-inconsistent stochastic control in discrete time.

As standard in this literature (see, for instance, Björk and Murgoci [2013] and Björk et al. [2014]), we view the problem as a multi-player game, with one player at each time $n \in \{0,1,\ldots,T-1\}$ thought of as a future self of the client. Player $n$ then wishes to maximize the objective function $J_{n,T}^{(\phi)}$, but decides only the control $\pi_{n,T}$, while $\{\pi_{n+1,T},\ldots,\pi_{T-1,T}\}$ are chosen by her future selves. The resulting optimal control strategy, $\pi^*$, is the subgame perfect equilibrium of this dynamic game, and can be computed using backward induction. At time $n = T-1$, the equilibrium control $\pi_{T-1,T}^*$ is obtained by maximizing $J_{T-1,T}^{(\phi)}$ over $\pi_{T-1,T}$, which is a standard optimization problem. For $n < T-1$, the equilibrium control $\pi_{n,T}^*$ is then obtained by letting player $n$ choose $\pi_{n,T}$ to maximize $J_{n,T}^{(\phi)}$, given that player $n'$ will use $\pi_{n',T}^{\pi^*}$, for $n' = n+1,n+2,\ldots,T-1$.

The unique structure of our problem, where the client and the robo-advisor interact, implies that the optimal control $\pi^*$ depends also on $\phi$, the updating rule used to solicit the client’s risk preferences, as described in Section 3.2. We first present the solution in the case of a general $\phi$, and then discuss properties of the solution by focusing on the two boundary cases of full information, i.e., $\phi \equiv 1$, and no information, i.e., $\phi \equiv \infty$. We will see that for a general updating rule $\phi$ the solution exhibits properties of the solutions in those two extreme cases, and in Section 5 we will further analyze how the updating rule $\phi$ determines the client’s regret associated with the optimal investment strategy. For $\phi \equiv 1$ the regret is zero, while for any other updating rule the regret is nonzero because the robo-advisor faces uncertainty in the client’s true risk aversion, and it is maximized if $\phi \equiv \infty$. 

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Proposition 4.1. The optimal control for the criterion (3.9) is given by

\[ \pi_{n,T}(X_n, \gamma_n, \tau_n; \theta) = \pi_{n,T}^*(\gamma_n, \tau_n; \theta) \times X_n, \quad 0 \leq n < T, \tag{4.1} \]

where the proportion of wealth allocated to the risky asset is given by

\[ \pi_{n,T}^*(\gamma_n, \tau_n; \theta) = \frac{\bar{\mu}/\sigma^2}{\Delta_n} \times \frac{\mu_n^a(\gamma_n, \tau_n) - R \times \Delta_n \times (\mu_n^b(\gamma_n, \tau_n) - (\mu_n^a(\gamma_n, \tau_n))^2)}{\mu_n^b(\gamma_n, \tau_n) + \left(\frac{\bar{\mu}}{\sigma}\right)^2 \times (\mu_n^b(\gamma_n, \tau_n) - (\mu_n^a(\gamma_n, \tau_n))^2)}. \tag{4.2} \]

Above, for each \( \gamma_n \in \Gamma \), and each \( 0 \leq \tau_n \leq n \), we have defined

\[ \mu_n^a(\gamma_n, \tau_n) := E_n^{(\gamma)}[a_{n+1}(\gamma_{n+1}, \tau_{n+1})], \quad \mu_n^b(\gamma_n, \tau_n) := E_n^{(\gamma)}[b_{n+1}(\gamma_{n+1}, \tau_{n+1})], \tag{4.3} \]

with \( a_n(\gamma_n, \tau_n) \) and \( b_n(\gamma_n, \tau_n) \) satisfying the backward recursions

\[ a_n(\gamma_n, \tau_n) = \mu_n^a(\gamma_n, \tau_n) \times (R + \bar{\mu} \pi_n(\gamma_n, \tau_n)), \quad 0 \leq n < T, \]

\[ b_n(\gamma_n, \tau_n) = \mu_n^b(\gamma_n, \tau_n) \times (\sigma^2(\pi_n(\gamma_n, \tau_n))^2 + (R + \bar{\mu} \pi_n(\gamma_n, \tau_n))^2), \quad 0 \leq n < T, \]

with \( a_T(\gamma_T, \tau_T) = b_T(\gamma_T, \tau_T) = 1 \) for all \( \gamma_T \in \Gamma \) and \( 0 \leq \tau_T < T \). \( \square \)

In Appendix D, we provide pseudocode for the backward recursion used to compute the optimal risky asset allocation in the above proposition. Recall that \( R, \bar{\mu}, \) and \( \sigma \), are the market parameters, introduced in Section 3.4, and that \( \Delta_n \) is the risk-return trade-off coefficient, defined in (3.13). We also stress that the optimal solution \( \pi^* \) depends on the updating rule \( \phi \), but for readability purposes we omit this dependence in the notation. Similarly, the dependence of the \( a_n \)- and \( b_n \)-coefficients on \( \theta, \phi \), and the investment horizon \( T \), is not explicitly highlighted.

From (4.1) we see that the optimal solution is Markovian as the amount of wealth allocated to the risky asset at time \( n \) only depends on the current wealth, \( X_n \), the most recently communicated level of the client’s risk aversion, \( \gamma_n \), and the corresponding updating time, \( \tau_n \). Furthermore, the optimal proportion of wealth allocated to the risky asset is independent of the current wealth. In other words, the objective function (3.9) at time \( n \) is consistent with allocating the wealth at time \( n \) to maximize a particular one-period mean-variance utility function, and therefore consistent with a constant relative risk aversion criterion.

The coefficients \( a_n(\gamma_n, \tau_n) \) and \( b_n(\gamma_n, \tau_n) \) in the proposition are the first and second moment of the future value of one dollar invested optimally between time \( n \) and the terminal horizon \( T \), when the latest interaction between the client and the robo-advisor occurred at time \( \tau_n \) and the communicated risk aversion level was
There are two distinct cases. First, if $\gamma_n \in \{\gamma_n, \tau_n\}$. That is,

$$a_n(\gamma_n, \tau_n) = E_n^{(0)}[1 + r_{n,T}^{*}]$$
$$b_n(\gamma_n, \tau_n) = E_n^{(0)}[(1 + r_{n,T}^{*})^2],$$

where the simple return $r_{n,T}^{*}$ is defined in (3.10). It follows that the optimal value function (3.12) is given by

$$V_{n,T}^{(\phi)}(\gamma_n, \tau_n; \theta) = a_n(\gamma_n, \tau_n) - \frac{\Delta_n}{2} \times (b_n(\gamma_n, \tau_n) - a_n^2(\gamma_n, \tau_n)), \quad 0 \leq n \leq T,$$

(4.4)

which is independent of the current wealth $X_n$.

**Remark 4.2.** Observe that, consistently with intuition, the optimal risky asset allocation at time $n$ is increasing in $\mu_n^{\phi}(\gamma_n, \tau_n)$, the expected future value of one dollar at time $n + 1$, and decreasing in its variance, $\mu_n^{b}(\gamma_n, \tau_n) - (\mu_n^{\phi}(\gamma_n, \tau_n))^2$, where $\mu_n^{\phi}(\gamma_n, \tau_n)$ and $\mu_n^{b}(\gamma_n, \tau_n)$ have been defined in (4.3) and admit the explicit representation given by

$$\mu_n^{\phi}(\gamma_n, \tau_n) = \sum_{\gamma \in \Gamma} p_{\tau_{n+1}}(\gamma; \gamma_n, \tau_n) \times a_{n+1}(\gamma, \tau_{n+1}), \quad \mu_n^{b}(\gamma_n, \tau_n) = \sum_{\gamma \in \Gamma} p_{\tau_{n+1}}(\gamma; \gamma_n, \tau_n) \times b_{n+1}(\gamma, \tau_{n+1}).$$

The transition probabilities $p_{\tau_{n+1}}(\gamma; \gamma_n, \tau_n)$, for each $\gamma \in \Gamma$, are given explicitly in Lemma A.1-(a). There are two distinct cases. First, if $\tau_{n+1} = \tau_n < n + 1$, then the client’s risk-preferences are not solicited at time $n + 1$, and $p_{\tau_{n+1}}(\gamma_n; \gamma_n, \tau_n) = 1$, so $\mu_n^{a}(\gamma_n, \tau_n) = a_{n+1}(\gamma_n, \tau_n)$ and $\mu_n^{b}(\gamma_n, \tau_n) = b_{n+1}(\gamma_n, \tau_n)$.

Second, if $\tau_{n+1} = n + 1$, then the client’s risk preferences are solicited at time $n + 1$, and the probabilities $\{p_{n+1}(\gamma; \gamma_n, \tau_n), \gamma \in \Gamma\}$ link the optimal allocation at time $n$, corresponding to $\gamma_n$ and $\tau_n$, to the optimal allocations at time $n + 1$, corresponding to $\gamma$ and $n + 1$, for each $\gamma \in \Gamma$. In the latter case, the probability that the realized risk aversion at time $n + 1$ is different from $\gamma_n$ is increasing both in the time since the previous update, $(n + 1) - \tau_n$, and in the volatility of the random walk describing the client’s risk aversion, $\sigma_{\gamma_n}$ (see Lemma A.1-(a)).

Next, we present the solution to the optimization problem for the two extreme cases of $\phi = 1$ and $\phi = \infty$. The former implies that $\Delta_n = \gamma_n$, so there is no $\theta$-dependence, and we denote the optimal proportion of wealth allocated to the risky asset by $\pi_{n,T}^{*}(\gamma_n, n)$. The latter case implies that $\Delta_n = \gamma_n + \theta \times \delta_n(\gamma_n, 0)$, and we use the notation $\pi_{n,T}^{*}(\gamma_0; \theta)$. In this case, we also use the superscript $\infty$ in the $a_n$- and $b_n$-coefficients to distinguish them from the case when $\phi = 1$.

**Corollary 4.3.**
(a) Assume that $\phi \equiv 1$. Then the optimal proportion of wealth allocated to the risky asset at time $n$, if the risk-aversion level is $\gamma_n \in \Gamma$, is given by

$$
\pi^*_n(\gamma_n, n) = \frac{\bar{\mu}/\sigma^2}{\gamma_n} \times \frac{E_n^{(1)}[a_{n+1}(\gamma_{n+1})] - R \times \gamma_n \times (E_n^{(1)}[b_{n+1}(\gamma_{n+1})] - (E_n^{(1)}[a_{n+1}(\gamma_{n+1})])^2)}{E_n^{(1)}[b_{n+1}(\gamma_{n+1})] + \left(\frac{\bar{\mu}}{\sigma}\right)^2 \times (E_n^{(1)}[b_{n+1}(\gamma_{n+1})] - (E_n^{(1)}[a_{n+1}(\gamma_{n+1})])^2)},
$$

for $0 \leq n < T$, where for each $\gamma_n \in \Gamma$,

$$
a_n(\gamma_n) = E_n^{(1)}[a_{n+1}(\gamma_{n+1})] \times (R + \bar{\mu}\pi^*_{n}(\gamma_n, n)), \quad 0 \leq n < T,
$$

$$
b_n(\gamma_n) = E_n^{(1)}[b_{n+1}(\gamma_{n+1})] \times (\sigma^2(\pi^*_{n}(\gamma_n, n))^2 + (R + \bar{\mu}\pi^*_{n}(\gamma_n, n))^2), \quad 0 \leq n < T,
$$

with $a_T(\gamma_T) = b_T(\gamma_T) = 1$ for all $\gamma_T \in \Gamma$.

(b) Assume that $\phi \equiv \infty$. Then the optimal proportion of wealth allocated to the risky asset at time $n$, when the initial risk aversion level is $\gamma_0 \in \Gamma$, is given by

$$
\pi^*_n(\gamma_0; \theta) = \frac{\bar{\mu}/\sigma^2}{\Delta_n} \times \frac{a_n(\gamma_0) - R \times \Delta_n \times (b_n(\gamma_0) - (a_n(\gamma_0))^2)}{b_n(\gamma_0) + \left(\frac{\bar{\mu}}{\sigma}\right)^2 \times (b_n(\gamma_0) - (a_n(\gamma_0))^2)}, \quad 0 \leq n < T,
$$

where for each $\gamma_0 \in \Gamma$,

$$
a_n(\gamma_0) = a_n^{(\infty)}(\gamma_0) \times (R + \bar{\mu}\pi^*_{n}(\gamma_0)), \quad 0 \leq n < T,
$$

$$
b_n^{(\infty)}(\gamma_0) = b_n^{(\infty)}(\gamma_0) \times (\sigma^2(\pi^*_{n}(\gamma_0))^2 + (R + \bar{\mu}\pi^*_{n}(\gamma_0))^2), \quad 0 \leq n < T,
$$

with $a_T^{(\infty)}(\gamma_0) = b_T^{(\infty)}(\gamma_0) = 1$ for all $\gamma_0 \in \Gamma$.

Recall from Section 3.2 that if $\phi \equiv 1$, then the risk aversion level process $(\gamma_n)_{n \geq 0}$ is a Markov chain, with transition probability matrix $\Lambda$ defined in (A.8). The expected values in (4.6) thus become

$$
E_n^{(1)}[a_{n+1}(\gamma_{n+1})] = \sum_{\gamma \in \Gamma} \Lambda_{\gamma_n, \gamma} \times a_{n+1}(\gamma), \quad E_n^{(1)}[b_{n+1}(\gamma_{n+1})] = \sum_{\gamma \in \Gamma} \Lambda_{\gamma_n, \gamma} \times b_{n+1}(\gamma).
$$

In the case $\phi \equiv \infty$ all the mass of the distribution at time $n + 1$ is put on the risk aversion level $\gamma_n$, i.e.,

$$
E_n^{(\infty)}[a_{n+1}(\gamma_{n+1})] = a_{n+1}^{(\infty)}(\gamma_n), \quad E_n^{(\infty)}[b_{n+1}(\gamma_{n+1})] = b_{n+1}^{(\infty)}(\gamma_n),
$$

with $\gamma_n = \gamma_0$ (see (4.7)).

This holds for all times $n$, and it follows that if $\phi \equiv \infty$, then the sequence of optimal allocations, $(\pi^*_n(\gamma_0; \theta))_{0 \leq n < T}$, becomes “open-loop”, as the robo-advisor never observes the client’s risk aversion level.
Figure 2: For a fixed investment horizon $T$, the figure shows how $\pi_{n,T}(\gamma_n, \tau_n; \theta)$, the optimal proportion of wealth allocated to the risky asset, is determined in a piecewise manner. At the initial updating time, $T_0(\phi) = 0$, the optimal allocation until the updating time $T_1(\phi)$ is determined, so $\{\pi_{n,T}(\gamma_0, 0; \theta), 0 \leq n < T_1(\phi)\}$ is a deterministic sequence, parameterized by the initial risk aversion level $\gamma_0$, in addition to the parameter $\theta$ and the terminal date $T$. In general, given the $k$-th updating time $T_k(\phi)$ and the realized risk aversion level $\gamma_{T_k(\phi)}$, the optimal allocation $\{\pi_{n,T}(\gamma_{T_k(\phi)}, T_k(\phi); \theta), T_k(\phi) \leq n < T_{k+1}(\phi)\}$ is a deterministic sequence, parameterized by $T_k(\phi)$ and $\gamma_{T_k(\phi)}$.

Aversion after time zero. As a result, the optimal strategy throughout the investment horizon is determined with probability one at time zero, as opposed to depending on future risk aversion levels communicated by the client, which are random. The optimal value function (4.4) corresponding to $\phi \equiv 1$ and $\phi \equiv \infty$ becomes

$$V_{n,T}^{(1)}(\gamma_n) = a_n(\gamma_n) - 1 - \frac{a_n^2(\gamma_n)}{2} \times (b_n(\gamma_n) - b_n^2(\gamma_n)),$$

$$V_{n,T}^{(\infty)}(\gamma_0; \theta) = a_n^{(\infty)}(\gamma_0) - 1 - \frac{a_n^{(\infty)}(\gamma_0)}{2} \times (b_n^{(\infty)}(\gamma_0) - (a_n^{(\infty)}(\gamma_0))^2), \tag{4.8}$$

respectively, for $0 \leq n \leq T$.

From Remark 4.2, we can now see how the optimal solution for a general updating rule $\phi$ exhibits properties of the optimal solutions in the two extreme cases analyzed above: Between consecutive updating times, when there is no feedback received from the client, the optimal solution is open-loop, as in the case $\phi \equiv \infty$, while at updating times the optimal solution depends on the realized risk aversion level, as in the case $\phi \equiv 1$ (see Figure 2). Since the client’s risk aversion evolves like a random walk between consecutive updating times (see (3.1)), the variance of the realized risk aversion at an updating time is increasing both in the time since the previous update, and in the volatility of the random walk process describing the client’s risk aversion since the last update. This variance is minimized when $\phi \equiv 1$, i.e., when risk preferences are solicited at all times, while for any other updating rule the variance accumulates over time, and is therefore greater. The updating rule $\phi$ thus determines the level of uncertainty in the client’s risk aversion at updating times, and in Section 5.1 we quantify the effect of this uncertainty on the optimal investment strategy using a measure of regret. We show that for $\phi \equiv 1$ the regret is zero, while for any other updating rule the regret is nonzero, and it is maximized if $\phi \equiv \infty$.

We next provide a financial interpretation of the optimal portfolio allocation (4.2). Again, we first consider the cases $\phi \equiv \infty$ and $\phi \equiv 1$ before the most general case. We assume zero interest rates, i.e., $R = 1$, etc.
with the general case derived in Appendix B, and we set the machine-specific component of the risk-return trade-off to zero, i.e., θ = 0. The effect of a nonzero θ on the optimal investment strategy is studied in Section 5.2. We denote by Z a random variable with the same distribution as \((Z_n)_{n \geq 1}\) in (3.8), but independent of it. Hence, \(Z' := Z/\sigma\) has mean \(\bar{\mu}/\sigma\) and unit variance.

First, if \(\phi \equiv \infty\), the optimal allocation at time \(n\), given the initial risk aversion \(\gamma_0 \in \Gamma\), can be written as

\[
\pi^*_{n,T}(\gamma_0; 0) = \pi^*_{T-1,T}(\gamma_0; 0) \times \frac{1 + V^{(\infty)}_{n+1,T}(\gamma_0; 0) - \frac{2\gamma_0}{2} \times Var^{(\infty)}_{n+1}[1 + r^{\pi^*}_{n+1,T}]}{Var^{(\infty)}[Z' \times (1 + r^{\pi^*}_{n+1,T})]}, \tag{4.9}
\]

This shows that the risky asset allocation at time \(n\) is equal to the final period allocation, \(\pi^*_{T-1,T}(\gamma_0; 0)\), multiplied by the future value of one dollar received at time \(n + 1\), and invested optimally between time \(n + 1\) and the terminal date \(T\). We then subtract a term that quantifies the uncertainty in this future value, and scale everything by a factor that accounts for the current market scenario, via the random variable \(Z'\). This scaling factor captures the uncertainty in the investment return between time \(n\) and time \(n + 1\), and

\[Var^{(\infty)}_{n+1}[Z' \times (1 + r^{\pi^*}_{n+1,T})] \geq Var^{(\infty)}_{n+1}[1 + r^{\pi^*}_{n+1,T}].\]

This interpretation can be readily extended to the case \(\phi \equiv 1\). For \(\gamma_n \in \Gamma\), we note that \(1 - \Lambda_{\gamma_n, \gamma_n}\) is the probability that the risk aversion level at time \(n + 1\) will be different from \(\gamma_n\), and write

\[
\pi^*_{n,T}(\gamma_n; n) = \pi^*_{T-1,T}(\gamma_n; T - 1) \times \frac{1 + V^{(1)}_{n+1,T}(\gamma_n) - \frac{2\gamma_n}{2} \times Var^{(\infty)}_{n+1}[1 + r^{\pi^*}_{n+1,T} | \gamma_n = \gamma_n]}{Var^{(1)}_{n+1}[Z' \times (1 + r^{\pi^*}_{n+1,T}) | \gamma_n = \gamma_n]} + O(1 - \Lambda_{\gamma_n, \gamma_n}). \tag{4.10}
\]

In the above expression, the utility at time \(n + 1\) is evaluated at the current risk aversion level, \(\gamma_n\). The interpretation in (4.9) extends up to a \(O(\cdot)\) term which captures the probability that the risk aversion at time \(n + 1\) will be different from \(\gamma_n\). This probability is larger in risk aversion level where the transition probability \(1 - \Lambda_{\gamma_n, \gamma_n}\) is high, i.e., if the uncertainty about the risk aversion level at time \(n + 1\) is high. In practice, risk preferences are persistent and stable over time (Schildberg-Horisch [2018]), so the majority of the probability mass at time \(n + 1\) concentrates on the time \(n\) risk aversion level, \(\gamma_n\).

Finally, for the case of a general updating rule \(\phi\), the optimal allocation (4.2) may be written as

\[
\pi^*_{n,T}(\gamma_n; \tau_n; 0) = \pi^*_{T-1,T}(\gamma_n, \tau_n; 0) \times \frac{1 + V^{(\phi)}_{n+1,T}(\gamma_n, \tau_n; 0) - \frac{2\gamma_n}{2} \times Var^{(\phi)}_{n+1}[1 + r^{\pi^*}_{n+1,T} | \gamma_n = \gamma_n]}{Var^{(\phi)}_{n+1}[Z' \times (1 + r^{\pi^*}_{n+1,T}) | \gamma_n = \gamma_n]} + O(1 - p_{\tau_{n+1}}(\gamma_n; \gamma_n, \tau_n)), \tag{4.11}
\]
where the error term again comes from the fact that there is a nonzero probability that the risk aversion at time \( n + 1 \) will be different from \( \gamma_n \). As as in Remark 4.2, two cases arise. First, between updating times, i.e., if \( \tau_{n+1} = \tau_n < n + 1 \), the error term vanishes as in (4.9). Second, at updating times, i.e., if \( \tau_{n+1} = n + 1 \), the error term is nonzero as in (4.10). This further shows how the solution for a general updating rule \( \phi \) lies between the solutions in the two extreme cases of \( \phi \equiv \infty \) and \( \phi \equiv 1 \).

5 Performance of the Client/Robo-Advisor System

In this section we study the performance of the interaction system formed by the client and the robo-advisor, by considering the investment implications of the information asymmetry between the client and the robo-advisor. The form of the risk-return coefficient in the adaptive mean-variance criterion (see (3.13)) highlights the presence of two effects. First, the robo-advisor may determine the allocation at time \( n \) based on stale information, as the risk aversion \( \gamma_n \) is revealed at time \( \tau_n \leq n \). We quantify this client-specific effect in Section 5.1 using a measure of regret. We also quantify the extent to which the robo-advisor underestimates the true regret by using a discrete set of risk aversion levels rather than a continuum of them to categorize the client.\(^8\) Second, if \( \theta > 0 \), the robo-advisor is averse with respect to uncertainty in the client’s risk aversion, and thus picks a portfolio with a lower market risk. In Section 5.2 we quantify this machine-specific effect on the optimal portfolio allocation.

5.1 Client-Specific Effect: Portfolio Regret

The robo-advisor solves the investment problem in Section 4 optimally, given the form of the risk-return coefficient, but its investment decisions are subject to uncertainty in the client’s risk aversion. This uncertainty vanishes at the times when the client’s risk preferences are communicated, but at each time within an updating interval, defined by two consecutive communication times, the regret is a measure of the difference between the robo-advisor’s asset allocation and a benchmark allocation that corresponds to immediately updating the client’s risk preferences.

More specifically, consider a fixed investment horizon \( T \), and an updating rule \( \phi \). The regret at time \( n \in \{0, 1, \ldots, T - 1\} \), given that the previous communication of risk preferences took place at time \( \tau \leq n \)

\(^8\)While it is theoretically possible to use a continuum of risk aversion levels, it would not be possible to calibrate the model for such a fine granularity.
and resulted in risk aversion level \( \gamma \in \Gamma \), is then defined as

\[
\tilde{R}_{n,T}(\gamma, \tau) := \mathbb{E}_n^\phi \left[ \left| \frac{\pi_{n,T}^*(\gamma_n, \tau_n; 0)}{\pi_{n,T}(\xi_n, n; 0)} \right| \bigg| \gamma_n = \gamma, \tau_n = \tau \right].
\]

This is the expected relative difference between the robo-advisor’s strategy \( \pi_{n,T}^*(\gamma_n, \tau_n; 0) \), i.e., the optimal allocation without updating of risk preferences at time \( n \) (so \( \tau_n = \tau \) and \( \gamma_n = \gamma \)), and \( \pi_{n,T}(\xi_n, n; 0) \), i.e., the optimal allocation with updating of risk preferences at time \( n \) (so \( \tau_n = n \) and \( \gamma_n = \Psi(\xi_n) \)). In particular, if \( n \) is an updating time, so \( \tau = n \), then \( \tilde{R}_{n,T}^\phi(\gamma, \tau) = 0 \). Also observe that we set \( \theta = 0 \) to isolate the client-specific component of risk aversion uncertainty, as discussed at the beginning of Section 5. We then define a worst-case measure of regret as

\[
\tilde{R}_{T}^\phi(\gamma) := \sup_{0 \leq \tau \leq T-1} \sup_{\tau \leq n < n_{\tau}} \tilde{R}_{n,T}^\phi(\gamma, \tau),
\]

where \( n_{\tau} := (\tau + \phi(\gamma)) \wedge T \). For a fixed value of \( \tau \), the quantity \( \sup_{\tau \leq n < n_{\tau}} \tilde{R}_{n,T}^\phi(\gamma, \tau) \) is the worst-case regret when starting in risk aversion level \( \gamma \) at time \( \tau \), and following the optimal investment strategy, under the updating rule \( \phi \), until the risk preferences are updated again, or the investment horizon is reached, whichever comes first. Hence, (5.1) is a worst-case measure of regret both with respect to the starting point of the updating interval, \( \tau \), and with respect to the time point within the updating interval at which the regret is evaluated, \( n \).

In Appendix C, we show that

\[
\tilde{R}_{T}^\phi(\gamma) \leq \sup_{\tau \leq n < T} \tilde{R}_{n,T}^\phi(\gamma, \tau) \bigg|_{\tau = (T - \phi(\gamma)) \vee 0} + O(1 - p_\phi(\gamma))_{\gamma}, \quad (5.2)
\]

where we have used that \( n_{\tau} = T \) for \( \tau = (T - \phi(\gamma)) \vee 0 \). This means that the worst-case regret occurs for an updating interval whose right endpoint coincides with the investment horizon \( T \). The error term comes from the fact that the regret is local, i.e., corresponds to an updating interval bounded by two consecutive updating times, but at each time within this interval the optimal allocation depends on all future risk aversion levels communicated by the client, which are random. We also show that

\[
\sup_{\tau \leq n < T} \tilde{R}_{n,T}^\phi(\gamma, \tau) \bigg|_{\tau = (T - \phi(\gamma)) \vee 0} = \tilde{R}_{T-1,T}^\phi(\gamma, \tau) \bigg|_{\tau = (T - \phi(\gamma)) \vee 0}, \quad (5.3)
\]

which states that within the worst-case updating interval, the regret is maximized at the very last time point; this is also the point within the updating interval where the variance of the client’s risk aversion is the largest, due to the random walk assumption (3.1). Both of the above properties can be visualized in Figure

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Figure 3: The figure shows the regret $\tilde{R}_{n,T}^{(\phi)}(\gamma, \tau)$ for investment horizon $T = 24$ months and updating rule $\phi \equiv 3$ (months). For $n = 0, 1, \ldots, T - 1$ we consider $\gamma \in \{2.5, 3.5, 4.5\}$ and $\tau = \lfloor n/3 \rfloor \times 3$. The worst-case upper bound $R_T^{(\phi)}(\gamma)$ in (5.4) is equal to $\tilde{R}_{n,T}^{(\phi)}(\gamma, \tau)$ evaluated at $n = T - 1$. The set of risk aversion levels is $\Gamma = \{2, 2.5, 3, 3.5, 4, 4.5, 5\}$, and the risk aversion volatility is $\sigma_\gamma = 0.152$, for all $\gamma \in \Gamma$, which corresponds to a 10% probability of leaving the current risk aversion level in a single step (since $\sqrt{4 \times 0.152} \approx 0.30$, this is comparable to the volatility in Example 1, where the step size was four months). The market parameters on an annualized basis are $r = 0.04$, $\bar{\mu} = 0.08$, and $\sigma = 0.20$.

3, and combining them gives

$$\tilde{R}_T^{(\phi)}(\gamma) \leq \tilde{R}_{T-1,T}^{(\phi)}(\gamma, \tau) \bigg|_{\tau = (T-\phi(\gamma))\vee 0} + O(1 - p_{\phi(\gamma)}(\gamma; \gamma)) := R_T^{(\phi)}(\gamma) + O(1 - p_{\phi(\gamma)}(\gamma; \gamma)), \quad (5.4)$$

with $R_T^{(\phi)}(\gamma)$ given explicitly in Proposition 5.2. In words, at any time point within an updating interval starting in risk aversion level $\gamma$, the quantity $R_T^{(\phi)}(\gamma)$ is an upper bound for the expected relative change in asset allocation that would be observed if there were an update of the client’s risk preferences. Also, observe that $R_T^{(\phi)}(\gamma)$ only depends on the updating rule $\phi$ via $\phi(\gamma)$, i.e., the time spent in level $\gamma$ before an update of risk preferences. That is, $R_T^{(\phi)}(\gamma) = R_T^{(\phi')}(\gamma)$ for any updating rule $\phi'$ such that $\phi'(\gamma) = \phi(\gamma)$.

The regret $R_T^{(\phi)}(\gamma)$ is a measure of the allocation error suffered by a client in risk aversion level $\gamma \in \Gamma$. We also define a steady-state measure of regret,

$$\bar{R}_T^{(\phi)} := \mathbb{E}[R_T^{(\phi)}(\gamma) | \gamma \sim \lambda] = \sum_{\gamma \in \Gamma} \lambda(\gamma) \times R_T^{(\phi)}(\gamma), \quad (5.5)$$

which is the regret when starting in a risk aversion level drawn from the stationary distribution of the risk aversion level process $(\gamma_n)_{n \geq 0}$, denoted by $\lambda$, and guaranteed to be unique by Lemma A.1-(d).

**Remark 5.1.** The client base of a robo-advisor can be used to construct the stationary distribution $\lambda$,
where \( \lambda(\gamma) \) represents the proportion of clients belonging to risk aversion level \( \gamma \). For a large pool of clients, the quantity (5.5) can then be interpreted as the cross-sectional average regret suffered by clients, under the updating rule \( \phi \). Alternatively, for \( T \to \infty \) it can be interpreted as the average regret through time for a single client with a long investment horizon (e.g., a young client planning for retirement).

The following proposition provides a formula for the regret measures defined in (5.1) and (5.4). Recall from (3.6) that \( \mu_n(\gamma) \) is the expected change in the client’s risk aversion, in \( n \geq 0 \) time steps, starting in level \( \gamma \in \Gamma \) at time zero, and with no intermediate updates of risk preferences.

**Proposition 5.2.** Let \( \phi : \Gamma \to \mathbb{N} \) be an updating rule. For \( \gamma \in \Gamma \), the regret measure (5.1) satisfies

\[
\tilde{R}_T^{(\phi)}(\gamma) \leq R_T^{(\phi)}(\gamma) + O(1 - p_\phi(\gamma; \gamma)),
\]

where the upper bound is given by

\[
R_T^{(\phi)}(\gamma) = \frac{\mu(\phi(\gamma) \wedge T) - 1(\gamma)}{\gamma}.
\]

Furthermore, \( R_T^{(\phi)}(\gamma) \) is increasing in \( \phi(\gamma) \) and \( \sigma_\gamma \), and

\[
\lim_{\phi(\gamma) \wedge T \to \infty} R_T^{(\phi)}(\gamma) = \frac{\tilde{\gamma}_K - \tilde{\gamma}_1}{2\gamma}, \quad \lim_{\sigma_\gamma \to \infty} R_T^{(\phi)}(\gamma) = \frac{\tilde{\gamma}_K - \tilde{\gamma}_1}{2\gamma}.
\]

In the sequel we refer to the upper bound \( R_T^{(\phi)}(\gamma) \) as the regret of the robo-advisor. The proposition shows that the regret is increasing in \( \phi(\gamma) \), the time spent in risk aversion level \( \gamma \) before an update of risk preferences, and equal to zero if and only if \( \phi(\gamma) = 1 \). The regret is also increasing in the risk aversion volatility \( \sigma_\gamma \). In other words, the longer the time before the client’s risk preferences are communicated, the higher the client’s regret, and this effect is more pronounced in levels where the risk aversion is more volatile. Clearly, for any investment horizon the regret is maximized if the no-information updating rule \( \phi \equiv \infty \) is used.

Importantly, the regret measure \( R_T^{(\phi)} \) depends on the specific risk aversion levels in the discrete set \( \Gamma \). In particular, for a fixed \( \gamma \in \Gamma \) and volatility \( \sigma_\gamma \), the regret \( R_T^{(\phi)}(\gamma) \) gets larger if additional risk aversion levels are included in this set. To quantify the effect of this discretization on regret, recall from (3.1)-(3.2) that the client’s risk aversion evolves like a random walk that, at updating times, is reset to the nearest risk
Figure 4: The blue and red curves show the regret measures $R_T^{(φ)}(γ)$ and $R_{c,T}^{(φ)}(γ)$ in (5.6). The market and risk aversion parameters are the same as in Figure 3.

aversion level in $Γ$. For each $γ ∈ Γ$, we then have by Lemma A.1-(c),

$$μ_n(γ) = E[|Ψ(˘γ_n) − γ_0| | γ_0 = γ, τ_n = 0] ≤ E[|˘γ_n − γ_0| | γ_0 = γ, τ_n = 0] =: μ_c(γ),$$

where the difference between $μ_c(γ)$ and $μ_n(γ)$ is that the former is based on a continuous range of risk aversion levels, as $˘γ_n ∈ R$, while the latter is based on the discrete set of risk aversion levels employed by the robo-advisor, as $Ψ(˘γ_n) ∈ Γ$. From this it follows that

$$R_T^{(φ)}(γ) = \frac{μ(φ(γ)∧T)−1(γ)}{γ} ≤ \frac{μ_c(φ(γ)∧T)−1(γ)}{γ} := R_{T,c}^{(φ)}(γ),$$

(5.6)

where the regret $R_T^{(φ)}(γ)$ is based on a continuous range of risk aversion levels, and given in closed form in Lemma A.1-(c). In Figure 4, the two regret measures are compared, with $R_{T,c}^{(φ)}$ being larger than $R_T^{(φ)}$, as predicted by the inequality above. This is because even small changes in the underlying random walk model contribute to the regret $R_{T,c}^{(φ)}$, while these are truncated to zero in the discretization used to compute $R_T^{(φ)}$.

The regret measure $R_{T,c}^{(φ)}$ has the desirable property that it is not affected by the addition or removal of values from the set $Γ$. For $γ ∈ Γ$, we can then define

$$R_{T,d}^{(φ)}(γ) := R_{T,c}^{(φ)}(γ) − R_T^{(φ)}(γ) ≥ 0,$$

which is the additional regret incurred from using a finite number of risk aversion levels, as opposed to tailoring the risk aversion level to the personal profile of each client. This is a measure of how much the robo-advisor underestimates the regret by using the discrete set $Γ$, and $R_{T,d}^{(φ)}(γ)$ is strictly positive unless $φ(γ) = 1$, i.e., if the client’s risk aversion is immediately updated, in which case both regret measures are
zero. Furthermore, $R_{T,c}^{(\phi)}(\gamma)$ decreases as the grid of risk aversion levels in $\Gamma$ becomes more dense, as this increases the robo-advisor’s regret measure $R_{T}^{(\phi)}(\gamma)$, which in the limit converges to $R_{T,c}^{(\phi)}(\gamma)$.

**Example 1** (Continued). We continue Example 1 from Section 3.4 to discuss the regret associated with the robo-advisor’s optimal investment strategy and its dependence on the uncertainty in the client’s risk aversion, determined by the updating rule for the client’s risk preferences.

First, without any updating of the client’s risk preferences, then the entire strategy profile is determined at time zero by the client’s initial risk aversion level. That is, if $\tau_n = 0$, and thus $\gamma_n = \gamma_0$, for $n = 1, 2$, then the sequence $(\pi_0^*(\gamma_0,0;\theta), \pi_1^*(\gamma_0,0;\theta), \pi_2^*(\gamma_0,0;\theta))$, given in Table 1, is the optimal proportion of wealth allocated to the risky asset. Observe that this is also the optimal strategy of a robo-advisor that does not employ a stochastic model for the client’s risk aversion but rather assumes it to be constant throughout the investment horizon, and equal to the result of a one time interaction at the beginning of the investment horizon.

However, with the client’s risk aversion being stochastic, regret accumulates along the way by following this strategy, with the regret at time $n$ being defined in terms of the relative difference between $\pi_n^*(\gamma_0,0;\theta)$ and a benchmark allocation that corresponds to an immediate update of risk preferences. For instance, if $\gamma_0 = \bar{\gamma}_2$, then at times $n = 1, 2$ the probability of being in one of the other two levels is approximately $np/2$, based on a first order approximation of the transition probabilities corresponding to $\Lambda$. Hence, using the worst-case measure of regret in (5.1), the regret corresponding to the no-information rule $\phi \equiv \infty$ becomes

$$R_{T}^{(\phi)}(\bar{\gamma}_2) \approx \left[ \frac{\pi_2^*(\gamma_2,0;\theta) - \pi_2^*(\bar{\gamma}_1,0;\theta)}{\pi_2^*(\gamma_1,0;\theta)} \right] \times p = 0.114.$$  

At the other end of the spectrum, if the risk preferences are updated at all times, i.e., $\phi \equiv 1$, then the sequence of optimal allocations is given by $(\pi_0^*(\gamma_0,0;\theta), \pi_1^*(\gamma_1,1;\theta), \pi_2^*(\gamma_2,2;\theta))$, where $\gamma_n$ is the realized risk aversion value at time $n \in \{0,1,2\}$. In this case, the regret is zero, as the robo-advisor faces no risk of misclassifying the client in terms of risk aversion.

Finally, for the intermediate case where the risk preferences are updated only at time $n = 2$, i.e., $\phi \equiv 2$, the regret becomes

$$R_{T}^{(\phi)}(\gamma_2) \approx \left[ \frac{\pi_1^*(\gamma_2,0;\theta) - \pi_1^*(\gamma_1,0;\theta)}{\pi_1^*(\gamma_1,0;\theta)} \right] \times \frac{p}{2} = 0.055,$$

which is about a half of the regret in the case of zero updates.
5.2 Machine-Specific Effect: Portfolio Tilting

In this section we study how the robo-advisor’s aversion to uncertainty in the client’s risk aversion impacts the optimal portfolio allocation. Recall that at time $n$ the risk-return trade-off coefficient is given by

$$\Delta_n = \gamma_n + \theta \times \delta_n(\gamma_n, \tau_n),$$

where $\delta_n(\gamma_n, \tau_n)$ is the conditional standard deviation of the client’s (unknown) risk aversion, with the most recent risk aversion level $\gamma_n$ having been communicated at time $\tau_n$ (see (3.14)). If $n$ is an updating time, then $\delta_n(\gamma_n, \tau_n) = 0$, and $\Delta_n = \gamma_n$. However, between consecutive updating times we have $\delta_n(\gamma_n, \tau_n) > 0$, and thus $\Delta_n > \gamma_n$. The robo-advisor’s optimal portfolio is therefore tilted towards a less risky composition, relative to the portfolio corresponding to $\Delta_n = \gamma_n$. To quantify the magnitude of this change, we define

$$\tilde{R}_n^{(\phi)}(\gamma, \tau; \theta) := \left| \pi_{n,T}^*(\gamma_n, \tau_n; \theta) - \pi_{n,T}^*(\gamma_n, \tau_n; 0) \right|_{\gamma_n = \gamma, \tau_n = \tau},$$

which is the relative change in the optimal allocation at time $n \in \{0, 1, \ldots, T-1\}$, resulting from using a nonzero value of $\theta$, given that the previous risk aversion level $\gamma \in \Gamma$ was communicated at time $\tau \leq n$. In Appendix C we use similar arguments as in Section 5.1 to derive the following result.

**Proposition 5.3.** Let $\phi : \Gamma \mapsto \bar{\mathbb{N}}$ be an updating rule. Then,

$$\tilde{R}_n^{(\phi)}(\gamma, \tau; \theta) \leq R_n(\gamma, \tau; \theta) + O(\theta \times (1 - p_{\phi(\gamma)}(\gamma; \gamma))).$$

(5.7)

where the upper bound is given by

$$R_n(\gamma, \tau; \theta) = \frac{\theta \times \delta_n(\gamma, \tau)}{\gamma + \theta \times \delta_n(\gamma, \tau)},$$

(5.8)

and satisfies

$$\lim_{\theta \to 0} R_n(\gamma, \tau; \theta) = 0, \quad \lim_{\theta \to \infty} R_n(\gamma, \tau; \theta) = 1.$$

Observe that $\tilde{R}_n(\gamma, \tau; \theta) = 0$ if $\theta = 0$, and that the upper bound $R_n(\gamma, \tau; \theta)$ does not depend on the updating rule $\phi$. Using the approximation $\Psi(\gamma) \approx \gamma$ for the function $\Psi$ defined in (3.2), which is accurate if the client’s true risk aversion is close to a risk aversion level in the set $\Gamma$, we can write

$$\theta \times \delta_n(\gamma, \tau) \approx \theta \times \sigma_{\gamma} \times \sqrt{n - \tau}.$$  

(5.9)
Figure 5: The left panel shows the optimal proportion of wealth allocated to the risky asset, \( \pi^{*}_{n,T}(\gamma_n, \tau_n; \theta) \), with investment horizon \( T = 12 \) months, in risk aversion level \( \gamma_n = 3.5 \), and with no updating of risk preferences, so \( \tau_n = 0 \) for \( n = 0, 1, \ldots, T - 1 \). We consider \( \theta \in \{0, 0.25, 1\} \) and \( \sigma_\gamma = c \times 0.152 \), where \( c \in \{1/\sqrt{2}, 1, \sqrt{2}\} \), for all \( \gamma \in \Gamma \). The right panel shows the same for \( \sigma_\gamma = 0.152 \) and quarterly updates of risk preferences, i.e., \( \phi \equiv 3 \) (months), so \( \tau_n = \lfloor n/3 \rfloor \times 3 \). The risk aversion set \( \Gamma \) and the market parameters are the same as in Figure 3.

This explicitly shows that the magnitude of the allocation effect depends on \( \theta \), the risk aversion volatility, \( \sigma_\gamma \), and the time since the previous update, \( n - \tau \).

We visualize this effect in Figure 5 for \( \theta \in \{0, 0.25, 1\} \). In the left panel, the blue curve shows the optimal proportion of wealth allocated to the risky asset with no updating of risk preferences and \( \theta = 0 \). The red and green curves show how a larger value of \( \theta \) results in a larger shift in allocation, and for a fixed value of \( \theta \), a higher risk aversion volatility has the same effect. The black curves show the downward shift in allocation corresponding to (5.8), and as predicted by the inequality in (5.7), they provide an upper bound for the change in allocation resulting from \( \theta > 0 \). Finally, note that at the initial time, when the client’s risk aversion level is known, the allocations corresponding to a nonzero value of \( \theta \) are slightly larger than the one corresponding to \( \theta = 0 \), in anticipation of the fact that at future time points the allocation will be tilted towards a less risky portfolio if \( \theta > 0 \). The right panel shows the effect that a nonzero \( \theta \) has on the optimal allocation throughout the investment horizon with quarterly updates of risk preferences.

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\(^9\)In Section 7.4 we show how \( \theta = 0.25 \) and \( \theta = 1 \) correspond to setting \( \Delta_n \) to the right endpoint of a confidence interval for the client’s risk aversion, with significance levels 60% and 85%, respectively. The value \( \theta = 0 \) corresponds to taking the center of any confidence interval, as the client’s risk aversion evolves like a centered random walk.
6 Robo-Advisor or Human-Advisor?

In Section 6.1, we introduce a model of a traditional financial advisor, or human-advisor, and define a corresponding measure of regret. In Section 6.2, we compare the performance of the robo-advisor with that of the human-advisor.

6.1 Regret of Human-Advisor

Our model of the human-advisor is driven by the following observations about the skills and limitations of robo-advisor and human-advisors. On the one hand, robo-advisors have enhanced computational power, are able to process large amounts of historical data on investor behavior, model the evolving path of the client’s risk aversion, and to offer portfolios that are optimal given the client’s risk profile. Human-advisors, instead, have a limited capacity to process complex information, and may not be able to model the stochastic evolution of the client’s risk aversion and the impact it has on the optimal portfolio strategy. On the other hand, robo-advisors may not know the client as well as human-advisors. With a human-advisor there is room for personal interaction and tailored questions, while robo-advisors typically employ general purpose questionnaires to solicit information from the client. These questionnaires are unlikely to provide a complete overview of the client’s financial situation and needs. Robo-advisors also lack other important aspects of client-advisor relationships, such as when it comes to defining financial goals and offering counseling during market downturns (see Accenture [2015]).

We present a stylized model of a human-advisor that embodies some of the characteristics mentioned above. We assume that the human-advisor observes the client’s risk preferences at all times, but may lack the robo-advisor’s sophistication to optimally solve the investment problem. More precisely, the human-advisor is myopic, in the sense that he approaches the $T$-period investment problem by repeatedly solving an $m$-period optimization problem, where $1 \leq m \leq T$. If $m = 1$, then the human-advisor repeatedly solves a one-step optimization problem. As $m$ increases, the human-advisor becomes more sophisticated and if $m = T$, then he is as sophisticated as the robo-advisor.

Following the definition of the client’s regret with the robo-advisor in (5.1), we propose a definition of regret resulting from solving a $T$-period investment problem using subperiods of length $m$. For a fixed investment horizon $T \geq 1$, and $m$ such that $1 \leq m \leq T$, the subperiods used by the human-advisor are given by (see Figure 6)

\[ I_{m_0} := \{m_0 \times m, m_0 \times m + 1, \ldots, m_0 \times m + m\}, \quad m_0 = 0, 1, \ldots, [T/m] - 1, \]
and φ ≡ 1 because the human-advisor knows the client’s risk aversion at all times. In this case, the optimal allocation at time n, in risk aversion level γ_n ∈ Γ, is given by π^*_n,T(γ_n, n) (see Corollary 4.3-(a)). However, at the beginning of subperiod I_{m_0}, the human-advisor solves an m-period investment problem with terminal horizon m_0 × m + m, so the human-advisor’s allocation at time n such that m_0 × m ≤ n < m_0 × m + m is given by π^*_n,m_0×m+m(γ_n, n), which is optimal for an investment horizon of length m, starting at time m_0 × m.

For risk aversion level γ ∈ Γ, we then define the average regret for the investment horizon T as

\[
\overline{R}_{m,T}(γ) := \frac{1}{T} \sum_{n=0}^{T-1} \left| \frac{π^*_n,n/m×m+m(γ, n) - π^*_n,T(γ, n)}{π^*_n,T(γ, n)} \right|,
\]

which is the average relative difference between the human-advisor’s allocation, in level γ, and the optimal allocation, in level γ, throughout the investment horizon. Note that the above can also be written as

\[
\overline{R}_{m,T}(γ) = \frac{m}{T} \sum_{m_0=0}^{\lfloor T/m \rfloor - 1} \overline{R}_{m, m_0, T}(γ),
\]

where \( R_{m, m_0, T}(γ) \) is the average regret for the m_0-th subperiod,

\[
R_{m, m_0, T}(γ) := \frac{1}{m} \sum_{n=m_0×m}^{m_0×m+m-1} \left| \frac{π^*_n,m_0×m+m(γ, n) - π^*_n,T(γ, n)}{π^*_n,T(γ, n)} \right|.
\]

In Lemma A.3-(a) we show that for any m_0 ∈ \{0, 1, \ldots, \lfloor T/m \rfloor - 1\},

\[
R_{m,0,T}(γ) \geq R_{m,m_0,T}(γ) + O(1 - \Lambda_{γ, γ}),
\]

where 1 − \( \Lambda_{γ, γ} \) is the probability of leaving risk aversion level γ_n in a single time step, and we define worst-case regret of the client with the human-advisor as

\[
R_{m,T}(γ) := R_{m,0,T}(γ).
\]

In other words, for a fixed T and m, the worst-case regret with the human-advisor occurs in the first subinterval, when the difference between the actual investment horizon and the human-advisor’s horizon is the largest. This can intuitively be understood as follows: In the m_0-th subperiod, the human-advisor solves an optimization problem with a terminal date m_0 × m + m as opposed to solving a problem with a terminal date T, and the difference between the two is the greatest in the first subinterval, when m_0 = 0.
Figure 6: The figure shows the subperiods of length $m$ used by the human-advisor. In this example the endpoint of the final subperiod coincides with the investment horizon $T$ (i.e., $T/m$ is an integer).

Figure 7: The figure shows the value of each term in the regret measure $R_{m,T}(\gamma)$ in (6.1). The investment horizon is $T = 12$ months and $m = 3$. The market and risk aversion parameters are the same as in Figure 3.

Furthermore, in Lemma A.3-(b) we show that for $m' > m$ such that $m$ divides $m'$,

$$R_{m',T}(\gamma) \leq R_{m,T}(\gamma) + O(1 - \Lambda_{1,\gamma}), \quad R_{m',T}(\gamma) \leq R_{m,T}(\gamma) + O(1 - \Lambda_{\gamma,\gamma})$$

so the larger the human-advisor’s horizon, $m$, the smaller the average and worst-case regret measures. By contrast, we show in Lemma A.3-(c) that if $T = M \times m$, for some $M \geq 1$, and $T' = T + m$, then

$$R_{m,T'}(\gamma) \geq R_{m,T}(\gamma) + O(1 - \Lambda_{\gamma,\gamma}), \quad R_{m,T'}(\gamma) \geq R_{m,T}(\gamma) + O(1 - \Lambda_{\gamma,\gamma})$$

so extending the investment horizon, $T$, has the opposite effect of increasing $m$. These observations can be summarized as follows: As the difference between the human-advisor’s horizon, $m$, and the actual investment horizon, $T$, increases, the regret associated with the human-advisor’s investment decisions gets larger. This can also be visualized in Figure 7, where we display the value of each term in the sum (6.1). This highlights that the largest regret occurs in the first subperiod, and that during the final period there is zero regret, as the human-advisor’s horizon coincides with the actual investment horizon.
6.2 Comparison of Human-Advisor and Robo-Advisor

In this section we compare the client’s regret with the robo-advisor defined in Section 5.1, and the client’s regret with the human-advisor defined in Section 6.1.

For risk aversion level $\gamma \in \Gamma$, the regret with the robo-advisor, $R^{(\phi)}_{T}(\gamma)$, and the human-advisor, $R_{m,T}(\gamma)$ and $\bar{R}_{m,T}(\gamma)$, originate from different sources. For the robo-advisor, regret arises solely due to information asymmetry, i.e., the client’s risk aversion not being observable by the robo-advisor, as the robo-advisor solves the investment problem optimally for any updating rule $\phi$. On the contrary, the human-advisor continuously observes the client’s risk preferences, but incurs regret by making suboptimal investment decisions, due to being myopic when solving the investment problem.

There is another important difference between the two regret measures. For a fixed updating rule $\phi$, the client’s regret with the robo-advisor is independent of the investment horizon $T$ (as long as $T > \phi(\gamma)$), while for a fixed $m$, the client’s regret with the human-advisor is increasing in $T$, up to a $O(\cdot)$ term, as explained in the previous section. It follows that for any $\phi$ and $m$, there exists an investment horizon threshold $T_{\phi,m}(\gamma) \geq m$ such that the regret with the human-advisor is larger than with the robo-advisor if $T \geq T_{\phi,m}(\gamma)$. This break-even investment horizon is increasing in $m$, which reduces the regret with the human-advisor for any value of $T$, and increasing in $\phi(\gamma)$ and $\sigma_{\gamma}$, which increase the robo-advisor’s regret (see Proposition 5.2).

Furthermore, since $R^{(\phi)}_{T}(\gamma)$ is increasing in $\phi(\gamma)$, it follows that for fixed values of $m$ and $T$, one can find a threshold on the updating rule $\phi_{m,T}^{*}(\gamma) \geq 1$ such that the regret with the robo-advisor is lower if $\phi(\gamma) \leq \phi_{m,T}^{*}(\gamma)$, i.e., if the risk preferences are updated frequently enough, but higher if $\phi(\gamma) > \phi_{m,T}^{*}(\gamma)$. In the latter case, the use of stale information is more costly to the robo-advisor than the short-sightedness is to the human-advisor. The break-even value $\phi_{m,T}^{*}(\gamma)$ is decreasing in $m$ (increasing in $T$), because the regret with the human-advisor is decreasing in $m$ (increasing in $T$). It is also decreasing in $\sigma_{\gamma}$, as we recall that a larger $\sigma_{\gamma}$ implies a higher regret with the robo-advisor.

These properties are displayed in Figure 8 which shows an example of the client’s regret profiles with the robo-advisor and the human-advisor. To be conservative in this comparison we use the regret measure $R^{(\phi)}_{T,c}$ for the robo-advisor that is based on a continuous range of risk aversion values (see (5.6)), and for the human-advisor we display both the average and worst-case measures of regret. The first three panels show that as the investment horizon $T$ grows, the client needs less frequent updates with the robo-advisor to match the regret with the human-advisor, for a fixed value of $m$. Equivalently, the human-advisor requires a larger value of $m$ for a fixed time between updates of the client’s risk preferences. For instance, the dotted lines correspond to quarterly updates of risk preferences for the robo-advisor and show the corresponding value
of $m$ for the human-advisor so that the regret of the robo-advisor and the human-advisor are the same. As the investment horizon goes from $T = 12$ to $T = 24$, and to $T = 36$ months, then the value of $m$ goes from 33% (67%), to 63% (83%), and to 70% (89%) of the investment horizon $T$, based on the average (worst-case) regret measure of the human-advisor.

The final panel of Figure 8 shows $\phi_{m,T}^*(\gamma)$, the time between updates needed with the robo-advisor to match the regret of the human-advisor with $m = 12$. For an investment horizon $T$ such that $T \leq m$, the regret of the human-advisor is zero, so the client’s risk preferences need to be communicated at all times. For $T > m$ the robo-advisor can achieve a lower regret than the human-advisor by more frequent updating, and the dotted line shows that for $T \geq 16$ the robo-advisor needs at most quarterly updates to match the regret of the human-advisor.

In our model, the client faces a trade-off between having a low level of involvement in the investment process (i.e., low updating frequency), and having a good investment performance (i.e., low regret). At one end of the spectrum is a client that only cares about investment performance. This client will choose the robo-advisor and push the regret to zero by communicating with the robo-advisor at all times, unless she has access to a human-advisor that is as technologically proficient as the robo-advisor. At the other end of the spectrum is a client that does not care about investment performance, and only cares about delegating investment decisions to the financial advisor, and thus minimizing her own involvement. This client will choose the human-advisor over the robo-advisor. In between those extremes is a client that cares about investment performance in addition to wanting a limited level of involvement. This client specifies the time between consecutive communication times with the robo-advisor, and for a given skill level of the financial advisor, chooses the financial advisor with the lower regret. Our analysis shows that unless the human-advisor is near the level of sophistication of the robo-advisor, or if the client’s risk aversion is very volatile, then a moderate time between updates is sufficient for the client to choose the robo-advisor. For instance, most robo-advisors encourage their client’s to revisit their risk preferences at least quarterly, which in our case is generally sufficient.

This means that clients that want little involvement in the investment process (i.e., don’t want frequent interaction with the robo-advisor) are less likely to choose the robo-advisor, which is consistent with the empirical evidence presented in Rossi and Utkus [2019b] where individuals that put emphasis on delegating financial decisions (i.e., don’t want much involvement) are less likely to use the services of a robo-advisor. On the other hand, clients that care about investment performance (i.e., want low regret) are more likely to choose the robo-advisor, which is also consistent with the empirical results in Rossi and Utkus [2019b].
Figure 8: For investment horizon $T \in \{12, 24, 36\}$ months and $\gamma = 3.5$, the first three panels show the regret measure for the robo-advisor, $R_c^{(\gamma)}(\phi)$, as a function of $\phi(\gamma)$, and the worst-case and average regret measures for the human-advisor, $R_{m,T}(\gamma)$ and $\bar{R}_{m,T}(\gamma)$, as a function of $m$. The market and risk aversion parameters are the same as in Figure 3. For $\phi(\gamma) = 3$ (months), the dotted lines show the value of $m$ such that the regret of the human-advisor and the robo-advisor are the same. The final panel shows the break-even time between updates for the robo-advisor corresponding to $m = 12$ for the human-advisor.

In conclusion, for realistic parameter values, the robo-advisor does not require frequent updates of risk preferences to match the of a myopic human-advisor, unless the client has access to a sophisticated human-advisor, i.e., a human-advisor using a horizon $m$ very close to the investment horizon $T$. However, if the risk aversion volatility is large enough, there can be situations where the robo-advisor needs to solicit the client’s risk preferences at all times to match the performance of a short-sighted human-advisor, i.e., with a horizon $m$ significantly shorter than $T$. However, changes to the client’s risk aversion are driven by significant events and changes to the client’s personal or professional status, which are infrequent in nature. Hence, even if the risk aversion volatility is high, this corresponds to a highly unusual situation that may not be well suited for a robo-advisor, that is meant to serve clients with a “typical” risk profile. Rather, a fully dedicated human-advisor may better handle a client with such unstable risk preferences.

7 Towards the Design of Personalized Robo-Advisor

In this section we discuss how components of the human-machine interaction system can be calibrated. Section 7.1 shows how the updating rule for the client’s risk preferences can be tailored to the client’s risk
aversion level to maintain a target level of regret, and Section 7.2 discusses the implications of the robo-advisor using a less personalized updating rule. Section 7.3 shows how the robo-advisor’s set of risk aversion levels can be constructed to fix the value of communication for each risk aversion level, with value defined in terms of the probability of a false positive. Section 7.4 shows how the machine specific component of the risk-return trade-off can be calibrated to limit the probability of choosing a too risky portfolio composition for the client. Section 7.5 shows how the risk aversion volatility of each level can be estimated from historical data on client transitions between risk aversion levels.

7.1 Personalizing the Updating Rule

In Section 5.1 we saw that for a fixed time between communication of risk preferences, the portfolio regret depends on the prevailing risk aversion level, and the corresponding risk aversion volatility. In this section we show how the robo-advisor can calibrate the updating rule to maintain a given target level of regret across risk aversion states.

Denote by \( \kappa \in \mathbb{R}^+ \) the target level of regret, and assume for notational simplicity that the investment horizon \( T \) is such that \( R_T^{(\phi)}(\gamma) \mid_{\phi \equiv T} > \kappa \), for each \( \gamma \in \Gamma \). That is, without any updating of the client’s risk preferences, the regret in each level exceeds \( \kappa \) before the end of the investment horizon. Using Proposition 5.2 and the definition of the threshold updating rule (3.7), we can then write

\[
\phi_{b(\kappa)}(\gamma) := \sup \{ n \geq 1 : R_T^{(\phi)}(\gamma) \mid_{\phi \equiv n} < \kappa \} = \inf \{ n \geq 1 : \mu_n(\gamma) > b_\gamma(\kappa) \},
\]

where the updating threshold \( b(\kappa) \) is given by

\[
b_\gamma(\kappa) := \gamma \kappa. \tag{7.1}
\]

This shows that the threshold updating rule \( \phi_{b(\kappa)} \) corresponds to triggering an update of risk preferences right before the regret exceeds the level \( \kappa \). Notice that for a given risk aversion level \( \gamma \in \Gamma \), the threshold \( b_\gamma(\kappa) \) does not depend on the risk aversion volatility, \( \sigma_\gamma \), so from Lemma A.1-(b) we deduce that for a given target level of regret, the time between updates in a given risk aversion level needs to decrease to accommodate a higher risk aversion volatility. However, the threshold is increasing in the risk aversion level

\[\text{In exactly the same way, the regret measure } R_{T,c}^{(\phi)} \text{ can be related to the threshold updating rule } \phi_{c(\kappa)}(\gamma) := \inf \{ n \geq 1 : \mu_n(\gamma) > b_\gamma \}, \]

The threshold corresponding to regret \( \kappa \) is still given by (7.1), but from \( R_{T,c}^{(\phi)}(\gamma) \geq R_T^{(\phi)}(\gamma) \) it follows that \( \phi_{c(\kappa)}(\gamma) \leq \phi_{b(\kappa)}(\gamma) \), so a shorter time between updates is required to bound the regret measure \( R_{T,c}^{(\phi)}(\gamma) \).
itself, indicating that more frequent updating is required for lower levels of risk aversion, for a fixed level of volatility.\footnote{This can also be deduced from Figure 3 that shows the regret profile corresponding to different risk aversion states with the same risk aversion volatility, and the same time between updates.} Intuitively, this is because regret is defined in terms of relative risk aversion changes, and a given absolute risk aversion change results in a larger relative change if the client’s initial risk aversion is low.

At the beginning of the investment process the client specifies a desired updating frequency, depending on her willingness to participate in the investment process. Given the client’s initial risk aversion level, the robo-advisor can then compute the corresponding regret, and use the above result to compute an updating threshold that allows the client to maintain the same level of regret across risk aversion levels. When a new risk aversion level is communicated by the client, a higher updating frequency may be required to maintain the same investment performance; in the following section we consider the implications of not using such a personalized updating rule.

### 7.2 The Benefits of Personalization

In the previous section we saw how the robo-advisor could adjust the updating rule to maintain a given level of regret. In this section we analyze the costs imposed on the client by using a robo-advisor which provides a less personalized service. Specifically, we analyze the effect on regret from using an updating rule that is not tailored to the specific properties of each risk aversion level.

We consider a robo-advisor that sets the time between updates of risk preferences to be the same regardless of the risk aversion level of the client. This frequency might be set arbitrarily by the robo-advisor, without any notion of regret, or calibrated so that the steady-state regret measure $\bar{R}_T^{(\phi)}$, defined in (5.5) is equal to a specific value, without any regard to the heterogeneity of risk aversion levels. In Appendix C we use the approximation $\Psi(\gamma) \approx \gamma$, as in (5.9), to show that for a given level of regret $\kappa$, the updating rule

\[
\bar{\phi}_\kappa(\gamma) \approx \frac{\pi}{2} \kappa^2 \left( \sum_{\gamma' \in \Gamma} \lambda(\gamma') \times \frac{\sigma_{\gamma'}}{\gamma'} \right)^2, \quad \gamma \in \Gamma,
\]

yields $\bar{R}_T^{(\phi)} \approx \kappa$. In words, for a large pool of clients, the cross-sectional regret $\bar{R}_T^{(\phi)}$, with the same time between updates $\bar{\phi}_\kappa$ used in all risk aversion levels, is indeed approximately equal to $\kappa$. This updating rule can be compared to the fixed regret updating rule corresponding to the threshold $b(\kappa)$ in (7.1), for which we
have in the same way as (7.2),

\[
\phi_{\hat{b}(\kappa)}(\gamma) \approx \frac{\pi}{2} \kappa^2 \left( \frac{\sigma_\gamma}{\gamma} \right)^2, \quad \gamma \in \Gamma.
\]  

This shows that in determining the aggregate updating rule \( \tilde{\phi}_\kappa \), the volatility of relative risk aversion changes, \( \sigma_\gamma/\gamma \), of each risk aversion state is weighted by the stationary distribution \( \lambda \). In turn, this shows that as the heterogeneity of risk aversion levels increases, measured in terms of relative risk aversion volatility, the aggregate updating rule becomes less tailored to individual levels. Furthermore, the regret in risk aversion level \( \gamma \in \Gamma \) satisfies

\[
R_{T}^{(\hat{\phi}_\kappa)}(\gamma) \approx \kappa \times \sum_{\gamma' \in \Gamma} \lambda(\gamma') \times \left( \frac{\sigma_{\gamma'}}{\gamma'} \right),
\]

which shows that for a given risk aversion level \( \gamma \in \Gamma \), the regret is larger (smaller) than \( \kappa \) if the relative risk aversion volatility, \( \sigma_\gamma/\gamma \), is higher (lower) than the weighted average of those volatilities, with the weights determined by the steady-state distribution \( \lambda \). From the above identity we can for instance see that the regret in risk aversion levels with a low weight can deviate substantially from the target regret \( \kappa \), and that if all levels have the same risk aversion volatility, then the regret is greater (lower) than \( \kappa \) for low (high) risk aversion levels. Hence, by using this “one-size-fits-all” strategy, i.e., using the same updating frequency for all levels, the robo-advisor is implicitly providing a worse service to clients with certain characteristics. That is, lower regret in certain risk aversion levels comes at the expense of a higher regret in other levels.

The analysis above corresponds to a robo-advisor that uses the same updating frequency for all risk aversion state. Alternatively, one can consider a robo-advisor that employs a threshold updating rule with a uniform threshold, i.e., a threshold that does not depend on the risk aversion level. Again, this threshold might be set arbitrarily by the robo-advisor, or calibrated so that the steady-state regret measure in (5.5) is equal to a specific value. In this case one can similarly show that as the heterogeneity of risk aversion levels increases, both in terms of risk aversion levels and risk aversion volatilities, the aggregate threshold becomes less tailored to individual levels. This implies, again, that the robo-advisor is treating investors in different risk aversion levels differently, in the sense that a lower regret in certain levels comes at the expense of a higher regret in other levels.
7.3 Value of Communication

The robo-advisor can calibrate the updating rule to maintain a target level of regret across risk aversion states. In this section we consider the probability of an update of risk preferences leading to a new risk aversion level for the client. That is, we consider how the “value” of communication depends on the client’s risk aversion level, and how this value can also be leveled across risk aversion levels.

The threshold parameter $b_\kappa$ in (7.1) quantifies the trade-off between having low regret, $\kappa$, and having a low probability of initiating an update without the client’s risk aversion level actually having changed (false positive), defined for a general updating threshold $b$, and investment horizon $T$, as

$$
\alpha_{b,T}(\gamma) := \sup_{0 \leq n < T} \mathbb{P}_n(\Psi(\tilde{\gamma}(n+\phi_\kappa(\gamma))\wedge (T-1)) = \gamma | \gamma_n = \gamma, \tau_n = n) \times 1\{\phi_\kappa(\gamma) < T\}, \quad \gamma \in \Gamma.
$$

In Appendix C we first provide a closed form formula for this probability, and then use the same approximation as in (5.9) to show that

$$
\alpha_{b(\kappa),T}(\gamma) \approx \Phi\left(\sqrt{\frac{2}{\pi}} \frac{\gamma^+ - \gamma}{\kappa} \right) - \Phi\left(\sqrt{\frac{2}{\pi}} \frac{\gamma^- - \gamma}{\kappa} \right), \quad \gamma \in \Gamma,
$$

(7.5)

where $\gamma^\pm$ are defined in (3.3) as the midpoints between neighboring risk aversion levels. This highlights the aforementioned trade-off between having a small regret, $\kappa$, that requires frequent updating, and having a small probability of a false positive, $\alpha_{b(\kappa),T}(\gamma)$, that requires less frequent updating. We also observe that the above expression is independent of the risk aversion volatility, just like the threshold $b_\kappa$. However, it is evident that a uniformly spaced grid of risk aversion levels, i.e. such that $\gamma^\pm - \gamma$ is independent of $\gamma$, results in a higher false positive probability for low risk aversion levels.

Informally, this means that the value of updating is not the same for all risk aversion levels, in the sense that the probability of an update leading to the client being classified to a new risk aversion level is not the same. In order for that to be the case, the robo-advisor’s set of risk aversion levels, $\Gamma$, needs to be constructed so that the relative distances $(\gamma^+ - \gamma)/\gamma$ and $(\gamma - \gamma^-)/\gamma$ are independent of $\gamma$. That is, the grid of risk aversion levels needs to more dense for low risk aversion levels. For such a grid both the regret and the false positive probability are constant across risk aversion levels.

7.4 Calibration of $\theta$

The robo-advisor can use the machine-specific component of the risk-return trade-off coefficient in (3.13) to bound the probability of choosing a too risky portfolio for the client. By using the approximation $\Psi(\tilde{\gamma}_n) \approx \tilde{\gamma}_n$,
as in (5.9), and \( \gamma_n \sim \mathcal{N}(\gamma_n, \delta_n(\gamma_n, \tau_n)) \), from (3.2), it follows that
\[
\gamma_n \pm \theta \times \delta_n(\gamma_n, \tau_n),
\]
is an approximate 100(1 − \( \alpha \))% confidence interval for \( \Psi(\bar{\gamma}_n) \), the client’s (unknown) risk aversion level at time \( n \), where the significance level is \( \alpha = 2(1 - \Phi(\theta)) \).

By using the risk-return coefficient \( \Delta_n = \gamma_n + \theta \times \delta_n(\gamma_n, \tau_n) \) in the optimization problem, the robo-advisor uses the right endpoint of this confidence interval, i.e., the most risk averse value within the interval. Hence, the probability of the client in reality having a higher risk aversion is approximately \( \alpha/2 = 1 - \Phi(\theta) \).

Equivalently, this is approximately the probability that the optimal portfolio under information asymmetry is riskier than the optimal portfolio corresponding to the the robo-advisor knowing the client’s true risk aversion level. If \( \theta = 0 \) then \( \alpha/2 = 0.5 \), as the risk aversion evolves like a centered random walk, and \( \alpha/2 \) decreases to zero as \( \theta \to \infty \), i.e., as the robo-advisor becomes more conservative. For a given target probability \( \alpha^* / 2 \), the robo-advisor can set \( \theta = \Phi^{-1}(1 - \alpha^*/2) \).

### 7.5 Risk Aversion Volatility Estimation

The risk aversion volatilities in (3.1) can be estimated from historical data on client transitions between risk aversion level. Such data is likely to be at a lower frequency (e.g., quarterly or annually) than the frequency used for portfolio rebalancing (e.g., weekly or monthly). This is equivalent to observing trajectories of the risk aversion process \( (\gamma_{T_k}^{(o)})_{k \geq 0} \), corresponding to the updating rule \( \phi \equiv c \), for some \( c \geq 1 \), which is a time-homogeneous Markov chain, whose transition matrix can be estimated. For \( \gamma \in \Gamma \) we then let
\[
\sigma^2_{\gamma} = \frac{1}{c} \text{Var}[\gamma_{T_k}^{(o)} | \gamma_0 = \gamma],
\]
matching the \( c \)-step variance of the random walk (3.1) to the empirical \( c \)-step variance. Alternatively, we can set \( \sigma_\gamma \) such that
\[
\mathbb{P}(\sigma_\gamma \times (\tilde{Z}_1 + \cdots + \tilde{Z}_c) \notin (\gamma^-, \gamma^+)) = \hat{\mathbb{P}}(\gamma_{T_k}^{(o)} \neq \gamma | \gamma_0 = \gamma),
\]
which matches the probability of the random walk (3.1) leaving the initial level in \( c \) steps, to the corresponding empirical probability.
8 Conclusions and Future extensions

The past decade has witnessed the emergence and rapid ascension of robo-advisors, investment platforms where the client interacts directly with the investment algorithm without the intervention of a human. Recent work has provided empirical evidence on the implications of robo-advising on investment portfolios and what characteristics distinguish clients of robo-advisors from those of traditional financial advisors. In the present work we provide a modeling framework that corroborates those findings.

We present a dynamic investment model between a client and a robo-advisor, with the investment criterion automatically adapting to changes in the client’s risk profile. These changes are self-reported by the client via repeated interaction with the robo-advisor, throughout the investment horizon. The frequency of interaction is determined by the client’s trade-off between having low involvement in the investment process (“piece of mind”), and a low portfolio regret resulting from the robo-advisor not having access to up-to-date information and making suboptimal investment decisions. We compare the client’s regret with the robo-advisor to the client’s regret with a traditional financial advisor (human-advisor) that is consistently aware of the client’s risk profile, but lacks the robo-advisor’s technical sophistication to solve the investment problem optimally. We find that clients placing emphasis on investment performance are more likely to favor the robo-advisor, while clients seeking a high level of delegation are more likely to favor the human-advisor.

Our model can be extended along several directions. First, the client’s risk aversion process in Section 3.1 can be enhanced to include a component that captures the overall state of the economy, consistent with empirical studies documenting that individuals are willing to take substantially larger risks during periods of economic growth, and are more risk averse during periods of recession (Buccioli and Miniaci [2011], Sahm [2012]). Second, the risk aversion process can be allowed to depend on market returns, e.g., with risk aversion going down in market upswings and going up in market downturns, which is consistent with empirical evidence. In the comparison of the robo-advisor and the human-advisor in Section 6.2 this would place a constraint on the frequency of communication between the client and the robo-advisor. Namely, with risk aversion being affected by portfolio performance this is imperative to prevent market timing (i.e., buying high/selling low), and a client whose risk aversion is highly sensitive to short-term market swings would be better served by a human-advisor that offers a more personal relationship during times of distress.

The model for the market dynamics in Section 3.3 can be extended to time-dependency or even stochastic volatility, and the portfolio rebalancing times in the investment model of Section 3.4 can be allowed to be random. For instance, rebalancing could be triggered by the portfolio composition having drifted sufficiently much from the optimal composition, given the client’s risk aversion level. Robo-advisors generally use such threshold updating rules for rebalancing their portfolios (see Kaya [2017]), and this mirrors how a target
regret threshold was used to trigger a communication of the client’s risk preferences (see Section 7.1), with regret defined in terms of the expected change in portfolio composition.

We remark that various assumptions made in the analysis in Section 6.2 are favorable to the human-advisor. For the robo-advisor we use a worst-case measure of regret, that also accounts for the discretization used in the risk aversion levels, while for the human-advisor we consider both the average and worst-case regret measures. We also do not take into account the fact that human-advisors charge significantly higher fees than robo-advisors, which deteriorates their investment performance relative to robo-advisors. Furthermore, different ways to model the technical deficiency of the human-advisor in Section 6.1 do not have a qualitative effect on the analysis. For instance, one can consider a human-advisor that solves the dynamic mean-variance investment problem optimally, but rebalances the portfolio at a lower frequency than the robo-advisor. This pertains to the fact that robo-advisors monitor for rebalancing opportunities on a daily basis, minimizing expense ratios and maximizing tax efficiency, while maintaining investment discipline with the aid of automation. Human-advisor do this to a lesser extent, and they may or may not attempt to harvest tax losses.

A Technical Lemmas

Lemma A.1 contains results related to the client’s risk aversion process, introduced in Section 3.1. Recall that at time $n \geq 0$ the client’s (unknown) risk aversion is denoted by $\gamma_n \in \mathbb{R}$, while $\Psi(\gamma_n) \in \Gamma$ is the risk aversion level used by the robo-advisor (see (3.1)-(3.2)). Assume that at time $n$ the previous communication of risk preferences took place at time $\tau_n = \tau \leq n$, and resulted in risk aversion level $\gamma_n = \gamma \in \Gamma$. Then the distribution of $\Psi(\gamma_n)$ is characterized by the probabilities

$$p_n(\gamma'; \gamma, \tau) := \mathbb{P}(\Psi(\gamma_n) = \gamma' | \gamma_n = \gamma, \tau_n = \tau) = \mathbb{P}(\Psi(\gamma_{n-\tau}) = \gamma' | \gamma_{n-\tau} = \gamma, \tau_{n-\tau} = 0) =: p_{n-\tau}(\gamma'; \gamma), \quad (A.1)$$

which is given in closed form in Lemma A.1-(a), and in the second equality we used the (conditional) time-homogeneity of the process $(\gamma_n)_{n \geq 0}$. The expected absolute change in the client’s risk aversion since the previous updating time is given by

$$\mu_n(\gamma, \tau) := \mathbb{E}[|\Psi(\gamma_n) - \gamma_\tau| | \gamma_n = \gamma, \tau_n = \tau] = \mathbb{E}[|\Psi(\gamma_{n-\tau}) - \gamma_0| | \gamma_0 = \gamma, \tau_{n-\tau} = 0] =: \mu_{n-\tau}(\gamma), \quad (A.2)$$
which is given in closed form in Lemma A.1-(b). We similarly define the expected absolute change without projecting the client’s risk aversion to the nearest level in the set \( \Gamma \),

\[
\mu_n^c(\gamma, \tau) := E[|\bar{\gamma}_n - \gamma_n| | \gamma_n = \gamma, \tau_n = \tau] = E[|\bar{\gamma}_{n-\tau} - \gamma_0| | \gamma_0 = \gamma, \tau_{n-\tau} = 0] =: \mu_n^c(\gamma), \tag{A.3}
\]

which is given in Lemma A.1-(c). Finally, the conditional standard deviation of \( \Psi(\bar{\gamma}_n) \) is given by

\[
\delta_n(\gamma, \tau) := \sqrt{\text{Var}[\Psi(\bar{\gamma}_n) | \gamma_n = \gamma, \tau_n = \tau]} = \sqrt{\text{Var}[\Psi(\bar{\gamma}_{n-\tau}) | \gamma_{n-\tau} = \gamma, \tau_{n-\tau} = 0]} =: \delta_n(\gamma), \tag{A.4}
\]

which is given in Lemma A.1-(d).

**Lemma A.1.**

(a) For \( n \geq 0 \), and \( \gamma, \gamma' \in \Gamma \), we have for \( p_n(\gamma'; \gamma) \) defined in (A.1),

\[
p_n(\gamma'; \gamma) = \begin{cases} 
1_{\{\gamma' = \gamma\}}, & n = 0, \\
\Phi\left(\frac{\gamma' - \gamma}{\sigma_n\sqrt{n}}\right) - \Phi\left(\frac{\gamma' - \gamma}{\sigma_n\sqrt{n}}\right), & n \geq 1,
\end{cases} \tag{A.5}
\]

where the constants \((\gamma')^\pm\) are defined in (3.3) and \( \Phi \) is the standard normal cumulative distribution function. Furthermore, \( p_n(\gamma'; \gamma) \) is decreasing in \( n \geq 1 \) and \( \sigma_n \), and

\[
\lim_{n \to \infty} p_n(\gamma'; \gamma) = \lim_{\sigma_n \to \infty} p_n(\gamma'; \gamma) = \begin{cases} 
0, & \gamma' \notin \{\bar{\gamma}_1, \bar{\gamma}_K\}, \\
1/2, & \gamma' \in \{\bar{\gamma}_1, \bar{\gamma}_K\}.
\end{cases}
\]

(b) For \( n \geq 0 \), and \( \gamma \in \Gamma \), we have for \( \mu_n(\gamma) \), defined in (A.2),

\[
\mu_n(\gamma) = \sum_{\gamma' \in \Gamma} p_n(\gamma'; \gamma) \times |\gamma' - \gamma|.
\tag{A.6}
\]

where \( p_n(\gamma'; \gamma) \) given in (A.5). Furthermore, \( \mu_n(\gamma) \) is increasing in \( n \) and \( \sigma_n \), and

\[
\lim_{n \to \infty} \mu_n(\gamma) = \lim_{\sigma_n \to \infty} \mu_n(\gamma) = \frac{\bar{\gamma}_1 + \bar{\gamma}_K}{2}.
\]

(c) For \( n \geq 0 \), and \( \gamma \in \Gamma \), we have for \( \mu_n^c(\gamma) \), defined in (A.3),

\[
\mu_n^c(\gamma) = \sqrt{\frac{2}{\pi}} \sigma_n \sqrt{n}.
\]
Furthermore, for $\mu_n(\gamma)$ given in (A.6),

$$\mu_n(\gamma) \leq \mu_c^n(\gamma),$$

and the inequality is strict for $n > 0$.

(d) For $n \geq 0$, and $\gamma \in \Gamma$, we have for $\delta_n(\gamma)$, defined in (A.4),

$$\delta_n(\gamma) = \sqrt{\sum_{\gamma' \in \Gamma} p_n(\gamma'; \gamma) \times (\gamma')^2} - \left( \sum_{\gamma' \in \Gamma} p_n(\gamma'; \gamma) \times \gamma' \right)^2. \quad (A.7)$$

Furthermore,

$$\lim_{n \to \infty} \delta_n(\gamma) = \frac{(\bar{\gamma}_K - \bar{\gamma}_1)^2}{4}.$$  

(e) For any updating rule $\phi$, the process $(\gamma_{T_k(\phi)})_{k \geq 0}$ is an irreducible and aperiodic Markov chain on $\Gamma$, with respect to the filtration $(G_k)_{k \geq 0}$, where $G_k := F_{T_k(\phi)}$. It has a time-homogeneous transition matrix $\Lambda^{(\phi)}$, given by

$$\Lambda^{(\phi)}_{\gamma', \gamma} = p_{\phi(\gamma')}(\gamma'; \gamma), \quad \gamma, \gamma' \in \Gamma.$$

(f) If $\phi \equiv 1$, then $(\gamma_n)_{n \geq 0}$ is an irreducible and aperiodic Markov chain on $\Gamma$, with respect to the filtration $(F_n)_{n \geq 0}$, with a time-homogeneous transition matrix $\Lambda$, given by

$$\Lambda_{\gamma', \gamma} = p_1(\gamma'; \gamma), \quad \gamma, \gamma' \in \Gamma. \quad (A.8)$$

Furthermore, it has a unique stationary distribution, $\lambda := (\lambda(\gamma))_{\gamma \in \Gamma}$, such that for $\gamma, \gamma' \in \Gamma$,

$$\lambda(\gamma) = \lim_{n \to \infty} \Lambda_{\gamma', \gamma}^n.$$
Proof: For part (a) we have from the dynamics (3.1),

\[ p_n(\gamma'; \gamma) = \mathbb{P}(\Psi(\tilde{\gamma}_n) = \gamma' | \gamma_0 = \gamma, \tau_n = 0) = \mathbb{P}(\gamma + \sigma_\gamma \times (\bar{Z}_1 + \bar{Z}_2 + \cdots + \bar{Z}_n) \in ((\gamma')^+ , (\gamma')^-)) = \Phi\left(\frac{(\gamma')^+ - \gamma}{\sigma_\gamma \sqrt{n}}\right) - \Phi\left(\frac{(\gamma')^- - \gamma}{\sigma_\gamma \sqrt{n}}\right), \]

where we used that \( \bar{Z}_1 + \bar{Z}_2 + \cdots + \bar{Z}_n \sim \mathcal{N}(0, n) \). If \( \gamma' = \gamma \), then \( p_n(\gamma; \gamma) \) is decreasing in \( n \) and \( \sigma_\gamma \), as \( \gamma^+ - \gamma > 0 \) and \( \gamma^- - \gamma < 0 \). The limits follow from the fact that \( 0 < (\gamma')^+ - \gamma < \infty \) and \( -\infty < (\gamma')^- - \gamma < 0 \), unless \( \gamma' = \gamma_K \) in which case \( (\gamma')^+ - \gamma = \infty \), or \( \gamma' = \gamma_1 \), in which case \( (\gamma')^- - \gamma = -\infty \). For part (b), the identity (A.6) follows by definition, and the limits follow from part (a). To show that \( \mu_n(\gamma) \) is increasing in \( n \) and \( \sigma_\gamma \), it suffices to notice that for any \( c > 0 \), the probabilities

\[ \mathbb{P}(\tilde{\gamma}_n > \gamma_0 + c | \gamma_0 = \gamma, \tau_n = 0) = \mathbb{P}(Z > \frac{c}{\sigma_\gamma \sqrt{n}}), \quad \mathbb{P}(\tilde{\gamma}_n < \gamma_0 - c | \gamma_0 = \gamma, \tau_n = 0) = \mathbb{P}(Z < -\frac{c}{\sigma_\gamma \sqrt{n}}), \]

are increasing in \( n \) and \( \sigma_\gamma \). For part (c) we again use that \( \bar{Z}_1 + \bar{Z}_2 + \cdots + \bar{Z}_n \sim \mathcal{N}(0, n) \) to write

\[ \mu^c_n(\gamma) = \mathbb{E}[(\gamma_0 + \sigma_\gamma \times (\bar{Z}_1 + \bar{Z}_2 + \cdots + \bar{Z}_n) - \gamma_0)] = \sigma_\gamma \sqrt{n} \mathbb{E}|Z| = \sqrt{\frac{2}{\pi}} \sigma_\gamma \sqrt{n}, \]

using the absolute moment formula for the Gaussian distribution. To show the inequality, we denote by \( f_{\tilde{\gamma}_n} \) the probability density function of \( \tilde{\gamma}_n \), given \( \gamma_0 = \gamma \) and \( \tau_n = 0 \), and write

\[
\begin{align*}
\mu_n(\gamma) &= \sum_{k=1}^{K-1} \left[ \int_{\gamma_k}^{\gamma_{k+1}} f_{\tilde{\gamma}_n}(\gamma') d\gamma' \times |\gamma_k - \gamma_0| + \int_{\gamma_{k+1}}^{\gamma_{k+2}} f_{\tilde{\gamma}_n}(\gamma') d\gamma' \times |\gamma_{k+1} - \gamma_0| \right] \\
&\quad + \int_{\gamma_1}^{\gamma_2} f_{\tilde{\gamma}_n}(\gamma') d\gamma' \times (\gamma - \gamma_1) + \int_{\gamma_K}^{\infty} f_{\tilde{\gamma}_n}(\gamma') d\gamma' \times (\gamma_K - \gamma_0) \\
&\leq \sum_{k=1}^{K-1} \left[ \int_{\gamma_k}^{\gamma_{k+1}} f_{\tilde{\gamma}_n}(\gamma') |\gamma' - \gamma_0| d\gamma' + \int_{\gamma_{k+1}}^{\gamma_{k+2}} f_{\tilde{\gamma}_n}(\gamma') |\gamma' - \gamma_0| d\gamma' \right] + \int_{\gamma_1}^{\gamma_2} f_{\tilde{\gamma}_n}(\gamma') |\gamma' - \gamma_0| d\gamma' \int_{\gamma_K}^{\infty} f_{\tilde{\gamma}_n}(\gamma') |\gamma' - \gamma_0| d\gamma' \\
&= \mu^c_n(\gamma).
\end{align*}
\]
For part (d), the identity (A.7) follows by definition, and the limit follows from part (a). For part (e), recall from (3.4) that $T_{k+1}^{(\phi)} = T_k^{(\phi)} + \phi(\gamma_k^{(\phi)})$, so, using Lemma A.1-(a),

$$
\mathbb{P}^{(\phi)}(\gamma_{T_k^{(\phi)}}) = \gamma|\gamma_{T_k^{(\phi)}} = \gamma') = \mathbb{P}^{(\phi)}(\gamma_{T_k^{(\phi)}} + \phi(\gamma_k^{(\phi)}) = \gamma|\gamma_k^{(\phi)} = \gamma')
$$

$$
= p_{T_k^{(\phi)} + \phi(\gamma_k^{(\phi)})}(\gamma; \gamma', T_k^{(\phi)})
$$

$$
= p_{\phi(\gamma')}(\gamma; \gamma'),
$$

so the transition probabilities are time-homogeneous. The Markov chain is clearly irreducible and aperiodic as the random walk $(\gamma_n)_{n \geq 0}$ has Gaussian increments. Part (f) is a special case of part (e), and every irreducible and aperiodic Markov chain with a finite state space has a unique stationary distribution.

Lemma A.2 contains properties of the optimal portfolio strategy in Section 4. Recall that $\pi^{*}_{n,T}(\gamma, \tau; \theta)$ denotes the optimal proportion of wealth allocated to the risky asset at time $n \in \{0, 1, \ldots, T - 1\}$, with $\gamma \in \Gamma$ representing the most recent risk aversion level, communicated at time $\tau \in \{0, 1, \ldots, n\}$. If $\phi \equiv 1$, the optimal allocation is denoted by $\pi^{*}_{n,T}(\gamma, n)$, and if $\phi \equiv \infty$, it is denoted by $\pi^{*}_{n,T}(\gamma; \theta)$.

**Lemma A.2.**

(a) The optimal final period allocation, at time $n = T - 1$, is given by

$$
\pi^{*}_{T-1,T}(\gamma, \tau; \theta) = \frac{1}{\Delta_{T-1}(\gamma, \tau; \theta)} \times \frac{\mu - r}{\sigma^2},
$$

i.e., it is equal to the expected excess return of the risky asset, per unit of variance, with the variance scaled by the risk-return trade-off coefficient in (3.13).

(b) The optimal allocation at time $n \in \{0, 1, \ldots, T - 1\}$ depends only on the time since the previous update, $n - \tau$, and the time until the end of the investment horizon, $T - n$. That is,

$$
\pi^{*}_{n,T}(\gamma, \tau; \theta) = \pi^{*}_{n-\tau, T-\tau}(\gamma, 0; \theta).
$$

Furthermore, if $\phi \equiv \infty$ and $\theta = 0$, then

$$
\pi^{*}_{n,T}(\gamma; 0) = \pi^{*}_{0,T-n}(\gamma; 0),
$$
and, if $\phi \equiv 1$, then,

$$\pi^*_{n,T}(\gamma, n) = \pi^*_{0,T-n}(\gamma, 0).$$

(c) Let $\theta = 0$. Then the optimal allocation is increasing between updating times, and increasing up to a $O(\cdot)$ term at updating times. That is, for $n \in \{0, 1, \ldots, T - 2\}$,

$$\pi^*_{n,T}(\gamma, \tau; 0) \leq \pi^*_{n+1,T}(\gamma, \tau_{n+1}; 0) + O(1 - p_{\tau_{n+1}}(\gamma; \gamma, \tau)),$$

with $p_{\tau_{n+1}}(\gamma; \gamma, \tau)$ given in Lemma A.1-(a). In particular, if $\phi \equiv \infty$, then

$$\pi^*_{n,T}(\gamma; 0) \leq \pi^*_{n+1,T}(\gamma; 0),$$

and, if $\phi \equiv 1$, then

$$\pi^*_{n,T}(\gamma, n) \leq \pi^*_{n+1,T}(\gamma, n + 1) + O(1 - \Lambda_{\gamma,\gamma}),$$

with $\Lambda_{\gamma,\gamma}$ given in (A.8).

(d) Let $\theta = 0$. Then for any $0 \leq \tau < T - \phi(\gamma)$, the optimal allocation corresponding to the updating rule $\phi$, and the optimal allocation corresponding to the no-updating rule, satisfy

$$\pi^*_{n,T}(\gamma, \tau; 0) = \pi^*_{n,T}(\gamma; 0) + O(1 - p_{\phi(\gamma)}(\gamma; \gamma)), \quad \tau \leq n < \tau + \phi(\gamma).$$

For $T - \phi(\gamma) \leq \tau < T$, then the two allocations are the same,

$$\pi^*_{n,T}(\gamma, \tau; 0) = \pi^*_{n,T}(\gamma; 0), \quad \tau \leq n < T.$$

(e) For $n = 0, 1, \ldots, T - 1$ the optimal allocations corresponding to the updating rule $\phi \equiv \infty$ satisfy

$$\pi^*_{n,T}(\gamma; 0) = c_{n,T}(\gamma) \times \pi^*_{T-1,T}(\gamma; 0),$$

where $c_{n,T}(\gamma)$ is increasing in $\gamma$, and $c_{T-1,T}(\gamma) = 1$. 

\[\square\]
Proof: Part (a) follows directly from the optimal allocation formula (4.2), by using $a_T = b_T = 1$. Part (b) follows by showing inductively that $\pi_{n,T}(\gamma, \tau; \theta) = \pi_{n,T}^\ast(\gamma, 0; \theta)$, for $k = 1, 2, \ldots, T - n$, where the case $k = 1$ follows from part (a). We first show part (c) for $\phi \equiv 1$. In that case we have

$$a_{n+1}(\gamma_n) = \mathbb{E}_{n+1, \gamma_n}^{(1)}[a_{n+2}] \times (R + \bar{\mu} \pi_{n+1,T}^\ast(\gamma_n, n + 1)),$$

$$b_{n+1}(\gamma_n) = \mathbb{E}_{n+1, \gamma_n}^{(1)}[b_{n+2}] \times (\sigma^2(\pi_{n+1,T}^\ast(\gamma_n, n + 1))^2 + (R + \bar{\mu} \pi_{n+1,T}^\ast(\gamma_n, n + 1))^2) \geq \mathbb{E}_{n+1, \gamma_n}^{(1)}[b_{n+2}] \times (R + \bar{\mu} \pi_{n+1,T}^\ast(\gamma_n, n + 1))^2,$$

where $\mathbb{E}_{n+1, \gamma_n}[a_{n+2}]$ is shorthand notation for $\mathbb{E}_{n+1}[a_{n+2}(\gamma_n+2)|\gamma_{n+1} = \gamma_n]$, and $\mathbb{E}_{n+1, \gamma_n}[b_{n+2}]$ is shorthand notation for $\mathbb{E}_{n+1}[b_{n+2}(\gamma_n+2)|\gamma_{n+1} = \gamma_n]$. Using the above we have

$$\pi_{n,T}^\ast(\gamma_n, n)$$

$$= \frac{\bar{\mu}/\sigma^2}{\gamma_n} \times a_{n+1}(\gamma_n) - R \gamma_n (b_{n+1}(\gamma_n) - a_{n+1}^2(\gamma_n)) + R_n(\gamma_n)$$

$$\leq \frac{\bar{\mu}/\sigma^2}{\gamma_n (R + \bar{\mu} \pi_{n+1,T}^\ast(\gamma_n, n + 1))} \mathbb{E}_{n+1, \gamma_n}^{(1)}[a_{n+2}] - R \gamma_n (R + \bar{\mu} \pi_{n+1,T}^\ast(\gamma_n, n + 1)) (\mathbb{E}_{n+1, \gamma_n}^{(1)}[b_{n+2}] - (\mathbb{E}_{n+1, \gamma_n}^{(1)}[a_{n+2}]^2) + R_n(\gamma_n)$$

$$\leq \pi_{n+1,T}^\ast(\gamma_n, n + 1) + R_n(\gamma_n),$$

where $R_n(\gamma_n) = O(1 - A_{n, \gamma_n})$, and the equality above can be justified in the same way as (B.10). The result for $\phi = \infty$ is a special case of this, and the general result is shown in a similar way. In part (d) it is clear that if $\tau \geq T - \phi(\gamma)$, then $\pi_{n,T}^\ast(\gamma, \tau; 0) = \pi_{n,T}^\ast(\gamma; 0)$, for $n \geq \tau$. For $\tau \in \{T - 2\phi(\gamma), T - 2\phi(\gamma) + 1, \ldots, T - \phi(\gamma) - 1\}$, then we can show recursively that for $n = \tau + \phi(\gamma) - 1, \tau + \phi(\gamma) - 2, \ldots, \tau + 1, \tau$,

$$\pi_{n,T}^\ast(\gamma, \tau; 0) = \pi_{n,T}^\ast(\gamma; 0) + O(1 - p_{\phi(\gamma)}(\gamma; \gamma)),$$

$$a_n(\gamma, \tau) = a_n^{(\infty)}(\gamma) + O(1 - p_{\phi(\gamma)}(\gamma; \gamma)),$$

$$b_n(\gamma, \tau) = b_n^{(\infty)}(\gamma) + O(1 - p_{\phi(\gamma)}(\gamma; \gamma)),$$

where the $O(\cdot)$ terms can be justified in the same way as (B.10). In the same way one can then show the result for $\tau < T - 2\phi(\gamma)$ and $n = \tau + \phi(\gamma) - 1, \tau + \phi(\gamma) - 2, \ldots, \tau + 1, \tau$. In part (e) the result is trivial for $n = T - 1$. For $n = T - 2$ we can explicitly compute $\pi_{n,T}^\ast(\gamma; 0)$ and show that the result holds. Now assume that $c_{n,T}(\gamma)$ is increasing in $\gamma \geq 0$, for some $0 < n < T - 1$. Then, using the recursive equations for the $a_n$-
Lemma A.3 contains properties of the regret for the human-advisor in Section 6.1, with a horizon \( m \), where \( 1 \leq m \leq T \). For an investment horizon \( T \geq 1 \), recall the definition of the average regret, \( \bar{R}_{m,T}(\gamma) \), where \( \gamma \in \Gamma \), and the regret for the \( m_0 \)-th subperiod, \( R_{m,m_0,T}(\gamma) \), where \( 0 \leq m_0 \leq \lfloor T/m \rfloor \).

**Lemma A.3.**

(a) The subperiod regret \( R_{m,m_0,T}(\gamma) \) is decreasing in \( m_0 \). That is, for \( 0 \leq m_0 < m'_0 < \lfloor T/m \rfloor \),

\[
R_{m,m_0,T}(\gamma) \geq R_{m,m'_0,T}(\gamma) + O(p_{\gamma}).
\]

In particular, the worst-case regret occurs in the first subperiod,

\[
R_{m,0,T}(\gamma) \geq R_{m,m_0,T}(\gamma) + O(p_{\gamma}).
\]

(b) The average regret \( R_{m,T}(\gamma) \) is increasing in \( m \). That is, for \( m' > m \) such that \( m \) divides \( m' \),

\[
R_{m',T}(\gamma) \leq R_{m,T}(\gamma) + O(p_{\gamma}).
\]

The first subperiod’s regret is also increasing in \( m \),

\[
R_{m',0,T}(\gamma) \leq R_{m,0,T}(\gamma) + O(p_{\gamma}).
\]

(c) The average regret \( R_{m,T}(\gamma) \) is increasing in \( T \). That is, if \( T = Mm \), for some \( M \geq 1 \), and \( T' = T + m \),

\[
R_{m,T}(\gamma) \leq R_{m,T}(\gamma) + O(p_{\gamma}).
\]
then

\[ R_{m,T'}(\gamma) \geq R_{m,T}(\gamma) + O(p_\gamma). \]

The first subperiod’s regret is also increasing in \( T \),

\[ R_{m,0,T'}(\gamma) \geq R_{m,0,T}(\gamma) + O(p_\gamma). \]

\[ \square \]

**Proof:** For brevity we use \( \pi^*_n,T(\gamma) \) as shorthand notation for \( \pi^*_n,T(\gamma,n) \), for \( n \in \{0,1,\ldots,T-1\} \), and \( \gamma \in \Gamma \). For part (a), note that by Lemma A.2-(b),

\[ R_{m,m'_0,T}(\gamma) = \frac{1}{m} \sum_{n=m'_0}^{m+1} \frac{\pi^*_n,m_0 \times m(\gamma) - \pi^*_n,T(\gamma)}{\pi^*_n,T(\gamma)} \]

Then, by Lemma A.2-(c), and the same justification as for (B.10),

\[ R_{m,0,T}(\gamma) \geq \frac{1}{m} \sum_{k=0}^{m-1} \frac{\pi^*_T - m + k,T(\gamma) - \pi^*_m,m_k + k,T(\gamma)}{\pi^*_{m_0 \times m} + k,T(\gamma)} \]

For part (b) we use the same arguments as in part (a) to write

\[ R_{m',T}(\gamma) = \frac{1}{T} \sum_{m_0=0}^{[T/m']} \sum_{k=0}^{m'-1} \frac{\pi^*_T - m + k,T(\gamma) - \pi^*_m,m_k + k,T(\gamma)}{\pi^*_m + k,T(\gamma)} \]

with the inequality for the first subperiod’s regret shown in the same way. For part (c) we have

\[ R_{m,T'}(\gamma) = \frac{m}{T} \sum_{m_0=0}^{M+1} R_{m,m_0,T'}(\gamma) = \frac{m}{T} R_{m,0,T}(\gamma) + \frac{m}{T} \sum_{m_0=1}^{M+1} R_{m,m_0,T}(\gamma) \]
and then by part (b), and Lemma A.2-(b),

\[ R_{m,T'}(\gamma) \geq \frac{m}{T'} R_{m,T}(\gamma) + O(1 - \Lambda_{1,\gamma}) + \frac{T}{T'} R_{m,T}(\gamma), \]

from which it follows that

\[ R_{m,T'}(\gamma) \geq R_{m,T}(\gamma) + O(1 - \Lambda_{1,\gamma}). \]

Again, the inequality for the first subperiod’s regret is shown in the same way. \qed

## B Proofs of Results in Section 4

**Proof of Proposition 4.1:** We begin by deriving the HJB system of equations satisfied by any potential optimal control for the objective function (3.9). Recall that \( \mathbb{P}_n^{(\phi)}(\cdot) \) is shorthand notation for \( \mathbb{P}(\cdot|F_n) \), and that \( X_n = (X_k)_{0 \leq k \leq n} \), with analogue identities for \( \gamma_n \) and \( \tau_n \). Furthermore, given \( X_n \), then \( X_{n+1} = (X_n, X_{n+1}) \), where \( X_{n+1} \) is obtained by applying the control \( \pi_n \) to \( X_n \) at time \( n \).

**Proposition B.1.** Assume that an optimal control \( \pi^* \) for the objective function (3.9) exists. Then the optimal value function (3.12) satisfies

\[
V_{n,T'}(X_n, \gamma_n, \tau_n; \theta) = \sup_{\pi} \left\{ \mathbb{E}_n^{(\phi)}[V_{n+1,T'}(X_{n+1}^*, \gamma_{n+1}, \tau_{n+1}; \theta)] \right. \\
- \left( \mathbb{E}_n^{(\phi)}[f_{n+1,n+1}(X_{n+1}^*, \gamma_{n+1}, \tau_{n+1}; X_{n+1}^*, \gamma_{n+1}, \tau_{n+1}, \theta)] - \mathbb{E}_n^{(\phi)}[f_{n+1,n}(X_{n+1}^*, \gamma_{n+1}, \tau_{n+1}; X_n, \gamma_n, \tau_n, \theta)] \right) \\
- \left( \mathbb{E}_n^{(\phi)}\left[ \frac{\Delta_{n+1}}{2} \times \left( \frac{g_{n+1}(X_{n+1}^*, \gamma_{n+1}, \tau_{n+1})}{X_n} \right)^2 \right] \right) \leq \frac{\Delta_n}{2} \times \left( \mathbb{E}_n^{(\phi)}\left[ \frac{g_{n+1}(X_{n+1}^*, \gamma_{n+1}, \tau_{n+1})}{X_n} \right] \right)^2,
\]

for \( 0 \leq n < T \), with the terminal condition

\[
V_{T,T'}(X_T, \gamma(T), \tau(T); \theta) = 0.
\]

Herein, for any \( x' > 0 \), \( \gamma' > 0 \), and \( \tau' \geq 0 \), and any \( k = 0, 1, \ldots, T \), the function sequence \( (f_{n,k}(X_n, \gamma_n, \tau_n; x', \gamma', \tau', \theta))_{0 \leq n \leq T} \) is determined by the recursion

\[
f_{n,k}(X_n, \gamma_n, \tau_n; x', \gamma', \tau', \theta) = \mathbb{E}_n^{(\phi)}[f_{n+1,k}(X_{n+1}^*, \gamma_{n+1}, \tau_{n+1}; x', \gamma', \tau', \theta)], \quad 0 \leq n < T,
\]

\[
f_{T,k}(X_T, \gamma(T), \tau(T); x', \gamma', \tau', \theta) = \frac{X_T}{x'} - 1 - \frac{\Delta_k(\gamma', \tau'; \theta)}{2} \times \left( \frac{X_T}{x'} \right)^2,
\]

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Furthermore, we have, for any given control \( \pi \) the probabilistic interpretation, while the function sequence \( (g_n(X_n, \gamma(n), \tau(n)))_{0 \leq n \leq T} \) is determined by the recursion

\[
  g_n(X_n, \gamma(n), \tau(n)) = E_{n}^{(\phi)} [ g_{n+1}(X^\pi_{n+1}, \gamma(n+1), \tau(n+1)) ], \quad 0 \leq n < T, \\
  g_T(X(T), \gamma(T), \tau(T)) = X_T.
\]

Furthermore, we have, for \( 0 \leq n < T \), the probabilistic interpretation,

\[
  f_{n,k}(X_n, \gamma(n), \tau(n); x', \gamma', \tau', \theta) = E_{n}^{(\phi)} \left[ \frac{X^\pi_T}{x'} - 1 - \frac{\Delta_k(\gamma', \tau'; \theta)}{2} \times \left( \frac{X^\pi_T}{x'} \right)^2 \right], \\
  g_n(X_n, \gamma(n), \tau(n)) = E_{n}^{(\phi)} [X^\pi_T].
\]

**Proof of Proposition B.1:** We begin by deriving the HJB system for the general problem

\[
  J_{n,T}(X_n, \gamma(n), \tau(n); \pi, \theta) := E_{n}[F_n(X_n, \gamma_n, \tau_n, X^\pi_T; \theta)] + G_n(X_n, \gamma_n, \tau_n, E_n[X^\pi_T]; \theta), \quad 0 \leq n < T,
\]

where for simplicity we have dropped the superscript \( (\phi) \) from the notation. This system is given in Eqs. (B.5)-(B.6), and Proposition B.1 is then a special case with

\[
  F_n(x, \gamma, \tau, y; \theta) = \frac{y}{x} - 1 - \frac{\Delta_n(\gamma, \tau; \theta)}{2} \times \left( \frac{y}{x} \right)^2, \quad G_n(x, \gamma, \tau, y; \theta) = \frac{\Delta_n(\gamma, \tau; \theta)}{2} \times \left( \frac{y}{x} \right)^2. \quad (B.1)
\]

The proof consists of two part. First we derive the recursive equation satisfied by the objective function for any given control \( \pi \). Then we derive the system of equations necessarily satisfied by an optimal control \( \pi^* \).

**Step 1: Recursion for \( J_{n,T}(X_n, \gamma(n), \tau(n); \pi, \theta) \).** For a given control \( \pi \), we define the functions

\[
  f_{n,k}(x, \gamma, \tau; x', \gamma', \tau', \theta) := E_{n}[F_k(x', \gamma', \tau', X^\pi_T; \theta)]|_{X(n) = x, \gamma(n) = \gamma, \tau(n) = \tau}, \quad (B.2)
\]

and write the objective function at time \( n+1 \) as

\[
  J_{n+1,T}(X_{n+1}, \gamma(n+1), \tau(n+1); \pi, \theta) = E_{n+1}[F_{n+1}(X_{n+1}, \gamma_{n+1}, \tau_{n+1}, X^\pi_{n+1}; \theta)] + G_{n+1}(X_{n+1}, \gamma_{n+1}, \tau_{n+1}, E_{n+1}[X^\pi_{n+1}]; \theta) \]

\[
  = f_{n+1,k}(X_{n+1}, \gamma(n+1), \tau(n+1); X_{n+1}, \gamma_{n+1}, \tau_{n+1}, \theta) + G_{n+1}(X_{n+1}, \gamma_{n+1}, \tau_{n+1}, g_{n+1}(X_{n+1}, \gamma(n+1), \tau(n+1)); \theta).
\]
Taking expectations with respect to time $n$ information yields, with $X_{(n+1)}^\pi = (X(n), X_{n+1}^\pi)$,

$$\mathbb{E}_n[J_{n+1,T}(X_{(n+1)}^\pi, \gamma(n+1), \tau(n+1), \pi; \theta)] = \mathbb{E}_n[f_{n+1,n+1}(X_{(n+1)}^\pi, \gamma(n+1), \tau(n+1); X_{n+1}^\pi, \gamma(n+1), \tau(n+1); \theta)]$$

$$+ \mathbb{E}_n[G_{n+1}(X_{n+1}^\pi, \gamma(n+1), \tau(n+1); g_{n+1}(X_{n+1}^\pi, \gamma(n+1), \tau(n+1); \theta)].$$

Adding and subtracting $J_{n,T}(X(n), \gamma(n), \tau(n), \pi; \theta)$ then gives

$$\mathbb{E}_n[J_{n+1,T}(X_{(n+1)}^\pi, \gamma(n+1), \tau(n+1), \pi; \theta)] = J_{n,T}(X(n), \gamma(n), \tau(n), \pi; \theta)$$

$$+ \mathbb{E}_n[f_{n+1,n+1}(X_{(n+1)}^\pi, \gamma(n+1), \tau(n+1); X_{n+1}^\pi, \gamma(n+1), \tau(n+1); \theta)] - \mathbb{E}_n[F_{n}(X_{n}, \gamma_{n}, \tau_{n}, X_{n}^\pi, \pi)]$$

$$+ \mathbb{E}_n[G_{n+1}(X_{n+1}^\pi, \gamma(n+1), \tau(n+1); g_{n+1}(X_{n+1}^\pi, \gamma(n+1), \tau(n+1); \theta)) - G_{n}(X_{n}, \gamma_{n}, \tau_{n}, \mathbb{E}_n[X_{n}^\pi].$$

By the law of iterated expectations we have

$$f_{n,k}^\pi(X(n), \gamma(n), \tau(n); x', \gamma', \tau'; \theta) = \mathbb{E}_n[f_{n+1,n,k}(X_{(n+1)}^\pi, \gamma(n+1), \tau(n+1); x', \gamma', \tau'; \theta)],$$

$$g_{n}^\pi(X(n), \gamma(n), \tau(n)) = \mathbb{E}_n[g_{n+1,n}(X_{(n+1)}^\pi, \gamma(n+1), \tau(n+1); \theta)],$$

so the above becomes

$$J_{n,T}(X(n), \gamma(n), \tau(n), \pi; \theta) = \mathbb{E}_n[J_{n+1,T}(X_{(n+1)}^\pi, \gamma(n+1), \tau(n+1), \pi; \theta)]$$

$$= \left(\mathbb{E}_n[f_{n+1,n+1}(X_{(n+1)}^\pi, \gamma(n+1), \tau(n+1); X_{n+1}^\pi, \gamma(n+1), \tau(n+1); \theta)] - \mathbb{E}_n[f_{n+1,n}(X_{(n+1)}^\pi, \gamma(n+1), \tau(n+1); X_{n}, \gamma_{n}, \tau_{n}; \theta)]\right)$$

$$- \left(\mathbb{E}_n[G_{n+1}(X_{n+1}^\pi, \gamma(n+1), \tau(n+1); g_{n+1}(X_{n+1}^\pi, \gamma(n+1), \tau(n+1); \theta)) - G_{n}(X_{n}, \gamma_{n}, \tau_{n}, \mathbb{E}_n[X_{n}^\pi].

**Step 2: Recursion for $V_{n,T}(X(n), \gamma(n), \tau(n); \theta)$**. Assume that there exists an optimal strategy $\pi^*$, and consider a strategy $\pi$ that coincides with $\pi^*$ after time $n$, so $\pi_k(X(k), \gamma(k), \tau(k); \theta) = \pi^*_k(X(k), \gamma(k), \tau(k); \theta)$, for all $k = n + 1, \ldots, T - 1$. By definition, we then have

$$J_{n,T}(X(n), \gamma(n), \tau(n), \pi^*; \theta) = V_{n,T}(X(n), \gamma(n), \tau(n); \theta),$$

$$J_{n,T}(X(n), \gamma(n), \tau(n), \pi; \theta) \leq V_{n,T}(X(n), \gamma(n), \tau(n); \theta).$$

For the optimal strategy $\pi^*$, we define

$$f_{n,k}(x, \gamma, \tau; x', \gamma', \tau'; \theta) := f_{n,k}^\pi(x, \gamma, \tau; x', \gamma', \tau'; \theta),$$

$$g_{n}(x, \gamma, \tau) := g_{n}^\pi(x, \gamma, \tau).$$
Then, since \( \pi \) and \( \pi^* \) coincide after time \( n \), we have

\[
J_{n+1,n}(X_{n+1}, \gamma(n+1), \tau(n+1), \pi; \theta) = V_{n+1,n}(X_{n+1}, \gamma(n+1), \tau(n+1); \theta),
\]

\[
f_{n+1,k}(X_{n+1}, \gamma(n+1), \tau(n+1); x', \gamma', \tau', \theta) = f_{n+1,k}^*(X_{n+1}, \gamma(n+1), \tau(n+1); x', \gamma', \tau', \theta),
\]

\[
g_{n+1}(X_{n+1}, \gamma(n+1), \tau(n+1)) = g_n^*(X_{n+1}, \gamma(n+1), \tau(n+1)).
\]

In turn, using the recursion for \( J_{n,n}(X_{n+1}, \gamma(n+1), \tau(n+1), \pi; \theta) \), we can write

\[
V_{n,n}(X_{n+1}, \gamma(n+1), \tau(n+1); \theta) = \sup_{\pi} \left\{ \mathbb{E}_n[V_{n+1,n}(X_{n+1}, \gamma(n+1), \tau(n+1); \theta)] \right\} \tag{B.5}
\]

with the terminal condition

\[
V_{T,T}(X_{T}, \gamma(T), \tau(T); \theta) = F_T(X_T, \gamma_T, X_T; \theta) + G_T(X_T, \gamma_T, X_T; \theta).
\]

From (B.2), (B.3), and (B.4), we have that, for any \( x' > 0, \gamma' > 0, \) and \( \tau' \geq 0 \), and any \( k = 0, 1, \ldots, T \), the function sequence \( (f_{n,k}(X_{n+1}, \gamma(n+1), \tau(n+1); x', \gamma', \tau', \theta))_{0 \leq n \leq T} \) is determined by the recursion

\[
f_{n,k}(X_{n+1}, \gamma(n+1), \tau(n+1); x', \gamma', \tau', \theta) = \mathbb{E}_n[f_{n+1,k}(X_{n+1}, \gamma(n+1), \tau(n+1); x', \gamma', \tau', \theta)], \quad n = 0, \ldots, T - 1,
\]

\[
f_{T}(X_{T}, \gamma(T), \tau(T); x', \gamma', \tau', \theta) = F_n(x', \gamma', \tau', X_T; \theta).
\]

Furthermore, the function sequence \( (g_{n}(X_{n+1}, \gamma(n+1), \tau(n+1)))_{0 \leq n \leq T} \) is determined by the recursion

\[
g_{n}(X_{n+1}, \gamma(n+1), \tau(n+1)) = \mathbb{E}_n[g_{n+1}(X_{n+1}, \gamma(n+1), \tau(n+1))], \quad n = 0, \ldots, T - 1,
\]

\[
g_{T}(X_{T}, \gamma(T), \tau(T)) = X_T.
\]

We also have, for \( 0 \leq n \leq T \), the probabilistic representation

\[
f_{n,k}(X_{n+1}, \gamma(n+1), \tau(n+1); x', \gamma', \tau', \theta) = \mathbb{E}_n[F_k(x', \gamma', \tau', X_T^\gamma; \theta)], \quad g_{n}(X_{n+1}, \gamma(n+1), \tau(n+1)) = \mathbb{E}_n[X_T^\gamma].
\]

We easily conclude.

\[
\square
\]

\textit{Proof of Proposition 4.1:} Assuming the existence of an optimal control \( \pi^* \), the value function at time \( n + 1 \)
satisfies

\[
V_{n+1,T}(X_{n+1}, \gamma(n+1), \tau(n+1); \theta) = f_{n+1,n+1}(X_{n+1}, \gamma(n+1), \tau(n+1); X_{n+1}, \gamma(n+1), \tau(n+1); \theta) + \frac{\Delta_{n+1}}{2} \times \left( \frac{g_{n+1}(X_{n+1}, \gamma(n+1), \tau(n+1))}{X_{n+1}} \right)^2,
\]

and plugging this into the HJB equation in Proposition B.1 gives

\[
V_n, T(X_n, \gamma(n), \tau(n); \theta) = \sup_{\pi} \left\{ \mathbb{E}_n \left[ f_{n+1,n}(X^\pi_{n+1}, \gamma(n+1), \tau(n+1); X_n, \gamma(n), \tau(n); \theta) \right] + \frac{\Delta_n}{2} \times \left( \frac{g_n(X^\pi_{n+1}, \gamma(n+1), \tau(n+1))}{X_n} \right)^2 \right\}.
\]

(B.7)

Next, we look for a candidate optimal policy of the form

\[
\pi_n = \pi_n(X_n, \gamma(n), \tau(n); \theta) = \pi_n(X_n, \gamma(n), \tau(n); \theta) \times X_n.
\]

From such a policy we use the wealth dynamics (3.8) to show that (see Appendix B)

\[
\mathbb{E}_n[X^\pi_n] = a_n(\gamma(n), \tau(n)) \times X_n, \quad \mathbb{E}_n[(X^\pi_n)^2] = b_n(\gamma(n), \tau(n)) \times X^2_n,
\]

(B.8)

where the \(a_n\) and \(b_n\)-coefficients are \(\pi\)-dependent and satisfy the recursions

\[
a_n(\gamma(n), \tau(n)) = (R + \bar{\mu} \pi(n, \gamma(n), \tau(n); \theta)) \times \mathbb{E}_n[a_{n+1}(\gamma(n+1), \tau(n+1))],
\]

\[
b_n(\gamma(n), \tau(n)) = (\sigma^2 \pi_n^2(\gamma(n), \tau(n); \theta) + (R + \bar{\mu} \pi(n, \gamma(n), \tau(n); \theta))^2) \times \mathbb{E}_n[b_{n+1}(\gamma(n+1), \tau(n+1))].
\]

(B.9)

From (B.1), (B.2), and (B.4), it then follows that for an optimal policy,

\[
f_{n,k}(X(n), \gamma(n), \tau(n); x', \gamma', \tau', \theta) = a_n(\gamma(n), \tau(n)) \times \frac{X_n}{x'} - 1 - \frac{\Delta_k(\gamma', \tau'; \theta)}{2} \times b_n(\gamma(n), \tau(n)) \times \left( \frac{X_n}{x'} \right)^2,
\]

\[
g_n(X(n), \gamma(n), \tau(n)) = a_n(\gamma(n), \tau(n)) \times X_n.
\]

Plugging this into (B.7), using (3.8), and dropping the arguments from \(a_{n+1}, b_{n+1}, \) and \(\pi_n\), gives

\[
V_n, T(X_n, \gamma(n), \tau(n); \theta) = \sup_{\pi} \left\{ \mathbb{E}_n \left[ a_{n+1} \times \frac{X^\pi_{n+1}}{X_n} - 1 - \frac{\Delta_n}{2} \times b_{n+1} \times \left( \frac{X^\pi_{n+1}}{X_n} \right)^2 \right] + \frac{\Delta_n}{2} \times \left( \mathbb{E}_n \left[ a_{n+1} \times \frac{X^\pi_{n+1}}{X_n} \right] \right)^2 \right\}
\]

\[
= \sup_{\pi} \left\{ \mathbb{E}_n \left[ a_{n+1} \times (RX_n + Z_{n+1} \pi_n) - 1 - \frac{\Delta_n}{2} \times b_{n+1} \times (RX_n + Z_{n+1} \pi_n)^2 \right] + \frac{\Delta_n}{2} \times \left( \mathbb{E}_n \left[ a_{n+1} \times (RX_n + Z_{n+1} \pi_n) \right] \right)^2 \right\}.
\]
Recalling that $Z_{n+1}$ has mean $\bar{\mu}$ and variance $\sigma^2$ gives

\[
V_{n,T}(X(n), \gamma(n), \tau(n); \theta) = \sup_{\pi} \left\{ \mathbb{E}_n[a_{n+1}] \times (RX_n + \bar{\mu} \pi_n) - 1 - \frac{1}{2} \frac{\Delta_n}{X_n} \times \mathbb{E}_n[b_{n+1}] \times (R^2 X_n^2 + 2RX_n \bar{\mu} \pi_n + (\mu^2 + \sigma^2) \pi_n^2) \right. \\
\left. \quad + \frac{1}{2} \frac{\Delta_n}{X_n} \times (\mathbb{E}_n[a_{n+1}])^2 \times (R^2 X_n^2 + 2R \bar{\mu} \pi_n + \mu^2 \pi_n^2) \right\}.
\]

This can be rewritten as

\[
V_{n,T}(X(n), \gamma(n), \tau(n); \theta) = \sup_{\pi} \left\{ \bar{\mu} \times \left[ \mathbb{E}_n[a_{n+1}] - R \times \Delta_n \times (\mathbb{E}_n[b_{n+1}] - (\mathbb{E}_n[a_{n+1}])^2) \right] \times \pi_n - \frac{1}{2} \frac{\Delta_n}{X_n} \times \left[ \mathbb{E}_n[b_{n+1}] \times (\sigma^2 + \bar{\mu}^2) - (\mathbb{E}_n[a_{n+1}])^2 \times \bar{\mu}^2 \right] \times \pi_n^2 - 1 + RX_n \times \left[ \mathbb{E}_n[a_{n+1}] - \frac{\Delta_n}{2} \times \mathbb{E}_n[a_{n+1}] \times R + \frac{\Delta_n}{2} \times (\mathbb{E}_n[a_{n+1}])^2 \times R \right] \right\},
\]

and taking the derivative with respect to $\pi_n$ gives the optimal allocation (4.1). One can then easily check that the HJB equation in the proposition is satisfied by this solution.

Proof of (B.8)-(B.9): For $n = T - 1$ we have by (3.8)

\[
\mathbb{E}_{T-1} \left[ \frac{X^\tau}{X_{T-1}} \right] = \mathbb{E}_{T-1} [R + Z_T \pi_{T-1}(\gamma_{T-1}, \tau_{T-1})] = R + \bar{\mu} \pi_{T-1}(\gamma_{T-1}, \tau_{T-1}) = a_{T-1}(\gamma_{T-1}, \tau_{T-1}).
\]

Next, let $n \in \{0, 1, \ldots, T - 2\}$ and assume that the result holds for $n+1, n+2, \ldots, T - 1$. Then,

\[
\mathbb{E}_n \left[ \frac{X^\tau}{X_{n+1}} \right] = \mathbb{E}_n \left[ (R + Z_{n+1} \pi_n(\gamma_n, \tau_n)) \times \prod_{k=n+1}^{T-1} (R + Z_{k+1} \pi_k(\gamma_k, \tau_k)) \right] = (R + \bar{\mu} \pi_n(\gamma_n, \tau_n)) \times \mathbb{E}_n \left[ \prod_{k=n+1}^{T-1} (R + Z_{k+1} \pi_k(\gamma_k, \tau_k)) \right] = (R + \bar{\mu} \pi_n(\gamma_n, \tau_n)) \times \mathbb{E}_n[a_{n+1}(\gamma_{n+1}, \tau_{n+1})] =: a_n(\gamma_n, \tau_n).
\]

The result for the $b_n$-coefficients can be shown in the same way.

Proof of Corollary 4.3: Part (a) follows follow from Proposition 4.1 with $\tau_n = n$, for all $n \geq 0$. Part (b) follows from Proposition 4.1 with $\tau_n = 0$, for all $n \geq 0$.  

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**Proof of (4.9)–(4.11):** First consider the case $\phi \equiv 1$, and note that for $n = 0, 1, \ldots, T - 1$ there exist constants $K_a(n, T)$ and $K_b(n, T)$ such that,

$$1 \leq \sup_{\gamma \in \Gamma} a_n(\gamma) < K_a(n, T) < \infty, \quad 1 \leq \sup_{\gamma \in \Gamma} b_n(\gamma) < K_b(n, T) < \infty.$$  

Then we can write

$$\mathbb{E}_n^{(1)}[a_{n+1}(\gamma_{n+1})] = \Lambda_{\gamma_n, \gamma_n} \times a_{n+1}(\gamma_n) + \sum_{\gamma \neq \gamma_n} \Lambda_{\gamma, \gamma_n} \times a_{n+1}(\gamma) =: a_{n+1}(\gamma_n) + R_a(\gamma_n),$$

$$\mathbb{E}_n^{(1)}[b_{n+1}(\gamma_{n+1})] = \Lambda_{\gamma_n, \gamma_n} \times b_{n+1}(\gamma_n) + \sum_{\gamma \neq \gamma_n} \Lambda_{\gamma, \gamma_n} \times b_{n+1}(\gamma) =: b_{n+1}(\gamma_n) + R_b(\gamma_n),$$

where

$$|R_a(\gamma_n)| \leq 2K_a(n, T) \times (1 - \Lambda_{\gamma_n, \gamma_n}) = O(1 - \Lambda_{\gamma_n, \gamma_n}), \quad |R_b(\gamma_n)| \leq 2K_b(n, T) \times (1 - \Lambda_{\gamma_n, \gamma_n}) = O(1 - \Lambda_{\gamma_n, \gamma_n}).$$

Next, for the random variable $Z$, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, independently of $(Z_n)_{n \geq 1}$, but with the same distribution, we have that $Z' = Z/\sigma$ has mean $\bar{\mu}/\sigma$ and unit variance. Hence,

$$(\sigma^2 + \bar{\mu}^2) \times b_{n+1}(\gamma_n) - \bar{\mu}^2 \times a_{n+1}(\gamma_n) = \mathbb{E}_n^{(1)}[Z^2 \times (1 + r_{n+1, T})^2|\gamma_{n+1} = \gamma_n] + \mathbb{E}_n^{(1)}[Z \times (1 + r_{n+1, T})|\gamma_{n+1} = \gamma_n]^2$$

$$= \sigma^2 \times \text{Var}_n^{(1)}[Z' \times (1 + r_{n+1, T})|\gamma_{n+1} = \gamma_n].$$

Using the above we can write the optimal allocation in (4.5) as

$$\pi^*_n, T(\gamma_n, n) = \frac{\bar{\mu}}{\gamma_n} \times \frac{a_{n+1}(\gamma_n) - R \times \gamma_n \times (b_{n+1}(\gamma_n) - a_{n+1}^2(\gamma_n)) + O(1 - \Lambda_{\gamma_n, \gamma_n})}{(\sigma^2 + \bar{\mu}^2) \times b_{n+1}(\gamma_n) - \bar{\mu}^2 \times a_{n+1}^2(\gamma_n) + O(1 - \Lambda_{\gamma_n, \gamma_n})} \quad \text{(B.10)}$$

$$= \frac{\bar{\mu}/\sigma^2}{\gamma_n} \times \frac{\mathbb{E}_n^{(1)}[1 + r_{n+1, T}^2|\gamma_{n+1} = \gamma_n] - R \times \gamma_n \times \text{Var}_n^{(1)}[1 + r_{n+1, T}^2|\gamma_{n+1} = \gamma_n]}{\text{Var}_n^{(1)}[Z' \times (1 + r_{n+1, T})|\gamma_{n+1} = \gamma_n]} + O(1 - \Lambda_{\gamma_n, \gamma_n}),$$

and (4.10) follows by using Lemma A.1-(a) and (4.8), and by setting $R = 1$. To justify the second equality above, the following inequalities are sufficient. First, for $n = 0, 1, \ldots, T - 1$, there exists a constant $K(n, T) < \infty$ such that

$$\sup_{\gamma \in \Gamma} |a_{n+1}(\gamma) - R \times \gamma \times (b_{n+1}(\gamma) - a_{n+1}^2(\gamma))| < K(n, T) < \infty.$$
Second, from the recursive equation (4.6), it is clear that \( b_{n+1}(\gamma) \geq 1 \), so, by Jensen’s inequality,

\[
b_{n+1}(\gamma) \times (\sigma^2 + \mu^2) - \sigma_n^2(\gamma)\times \mu^2 \geq b_{n+1}(\gamma) \times \sigma^2 \geq \sigma^2 > 0, \quad \gamma \in \Gamma.
\]

The identity (4.9) is a special case of (4.10), and (4.11) is shown in a similar way.

\[\Box\]

C Proofs of Results in Section 5 and Section 7

Proof of (5.2)–(5.3): Recall from (A.1) that \( p_n(\gamma'; \gamma, \tau) \) is the probability of being in level \( \gamma' \) at time \( n \), after level \( \gamma \) being realized at time \( \tau \), with no intermediate risk aversion updates. We then have by Lemma A.2-(d),

\[
\tilde{R}_{n,T}(\gamma', \tau) = \sum_{\gamma' \in \mathcal{E}} p_n(\gamma'; \gamma, \tau) \times \frac{\left| \pi_{n,T}^*(\gamma, \tau; 0) - \pi_{n,T}^*(\gamma', \gamma; 0) \right|}{\pi_{n,T}^*(\gamma'; 0)} + O(1 - p_{\phi(\gamma)}(\gamma; \gamma)) + O(1 - p_{\phi(\gamma')}(\gamma'; \gamma)).
\]

The second equality above can be justified in the same way as (B.10) was justified, and for the third equality we used that for any \( \gamma' \in \Gamma, \)

\[
p_n(\gamma'; \gamma, \tau) \leq 1 - p_n(\gamma; \gamma, \tau) = 1 - p_{n-\tau}(\gamma; \gamma) \leq 1 - p_{\phi(\gamma)}(\gamma; \gamma),
\]

which follows from Lemma A.1-(a) and \( n - \tau < \phi(\gamma) \). We then have for any \( 0 \leq \tau < T \) and Lemma A.2-(e),

\[
\sup_{\tau \leq n < n+1} \tilde{R}_{n,T}(\gamma, \tau) = \sup_{\tau \leq n < n+1} \mathbb{E}_n^{(\phi)} \left[ \frac{c_n^{(\gamma)}(\gamma; \gamma)}{\pi_{n,T}^*(\Psi(\gamma); 0)} \pi_{n,T}^*(\gamma; 0) - \pi_{T-1,T}^*(\Psi(\gamma_n); 0) \right] \gamma_n = \gamma, \tau_n = \tau \right) + O(1 - p_{\phi(\gamma)}(\gamma; \gamma))
\]
The first inequality above follows from Lemma A.2-(e), because for \( \gamma < \Psi(\bar{\gamma}_n) \) we have \( c_{n,T}(\gamma) \leq c_{n,T}(\Psi(\bar{\gamma}_n)) \), so

\[
\pi^*_{T-1,T}(\Psi(\bar{\gamma}_n); 0) \leq \frac{c_{n,T}(\gamma)}{c_{n,T}(\Psi(\bar{\gamma}_n))} \times \pi^*_{T-1,T}(\gamma; 0) \leq \pi^*_{T-1,T}(\gamma; 0),
\]

with the inequalities flipping for \( \gamma > \Psi(\bar{\gamma}_n) \). The second equality follows from that conditionally on \( \gamma_n = \gamma \) and \( \tau_n = \tau \) we have \( \bar{\gamma}_n \sim \mathcal{N}(\gamma, (n - \tau) \times \sigma^2) \), so \( \mathbb{P}_n(\psi(\bar{\gamma}_n) > c) = \mathbb{P}_n(\psi(\bar{\gamma}_n) > c) \) is increasing in \( n \) for all \( c > 0 \). The second inequality and third equality also follow from this and the definition of \( \tilde{R}_{n,T}(\gamma, \tau) \).

Now (5.2) and (5.3) follow from the above and the fact that the \( O(\cdot) \) term vanishes and the inequalities becoming equalities for \( \tau = (T - \phi(\gamma)) \lor 0 \).

\( \square \)

**Proof of Proposition 5.2:** By Lemma A.2-(a) we have

\[
\mathcal{R}_T(\phi)(\gamma) = \mathbb{E}_n(\phi) \left[ \frac{\pi^*_{n,T}(\gamma_n; 0) - \pi^*_{n,T}(\Psi(\bar{\gamma}_n); 0)}{\pi^*_{n,T}(\Psi(\bar{\gamma}_n); 0)} \left| \gamma_n = \gamma, \tau_n = \tau \right|_{\tau=(T-\phi(\gamma))\lor0, n=T-1} \right] = \mathbb{E}_n(\phi) \left[ \frac{\psi(\bar{\gamma}_n) - \gamma_n}{\gamma_n} \left| \gamma_n = \gamma, \tau_n = \tau \right|_{\tau=(T-\phi(\gamma))\lor0, n=T-1} \right].
\]

The distribution of \( \bar{\gamma}_n \) under \( \mathbb{P}_n(\phi) \), given that \( \gamma_n = \gamma \) and \( \tau_n = \tau \), is the same as the distribution of \( \bar{\gamma}_{n-\tau} \) under \( \mathbb{P} \), given that \( \gamma_0 = \gamma \). Hence, noting that \( n - \tau = (\phi(\gamma) \land T) - 1 \) for \( \tau = (T - \phi(\gamma)) \lor 0 \) and \( n = T - 1 \), we have

\[
\mathcal{R}_T(\phi)(\gamma) = \mathbb{E} \left[ \frac{\psi(\bar{\gamma}(\phi(\gamma) \land T) - 1) - \gamma_0}{\gamma_0} \left| \gamma_0 = \gamma \right. \right] = \frac{\mu(\phi(\gamma) \land T) - 1(\gamma)}{\gamma},
\]

where the second equality follows from the definition of \( \mu_n \) in (3.6). The monotone properties and limits then follow from Lemma A.1-(b).
Proof of Proposition 5.3: We have

\[ \mathcal{R}_{n}^{(\phi)}(\gamma, \tau; \theta) = \frac{\left| \pi_{n,T}^{*}(\gamma, \tau; \theta) - \pi_{n,T}^{*}(\gamma; 0) \right|}{\pi_{n,T}^{*}(\gamma; 0)} + O(1 - p_{\phi(\gamma)}(\gamma; \gamma)) \]

\[ = \frac{\left| \pi_{n,T}^{*}(\gamma + \theta \times \delta_{n}(\gamma, \tau); 0) - \pi_{n,T}^{*}(\gamma; 0) \right|}{\pi_{n,T}^{*}(\gamma; 0)} + O(\theta \times (1 - p_{\phi(\gamma)}(\gamma; \gamma))) + O(1 - p_{\phi(\gamma)}(\gamma; \gamma)) \]

\[ \leq \frac{\left| \pi_{T-1,T}^{*}(\gamma + \theta \times \delta_{n}(\gamma, \tau); 0) - \pi_{T-1,T}^{*}(\gamma; 0) \right|}{\pi_{T-1,T}^{*}(\gamma; 0)} + O(\theta \times (1 - p_{\phi(\gamma)}(\gamma; \gamma))) \]

\[ \leq \frac{\theta \times \delta_{n}(\gamma, \tau)}{\gamma + \theta \times \delta_{n}(\gamma, \tau)} + O(\theta \times (1 - p_{\phi(\gamma)}(\gamma; \gamma))). \tag{C.2} \]

The first equality above follows from Lemma A.1-(d) and similar steps as those used to justify (B.10). For the second equality, first consider \( n \geq \tau \) where \( \tau \) is such that \( \tau + \phi(\gamma) > T - 1 \). Then we can show recursively for \( n' = T - 1, T - 2, \ldots, n \) that

\[ \pi_{n',T}^{*}(\gamma, \tau; \theta) = \pi_{n',T}^{*}(\gamma + \theta \times \delta_{n}(\gamma, \tau); 0) + O(\theta \times (\delta_{n'}(\gamma, \tau) - \delta_{n}(\gamma, \tau))), \]

\[ a_{n'}(\gamma, \tau) = a_{n'}^{(\infty)}(\gamma + \theta \times \delta_{n}(\gamma, \tau)) + O(\theta \times (\delta_{n'}(\gamma, \tau) - \delta_{n}(\gamma, \tau))), \]

\[ b_{n'}(\gamma, \tau) = b_{n'}^{(\infty)}(\gamma + \theta \times \delta_{n}(\gamma, \tau)) + O(\theta \times (\delta_{n'}(\gamma, \tau) - \delta_{n}(\gamma, \tau))). \]

Working backwards one can show that for a general \( n \) and \( \tau \), and \( n' \geq n \),

\[ \pi_{n,T}^{*}(\gamma, \tau; \theta) = \pi_{n,T}^{*}(\gamma + \theta \times \delta_{n}(\gamma, \tau); 0) + O(1 - p_{\phi(\gamma)}(\gamma; \gamma)) + O(\theta \times (\delta_{n'}(\gamma, \tau') - \delta_{n}(\gamma, \tau))), \]

\[ a_{n'}(\gamma, \tau) = a_{n'}^{(\infty)}(\gamma + \theta \times \delta_{n}(\gamma, \tau)) + O(1 - p_{\phi(\gamma)}(\gamma; \gamma)) + O(\theta \times (\delta_{n'}(\gamma, \tau') - \delta_{n}(\gamma, \tau))), \]

\[ b_{n'}(\gamma, \tau) = b_{n'}^{(\infty)}(\gamma + \theta \times \delta_{n}(\gamma, \tau)) + O(1 - p_{\phi(\gamma)}(\gamma; \gamma)) + O(\theta \times (\delta_{n'}(\gamma, \tau') - \delta_{n}(\gamma, \tau))), \]

where the additional \( O(1 - p_{\phi(\gamma)}(\gamma; \gamma)) \) term arises because of future updating times at which the realized risk aversion level is random. Now note that there exists a constant \( C < \infty \) such that

\[ |\delta_{n'}(\gamma, \tau') - \delta_{n}(\gamma, \tau)| \leq C \times ((1 - p_{n'-\tau}(\gamma)) + (1 - p_{n'-\tau_{n'}}(\gamma))) \leq C \times (1 - p_{\phi(\gamma)}(\gamma)), \]

by Lemma A.1-(a), as \( n' - \tau_{n'} \leq \phi(\gamma) \), and \( n - \tau \leq \phi(\gamma) \). The second equality in (C.2) now follows. The inequality in (C.2) follows from Lemma A.2-(c) as the first inequality in (C.1). The final equality follows from Lemma A.2-(a). \( \square \)
Proof of (5.9): Using $\Psi(\tilde{\gamma}_n) \approx \tilde{\gamma}_n$ we have

$$\theta \times \delta_n(\gamma) = \theta \times \sqrt{\text{Var}[\Psi(\tilde{\gamma}_n) | \gamma_n = \gamma, \tau_n = 0]} \approx \theta \times \sqrt{\text{Var}[\tilde{\gamma}_n | \gamma_n = \gamma, \tau_n = 0]} = \theta \times \sigma_\gamma \times \sqrt{n},$$

where the final equality follows from $\tilde{\gamma}_n \sim \mathcal{N}(\gamma_n \times \sigma_\gamma^2)$, conditionally on $\gamma_n = \gamma$ and $\tau_n = 0$.

Proof of (7.2)-(7.4): For (7.2) we used the approximation $\Psi(\gamma) \approx \gamma$ and Lemma A.1-(c) to write

$$\bar{R}_T^{(\phi)} = \sum_{\gamma \in \Gamma} \lambda(\gamma) \times R_T^{(\phi)}(\gamma) \approx \sum_{\gamma \in \Gamma} \lambda(\gamma) \times \sqrt{\frac{2 \sigma_\gamma}{\pi \gamma} \sqrt{\phi(\gamma)}},$$

and for the smallest natural number $c$ such that the right-hand side is greater than $\kappa$, with $\phi \equiv c$, we have

$$c = \left[ \frac{\kappa^2 \pi}{2} \left( \sum_{\gamma \in \Gamma} \lambda(\gamma) \times \frac{\sigma_\gamma}{\gamma} \right)^2 \right] \approx \frac{\kappa^2 \pi}{2} \left( \sum_{\gamma \in \Gamma} \lambda(\gamma) \times \frac{\sigma_\gamma}{\gamma} \right)^2.$$

For (7.4) we plug this value into the approximation for $\bar{R}_T^{(\phi)}$ above.

Proof of (7.5): By Lemma A.1-(a) we have

$$\alpha_{b,T}(\gamma) = \sup_{0 \leq n < T} \mathbb{P}_n^{(\phi_b)}(\Psi(\tilde{\gamma}_n + \phi_b(\gamma)) = \gamma | \gamma_n = \gamma, \tau_n = n) \times 1_{\{n + \phi_b(\gamma) < T\}}$$

$$= \sup_{0 \leq n < T} \mathbb{P}(\Psi(\tilde{\gamma}_n + \phi_b(\gamma)) = \gamma | \gamma_n = \gamma, \tau_n + \phi_b(\gamma) = n) \times 1_{\{n + \phi_b(\gamma) < T\}}$$

$$= \sup_{0 \leq n < T} p_{\phi_b(\gamma)}(\gamma; \gamma) \times 1_{\{n + \phi_b(\gamma) < T\}}$$

$$= p_{\phi_b(\gamma)}(\gamma; \gamma) \times 1_{\{\phi_b(\gamma) < T\}}.$$  

Notice, again by Lemma A.1-(a), that $\alpha_{b,T}(\gamma)$ is decreasing in $\phi_b(\gamma)$, the time spent in risk aversion level $\gamma$ before an update of risk preferences is triggered, and therefore decreasing in the threshold $b_\gamma$. Also, the probability of a false positive is zero if $\phi_b(\gamma) \geq T$, as in that case there will be no updates prior to the terminal date $T$. For (7.5), we first use Lemma A.1-(a) to write

$$\alpha_{b(\kappa),T}(\gamma) = \left[ \Phi\left( \frac{\gamma^+ - \gamma}{\sigma_\gamma \sqrt{\phi_{b(\kappa)}(\gamma)}} \right) - \Phi\left( \frac{\gamma^- - \gamma}{\sigma_\gamma \sqrt{\phi_{b(\kappa)}(\gamma)}} \right) \right] \times 1_{\{\phi_{b(\kappa)}(\gamma) < T\}}.$$ 

The result then follows from (7.3), which is based on the approximation $\Psi(\gamma) \approx \tilde{\gamma}$. 

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D Pseudocode

Following is a pseudocode for the backward recursion to compute the optimal risky asset allocations in Proposition 4.1. For any updating rule $\phi$, and $\theta \geq 0$, we compute

$$\pi^*_{n,k,\tau} := \pi_n(\bar{\gamma}_k, \tau; \theta), \quad a_{n,k,\tau} := a_n(\bar{\gamma}_k, \tau), \quad b_{n,k,\tau} := b_n(\bar{\gamma}_k, \tau),$$

for $0 \leq n \leq T$, $1 \leq k \leq K$, and $0 \leq \tau \leq n$. Recall that $R = 1 + r$, where $r \geq 0$ is the risk-free rate, $\bar{\mu} = \mu - r$ and $\sigma$ are the excess return and volatility of the risky asset, and the transition probability $p_n(\gamma'; \gamma)$ is defined in (A.5).

1. Set $a_{T,k,\tau} = 1$, $b_{T,k,\tau} = 1$, and $\pi^*_{T,k,\tau} = 0$, for $k = 1, 2, \ldots, K$ and $\tau = 0, 1, \ldots, T$.

2. For $n = T - 1, T - 2, \ldots, 0$:
   For $k = 1, 2, \ldots, K$:
     For $\tau = n, n - 1, \ldots, 0$:

     $$\tau_{n+1} = 1_{(\tau + \phi(\bar{\gamma}_k) > n+1)} \times \tau + 1_{(\tau + \phi(\bar{\gamma}_k) \leq n+1)} \times (n + 1),$$

     $$\mu_a = \sum_{k' = 1}^{K} p_{\tau_{n+1} - \tau}(\bar{\gamma}_{k'}; \bar{\gamma}_k) \times a_{n+1,k',n+1},$$

     $$\mu_b = \sum_{k' = 1}^{K} p_{\tau_{n+1} - \tau}(\bar{\gamma}_{k'}; \bar{\gamma}_k) \times b_{n+1,k',n+1},$$

     $$\delta = \sum_{k' = 1}^{K} p_{n-\tau}(\bar{\gamma}_{k'}; \gamma_k) \times \bar{\gamma}_{k'}^2 - \left( \sum_{k' = 1}^{K} p_{n-\tau}(\bar{\gamma}_{k'}; \gamma) \times \bar{\gamma}_{k'} \right)^2,$$

     $$\pi^*_{n,k,\tau} = \frac{\bar{\mu}}{\bar{\gamma}_k + \theta \times \delta} \times \mu_a \times \frac{R \times (\bar{\gamma}_k + \theta \times \delta) \times (\mu_b - \mu_a^2)}{(\sigma^2 + \bar{\mu}^2) \times \mu_b - \bar{\mu}^2 \times \mu_a^2},$$

     $$a_{n,k,\tau} = \mu_a \times (R + \bar{\mu} \times \pi^*_{n,k,\tau}),$$

     $$b_{n,k,\tau} = \mu_b \times (\sigma^2 \times (\pi^*_{n,k,\tau})^2 + (R + \bar{\mu} \times (\pi^*_{n,k,\tau})^2).$$

End

End

End

References


