Proceedings of the 29th Conference
on Decision and Control
Honolulu, Hawaii • December 1990

Investment-Consumption Models with Transaction Costs and Markov-Chain Parameters

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Abstract

This paper considers an infinite horizon investment-consumption model in which a single agent consumes and distributes his wealth in two assets, a bond and a stock. The problem of maximization of the total utility from consumption is treated. State (amount allocated in assets) and control (consumption, rates of trading) constraints are present. It is shown that the value function is the unique viscosity solution of a system of variational inequalities with gradient constraints.

1. INTRODUCTION

In this paper we examine a general investment and consumption decision problem for a single agent. The investor consumes at a nonnegative rate and he distributes his current wealth between two assets continuously in time. One asset is a bond, i.e. a riskless security with instantaneous rate of return r. The other asset is a stock, which rate of return z_t is a continuous time Markov chain. In our version of the model the investor *cannot borrow* money to finance his investment in bond and he *cannot short-sell* the stock. In other words, the amount of money allocated in bond and stock must stay nonnegative.

When the investor makes a transaction, he pays transaction fees which are assumed to be proportional to the amount transacted. The control objective is to maximize, in an infinite horizon, the expected discounted utility which comes only from consumption. Due to the presence of the transaction fees, this is a singular control problem.

II. The Financial Model with Transaction Fees

We consider a market with two assets: a *bond* and a *stock*. The price P_t^0 of the bond is given by

$$\begin{aligned} dP_t^0 &= rP_t^0 dt \\ P_0^0 &= p_0, \end{aligned} \qquad (1)$$

where $r > 0$. The price P_t of the stock satisfies

$$\begin{aligned} dP_t &= z(t)P_t dt \\ P_0 &= p. \end{aligned} \qquad (2)$$

The rate of return z is a *finite state continuous time Markov chain* defined on some underlying probability space (Ω, F, P) with jumping rate $q_{zz'}$ from state z to state z'. The state space is denoted by Z. The associated generator $\mathcal{L}$ of the Markov chain has the form

$$\mathcal{L}v(z) = \sum_{z' \neq z} q_{zz'}[v(z\ ) - v(z)].$$

Let $K = \max_{z \in Z} z$. A natural assumption is $K \geq r$. The amount of wealth x_t and y_t, invested at time t in bond and stock respectively, are the state variables and they evolve (see [17]) according to the equations

$$\begin{aligned} dx_t &= (rx_t - C_t)dt - (1+\lambda)dM_t + (1-\mu)dN_t \\ dy_t &= z(t)y_t dt + dM_t - dN_t \\ x_0 &= x, y_0 = y, z(0) = z\ . \end{aligned} \qquad (3)$$

The numbers λ and μ represent the proportional transaction fees; they are assumed to be nonnegative and one of them must always be positive. For simplicity we assume here that all financial charges are paid from the holdings in bond. The investor cannot borrow money or short sell the stock. The control processes are the *consumption rate* C_t and the processes M_t and N_t which represent *the cumulative purchases* and *sales of stock* respectively. The controls (C_t, M_t, N_t) are *admissible* if:

(i) C_t is F_t-measurable where $F_t = \sigma(z_s : 0 \leq s \leq t)$ and $C_t \geq 0$ a.e. $t \geq 0$.

(ii) M_t, N_t are F_t-measurable, right continuous and nondecreasing processes.

(iii) $x_t \geq 0$, $y_t \geq 0$ a.e. $t \geq 0$, where x_t, y_t are the trajectories given by the state equation (3) using the controls (C_t, M_t, N_t).

We denote by A the set of admissible controls.

The total expected discounted utility J coming from consumption is given by

$$J(x,y,z,C,M,N) = E\int_0^{+\infty} e^{-\beta t} U(C_t)dt$$

with $(C,M,N) \in A$ and $z(0) = z$, where the utility function $U:[0,+\infty) \to [0,+\infty)$ is assume to have the following properties:

U is strictly increasing, bounded, concave, C^1 function and

$$U(0) = 0, \lim_{c \to 0} U'(c) = +\infty, \lim_{c \to +\infty} U'(c) = 0.$$

The discount factor $\beta > 0$ weights consumption now versus consumption later, large β denoting instant gratification. Note that the controls M and N are acting implicitly through the constraint (iii).

The value function u is given by

$$u(x,y,z) = \sup_A E\int_0^{+\infty} e^{-\beta} U(C_t)dt.$$

Our goal is to derive the Bellman equation associated with this singular control problem and to characterize u as its unique solution. It turns out that the Bellman equation here is a system of variational inequalities.

Transaction costs are an essential feature of some economic theories. In [2], [3] Constantinides assumes that the transaction costs deplete only the riskless asset and that the stock price is a logarithmic Brownian motion. He shows that if an optimal policy exists, it is characterized by two reflecting barriers $\underline{\lambda}, \overline{\lambda}$ with $\underline{\lambda} \leq \overline{\lambda}$, such that the investor does not trade as long as the ratio y_t/x_t lies in $[\underline{\lambda}, \overline{\lambda}]$ and transacts to the closest boundary of the region of no transactions $[\underline{\lambda}, \overline{\lambda}]$, whenever this ratio lies outside this interval. Constantinides's work was generalized by Davis and Norman [5] .

Different criteria were used by Taksar, Klass and Assaf [16] and, under more general assumptions by Fleming, Grossman, Vila and Zariphopoulou [6].

Single-period models with fixed transaction costs are discussed in Leland [11], Mukherjee and Zabel [14], Brennan [1], Goldsmith [8], Levy [12] and Mayshar [13]. Finally, Kandel and Ross [10] introduce quasi-fixed transaction costs.

We now consider a similar control problem in which the controls, which represent the rates of trading, are assumed to be absolutely continuous processes. More precisely, we consider a market which offers a bond and a stock with prices evolving according to equations (1) and (2) respectively. The state variables x_t and y_t, which are the amount of money invested in bond and stock, obey the state equations

$$dx_t = (rx_t - C_t)dt - (1+\lambda)m_t dt + (1-\mu)n_t dt$$

$$dy_t = z(t)y_t dt + m_t dt - n_t dt \qquad (4)$$

$$x_0 = x_t y_0 = y, z(0) = z \quad .$$

The controls of the investor are the *consumption rate* C_t and the *rates of trading* m_t and n_t. The set of admissible controls A_L consists of controls (C,m,n) such that

(i) C_t is F_t-measurable where $F_t = \sigma(z_s : 0 \le s \le t)$, $C_t \ge 0$ a.e. $t \ge 0$.

(ii) m_t, n_t are F_t-measurable right continuous and nonnegative processes.

(iii) $0 \le m_t, n_t \le L$ a.e. $t \ge 0$ for some positive constant L.

(iv) $x_t \ge 0$, $y_t \ge 0$ a.e. $t \ge 0$, where $x_t y_t$ are the solutions of (4) using the controls (C,m,n).

The control objective is to maximize the expected discounted utility from consumption over the set of admissible controls. For each fixed $L > 0$, the value function is given by

$$u^L(x,y,z) = \sup_{A_L} E \int_0^{+\infty} e^{-\beta t} U(C_t) dt,$$

where U is the usual utility function and $\beta > 0$ is the discount factor.

III. Preliminaries

Proposition 2.1: The value functions u and u^L are increasing, concave and uniformly continuous functions on $\bar{\Omega} = [0,+\infty] \times [0,+\infty)$.

In the sequel we will need the following definition:

Definition 2.1: We consider a nonlinear partial differential equation of the form

$$F(X,z,u(X,z), Du(X,z)) = 0 \qquad (5)$$

where $z \in Z$, $X = (x,y)$ with $(x,y) \in \bar{\Omega}$, $Du(x,z) = (\frac{\partial u(X,z)}{\partial x}, \frac{\partial u(X,z)}{\partial y})$ and $F: \bar{\Omega} \times Z \times R \times R^2 \to R$ is continuous, for each $z \in Z$. A continuous function $u: \bar{\Omega} \times Z \to R$ is a constrained viscosity solution of (5) if

i) u is a viscosity subsolution of (5) on $\bar{\Omega}$, i.e. for each $z \in Z$

$$F(X,z,u(X,z),r) \le 0$$

$$X = (x,y) \in \bar{\Omega} \quad \text{and} \quad r \in D^+_{(x,y)} u(X,z)$$

where

$$D^+_{(x,y)} u(X,z) = \{r \in \mathbb{R}^2 : \limsup_{h \to 0} \frac{u(X+h,z) - u(X,z) - r \cdot h}{|h|} \le 0\},$$

ii) u is a viscosity supersolution of (5) in Ω, i.e. for each $z \in Z$

$$F(X,z,u(X,z),r) \ge 0$$

where

$$D^-_{(x,y)} u(X,z) = \{r \in \mathbb{R}^2 : \liminf_{h \to 0} \frac{u(X+h,z) - u(X,z) - r \cdot h}{|h|} \ge 0\}.$$

IV. Results

In the sequel, we characterize the value function u as the unique constrained *viscosity solution of the* associated Bellman equation. Some results about u^L are first stated.

Theorem 3.1: The value function u^L is a constrained viscosity solution of

$$\beta u^L = rxu^L_x + zyu^L_y + \max_{c \ge 0} [-cu^L_x + U(c)] + \mathcal{L}u^L(z)$$
$$+ \max_{0 \le m \le L} [-(1+\lambda)u^L_x + u^L_y]m + \max_{0 \le n \le L} [(1-\mu)u^L_x - u^L_y]n \qquad (6)$$
$$(x,y) \in \bar{\Omega}, \; z \in Z \; .$$

The proof follows along the results of Fleming, Sethi, and Soner [7]. It is essentially based on the dynamic programming principle and Dynkin's formula.

Theorem 3.2: The value function u^L is the unique constrained viscosity solution of (6) in the class of bounded and uniformly continuous functions.

Proof: We show that if u and v are respectively *a viscosity subsolution of* (6) *on* $\bar{\Omega}$ *and a viscosity* supersolution of (6) in Ω, then $u \le v$ on $\bar{\Omega}$. We argue by contradiction, i.e. we assume that

$$\max_{z \in Z} \sup_{X \in \bar{\Omega}} [u(X,z) - v(X,z)] > 0 \qquad (7)$$

which implies that for sufficiently small $\theta > 0$

$$\max_{z \in Z} \sup_{X \in \bar{\Omega}} [u(X,z) - v(X,z) - \theta |X|^2] > 0. \qquad (8)$$

We can find points $z_0 \in Z$ and $\bar{X} \in \bar{\Omega}$ such that

$$u(\bar{X},z_0) - v(\bar{X},z_0) - \theta|\bar{X}|^2 = \max_{z \in Z} \sup_{X \in \bar{\Omega}} [u(X,z) - v(X,z) - \theta|X|^2].$$

In the sequel we omit z_0. Next, for $\varepsilon > 0$ we consider the auxiliary function $\psi: \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$ given by

$$\psi(X,Y) = u(X) - \left|\frac{Y-X}{\varepsilon} - 4(1,1)\right|^2 - \theta|X|^2 - v(Y).$$

We show that if its maximum is achieved at (X_0,Y_0) then $Y_0 \in \Omega$ and

$$|Y_0 - X_0| \le \ell\varepsilon. \qquad (9)$$

We now consider the functions

$$\phi(Y) = u(X_0) - \left|\frac{Y-X_0}{\varepsilon} - 4(1,1)\right|^2 - \theta|X_0|^2$$

$$\bar{\phi}(Y) = v(Y_0) + \left|\frac{Y_0-X}{\varepsilon} - 4(1,1)\right|^2 + \theta|X|^2 .$$

We observe that $u - \bar{\phi}$ has a maximum at $X_0 \in \bar{\Omega}$ and $v - \phi$ has a minimum at $Y_0 \in \Omega$. Applying the definition of viscosity solution and using (9) we get

$$\beta[u(X,z) - v(X,z) - \theta|X|^2] \le \frac{C^2 L^2}{\beta}\theta, \; X \in \bar{\Omega} \text{ and } z \in Z.$$

Sending $\theta \to 0$, contradicts (8).

We now state the main theorems.

Theorem 3.3: The value function u is a constrained viscosity solution of

$$\min[(1+\lambda)u_x - u_y,\ -(1-\mu)u_x + u_y,\ \beta u - rxu_x - zyu_y - \max_{c\geq 0}[-cu_x + U(c)] - \mathcal{L}u(z)] = 0 \tag{10}$$

The proof is based on the Dynamic Programming Principle and the generalized Dynkin's formula. The presence of singular controls and the fact that the Bellman equation is actually a Variational Inequality make the proof rather technical.

Theorem 3.4: The value function u is the unique constrained viscosity solution of (10) in the class of bounded and uniformly continuous functions.

Proof: We are going to show that if $\underline{u}$ and U are respectively a subsolution of (10) on $\bar{\Omega}$ and a supersolution of (10) in Ω, then $U \geq \underline{u}$ on $\bar{\Omega}$. We follow the strategy of Ishii [9]. Let $\phi: \bar{\Omega} \to R$ be defined by $\phi(x,y) = C_1x + C_2y + k$, where C_1, C_2 and k are positive constants satisfying $(1-\mu)C_1 < C_2 < (1+\lambda)C_1$ and $\beta k > rC_1 + KC_2 + \max_{c\geq 0}[-cC_1 + U(c)]$. Let $X = (x,y) \in \bar{\Omega}$, $P = (p,q) \in R\times R$ and $H: \bar{\Omega}\times Z\times R\times R_2 \to \mathbb{R}$ given by

$$H(X,z,v,P) = \min[(1+\lambda)p - q,\ -(1-\mu)p + q,\ \beta u - rxp - zyq - \max_{c\geq 0}[-cp + U(c)] - \mathcal{L}u(z)].$$

Let $U_\theta = \bar{\theta}U + (1-\bar{\theta})\phi$ for $\bar{\theta}\in(0,1)$. Then there exists a positive constant M such that

$$H(X,z,U_\theta,\nabla U_\theta) \geq M(1-\theta) > 0 \quad X\in\bar{\Omega},\ z\in Z.$$

We next show that $U_\theta \geq u$. We work as in Theorem 3.1, we assume that for sufficiently small $\theta > 0$,

$$\max_{z\in Z}\ \sup_{X\in\bar{\Omega}}[u(X,z) - U_\theta(X,z) - \theta|X|^2] > 0. \tag{11}$$

We can find points $z_0 \in Z$ and $\bar{X}\in\bar{\Omega}$ such that

$$u(\bar{X},z_0) - U_\theta(X,z_0) - \theta|\bar{X}|^2 = \max_{z\in Z}\ \sup_{X\in\bar{\Omega}}[u(X,z) - U_\theta(X,z) - \theta|X|^2].$$

We look at

$$\psi(X,Y,z_0) = u(X,z_0) - U_\theta(Y,z_0) - \left|\frac{Y-X}{\varepsilon} - 4(1,1)\right|^2 - \theta|X|^2.$$

If its maximum is achieved at (X_0,Y_0) then $Y_0 \in \Omega$ and

$$|Y_0 - X_0| \leq \ell\varepsilon \tag{12}$$

and

$$\lim_{\theta\downarrow 0}\lim_{\varepsilon\downarrow 0} \theta|X_0|^2 = 0 \tag{13}$$

We now consider the functions

$$\psi(Y) = u(X_0) - \left|\frac{Y-X_0}{\varepsilon} - 4(1,1)\right|^2 - \theta|X_0|^2$$

$$\bar{\psi}(X) = U_\theta(Y_0) - \left|\frac{Y_0-X}{\varepsilon} - 4(1,1)\right|^2 + \theta|X|^2$$

We observe that $u - \bar{\psi}$ has a maximum at X_0 and $U_\theta - \psi$ has a minimum at Y_0. Using the viscosity property we get

$$H(X_0,z_0,u(X_0,z_0),\ P_\varepsilon + 2\theta X_0) - H(Y_0,z_0,U_\theta(Y_0,z_0),P_\varepsilon) \leq -M(1-\theta). \tag{14}$$

where $P_\varepsilon = -\frac{2}{\varepsilon}(\frac{Y_0-X_0}{\varepsilon} - 4(1,1))$.

Let $Y_0 = (x_0,y_0)$ and $P_\varepsilon = (p_\varepsilon,q_\varepsilon)$. We now look at different cases depending on the form of $H(Y_0,z_0,U_\theta(Y_0z_0),\ P_\varepsilon)$. If $H(Y_0,z_0,U_\theta(Y_0,z_0),\ P_\varepsilon) = \beta U_\theta - rx_0p_\varepsilon - z_0y_0q_\varepsilon - \max_{c\geq 0}[-cp_\varepsilon + U(c)] - \mathcal{L}U_\theta(Y_0,z_0)$

we work as in Theorem 3.2 and we contradict (3.6). In the other cases (14) yields

$$-C_1\theta|X_0| \leq -M(1-\theta)$$

for some $C_1 > 0$. Sending $\theta\downarrow 0$ and using (13) we again contradict (11). Finally we send $\bar{\theta}\uparrow 1$.

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