



Hedging in incomplete markets with HARA utility

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(Received 17 February 1993; final version received 29 May 1996)

Abstract

In the context of Merton's original problem of optimal consumption and portfolio choice in continuous time, this paper solves an extension in which the investor is endowed with a stochastic income that cannot be replicated by trading the available securities. The problem is treated by demonstrating, using analytic and, in particular, 'viscosity solutions' techniques, that the value function of the stochastic control problem is a smooth solution of the associated Hamilton–Jacobi–Bellman (HJB) equation. The optimal policy is shown to exist and given in a feedback form from the optimality conditions in the HJB equation. At zero wealth, a fixed fraction of income is consumed. For 'large' wealth, the original Merton policy is approached. We also give a sufficient condition for wealth, under the optimal policy, to remain strictly positive.

Keywords: Optimal portfolio choice; Incomplete markets; Hamilton–Jacobi–Bellman equation; Viscosity solution

JEL classification: G11

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Duffie acknowledges the financial support of the National Science Foundation under NSF SES 90-10062 and the Financial Research Initiative at the Graduate School of Business, Stanford University. H. Mete Soner acknowledges support from the National Science Foundation under NSF DMS-9500940. Wendell Fleming acknowledges financial support from the National Science Foundation under NSF DMS 900038. Thaleia Zariphopoulou acknowledges support from the National Science Foundation under NSF DMS 920486. We are grateful for research assistance from Robert Ashcroft and Flavio Auler and for comments from Ingrid Werner, Ming Huang, and an anonymous referee. We would especially like to thank Steve Shreve for many valuable discussions, and are grateful to Nicole El Karoui, Monique Jeanblanc-Picqué, and Hyeng Keun Koo for pointing out errors in an earlier version.

1. Introduction

In the context of Merton's (1971) original problem of optimal consumption and portfolio choice in continuous time, this paper treats an extension in which the investor is endowed with a stochastic income that cannot be replicated by trading the available securities. In other words, markets are incomplete in an essential way. The value function of the stochastic control problem is a smooth solution of the associated Hamilton–Jacobi–Bellman (HJB) equation. Optimal policies are derived in feedback form, and characterized, using the optimality conditions in the HJB equation. At zero wealth, a fixed fraction of wealth is consumed, the remainder being saved in the riskless asset. For 'large wealth', the original Merton consumption–investment policy is approached. We also give a sufficient condition for wealth, under the optimal policy, to remain strictly positive.

In the case of general time-additive utilities, studied in Duffie and Zariphopoulou (1993), the value function is characterized as the unique constrained viscosity solution of the HJB equation. Because of market incompleteness, in evidence in the stochastic income stream and the imperfect correlation of its source of noise with that of the stock price, the HJB equation can be degenerate and the value function therefore need not be smooth. It is highly desirable then, to obtain numerical approximations for the value function and the optimal policies. This can be successfully done by implementing a wide class of monotone and consistent schemes whose convergence is obtained via the strong stability properties of viscosity solutions. Considerable simplification is obtained by assuming HARA utility, whose homogeneity allows a reduction of the dimension of the state space from two (one state x for wealth and one state y for the stochastic income rate) to one (for the ratio $z = x/y$ of wealth to income). In this case, the HJB equation becomes a one-dimensional second-order ordinary differential equation, although it can still be degenerate. For this reason, the classical results for uniformly elliptic equations cannot be directly applied. The approach taken here is first to approximate the value function by a sequence of smooth functions that are value functions of non-degenerate stochastic income problems. Then, the limit of this sequence, which turns out in fact to be smooth, is thereby identified with the value function. This is done using the strong stability properties of viscosity solutions and the fact that the value function is the unique viscosity solution of the HJB equation. The reduced state variable z for the original income-hedging problem can also be viewed as the wealth state variable for a new investment–consumption problem, in which the utility function is not HARA and in which a fixed fraction of wealth must be held in an asset whose returns are uncorrelated with the returns from the available risky security. The 'duality' between these two hedging problems is also a focus of this paper.

Aside from its role in obtaining smooth solutions, the reduction to a one-dimensional HJB equation facilitates the characterization and numerical com-

putation of the optimal policy. In independent work on the same¹ problem, Koo (1991) also uses the homogeneity of the problem in order to reduce it to a simpler problem. His methods are quite different.

The reader is invited to draw comparisons between the PDE-approach taken here and the martingale-measure duality approach used to address similar hedging problems by Cuoco (1995), He and Pearson (1991), and Karatzas et al. (1991). Related recent literature includes Duffie and Richardson (1991), Cvitanic and Karatzas (1991), Dybvig (1992), El Karoui and Jeanblanc-Picqué (1994), He and Pagès (1993), and Svensson and Werner (1993).

2. The problem

On a given probability space is a standard Brownian motion $W = (W^1, W^2)$ in \mathbb{R}^2 . The standard augmented filtration $\{\mathcal{F}_t: t \geq 0\}$ generated by W is fixed. Riskless borrowing or lending is possible at a constant continuously compounding interest rate r . A given investor receives income at time t at the rate Y_t , where

$$\begin{aligned} dY_t &= \mu Y_t dt + \sigma Y_t dW_t^1 \quad (t \geq 0), \\ Y_0 &= y \quad (y > 0), \end{aligned} \tag{2.1}$$

where μ and σ are positive constants and y is the initial level of income. A traded security has a price process S given by

$$\begin{aligned} dS_t &= aS_t dt + \bar{\sigma} S_t dB_t \quad (t \geq 0), \\ S_0 &= S_0 \quad (S_0 > 0), \end{aligned}$$

for positive constants a and $\bar{\sigma}$, where B is a standard Brownian motion having correlation $\rho \in (-1, 1)$ with W^1 . For this, we can take $B = \rho W^1 + \sqrt{1 - \rho^2} W^2$. The risky asset pays dividends at current rate δS_t for some constant δ . The total expected rate of return of the risky asset is therefore $b = a + \delta$.

A consumption process is an element of the space \mathcal{L}_+ consisting of any non-negative $\{\mathcal{F}_t\}$ -progressively measurable process C such that $E(\int_0^T C_t dt) < \infty$ for any $T > 0$. The agent's utility function $\mathcal{J}: \mathcal{L}_+ \rightarrow \mathbb{R}^+$ for

¹In his most recent revision, Koo changes the problem formulation somewhat, assuming the investor must maintain portfolio fractions in a compact set.

consumption is given by

$$\mathcal{J}(C) = E \left(\int_0^{+\infty} e^{-\beta t} C_t^\gamma dt \right)$$

for some risk-aversion measure $\gamma \in (0, 1)$ and discount factor $\beta > r$. As stated above, we assume throughout that $\beta > r$, that $|\rho| \neq 1$, and that the volatility coefficient σ is strictly positive. Cases in which $\beta \leq r$, $\sigma = 0$, or $|\rho| = 1$ are not treated here, and may lead to a different characterization of optimal policies than shown here.

The agent’s *wealth process* X evolves according to the equation

$$dX_t = [rX_t + (b - r)\Pi_t - C_t + Y_t] dt + \bar{\sigma}\Pi_t dB_t \quad (t \geq 0),$$

$$X_0 = x \quad (x \geq 0), \tag{2.2}$$

where x is the *initial wealth endowment*, and the *control processes* C and Π represent the consumption rate C_t and investment Π_t in the risky asset, with the remainder of wealth held in riskless borrowing or lending. The controls C and Π are drawn, respectively, from the spaces $\mathcal{C} = \{C \in \mathcal{L}_+ : \mathcal{J}(C) < \infty\}$ and $\Phi = \{l : l \text{ is } \mathcal{F}_t\text{-progressively measurable and } \int_0^t l_s^2 ds < \infty \text{ a.s. } (t \geq 0)\}$. The set $\mathcal{A}(x, y)$ of *admissible controls* consists of pairs (C, Π) in $\mathcal{C} \times \Phi$ such that $X_t \geq 0$ a.s., $(t \geq 0)$, where X is given by the state equation (2.2) using the controls (C, Π) .

The agent’s *value function* v is given by

$$v(x, y) = \sup_{(C, \Pi) \in \mathcal{A}(x, y)} \mathcal{J}(C). \tag{2.3}$$

The goal is to characterize v as a classical solution of the HJB equation associated with this control problem, and then to use the regularity of v to prove the existence of optimal policies and to provide feedback formulae for them.

3. The Hamilton–Jacobi–Bellman equation

In this section we use the special form of the agent’s utility function to reduce the dimensionality of the problem.

Assuming, formally for the moment, that the value function v is finite-valued and twice continuously differentiable on $D \equiv (0, \infty) \times (0, \infty)$, it is natural to conjecture that v solves the HJB equation

$$\beta v = \max_{\pi} G^v(\pi) + \frac{1}{2} \sigma^2 y^2 v_{yy} + \max_{c \geq 0} H^v(c) + (rx + y)v_x + \mu y v_y \tag{3.1}$$

for $(x, y) \in D$, where subscripts indicate the obvious partial derivatives and

$$G^v(\pi) = \frac{1}{2} \bar{\sigma}^2 \pi^2 v_{xx} + \rho \pi \gamma \sigma \bar{\sigma} v_{xy} + (b - r) \pi v_x,$$

$$H^v(c) = -cv_x + c^\gamma.$$

It can be shown directly from (2.3) that if v is finite-valued then it is concave and is homogeneous with degree γ ; that is, for any (x, y) and positive constant k we have $v(kx, ky) = k^\gamma v(x, y)$. It therefore makes sense to define $u: [0, +\infty) \rightarrow [0, +\infty)$ by $u(z) = v(z, 1)$, so that knowledge of u recovers v from the fact that $v(x, y) = y^\gamma u(x/y)$ for $y > 0$. The same idea is used, for example, in Davis and Norman (1990). This does not recover $v(x, 0)$, which is known nevertheless to be Merton’s original solution without stochastic income.

If v satisfies (3.1) then, in $\Omega = (0, +\infty)$, u solves

$$\hat{\beta}u = \frac{1}{2} \sigma^2 z^2 u'' + \max_{\pi} [(\frac{1}{2} \bar{\sigma}^2 \pi^2 - \rho \sigma \bar{\sigma} \pi z) u'' + k_1 \pi u'] + k_2 z u' + F(u'), \quad (3.2)$$

where

$$\hat{\beta} = \beta - \mu \gamma + \frac{1}{2} \sigma^2 \gamma (1 - \gamma),$$

$$k_1 = b - r - (1 - \gamma) \rho \sigma \bar{\sigma}, \quad (3.3)$$

$$k_2 = \sigma^2 (1 - \gamma) + r - \mu,$$

and $F: [0, +\infty) \rightarrow [0, +\infty)$ is given by

$$F(p) = \max_{c \geq -1} [-cp + (1 + c)^\gamma]. \quad (3.4)$$

After performing the (formal) maximization in (3.2) (assuming that u is smooth and strictly concave), we get

$$\hat{\beta}u = \frac{1}{2} \sigma^2 z^2 (1 - \rho^2) u'' - \frac{k_1^2}{2 \bar{\sigma}^2} \frac{(u')^2}{u''} + k z u' + F(u') \quad (z > 0), \quad (3.5)$$

where

$$k = \frac{\rho k_1 \sigma}{\bar{\sigma}} + k_2. \quad (3.6)$$

In Sections 4 and 5, we show that u can be characterized as the value function of a so-called ‘dual’ investment-consumption problem. That is,

$$u(z) = \sup_{(C, \Pi) \in \mathcal{A}(z)} E \left[\int_0^{+\infty} e^{-\beta t} (1 + C_t)^\gamma dt \right], \tag{3.7}$$

where the set $\mathcal{A}(z)$ of admissible policies is defined in the next section. It turns out that this characterization of u is crucial for proving regularity results for the value function v as well as for obtaining feedback forms for the optimal policies. By a ‘feedback policy’, we mean, as usual, a pair (g, h) of measurable real-valued functions on $[0, \infty) \times [0, \infty)$ defining, with current wealth x and income rate y , the risky investment $h(x, y)$ and consumption rate $g(x, y)$. Such a feedback policy (g, h) determines the stochastic differential equation for wealth given by

$$\begin{aligned} dX_t &= [rX_t + (b - r)h(X_t, Y_t) - g(X_t, Y_t) + Y_t] dt + \bar{\sigma}h(X_t, Y_t) dB_t \\ (t \geq 0), \\ X_0 &= x \quad (x > 0). \end{aligned} \tag{3.8}$$

If there is a non-negative solution X to (3.8) and if the policy (C, Π) defined by

$$C_t = g(X_t, Y_t), \quad \Pi_t = h(X_t, Y_t),$$

are in \mathcal{C} and Φ , respectively, then (C, Π) is an admissible policy by definition of $\mathcal{A}(x, y)$.

Before stating our main conclusions, we recall that for the case $y = 0$ (implying $Y_t = 0$ for $t \geq 0$), the value function v is given from Merton’s (1971) work as follows. Provided that the constant

$$K = \frac{\beta - r\gamma}{1 - \gamma} - \frac{\gamma(b - r)^2}{2(1 - \gamma)^2 \bar{\sigma}^2} \tag{3.9}$$

is strictly positive, we have

$$v(x, 0) = K^{\gamma-1} x^\gamma, \tag{3.10}$$

with optimal policies given in feedback form by

$$g(x, 0) = Kx, \quad h(x, 0) = \frac{x(b - r)}{\bar{\sigma}^2(1 - \gamma)}. \tag{3.11}$$

For $x > 0$ and $y > 0$, the feedback policy functions g and h defined by the first-order optimality conditions for (3.1), in light of the homogeneity property

$v(x, y) = y^\gamma u(x/y)$, are given by

$$g(x, y) = y \left(\frac{u'(x/y)}{\gamma} \right)^{1/(\gamma-1)}, \tag{3.12}$$

$$h(x, y) = \frac{\rho\sigma}{\bar{\sigma}} x - y \frac{k_1}{\bar{\sigma}^2} \frac{u'(x/y)}{u''(x/y)}. \tag{3.13}$$

One of our results will be the regularity under which the policy (g, h) defined by (3.12) and (3.13) makes sense (and indeed is optimal) when applied to $\{(0, y): y \geq 0\}$, the zero-wealth boundary of the state space. To this end, we will prove:

Proposition 1. $u'(0) = \lim_{z \downarrow 0} u'(z)$ exists and is in the interval $[\gamma, \infty)$. Also,

$$\lim_{z \downarrow 0} z^2 u''(z) = 0.$$

Suppose that $k_1 \neq 0$. Then $u'(0) > \gamma$ and

$$\lim_{z \downarrow 0} zu''(z) = 0, \quad \limsup_{z \downarrow 0} \sqrt{z} u''(z) < 0.$$

A proof of Proposition 1 is given in Section 7. This behaviour of the derivatives of u near the origin implies that the risky investment policy h defined by (3.13), and the feedback consumption policy g defined by (3.12), uniquely extend continuously to the whole state space \bar{D} with $h(0, y) = 0$ and $g(0, y) \leq y$. Note that these inequalities are consistent with the budget-feasibility constraint, $X_t \geq 0$. In Section 8, we will show that there is a unique solution of (3.8) with (3.11)–(3.13) satisfying the constraint and then show that this process is optimal.

Our main results are thus as follows.

Theorem 1. Suppose $\hat{\beta}, K$, and $r - \mu$ are all strictly positive.

- (i) There is a unique $C^1([0, +\infty)) \cap C^2((0, +\infty))$ solution u of the ordinary differential equation (3.5) in the class of concave functions.
- (ii) The value function v is given by

$$v(x, 0) = K^{\gamma-1} x^\gamma,$$

$$v(x, y) = y^\gamma u\left(\frac{x}{y}\right), \quad y > 0. \tag{3.14}$$

- (iii) There is a unique solution X_t of (3.8) with (3.11)–(3.13) satisfying the budget feasibility constraint $X_t \geq 0$, and an optimal policy (C^*, Π^*) is given by

- $C_t^* = g(X_t, Y_t)$ and $\Pi_t^* = h(X_t, Y_t)$ where g , and h , are given by (3.11)–(3.13), with $h(0, y) = 0$ for all y and $g(0, y) = \alpha y$ for all y , where $\alpha = (u'(0)/\gamma)^{1/(\gamma-1)}$.
- (iv) If $k_1 \neq 0$, starting from strictly positive wealth ($x > 0$), the optimal wealth process, almost surely, will never hit zero, and starting from zero, almost surely, the optimal wealth process will instantaneously become strictly positive. The same conclusion holds if $k_1 = 0$ and $u'(0) > \gamma$.

We do not know whether the case $k_1 = 0, u'(0) = \gamma$ is possible for a particular choice of the parameters. If this case occurs, it might be possible that the optimal wealth process, starting from strictly positive wealth, hits zero in finite time. Analysis of this case is an interesting open question.

A proof of Theorem 1 is presented in Section 8. The idea is to use an auxiliary problem and techniques from the theory of viscosity solutions to prove the existence of a classical solution to the HJB equation in Sections 4–6, and then to use a verification approach to confirm the form of the optimal policy implied in feedback form by the HJB equation. A detailed analysis of u near the origin is given in Section 7 and the asymptotic behavior of u at infinity is analyzed in Section 8. Here, we characterize the behavior of the optimal policy as the ratio of wealth to income becomes large, showing it to converge to the optimal behavior in the original Merton (1971) problem with no stochastic income. The verification proof of Theorem 1 is also included in Section 8.

The technical conditions on parameters given in the theorem are maintained for the remainder of the paper.

In an extension of the problem to multiple risky assets, it is easy to see from the extension of (3.13) that we will not generally obtain portfolio separation, under which one could, without loss of generality, replace the collection of risky assets with a single risky asset.

The main results are obtained by studying a related hedging problem whose HJB equation reduces to (3.5). The new problem is stated in the following section and analyzed in Sections 5 and 6. The reader who is not interested in the technical points of the proof can skip Section 5 and follow only the main arguments of Sections 6–8.

4. A ‘dual’ hedging problem

We now consider a consumption–investment problem of an agent whose current wealth Z_t evolves, using a consumption process C and risky investment process Π , according to the equation

$$dZ_t = [kZ_t + k_1\Pi_t - C_t] dt + \bar{\sigma}\Pi_t dW_t^1 + \sigma Z_t \sqrt{1 - \rho^2} dW_t^2 \quad (t \geq 0),$$

$$Z_0 = z \quad (z \geq 0), \tag{4.1}$$

where z is the initial endowment and k_1 and k are given, respectively, by (3.3) and (3.6). The set \mathcal{L} of consumption processes consists of any progressively measurable process C such that $C_t \geq -1$ almost surely for all t , with $E(\int_0^t C_s ds) < \infty$ for all t . A control pair (C, Π) for (4.1) consists of a consumption process C in \mathcal{L} and a risky investment process $\Pi \in \Phi$. A control pair (C, Π) for (4.1) is admissible if $Z_t \geq 0$ a.s., ($t \geq 0$), where Z_t is given by (4.1). We denote by $\hat{\mathcal{A}}(z)$ the set of admissible controls. Observe that on the one hand, the agent is forced to invest a fixed multiple of wealth in a risky asset with expected return k and ‘volatility’ $\sigma\sqrt{(1 - \rho^2)}$. On the other hand, he chooses the amount Π invested in another risky asset with mean return k_1 and volatility $\bar{\sigma}$.

The agent’s utility $\hat{\mathcal{J}}: \mathcal{L} \rightarrow \mathbb{R}^+$ is given by

$$\hat{\mathcal{J}}(C) = E \left[\int_0^{+\infty} e^{-\beta t} (1 + C_t)^\gamma dt \right].$$

The value function $w: [0, \infty) \rightarrow \mathbb{R}^+$ is defined by

$$w(z) = \sup_{(C, \Pi) \in \hat{\mathcal{A}}(z)} \hat{\mathcal{J}}(C). \tag{4.2}$$

The HJB equation associated with this stochastic control problem is

$$\begin{aligned} \hat{\beta}w &= \frac{1}{2} \sigma^2 (1 - \rho^2) z^2 w_{zz} + \max_{\pi} \left[\frac{1}{2} \bar{\sigma}^2 \pi^2 w_{zz} + k_1 \pi w_z \right] \\ &+ \max_{c \geq -1} [-cw_z + (1 + c)^\gamma] + kz w_z \quad (z > 0). \end{aligned} \tag{4.3}$$

We observe that (4.3) reduces (at least formally) to (3.5) for smooth concave solutions. In the sequel, we show that (4.3) has a smooth concave solution w , which will ensure that u is also smooth.

We call problems (4.2) and (2.3) ‘dual’ to each other because one hedges an income stream and the other hedges an investment, and because of the relationship between their value functions: The reduced value function u of problem (2.3) for HARA utility reduces to the value function w of (4.2) for non-HARA utility. Conversely, the reader can show that problem (4.2), after substituting the HARA utility function \mathcal{J} for $\hat{\mathcal{J}}$ and substituting the correlated Brownian motion B for W^2 in (4.1), has a value function equivalent to that of problem (2.3), after making the opposite substitutions. Thus either of these dual problems can be reduced to a version of the other with a single-state variable.

5. Viscosity solutions of the Hamilton–Jacobi–Bellman equation

In this section we analyze the Hamilton–Jacobi–Bellman equation (4.3), using results from the theory of *viscosity solutions*. In particular, we show that the value function w is the *unique constrained viscosity solution* of (4.3). This characterization of w is natural because of the state constraint $Z_t \geq 0$.

The notion of *viscosity solutions* was introduced by Crandall and Lions (1983) for first-order equations and by Lions (1983) for second-order equations. For a general overview of the theory of viscosity solutions, we refer to the *User's Guide* of Crandall et al. (1992) as well as Fleming and Soner (1993). We now review the notion of constrained viscosity solutions, introduced by Soner (1986) for first-order equations (see also Capuzzo-Dolcetta and Lions (1990), Ishii and Lions (1990)).

To this end, consider a nonlinear second-order partial differential equation of the form

$$F(x, u, u_x, u_{xx}) = 0 \quad \text{in } \Omega, \quad (5.1)$$

where Ω is an open subset of \mathbb{R} and $F: \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and (degenerate) elliptic, meaning that $F(x, t, p, X + Y) \leq F(x, t, p, X)$ if $Y \geq 0$.

Definition. A continuous function $u: \bar{\Omega} \rightarrow \mathbb{R}$ is a constrained viscosity solution of (5.1) if

- (i) u is a viscosity subsolution of (5.1) on $\bar{\Omega}$, that is, for any $\varphi \in C^2(\bar{\Omega})$ and any local maximum point $x_0 \in \bar{\Omega}$ of $u - \varphi$, $F(x_0, u(x_0), \varphi_x(x_0), \varphi_{xx}(x_0)) \leq 0$, and
- (ii) u is a viscosity supersolution of (5.1) in Ω , that is, for any $\varphi \in C^2(\bar{\Omega})$ and any local minimum point $x_0 \in \Omega$ of $u - \varphi$, $F(x_0, u(x_0), \varphi_x(x_0), \varphi_{xx}(x_0)) \geq 0$.

The first result of this section characterizes w as a constrained viscosity solution of the associated HJB equation (4.3) on $\bar{\Omega} = [0, \infty)$.

Theorem 2. *The value function w is finite and is a constrained viscosity solution of (4.3) on $\bar{\Omega}$.*

Finiteness of w follows from an argument similar to Proposition A.2. Alternatively, the function

$$W(z) := c_0(z + c_1)^2, \quad z \geq 0$$

is a smooth supersolution of (4.3) for all sufficiently large c_0 and c_1 . Then a verification argument shows that $w \leq W$, hence, w is finite.

The fact that, in general, value functions of (stochastic) control problems and differential games turn out to be viscosity solutions of the associated PDEs follows directly from the dynamic programming principle and the theory of viscosity solutions (see, for example, Lions, 1983; Evans and Souganidis, 1984; Fleming and Souganidis, 1989; Fleming and Soner, 1993). The main difficulty for the problem at hand is that neither control, consumption rate nor risky investment, is uniformly bounded. In order to overcome this difficulty, we approximate the value function with a sequence of functions that are viscosity solutions of modified HJB equations. We repeatedly use the *stability properties* of viscosity solutions (see Lions, 1983) in order to pass to limits. Since the arguments are lengthy and also similar to those in Theorem 3.1 of Zariphopoulou (1993) and Theorem 4.2 of Duffie and Zariphopoulou (1993), they are not presented here.

Theorem 3. Suppose that u is an upper-semicontinuous concave viscosity subsolution of the HJB equation (4.3) on $[0, +\infty)$ and $u(z) \leq c_0(z^\gamma + 1)$ for some $c_0 > 0$, and suppose that v is bounded from below, uniformly continuous on $[0, +\infty)$ and locally Lipschitz in $(0, +\infty)$, and a viscosity supersolution of (4.3) in $(0, +\infty)$. Then $u \leq v$ on $[0, +\infty)$.

The proof follows in the lines of the proof of Theorem 4.1 in Zariphopoulou (1993). The latter is an adaptation of Theorem II.2 of Ishii and Lions (1990) for the case of controls that are not uniformly bounded, which is the case on hand.

The next theorem will be needed for the characterization and recovery of the value function of the original two-dimensional problem from that of the reduced problem. The proof is presented in Duffie and Zariphopoulou (1993), in a setting for general utility functions and income processes.

Theorem 4. The value function v is the unique constrained viscosity solution of the HJB equation (3.1) on \bar{D} in the class of concave functions.

6. Classical solutions of the HJB equation

In this section we show that w is a classical solution of (4.3). We begin with some useful basic properties.

Proposition 2. The value function w is concave, increasing, and continuous on $[0, \infty)$.

The arguments supporting concavity, monotonicity, and continuity are standard and are similar to those found, for example, in Karatzas et al. (1986) and Zariphopoulou (1993).

Theorem 5. The value function w is the unique $C[0, +\infty) \cap C^2(0, +\infty)$ solution of (4.3) in the class of concave functions.

Before going into the details of the proof, we describe the main ideas. Although the HJB equation (4.3) is an ordinary differential equation, it is not at all trivial to prove that it has a smooth solution. The main difficulty stems from the fact that (4.3) is not uniformly elliptic since, it is not a priori known if the optimal π^* in (4.3), given formally by $\pi^* = -kw_z/(\bar{\sigma}^2 w_{zz})$, lies in a compact set (see Krylov, 1987). To overcome this difficulty, we will first work in an interval $(z_1, z_2) \subset [0, +\infty)$ and show that w solves a *uniformly elliptic* HJB equation in (z_1, z_2) with boundary conditions $w(z_1)$ and $w(z_2)$. Standard elliptic regularity theory (see Krylov, 1987) and the uniqueness of viscosity solutions will then yield that w is smooth in (z_1, z_2) .

An important feature of the proof is the approximation of w by a family of smooth functions $\{w^L: L > 0\}$ that are solutions of a suitably regularized equation. For fixed $L > 0$, we first turn to w^L and its properties. We let

$$w^L(z) = \sup_{\mathcal{A}^L(z)} \mathbb{E} \left[\int_0^{+\infty} e^{-\beta t} (1 + C_t)^\gamma dt \right],$$

where

$$\mathcal{A}^L(z) = \{(C, \Pi) \in \hat{\mathcal{A}}(z): -L \leq \Pi_t \leq L \text{ a.s., } t \geq 0\}.$$

The concavity of the utility function as well as the linearity of the state equation (4.1) with respect to the state Z and controls (C, Π) , implies that w^L is concave, strictly increasing, and continuous on $\bar{\Omega}$. Moreover, using arguments similar to those used in Theorems 3.1 and 4.1 in Zariphopoulou (1993), we get that, for every $L > 0$, w^L is the unique constrained viscosity solution, in the class of concave functions on $\bar{\Omega}$, of the HJB equation

$$\begin{aligned} \beta w^L &= \frac{1}{2} \sigma^2 (1 - \rho^2) z^2 w_{zz}^L + \max_{-L \leq \pi \leq L} \left[\frac{1}{2} \bar{\sigma}^2 \pi^2 w_{zz}^L + k_1 \pi w_z^L \right] \\ &+ \max_{c \geq -1} \left[-cw_z^L + (1 + c)^\gamma \right] + kz w_z^L. \end{aligned} \tag{6.1}$$

Since $K > 0$ (cf. (3.9)), we see that for sufficiently large constants c_0 and c_1 ,

$$W(z) := c_0(z + c_1)^\gamma$$

is a supersolution of (6.1) and w^L is bounded by W , for every L . Then, we observe that there exists \hat{w} concave such that $w^L \rightarrow \hat{w}$ as $L \rightarrow \infty$, locally uniformly in $\bar{\Omega}$. The stability property of viscosity solutions yields that \hat{w} is a constrained

viscosity solution of (4.3) and therefore, by uniqueness, it coincides with w . Therefore, as $L \rightarrow \infty$,

$$w^L \rightarrow w, \quad \text{locally uniformly in } \bar{\Omega}. \tag{6.2}$$

We next show that w^L is smooth in any interval $[z_1, z_2]$ with $z_1 > 0$. Without loss of generality, due to concavity, we can choose the points z_1 and z_2 such that $w_z(z_1)$ and $w_z(z_2)$ exist; the reason will be apparent in the sequel.

We have that w^L is the unique viscosity solution (see Lions, 1983; Ishii and Lions, 1990) of the boundary value problem

$$\begin{aligned} \hat{\beta}u &= \frac{1}{2}\sigma^2(1 - \rho^2)z^2u_{zz} + \max_{-L \leq \pi \leq L} [\frac{1}{2}\bar{\sigma}^2\pi^2u_{zz} + k_1\pi u_z] \\ &+ \max_{c \geq -1} [-cu_z + (1 + c)^y] + kzu_z, \quad z \in (z_1, z_2), \\ u(z_1) &= w^L(z_1), \quad u(z_2) = w^L(z_2). \end{aligned} \tag{6.3}$$

From the theory of elliptic second-order equations (as in Krylov, 1987), (6.3) has a unique smooth solution u , which coincides with the unique viscosity solution w^L . Therefore, w^L is smooth in (z_1, z_2) .

In the sequel, we show that the artificial investment constraint $-L \leq \pi \leq L$ can be eliminated.

First, we observe that the concavity and monotonicity of w^L imply that w^L is a smooth solution of

$$\begin{aligned} \hat{\beta}u &= \frac{1}{2}\sigma^2(1 - \rho^2)z^2u_{zz} + \max_{\pi \leq L} [\frac{1}{2}\bar{\sigma}^2\pi^2u_{zz} + k_1\pi u_z] \\ &+ \max_{c \geq -1} [-cu_z + (1 + c)^y] + kzu_z, \quad z \in (z_1, z_2), \\ u(z_1) &= w^L(z_1), \quad u(z_2) = w^L(z_2). \end{aligned} \tag{6.4}$$

We next claim that, for sufficiently large L ,

$$\sup_{z \in (z_1, z_2)} \left[-\frac{k_1}{\bar{\sigma}^2} \frac{w_z^L(z)}{w_{zz}^L(z)} \right] \leq L. \tag{6.5}$$

Indeed, if there is no L satisfying (6.5), then there exist sequences (L_n) and (z_n) with $L_n \rightarrow \infty$ and $z_n \in (z_1, z_2)$, $n \in \mathbb{N}$, such that

$$-\frac{k_1}{\bar{\sigma}^2} \frac{w_z^{L_n}(z_n)}{w_{zz}^{L_n}(z_n)} > L_n. \tag{6.6}$$

Combining (6.6) with (6.4), we get

$$\begin{aligned} \hat{\beta}w^{L_n}(z_n) &\geq \frac{1}{2}\sigma^2(1 - \rho^2)z_n^2w_{zz}^{L_n}(z_n) + \frac{1}{2}\bar{\sigma}^2L_n^2w_{zz}^{L_n}(z_n) + k_1L_nw_z^{L_n}(z_n) \\ &\quad + \max_{c \geq -1} [-cw_z^{L_n}(z_n) + (1 + c)^\gamma] + kz_nw_z^{L_n}(z_n) \\ &\geq \frac{1}{2}\sigma^2(1 - \rho^2)z_n^2w_{zz}^{L_n}(z_n) + \frac{k_1}{2}L_nw_z^{L_n}(z_n) \\ &\quad + \max_{c \geq -1} [-cw_z^{L_n}(z_n) + (1 + c)^\gamma] + kz_nw_z^{L_n}(z_n). \end{aligned} \tag{6.7}$$

From (6.2), the concavity of w and w^L , and the given choices of z_1 and z_2 , we get the existence of positive constants C_1 and C_2 , independent of L , such that for L sufficiently large

$$C_1 \leq w_z^L(z) \leq C_2, \quad z \in (z_1, z_2). \tag{6.8}$$

We now send $n \rightarrow \infty$. From (6.6) and (6.8),

$$\lim_{n \rightarrow \infty} w_{zz}^{L_n}(z_n) = 0. \tag{6.9}$$

Since $z_n \in (z_1, z_2)$, $n \in \mathbb{N}$, there exists $\bar{z} \in [z_1, z_2]$ such that $\lim_{n \rightarrow \infty} z_n = \bar{z}$. Combining (6.7)–(6.9), and sending $L_n \rightarrow \infty$, we get a contradiction.

We now observe that (6.5) implies that w^L is a smooth concave solution of

$$\begin{aligned} \hat{\beta}u &= \frac{1}{2}\sigma^2(1 - \rho^2)z^2u_{zz} + \max_{\pi} [\frac{1}{2}\bar{\sigma}^2\pi^2u_{zz} + k_1\pi u_z] \\ &\quad + \max_{c \geq -1} [-cu_z + (1 + c)^\gamma] + kz u_z, \\ u(z_1) &= w^L(z_1), \quad u(z_2) = w^L(z_2). \end{aligned} \tag{6.10}$$

We next show that there exists a constant $\bar{K} < 0$, independent of L , such that

$$w_{zz}^L(z) < \bar{K}, \quad z \in (z_1, z_2). \tag{6.11}$$

Indeed, after performing the maximization in (6.10) with respect to π and c , we get

$$\hat{\beta}w^L = \frac{1}{2}\sigma^2(1 - \rho^2)z^2w_{zz}^L - \frac{k_1^2}{2\bar{\sigma}^2} \frac{(w_z^L)^2}{w_{zz}^L} + h(w_z^L) + kz w_z^L, \quad z \in (z_1, z_2) \tag{6.12}$$

for some $h: \mathbb{R} \rightarrow (0, \infty)$ whose calculation is left to the reader. Multiplying both sides of (6.12) by w_{zz}^L and using the fact that $h > 0$ we have

$$Q(w_{zz}^L) \equiv \frac{1}{2}\sigma^2(1 - \rho^2)z^2(w_{zz}^L)^2 + w_{zz}^L(kzw_z^L - \hat{\beta}w^L) - \frac{k_1^2}{2\bar{\sigma}^2}(w_z^L)^2 > 0.$$

It is immediate that the quadratic equation $Q(\lambda) = 0$ has two real roots λ_+ and λ_- of opposite signs. Therefore, $w_{zz}^L < \lambda_-$. Using the expression for λ_- , (6.2), and (6.8), we get the existence of a constant $\bar{K} < 0$, independent of L , such that (6.11) holds.

We now conclude as follows. Using the concavity of w^L , (6.8) and (6.11), we get that w^L is a smooth solution of

$$\begin{aligned} \hat{\beta}u &= \frac{1}{2}\sigma^2(1 - \rho^2)z^2u_{zz} + \max_{0 \leq \pi \leq M} [\frac{1}{2}\bar{\sigma}^2\pi^2u_{zz} + k_1\pi u_z] \\ &+ \max_{c \geq -1} [-cu_z + (1 + c)^y] + kzu_z, \quad z \in (z_1, z_2), \end{aligned}$$

$$u(z_1) = w^L(z_1), \quad u(z_2) = w^L(z_2),$$

where

$$M = -\frac{k_1 C_2}{\bar{\sigma}^2 \bar{K}}.$$

From (6.2) and the stability property of viscosity solutions, we have that w is a viscosity solution of

$$\begin{aligned} \hat{\beta}u &= \frac{1}{2}\sigma^2(1 - \rho^2)z^2u_{zz} + \max_{0 \leq \pi \leq M} [\frac{1}{2}\bar{\sigma}^2\pi^2u_{zz} + k_1\pi u_z] \\ &+ \max_{c \geq -1} [-cu_z + (1 + c)^y] + kzu_z, \quad z \in (z_1, z_2), \end{aligned}$$

$$u(z_1) = w(z_1), \quad u(z_2) = w(z_2). \tag{6.13}$$

On the other hand, (6.13) has a unique smooth solution (as in Krylov, 1987) and a unique viscosity solution (as shown by Ishii and Lions, 1990). Therefore, w is smooth in (z_1, z_2) . We can always choose z_1 and z_2 so that $(a, b) \subset (z_1, z_2)$ for any interval (a, b) . This completes the proof of Theorem 5.

7. Proof of Proposition 1

Let u be the unique solution of (3.5) constructed in Section 4. In this section, we study the behavior of u near the origin in several steps.

Step 1: In this step, we analyze $u'(z)$, as $z \downarrow 0$. By concavity and monotonicity,

$$u'(0) := \lim_{z \downarrow 0} u'(z)$$

exists and is non-negative, however, it may take the value of ∞ . Suppose that $u'(0) = \infty$. Then,

$$\lim_{z \downarrow 0} \frac{F(u'(z))}{u'(z)} = 1,$$

and, therefore, there exists $z_0 > 0$ satisfying

$$F(u'(z)) + kzu'(z) - \beta u(z) \geq \frac{1}{2}u'(z), \quad \forall z \in (0, z_0].$$

By (3.5) and the concavity of u ,

$$\begin{aligned} 0 &\geq \frac{1}{2}\sigma^2(1 - \rho^2)z^2u''(z) + F(u'(z)) + kzu'(z) - \beta u(z) \\ &\geq \sigma^2z^2u''(z) + \frac{1}{2}u'(z), \quad \forall z \in (0, z_0]. \end{aligned}$$

Set

$$y(\xi) := u'(z_0 - \xi), \quad 0 \leq \xi < z_0,$$

so that

$$\begin{aligned} y(\xi) &= y(0) + \int_0^\xi y'(s) ds = y(0) - \int_{z_0-\xi}^{z_0} u''(s) ds \leq y(0) + \int_{z_0-\xi}^{z_0} \frac{u'(s)}{\sigma^2 s^2} ds \\ &\leq y(0) + \int_0^\xi \frac{y(s)}{\sigma^2(z_0 - s)^2} ds \quad \forall \xi \in (0, z_0]. \end{aligned}$$

By Gronwall's inequality,

$$y(\xi) \geq y(0)\exp\left[\frac{1}{\sigma^2}\left[\frac{1}{(z_0 - \xi)^2} - \frac{1}{z_0^2}\right]\right] \quad \forall \xi \in (0, z_0]$$

and

$$0 \leq u(z) = u(z_0) - \int_z^{z_0} u'(s) \, ds \leq u(z_0) - \int_z^{z_0} C \exp(\sigma^{-2}s^{-2}) \, ds.$$

Since $\exp(s^{-2})$ is not integrable on $[0, z_0]$, we conclude that $u'(0)$ is finite.

Step 2: In this step, we will show that

$$\lim_{z \downarrow 0} z^2 u''(z) = 0. \tag{7.1}$$

Set $a := -\limsup_{z \downarrow 0} zu''(z)$, so that, for all sufficiently small $z > 0$,

$$\infty > u'(0) - u'(z) = -\int_0^z u''(s) \, ds \geq \int_0^z \frac{a}{2s} \, ds.$$

Hence $a = 0$. Set

$$A(z) := \hat{\beta}u(z) - F(u'(z)) - kzu'(z),$$

$$B(z) := \frac{1}{2} \sigma^2 z^2 (1 - \rho^2) u''(z) - \frac{(k_1 u'(z))^2}{2\bar{\sigma}^2 u''(z)},$$

$$b := -\liminf_{z \downarrow 0} z^2 u''(z).$$

By (3.5), $A(z) = B(z)$ and, since A is continuous on $[0, \infty)$, so is B . Suppose that $b > 0$. Since $a = 0$ and u'' is continuous on $(0, \infty)$, there are two sequences $z_n \downarrow 0$, and $\hat{z}_n \downarrow 0$ satisfying

$$\lim_{n \rightarrow \infty} (z_n)^2 u''(z_n) = -b, \quad \lim_{n \rightarrow \infty} (\hat{z}_n)^2 u''(\hat{z}_n) = -\frac{b}{2}.$$

Then,

$$\lim_{n \rightarrow \infty} B(z_n) = -\frac{b}{2} \sigma^2 (1 - \rho^2), \quad \lim_{n \rightarrow \infty} B(\hat{z}_n) = -\frac{b}{4} \sigma^2 (1 - \rho^2).$$

Since B is continuous, we conclude that $b = 0$.

Step 3: In this step, we will show that $u'(0) \geq \gamma$ and $\hat{\beta}u(0) = F(u'(0))$. Recall that u is a constrained viscosity solution of (3.5) in $[0, \infty)$. Hence,

$$\hat{\beta}u(0) - F(\varphi'(0)) + \frac{(k_1\varphi'(0))^2}{2\bar{\sigma}^2\varphi''(0)} \leq 0, \tag{7.2}$$

for every test function $\varphi \in C^2([0, \infty))$ satisfying $\varphi''(0) < 0$ and

$$(u - \varphi)(0) = \max_{z \geq 0} (u - \varphi)(z). \tag{7.3}$$

(See, for instance, Soner, 1986 or Fleming and Soner, 1993, Section II.12.)

For every $\varepsilon > 0$ and $R > 0$, there exists a smooth test function $\varphi_{\varepsilon,R}$ satisfying (7.3) and

$$\varphi'_{\varepsilon,R}(0) = u'(0) + R, \quad \varphi''_{\varepsilon,R}(0) = -\frac{1}{\varepsilon}.$$

We use $\varphi_{\varepsilon,R}$ in (7.2) and let $\varepsilon \downarrow 0$. The result is

$$\hat{\beta}u(0) - F(u'(0) + R) \leq 0 \quad \forall R \geq 0.$$

Moreover, by (7.1) and (3.5),

$$\hat{\beta}u(0) - F(u'(0)) = \lim_{z \downarrow 0} -\frac{(k_1u'(z))^2}{\bar{\sigma}^2u''(z)} \geq 0.$$

Hence,

$$\hat{\beta}u(0) = F(u'(0)) \leq F(u'(0) + R) \quad \forall R \geq 0 \tag{7.4}$$

and

$$\lim_{z \downarrow 0} \frac{(k_1u'(z))^2}{\bar{\sigma}^2u''(z)} = 0. \tag{7.5}$$

Inequality (7.4) implies that $u'(0)$ is greater than or equal to the minimizer of F ; hence $u'(0) \geq \gamma$. When, $k_1 \neq 0$, (7.5) yields $\lim_{z \downarrow 0} u''(z) = -\infty$.

In the remainder of this section, we assume that

$$k_1 \neq 0.$$

Step 4: In this step we show that $u'(0) > \gamma$. In Step 2, we have shown that $\limsup_{z \downarrow 0} zu''(z) = 0$. Therefore, there exists a sequence $z_n \downarrow 0$ such that $z_n u''(z_n)$

tends to zero as $n \rightarrow \infty$. We argue by contradiction, supposing that $u'(0) = \gamma$. Then, for every $z \geq 0$, $u'(z) \leq u'(0) = \gamma$ and

$$\hat{\beta}u(z) - F(u'(z)) = \hat{\beta}(u(z) - u(0)) - F(u'(z)) + F(u'(0)) \leq \hat{\beta}(u(z) - u(0)).$$

We use this in (3.5):

$$\begin{aligned} \infty &= - \lim_{n \rightarrow \infty} \frac{(k_1 u'(z_n))^2}{2z_n \bar{\sigma}^2 u''(z_n)} = \lim_{n \rightarrow \infty} \frac{\hat{\beta}u(z_n) - F(u'(z_n))}{z_n} - ku'(z_n) \\ &\leq \lim_{n \rightarrow \infty} \frac{\hat{\beta}(u(z_n) - u(0))}{z_n} - ku'(z_n) = (\hat{\beta} - k)u'(0). \end{aligned}$$

Hence $u'(0) > \gamma$.

Step 5. We claim that $\lim_{z \downarrow 0} zu''(z) = 0$. Let A and B be as in Step 2. Since $u'(0) > \gamma$ and $u''(0) = -\infty$,

$$\begin{aligned} \lim_{z \downarrow 0} \frac{A(z)}{z} &= \lim_{z \downarrow 0} \hat{\beta} \frac{u(z) - u(0)}{z} - ku'(z) - \frac{F(u'(z)) - F(u'(0))}{z} \\ &= (\hat{\beta} - k)u'(0) - F'(u'(0))u''(0) = \infty. \end{aligned}$$

Choose a sequence $z_n \downarrow 0$ such that the following limit exists:

$$v := \lim_{n \rightarrow \infty} z_n u''(z_n).$$

Then, by (3.5),

$$\infty = \lim_{z \downarrow 0} \frac{A(z)}{z} = \lim_{n \rightarrow \infty} \frac{B(z_n)}{z_n} = - \frac{(k_1 u'(0))^2}{2\bar{\sigma}^2 v} - \frac{1}{2} \sigma^2 (1 - \rho^2) v.$$

Hence $v = 0$, and the claim is proved.

Step 6: We finally show that $\lim \sqrt{z}u''(z) < 0$. Because

$$B(z) = \left[\frac{k_1 u'(z)^2}{2\bar{\sigma}^2} - \frac{(\sigma z u''(z))^2 (1 - \rho^2)}{2} \right] \left(- \frac{1}{u''(z)} \right),$$

$k_1 \neq 0$, $u'(0) \geq \gamma$, and $zu''(z) \rightarrow 0$ as $z \downarrow 0$, there are $z_1 > 0$ and $c_0 > 0$ satisfying

$$B(z) \geq -\frac{c_0}{u''(z)}, \quad z \in (0, z_1].$$

By Step 2,

$$\begin{aligned} A(z) &= \hat{\beta}[u(z) - u(0)] - zku'(z) - F(u'(z)) + F(u'(0)) \\ &= \int_0^z [\hat{\beta}u'(s) - ku'(z)] ds + \int_0^z F'(u'(s))(-u''(s)) ds. \end{aligned}$$

Because $u'(0) > \gamma$, there are $z_2 > 0$ and $c_1 > 0$ such that

$$A(z) \leq c_1 \left[z - \int_0^z u''(s) ds \right], \quad z \in (0, z_2].$$

We set $z_0 = \min\{z_1, z_2\}$ and $y(z) := -1/u''(z)$, so that

$$y(z) \leq \frac{B(z)}{c_0} = \frac{A(z)}{c_0} \leq \frac{c_1}{c_0} \left[z + \int_0^z \frac{1}{y(s)} ds \right] \quad \forall z \in [0, z_0].$$

Because $y(z) \rightarrow 0$ as $z \downarrow 0$, there is some $z^* > 0$ satisfying

$$y(z) \leq \frac{zc_1}{c_0} \cdot \int_0^z \frac{1}{y(s)} ds, \quad z \in (0, z^*].$$

Then, by Gronwall's inequality, there is a constant c^* such that $y(z) \leq c^* \sqrt{z}$ for all z near zero. Hence,

$$\limsup_{z \downarrow 0} \sqrt{zu''(z)} \leq -\frac{1}{c^*}.$$

This completes the proof of Proposition 1.

8. Verification results and optimal policies

In this section we prove the main Theorem 1, and describe the asymptotic behavior of the value function and the optimal policies as the ratio of wealth to income becomes large.

We start with a description of the asymptotic behavior of the value function v and the optimal policy as the ratio x/y of wealth to income becomes large. By ‘ $F(x, y) \sim \bar{F}$ ’, in the following theorem, we mean that ‘ $F(x_n, y_n) \rightarrow \bar{F}$ for any strictly positive sequence $\{(x_n, y_n)\}$ with $x_n/y_n \rightarrow \infty$ ’. As one can see from the following result, the optimal behavior is asymptotically that of the Merton problem with no stochastic income.

Theorem 6. As $x/y \rightarrow \infty$, we have:

- (i) $v(x, y) \sim K^{\gamma-1}x^\gamma$, for K given by (3.9).
- (ii) $g(x, y)/x \sim K$.
- (iii) $h(x, y)/x \sim (b - r)/[(1 - \gamma)\bar{\sigma}^2]$.

Proof. Part (i) is immediate from Proposition A.2 (Appendix A).

For Part (ii), we can use Part (i) and the relationship between u and v to see that

$$\lim_{z \rightarrow \infty} \frac{u(z)}{z^\gamma} = K^{\gamma-1}.$$

For $\lambda > 0$, we set

$$U_\lambda(z) := \lambda^{-\gamma}u(\lambda z),$$

so that U_λ solves (3.5) with $F(p)$ replaced by

$$F_\lambda(p) := \max_{c \geq \lambda^{-1}} \left[-cp + \left(\frac{1}{\lambda} + c \right)^\gamma \right], \quad p > 0.$$

Then, by Part (i), as $\lambda \uparrow \infty$, $U_\lambda(\cdot)$ converges locally uniformly to the Merton value function

$$v(z) = K^{\gamma-1}z^\gamma,$$

where K is as in (3.9). We note that v solves (3.5) with $F(p)$ replaced by

$$F_\infty(p) = \lim_{\lambda \uparrow \infty} F_\lambda(p) = (1 - \gamma) \left(\frac{p}{\gamma} \right)^{1/(1-\lambda)}.$$

Because U_λ is concave, the uniform convergence of U_λ to v implies the convergence of the derivatives:

$$\lim_{\lambda \uparrow \infty} U'_\lambda(z) = v'(z) = \gamma K^{\gamma-1}z^{\gamma-1}, \quad z \geq 0.$$

Hence,

$$\lim_{\lambda \uparrow \infty} U'_\lambda(1) = \lim_{\lambda \uparrow \infty} \lambda^{1-\gamma} u'(\lambda) = \gamma K^{\gamma-1},$$

and, therefore,

$$\lim_{x/y \rightarrow \infty} \frac{u'(x/y)}{\gamma(x/y)^{\gamma-1}} = K^{\gamma-1}. \tag{8.1}$$

Combining with (3.12) gives the result.

For Part (iii), the first-order conditions for (4.3) give us

$$\hat{\beta}w = \frac{1}{2} \sigma^2 (1 - \rho^2) z^2 w_{zz} - \frac{k^2}{2\bar{\sigma}^2} \frac{w_z^2}{w_{zz}} + w_z + w_z^{\gamma/(\gamma-1)} + kz w_z.$$

Solving with respect to w_{zz} , and taking into account that $w_{zz} < 0$, gives

$$w_{zz} = \frac{-(w_z + w_z^{\gamma/(\gamma-1)} + kz w_z - \hat{\beta}w) - \sqrt{(w_z w_z^{\gamma/(\gamma-1)} + kz w_z - \hat{\beta}w)^2 + k^2 \frac{\sigma^2}{\bar{\sigma}^2} (1 - \rho^2) z^2}}{\sigma^2 (1 - \rho^2) z^2}$$

Dividing by $z^{\gamma-2}$, using Part (i), (8.1), and $u = w$, and sending $z \rightarrow \infty$ yields

$$\lim_{z \rightarrow \infty} \frac{w_{zz}(z)}{z^{\gamma-2}} = K^{\gamma-1} \gamma (\gamma - 1),$$

which, combined with (8.1), yields

$$\lim_{z \rightarrow \infty} \frac{w_z(z)}{z w_{zz}(z)} = \frac{1}{\gamma - 1}.$$

Combining with (3.13) and the fact that $u = w$ gives the result. \square

Let X_t be a solution of (3.8) with g and h given by (3.11)–(3.13). (In the proof of Theorem 1, below, we will show that there is a unique solution X_t .) We continue by considering the question of whether, beginning with strictly positive wealth ($X_0 = x > 0$), the optimal policy allows zero wealth to be attained. We can address this issue by applying the boundary classification Lemma 6.1 of Karlin

and Taylor (1981, p. 228) to the process $Z_t = X_t/Y_t$. From (4.1),

$$dZ_t = \delta(z)dt + v(Z_t)d\zeta_t, \quad Z_0 > 0,$$

where

$$\delta(z) = kz - \frac{k_1^2}{\sigma^2} b(z) - a(z),$$

$$v(z) = (k_1^2 b^2(z) + \sigma^2 z^2 (1 - \rho^2))^{1/2},$$

and

$$a(z) = \left(\frac{u'(z)}{\gamma} \right)^{1/(\gamma-1)} - 1 \quad \text{and} \quad b(z) = \frac{u'(z)}{u''(z)}, \quad z \geq 0.$$

and where ζ is a standard Brownian motion. The scale measure S associated with Z is defined by

$$S(z) = \int_{x_0}^z s(\xi) d\xi,$$

where

$$s(\xi) = \exp \left[- \int_{\xi_0}^{\xi} \left(\frac{2\delta(z)}{v^2(z)} \right) dz \right],$$

where $x_0 > 0$ and $\xi_0 > 0$ are arbitrary. By Lemma 6.1 of Karlin and Taylor, if

$$\lim_{\xi \downarrow 0} S(\xi) = -\infty, \tag{8.2}$$

then $Z_t > 0$ for all t almost surely. This in turn would imply that $X_t > 0$ for all t almost surely.

When $k_1 \neq 0$, by Proposition 1, $a(0) < 0$, $b(0) = 0$, and for small z , $v^2(z)$ is of the order of z . Then, on an interval $(0, \varepsilon)$ for ε sufficiently small, $\delta(\cdot)$ is non-negative and bounded away from zero, so that indeed (8.2) holds, and $X_t > 0$ for all t almost surely. When $k_1 = 0$, $v^2(z)$ is of the order of z^2 . Therefore, if $u'(0) > \gamma$, (8.2) holds, and $X_t > 0$ for all t almost surely.

Proof of Theorem 1. We now prove the remaining part of Theorem 1.

Part (i) follows from Theorem 5 and the matching of (3.5) and (4.3). For part (ii), the case of $y = 0$ is handled by the fact that, if $Y_0 = y = 0$, then $Y_t = 0$ for all t almost surely, reducing the problem to that of Merton (1971). When $y > 0$, we first observe that the candidate value function $\hat{v}(x, y) = y^\gamma u(x/y)$ is smooth

because u coincides with w , which is smooth. Moreover, due to the properties of u , \hat{v} is concave and continuous to the boundary. Using the form of \hat{v} and Theorem 2, it immediately follows from the definition of viscosity solutions that \hat{v} is a constrained viscosity solution of the original HJB equation (3.1). Thus, \hat{v} coincides with the value function v since the latter, by Theorem 4, is the unique concave constrained viscosity solution of (3.1). Therefore, we conclude that v is given by (3.17) and it is also smooth.

We next continue with the verification of the candidate optimal policy, part (iii) of Theorem 1. Once we establish the existence of an optimal policy and an optimal wealth process, part (iv) follows from the argument given before this proof.

In order to show that (C^*, Π^*) , as given by the feedback policy (g, h) is optimal, we first show that it exists and is admissible under the assumptions of the theorem, and then show that $\mathcal{J}(C^*) = v(x, y)$.

We extend u to the real line by defining $u(z) = u(0) + u'(0)z$ for $z \leq 0$. As such, u is a concave function that is differentiable at 0. By Rockafellar (1970), Theorem 25.3, u' is continuous at 0. From Proposition 1, we know that h , given by (3.13), extends continuously to $h(0, y) = 0$ for all y . Also g given by (3.12) extends continuously to $g(0, y) = \alpha y$ for all y , where $\alpha \leq 1$ and $\alpha < 1$ if $u'(0) > \gamma$. Recall that this is indeed the case if $k_1 \neq 0$.

We can now show that X , as given by (3.8) and (3.11)–(3.13), is uniquely well defined, taking $g(0, y)$ as defined by (3.12) and $h(0, y) = 0$. As such, g and h are continuous. The existence of a solution X to (3.8) follows from the proof of existence of $Z = X/Y$ established in Appendix B. The uniqueness follows from the fact that, when $k_1 = 0$, h is locally Lipschitz on the whole real line and g is monotone. When $k_1 \neq 0$, the coefficients of (3.8) are locally Lipschitz only on $(0, \infty)$, but the solutions of (3.8) are, almost surely, positive. Since the sample paths of C^* and Π^* are continuous, it follows that $\int_0^t C_s^* ds$ and $\int_0^t \Pi_s^2 ds$ are finite for all t . That $u'(0) > \gamma$ is crucial in the foregoing argument, for this implies that $g(0, y) \leq y$, so that the drift $(1 - \alpha)Y_t$ of X at the zero-wealth boundary is non-negative. Since the diffusion $h(0, y)\bar{\sigma}$ is zero at the zero-wealth boundary, the solution to (3.8) for X is therefore non-negative.

Admissibility then follows from the fact that $\mathcal{J}(C^*) \leq v(x, y) < \infty$, which is true by the arguments in the proof of Proposition A.2.

We continue by showing that the policy (C^*, Π^*) is optimal. For given (x, y) with $y > 0$, let X be defined by the proposed policy (C^*, Π^*) . Since $Z = X/Y$ is a well-defined semimartingale and u is a concave function, the process U defined by $U_t = u(Z_t)$ is a well-defined semimartingale by application of Ito's Lemma for convex functions of continuous semimartingales, for example, Karatzas and Shreve (1988, Theorem 7.1, p. 218). By this result, we can ignore the lack of differentiability of u' at zero, and use the usual 'naive' form of Ito's lemma, ignoring the term $u''(z)$ where, at $z = 0$, it may not be defined. (See, for example, Karatzas and Shreve, 1988, Problem 7.3, p. 219.) From this, using Ito's Lemma

to expand the process V defined by $V_t = Y_t^\gamma U_t$ as the product of two continuous semimartingales, we can see that the usual form of Ito’s Lemma for $v(X_t, Y_t)$ applies, simply leaving out the second-order term v_{xx} wherever (at $X_t = 0$) it does not exist anyway!

By this application of Ito’s Lemma, the fact that v satisfies the HJB equation (3.1), and the fact that $(g(x, y), h(x, y))$ satisfies the first-order necessary and sufficient conditions for the maximization indicated in the HJB, we have, for any $T > 0$,

$$\mathbb{E} \left[\int_0^{\tau(n) \wedge T} e^{-\beta t} (C_t^*)^\gamma dt \right] = - \mathbb{E} [e^{-\beta \tau(n) \wedge T} v(X_{\tau(n) \wedge T}, Y_{\tau(n) \wedge T})] + v(x, y), \tag{8.3}$$

where, for any positive integer n ,

$$\begin{aligned} \tau(n) &= \inf\{t: Y_t = n\} \wedge \inf\{t: Y_t = n^{-1}\} \\ &\wedge \inf\left\{t: \int_0^t \Pi_s^2 ds = n\right\} \wedge \inf\{t: X_t = n\}. \end{aligned}$$

Letting $n \rightarrow \infty$, we have $\tau(n) \rightarrow +\infty$ almost surely. By Proposition A.2,

$$\begin{aligned} v(X_{\tau(n) \wedge T}, Y_{\tau(n) \wedge T}) &\leq K^{\gamma-1} (X_{\tau(n) \wedge T} + \phi^\gamma \tau(n) \wedge T)^\gamma \\ &\rightarrow K^{\gamma-1} (X_T + \phi Y_T)^\gamma, \end{aligned}$$

as $n \rightarrow \infty$. We let $n \rightarrow \infty$ in (8.3). By dominated convergence,

$$\mathbb{E} \left[\int_0^T e^{-\beta t} (C_t^*)^\gamma dt \right] = - \mathbb{E} [e^{-\beta T} v(X_T, Y_T)] + v(x, y) \tag{8.4}$$

for every $T > 0$. Applying monotone convergence and $\mathcal{J}(C^*) < \infty$, the left-hand side converges to $\mathcal{J}(C^*)$. If, in addition, we have

$$\liminf_{T \rightarrow \infty} \mathbb{E} [e^{-\beta T} v(X_T, Y_T)] = 0, \tag{8.5}$$

then we are finished, for this implies that $v(x, y) = \mathcal{J}(C^*)$. By Proposition A.2,

$$v(X_T, Y_T) \leq K^{\gamma-1} (X_T + \phi Y_T)^\gamma \leq K^{\gamma-1} (X_T)^\gamma + (\phi Y_T)^\gamma,$$

and, in view of (2.1),

$$e^{-\beta T} \mathbb{E} [(Y_T)^\gamma] = y^\gamma e^{-\hat{\beta} T}.$$

Since $\hat{\beta} > 0$, this term converges to zero, as $T \rightarrow \infty$.

We will use Theorem 6 to estimate $E[(X_T)^\gamma]$. Indeed, by Theorem 6(ii), there is a constant c_0 satisfying

$$C_t^* = g(X_t, Y_t) \geq KX_t - c_0Y_t.$$

Therefore, $(C_t^*)^\gamma \geq K^\gamma X_t^\gamma - c_0^\gamma Y_t^\gamma$, and

$$E \left[\int_0^\infty e^{-\beta t} K^\gamma X_t^\gamma - c_0^\gamma Y_t^\gamma dt \right] \leq \mathcal{J}(C^*) < \infty.$$

Since

$$E \left[\int_0^\infty e^{-\beta t} Y_t^\gamma dt \right] < \infty,$$

there exists a sequence $T_n \rightarrow \infty$, such that

$$\lim_{n \rightarrow \infty} E[e^{-\beta T_n} X_{T_n}^\gamma] = 0,$$

and, consequently,

$$\liminf_{T \rightarrow \infty} E[e^{-\beta T} v(X_T, Y_T)] \leq \lim_{T \rightarrow \infty} K^{\gamma-1} (\phi Y_T)^\gamma + \lim_{n \rightarrow \infty} K^{\gamma-1} (X_{T_n})^\gamma = 0.$$

This completes the proof of Theorem 1. \square

Appendix A. A pseudo-complete markets problem

In order to obtain convenient bounds on the value function v and characterize its asymptotic behavior, we consider a fictitious consumption–investment problem with the same objective function \mathcal{J} considered in Section 2, with no stochastic income, and with an additional ‘pseudo-asset’ with price process S' given by

$$dS'_t = b'S'_t dt + S'_t \sigma' dW_t, \quad S'_0 > 0,$$

where $b' \in \mathbb{R}$ and $\sigma' \in \mathbb{R}^2$ are coefficients to be appropriately chosen. We always take

$$\Sigma = \begin{pmatrix} \bar{\sigma}\rho & \sigma\sqrt{1-\rho^2} \\ \sigma'_1 & \sigma'_2 \end{pmatrix}$$

non-singular, implying effectively complete markets. We denote by U the value function of this ‘pseudo-problem’. That is,

$$U(x) = \sup_{(C, \Pi, \Pi') \in \mathcal{A}'(w)} \mathcal{J}(C),$$

where $\mathcal{A}'(x)$ is the set of (C, Π, Π') in $\mathcal{C} \times \Phi \times \Phi$ such that there is a non-negative solution X' to the stochastic differential equation

$$dX'_t = [rX'_t - C_t + (b - r)\Pi_t + (b' - r)\Pi'_t] dt + \bar{\sigma}\Pi_t dB_t + \Pi'_t \sigma' dW_t,$$

$$X'_0 = x \geq 0.$$

Proposition A.1. The constants b' and σ' can be chosen so that, for all x , we have $v(x, 0) = U(x)$.

Proof. We know from Merton’s original work that

$$U(x) = k^\gamma x^\gamma, \tag{A.1}$$

where

$$k = \frac{\beta - r\gamma}{1 - \gamma} - \frac{\gamma \lambda^T (\Sigma \Sigma^T)^{-1} \lambda}{2(1 - \gamma)^2}, \tag{A.2}$$

with $\lambda^T = (b - r, b' - r)$.

In view of the similar explicit solution for $v(x, 0)$ given by Theorem 1, is enough to choose b' and σ' so that

$$\lambda^T (\Sigma \Sigma^T)^{-1} \lambda = \frac{(b - r)^2}{\bar{\sigma}^2},$$

maintaining non-singularity of Σ . By expanding the matrix $(\Sigma \Sigma^T)^{-1}$ in terms of b' and σ' , this can be done by solving a quadratic equation for b' , fixing $\Gamma = \Sigma \Sigma^T$. We have

$$b' = r + (b - r) \frac{\Gamma_{12}}{\Gamma_{11}}. \tag{A.3}$$

This completes the proof. \square

Proposition A.2. *There exists a strictly positive constant ϕ such that, for all (x, y) ,*

$$K^{\gamma-1}x^\gamma \leq v(x, y) \leq K^{\gamma-1}(x + \phi y)^\gamma,$$

where K is given by (3.9).

Proof. Consider the pseudo-problem described above. Let $f: [0, +\infty) \rightarrow [0, +\infty]$, representing the ‘initial wealth equivalent’ of the stochastic income Y , be defined by

$$f(y) = E\left(\int_0^{+\infty} e^{-rt} \xi_t Y_t dt\right),$$

where

$$\xi_t = \exp(-\frac{1}{2}(\theta \cdot \theta)t + \theta \cdot W_t),$$

with $\theta = \Sigma^{-1}\lambda$. We note that f is finite-valued if

$$\theta_1\sigma < r - \mu. \tag{A.4}$$

Since $\mu - r < 0$ by assumption in Theorem 1, (A.4) holds provided Γ_{22} is sufficiently large, fixing b' and $\Gamma_{12} = \Gamma_{21}$. We can thus choose b' and σ' so that both (A.3) (which does not depend on Γ_{22}) and (A.4) hold. With this, $f(y) = \phi y < \infty$ for a constant $\phi > 0$.

The stochastic income Y can be replicated by a trading strategy involving the riskless asset, the original risky asset with price process S , and the pseudo-asset with price process S' . The associated initial investment required is ϕy . Because of the non-negative wealth constraint in the pseudo-problem, the optimal utility with stochastic income is smaller than the optimal utility in which the stochastic income is replaced with its wealth-equivalent, ϕy . The latter optimal utility is $U(x + \phi y)$. This approach is well known by now; for a standard reference, see Huang and Pagès (1992).

The utility $v(x, y)$ for the incomplete markets original problem is certainly no larger than the utility $U(x + \phi y)$ that obtains when one is allowed to invest in both the original assets and the pseudo-asset, and is also allowed to replace stochastic income with its wealth equivalent, establishing that $v(x, y) \leq U(x + \phi y)$. Combining this with Proposition A.1 and (A.1) gives the result. \square

It is worth noting from the construction in the proof that $\mu > r$ can be accommodated with slightly more complicated conditions.

Appendix B

Proposition B.1. The process Z given by (4.1) is uniquely well defined.

Proof. It suffices to show that the square of the coefficients of the stochastic differential equation for Z grow at most quadratically. (See Theorem 3 in Chapter 6 of Gikhman and Skorohod, 1972.)

First, we observe that the optimal policy (C^*, Π^*) of the reduced one-dimensional ‘dual’ problem are given in the feedback form

$$C_t^* = A(Z_t), \quad \Pi_t^* = -\frac{k_1}{\sigma^2} B(Z_t)$$

with A and B defined in Section 7. We next show that

$$A^2(z) + \frac{k_1^2}{\sigma^4} B^2(z) \leq C_1(1 + z^2) \quad (\text{B.1})$$

for some constant C_1 .

First, observe that $A(z) \geq 0$ and that $A(z)$ is strictly increasing in z since

$$A'(z) = \frac{1}{\gamma - 1} u''(z)u'(z)^{(2-\gamma)/(\gamma-1)} > 0.$$

The quadratic growth of $A^2(z)$ follows then from the above properties together with the continuity of A and Theorem 6, Part (i). Similarly, the quadratic growth of $B^2(z)$ follows from the positivity and finiteness of B together with (7.7) which gives

$$\lim_{z \rightarrow \infty} \frac{B(z)}{z} = \frac{1}{\gamma - 1},$$

from which (B.1) follows. \square

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