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Turnpike behavior of long-term investments*

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Abstract. We study the behavior of the optimal portfolio policy of a long-run investor in markets with stationary investment opportunity sets. We provide conditions on the utility function, for large wealth levels, which are sufficient for the optimal portfolio policy to approximate, as the trading horizon becomes very long, the policy of investing a constant proportion of wealth in the various assets. The analysis is carried out by employing the associated HJB equation and recent advances in the area of viscosity solutions.

Key words: Turnpike portfolios, stochastic control, viscosity solutions

JEL classification: D9, G1.

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1 Introduction

The optimal portfolio policy for a long-run investor has been a classical topic in financial economics; see Cox and Huang (1992), Hakansson (1974), Huberman and Ross (1983), Leland (1972), and Mossin (1968). The central question asked in these papers is whether in an economy with a stationary investment opportunity set, there are necessary and sufficient conditions on a long-run investor's utility

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function for final wealth, so that the optimal portfolio policy at the very beginning of the investment horizon can be approximated arbitrarily closely by the policy of investing constant proportions of wealth in the assets or in other words, if it exhibits the *portfolio turnpike property*?

It is well known that in an economy with a stationary investment environment, the optimal portfolio policy for an investor whose utility function exhibits a constant coefficient of the Arrow-Pratt measure of relative risk aversion (CRRA) is to invest constant proportion of wealth in the risky asset. The results in this literature thus lie in showing the turnpike property when the utility function of the long-run investor behaves almost like a CRRA utility function for large wealth levels.

In discrete-time models, Hakansson (1974) shows the turnpike property for a utility function U so that $\gamma^{-1}(x-a)^{\gamma}/ \leq U(x) \leq \gamma^{-1}(x+a)^{\gamma}$ $\forall x \geq a$ and for some $\gamma < 1$, and Huberman and Ross (1983) show this for U which is bounded from below and satisfies, for some $a \in (0,1)$,

$$\lim_{x\to\infty} \frac{-U''(x)x}{U'(x)} = a\,.$$

Huberman and Ross (1983), Leland (1972), Hakansson (1974), and Mossin (1968) all use dynamic programming in their discrete time model.

In a continuous-time model, Cox and Huang (1992) demonstrate, using probabilistic methods, that the portfolio turnpike property holds if there exist $A_1 > 0$, $A_2 > 0$, $b > 0$ and $y^* > 0$ with

$$(U')^{-1}(y) - A_1 y^{-\frac{1}{b}}| \leq A_2 y^{-a} \quad \forall y \leq y^*$$

for some $a \in [0, 1/b)$, where U' and $(U')^{-1}$ denotes the derivative of U and the inverse of U', respectively. They also bring out the economics of the problem most clearly: when the interest rate is strictly positive, the present value of any contingent claim having payoffs bounded from above can be made arbitrarily small when the investment horizon increases. Thus an investor concentrates his wealth in buying contingent claims that have payoffs unbounded from above at the very beginning of his horizon. As a consequence, *it is the asymptotic property of his utility function as wealth goes to infinity, that determines the optimal investment strategy at the very beginning of his horizon*.

The condition of Huberman and Ross (1983) is neither more general or more restrictive than that of Hakansson's. Cox and Huang's condition implies that the relative risk aversion of the utility function converges to a constant not necessarily between 0 and 1. Thus their condition is also neither more general nor more restrictive than that of Huberman and Ross'.

One open question remains: Can one find *necessary* and *sufficient* conditions for the turnpike property that include all the existing known conditions as special cases? We do not have a complete answer to this question in this paper, however; instead, we show that the turnpike property holds under a condition that neither includes as special cases all existing conditions, nor is included as a special case of one of the known conditions. More specifically, we show that in a continuous time economy the condition

$$\lim_{x\to\infty} \frac{U'(x)}{x^{\gamma-1}} = K \tag{1.1}$$

for some strictly positive scalar K for a utility function U is *sufficient* for the turnpike property. Nevertheless, we still contribute to the answering of the open question in two ways. First, we learn that the turnpike property holds under some conditions that were not previously known. Second, we use dynamic programming in our proofs and thus the sufficient conditions for the turnpike property are conditions on the marginal utility function. This contrasts with the conditions on the "inverse" of the marginal utility function of Cox and Huang (1992). Given that, in discrete time models, all existing conditions for the turnpike property are directly imposed on the utility function and the proofs depend on dynamic programming, it is useful to have a proof of the turnpike property in a *continuous time framework* which also uses dynamic programming. This makes it easier to compare the proofs of the turnpike property under different conditions and can potentially give rise to a general condition on the utility function for the turnpike property to hold.

Besides the issues discussed above, this paper also makes a methodological contribution that is of independent interest. We show how the recent advances in the theory of nonlinear partial differential equations and, in particular, in the theory of *viscosity solutions*, can be extremely useful in analyzing the asymptotic properties of the optimal portfolio policy. Similar techniques may also be useful in other contexts.

The rest of this paper is organized as follows. In Sect. 2 we formulate a continuous-time securities market economy with a stationary investment opportunity set. We analyze an investor's optimal portfolio decision when he invests his wealth over time in order to maximize the expected utility of wealth at the end of his investment horizon. We also show that the optimal amount invested in the risky asset solves a time homogeneous second order nonlinear ordinary differential equation. In Sect. 3 we first show that, independently of the length of the investment horizon, the optimal dollar amount invested in the risky asset is approximately a linear function of wealth, at high levels of wealth, when the marginal utility behaves asymptotically like a power function. Sect. 3 goes on to demonstrate that, given the power asymptotic behavior of the utility function, the optimal dollar amount invested in the risky asset for all levels of wealth and all investment horizons, must lie in a time independent "window." It then follows that the optimal dollar amount invested in the risky asset is approximately a linear function of wealth for all levels of wealth at the very beginning of a very long investment horizon, provided that the marginal utility function satisfies the aforementioned asymptotic condition. Finally, in Sect. 4 we state some concluding remarks.

After this work was completed, a similar class of investment problems with turnpike behavior was studied, using entirely different methodology, by Back et al. (1994).

2 The formulation

We start this section by describing the underlying financial model that we study herein. This is the classical optimal investment model, introduced by Merton (1971), in a market with a stationary opportunity set and a finite investment horizon. Since this model is widely known and extensively studied, we only present the main results without going into a detailed discussion about the technical conditions at this point. Some of the arguments, presented in this section, will be further discussed later on when we specify the class of preferences in the context of the turnpike behavior for the optimal investment policies. To this end, we consider a securities market economy with one long-lived asset and a riskless lending and borrowing opportunity. Denote by $S(t)$ the risky asset price at time t. Assume that the risky asset price follows a geometric Brownian motion; that is, in the short-hand differential form,

$$dS(t) = \mu S(t)dt + \sigma S(t)dw(t), \quad t \in \Re_+,$$

where μ and σ are two positive constants, $w = \{w(t); t \in \Re_+\}$ is a standard Brownian motion defined on some probability space $(\Omega, \mathscr{F}, P)$, and $\Re_+$ is the positive real line. The riskless interest rate is a constant denoted by r. For convenience, we assume that $\mu > r$; the case $\mu \leq r$ can be treated similarly.

For a given time horizon $T > 0$, consider at any $t \in [0, T]$ the problem of maximizing the expected utility of one's terminal wealth by dynamically investing in the risky asset and in the riskless one. Let A_τ denote the dollar amount invested in the risky asset when there are τ periods to T. Then, according to the budget constraint, the dollar amount invested in the riskless rate is $W_\tau - A_\tau$ where W_τ is the investor's wealth at time τ.

The wealth dynamics must satisfy the stochastic differential equation (SDE):

$$\begin{cases} dW_s = [rW_s + (\mu - r)A_s]dt + \sigma A_s dw(s), \quad s \in (t, T] \\ W_t = x, \quad x \geq 0. \end{cases} \tag{2.1}$$

We assume that the portfolio processes A_s satisfy the admissibility constraints:

i) A_s is $\mathscr{F}_s$-progressively measurable, where $\mathscr{F}_s = \sigma\{w_u; t \leq u \leq s\}$.

ii) A_s satisfies the integrability condition $E\int_t^T A_s^2 ds < +\infty$, a.s.

iii) the budget (state) constraint $W_s \geq 0$; a.s. $t \leq s \leq T$, holds.

We denote by $\mathscr{A}$ the set of admissible policies.

Given the initial level of wealth at t, $W_t = x$, we want to find the policy $A \in \mathscr{A}$ that solves the portfolio problem

$$J(x,\tau) \equiv \sup_{A\in\mathscr{A}} E[U(W_T)|W_t = x]; \quad a.s.\ \tau = T - t, \tag{2.2}$$

where the wealth dynamics satisfy (2.1) and U is the utility function of final wealth assumed to be increasing and strictly concave. The function $J(x,\tau)$ is called the indirect utility function or the *value function*.

Using the properties of U and the linearity of the state dynamics it can be shown (see, for example, Merton 1971) that J is strictly increasing and concave in x. Moreover, it follows (see Merton 1971) from the dynamic programming principle that if $J(x,\tau)$ is $C^{2,1}(\Re_+,\Re_+)$,[1] it satisfies the Bellman equation:

$$\max_A \left\{ \frac{1}{2}\sigma^2 A^2 J_{xx} + (\mu - r)AJ_x \right\} + rxJ_x - J_\tau = 0, \quad (x,\tau) \in (0,\infty)\times(0,T] \tag{2.3}$$

with initial and boundary conditions

$$\left\{ \begin{array}{lll} J(x,0) & = U(x) & \forall x \geq 0 \\ J(0,\tau) & = U(0) & \forall \tau \in (0,T] \end{array} \right\}. \tag{2.4}$$

Also, the first-order conditions for maximality in (2.3) imply that the optimal policy A^*_τ can be expressed in the feedback form $A^*_\tau = A(W^*_\tau,\tau)$ where W^*_τ is the optimal wealth trajectory and

$$A(x,\tau) = -\frac{\mu - r}{\sigma^2}\frac{J_x(x,\tau)}{J_{xx}(x,\tau)}, \quad (x,\tau) \in (0,\infty)\times(0,T]. \tag{2.5}$$

The fact that $\mu > r$ and that J is strictly increasing and concave in x yields

$$A(x,\tau) \geq 0, \quad \forall (x,\tau) \in \Re_+ \times (0,T]. \tag{2.6}$$

Substituting (2.5) into (2.3) gives the nonlinear partial differential equation

$$J_\tau = -\frac{(\mu - r)^2}{2\sigma^2}\frac{J_x^2}{J_{xx}} + rxJ_x, \quad (x,\tau) \in (0,\infty)\times(0,T]. \tag{2.7}$$

We impose the following assumption throughout the paper

Assumption 2.1: *The utility function* $U : \Re_+ \to \Re_+$ *is assumed to be increasing, concave and twice continuously differentiable. Moreover, it is assumed that* U'' *is nondecreasing and that* U *satisfies* $U(0) = 0$ *and also, the growth condition* $U(x) \leq K(1+x)^\gamma$ *for* $0 < \gamma < 1$ *and* $K > 0$.

Remark 2.1: In the next section, we will see that the above growth condition of the utility function follows from the conditions to be imposed on its asymptotic behavior for large wealth (see Lemma 3.1). At this point we impose this assumption, albeit redundant, in order to guarantee that the value function is well defined.[2] In fact, one can compute explicitly the value function, corresponding to the utility $U_1(x) = K(1+x)^\gamma$, which dominates the value function J.

The following proposition records that under Assumption 2.1, the value function solves (2.3) and the optimal control satisfies some nice properties.

Proposition 2.1: *The value function J is the unique* $C^{2,1}(\Re_+, \Re_+)$ *increasing and concave solution of the Bellman equation (2.3), also satisfying (2.4). Moreover, the optimal dollar amount invested in the risky asset* $A(x,\tau)$ *is* $C^{2,1}(\Re_+, \Re_+)$ *and satisfies the quasilinear parabolic equation*

$$\frac{1}{2}\sigma^2 A^2 A_{xx} + rxA_x - rA - A_\tau = 0, \quad (x,\tau) \in (0,\infty) \times (0,T] \tag{2.8}$$

with initial and boundary conditions

$$\left\{ \begin{array}{ll} A(x,0) & = -\frac{\mu - r}{\sigma^2}\frac{U'(x)}{U''(x)}, \quad \forall x > 0 \\ A(0,\tau) & = 0, \quad \forall \tau \in [0,T] \end{array} \right\} \tag{2.9}$$

The proof of the proposition is presented in the Appendix.

Remark 2.1. Note that the boundary conditions, at $x = 0$, for the value function and the optimal policy, given respectively in (2.4) and (2.9), are *not* explicitly given by (2.2). Actually, one can show that they are equivalent to the nonnegativity of the wealth state process (see, for example, He and Huang 1993).

3 Optimal portfolio for long horizons and sufficient conditions for the turnpike property

We are interested in the behavior of $A(x,\tau)$ when τ is large. Given that A satisfies (2.8), the behavior of $A(x,\tau)$ for very large τ should be approximated by the solution of the time independent version of (2.8):

$$\frac{1}{2}\sigma^2 A^2 A_{xx} + rxA_x - rA = 0, \quad x > 0. \tag{3.1}$$

Two problems arise, however. First, $\lim_{\tau\to\infty} A(x,\tau)$ may not exist. Second, it is easily verified that (2.8) does not have a unique solution; for example, any time-independent linear function of x satisfies (3.1). Thus, even if $\lim_{\tau\to\infty} A(x,\tau)$ exists, we won't get much information out of just (3.1). (We get some information; see Theorem 3.1.)

We will tackle the first problem using recent advances in the theory of viscosity solutions, namely the limit supremum and the limit infimum operations introduced by Barles and Perthame (1988). As for the second problem, after getting as much out of (3.1) as possible, we will use the Bellman equation and the relation between A and the function J described in (2.5).

Before we present the technique of Barles and Perthame, we state the definition of *viscosity solutions*.

The notion of viscosity solutions was introduced by Crandall and Lions (1984) for second order equations. For a general overview of the theory we refer the reader to the "User's Guide" by Crandall et al. (1992) and the book of Fleming and Soner (1993). To this end, consider a nonlinear second order partial differential equation of the form

$$F(z, u, Du, D^2u) = 0, \quad z \in \Omega, \tag{3.2}$$

where Ω is an open subset of $\Re^2$ and $F : \Omega \times R \times R^2 \times M_{2\times 2} \to R$ is continuous and (possibly degenerate) elliptic, i.e.,

$$F(z, u, Du, A + B) \leq F(z, u, Du, A) \quad \text{if } B \geq 0.$$

Moreover, Du and D^2u are, respectively, the gradient and the second-order derivative matrix of u and $M_{2\times 2}$ is the space of 2×2 matrices. Finally in the sequel we will denote the closure of Ω by $\overline{\Omega}$.

Definition 3.1:

1. *An upper semi-continuous function $u : \overline{\Omega} \to \Re$ is a viscosity subsolution of (3.2), if for any $\phi \in C^2(\overline{\Omega})$ and any local maximum point $z_0 \in \Omega$ of $u - \phi$,*

$$F(z_0, u(z_0), D\phi(z_0), D^2\phi(z_0)) \leq 0.$$

2. *A lower semi-continuous function $u : \overline{\Omega} \to \Re$ is a viscosity supersolution of (3.2), if for any $\phi \in C^2(\overline{\Omega})$ and any minimum point $z_0 \in \Omega$ of $u - \phi$,*

$$F(z_0, u(z_0), D\phi(z_0), D^2\phi(z_0)) \geq 0.$$

Definition 3.2: *A continuous function $u : \overline{\Omega} \to \Re$ is a viscosity solution of (3.2) if and only if it is both a sub- and a super-viscosity solution in Ω.*

3.1 Properties of the limsup and liminf of $A(W, \tau)$

We define the limsup and liminf of $A(x, \tau)$ as the functions

$$A^*(x) \equiv \limsup_{\substack{\tau\to\infty \\ y\to x}} A(y, \tau), \tag{3.3}$$

and

$$A_*(x) \equiv \liminf_{\substack{\tau\to\infty \\ y\to x}} A(y, \tau). \tag{3.4}$$

The following theorem gives useful properties of A^* and A_* in relation to the limit equation (3.1).

Theorem 3.1:

(i) The functions A^* and A_* are respectively upper and lower semicontinuous and sub- and super-viscosity solutions of the stationary equation (3.1)
(ii) The functions A^* and A_* satisfy

$$\begin{cases} 0 \leq A_*(x) \leq A^*(x) \\ A^*(0) = A_*(0) = 0. \end{cases} \tag{3.5}$$

The proof of (i) follows along the lines of Theorem 3.1 of Barles and Perthame (1988) and therefore it is omitted. Assertion (ii) follows from the definition of A^* and A_* and the budget (state) constraint.

The information contained in the above theorem, albeit valuable, does not suffice to pin down the asymptotic property of A when the investment horizon is long. This should not come as a surprise. In Bellman's equation (2.3) or (2.7), the information of the utility function comes into the analysis as a boundary condition when $\tau = 0$. Given the time independent nature of (2.8), however, there is no natural way to bring the information regarding the utility function into the analysis. How then can we expect (3.1) to give us very precise information about the asymptotic property of A? We have to, instead, bring the information on U into our analysis for finite τ through the Bellman equation and examine the properties of A in detail when τ increases. This is the subject to which we now turn. The reader will find out later that the HJB equation alone won't do the job either. It is a delicate interplay between the HJB equation and the portfolio equation (2.8) that makes one of the main results of this paper possible.

3.2 Asymptotic properties of $A(x, \tau)$ for high levels of wealth

We consider a class of utility functions that satisfy the following assumption

Assumption 3.1: *For some $\gamma < 1$,*

$$\lim_{x \to \infty} \frac{U'(x)}{x^{\gamma - 1}} = 1.$$

Under the above assumption, we will show that for *high levels of wealth*, $A(x, \tau)$ is *approximately linear* regardless of whether τ is large. To this end, we will utilize the form (2.5) of the optimal portfolio and thus need to examine the behavior of the derivatives of J when x is large.

Note that our results to follow apply to utility function that satisfy Assumption 3.1 with the right-hand-side of the equality replaced by any strictly positive scalar. Our choice of setting this scalar to be unity is for notational simplicity.

We begin with a lemma that characterizes all the utility functions U that satisfy Assumption 3.1.

Lemma 3.1: *Under Assumption 3.1, the utility function U is unbounded from above if $0 \le \gamma < 1$ and is bounded from above if $\gamma < 0$. In particular,*

(i) if $0 \le \gamma < 1$, Assumption 3.1 is equivalent to

$$\lim_{x \to \infty} \frac{U(x)}{\frac{1}{\gamma} x^{\gamma}} = 1, \tag{3.6}$$

where we understand that when $\gamma = 0$, $\frac{1}{\gamma} x^{\gamma} = \ln x$;

(ii) if $\gamma < 0$, Assumption 3.1 is equivalent to

$$\lim_{x\to\infty} \frac{U(x) - N}{\frac{1}{\gamma}x^{\gamma}} = 1$$

for some constant N;

(iii) for all values of $\gamma < 1$,

$$\lim_{x\to\infty} \frac{-U''(x)x}{U'(x)} = 1 - \gamma. \tag{3.7}$$

Proof. We only provide the proof of parts (i) and (iii) since part (ii) follows along the lines of (i). To this end, observe that from Assumption 3.1 we have that for $\epsilon > 0$, $\exists R_\epsilon > 0$ such that

$$(1-\epsilon)x^{\gamma-1} \le U'(x) \le (1+\epsilon)x^{\gamma-1}, \quad \text{for } x \ge R_\epsilon.$$

Integrating yields,

$$\begin{aligned}(1-\epsilon)\Big(\frac{1}{\gamma}x^{\gamma} - \frac{1}{\gamma}R_\epsilon^{\gamma}\Big) \quad &\le \quad U(x) - U(R_\epsilon)\\ &\le \quad (1+\epsilon)\Big(\frac{1}{\gamma}x^{\gamma} - \frac{1}{\gamma}R_\epsilon^{\gamma}\Big)\end{aligned}$$

and, in turn,

$$\begin{aligned}(1-\epsilon)\Big(1 - \Big(\frac{R_\epsilon}{x}\Big)^{\gamma}\Big) \quad + \quad \frac{U(R_\epsilon)}{\frac{1}{\gamma}x^{\gamma}} &\le \frac{U(x)}{\frac{1}{\gamma}x^{\gamma}}\\ &\le (1+\epsilon)\Big(1 - \Big(\frac{R_\epsilon}{x}\Big)^{\gamma}\Big).\end{aligned}$$

Sending $x \to \infty$ we have

$$(1-\epsilon) \le \varliminf_{x\to\infty} \frac{U(x)}{\frac{1}{\gamma}x^{\gamma}} \le \varlimsup_{x\to\infty} \frac{U(x)}{\frac{1}{\gamma}x^{\gamma}} \le 1 + \epsilon$$

and by letting $\epsilon \to 0$ we conclude.

For part (iii), we show (3.7) for $0 < \gamma < 1$ since the proof for the other values of γ is similar. To this end, we first show

$$\lim_{x\to\infty} \frac{U''(x)}{(\gamma-1)x^{\gamma-2}} = 1. \tag{3.8}$$

In fact, by Assumption 2.1, U' is convex which implies

$$U'(x+h) \ge U'(x) + hU''(x) \qquad \forall h > 0,\ x > 0.$$

Dividing by $(\gamma-1)hx^{\gamma-2}$, the above inequality yields

$$\frac{U'(x+h) - U'(x)}{(\gamma-1)hx^{\gamma-2}} \le \frac{U''(x)}{(\gamma-1)x^{\gamma-2}} \tag{3.9}$$

which in turn yields, for $h = \delta x$ with $\delta > 0$,

$$\frac{1}{\gamma - 1}\left[\frac{U'((1+\delta)x)}{(1+\delta)^{\gamma-1}x^{\gamma-1}}\frac{(1+\delta)^{\gamma-1}}{\delta} - \frac{U'(x)}{\delta x^{\gamma-1}}\right] \leq \frac{U''(x)}{(\gamma-1)x^{\gamma-2}}. \tag{3.10}$$

We now use Assumption 3.1 and, keeping δ fixed, inequality (3.10) implies

$$\frac{1}{(\gamma-1)}\frac{(1+\delta)^{\gamma-1}-1}{\delta} \leq \varliminf_{x\to\infty} \frac{U''(x)}{(\gamma-1)x^{\gamma-2}}.$$

The above inequality implies, as $\delta \to 0$,

$$\lim_{\delta\to 0}\frac{1}{\gamma-1}\frac{(1+\delta)^{\gamma-1}-1}{\delta} \leq \varliminf_{x\to\infty}\frac{U''(x)}{(\gamma-1)x^{\gamma-2}}$$

or,

$$1 \leq \varliminf_{x\to\infty}\frac{U''(x)}{(\gamma-1)x^{\gamma-2}}. \tag{3.11}$$

Applying the above arguments for $h = -\delta x$, starting with

$$U'(x(1-\delta)) \geq U'(x) - \delta x U''(x)$$

and passing to the limit yields

$$\varlimsup_{x\to\infty}\frac{U''(x)}{(\gamma-1)x^{\gamma-2}} \leq 1$$

which combined with (3.11) yields (3.8).

The assertion then follows since

$$\begin{aligned}\lim_{x\to\infty}\frac{-U''(x)x}{U'(x)} &= \lim_{x\to\infty}\frac{-U''(x)/x^{\gamma-2}}{U'(x)/x^{\gamma-1}} \\ &= -\frac{\lim_{x\to\infty} U''(x)/x^{\gamma-2}}{\lim_{x\to\infty} U'(x)/x^{\gamma-1}} \\ &= 1-\gamma.\end{aligned}$$

□

Even though the above lemma shows that the class of utility functions satisfying Assumption 3.1 includes those that are bounded as well as the logarithmic case ($\gamma = 0$), we will only deal with the cases where $0 < \gamma < 1$. This is assumed only for notational simplicity since all the proofs apply to the excepted cases with slight modification of some arguments.

We will next show that under the aforementioned assumption, the indirect utility function $J(x,\tau)$ inherits the asymptotic behavior of U for large x and for *all* finite times τ, not necessarily just for large τ. Instead of working directly with the function $J(x,\tau)$, we will first work with a family of auxiliary functions J^{ϵ}'s, given by

$$J^{\epsilon}(x,\tau) \equiv \epsilon^{\gamma} J\left(\frac{x}{\epsilon},\tau\right) \quad \forall x \geq 0, \epsilon > 0, \tau \in [0,T], \tag{3.12}$$

and examine the behavior of J^ϵ as $\epsilon \to 0$.

Before proceeding, we record the following two lemmas.

Lemma 3.2: *Assume that there exists a function w increasing and concave, such that $J^\epsilon(x,\tau) \leq w(x,\tau)\ \forall x \geq 0$, $\epsilon > 0$, $\tau \in [0,T]$. Then the functions $J^* = \overline{\lim}_{\epsilon\to 0} J^\epsilon$ and $J_* = \underline{\lim}_{\epsilon\to 0} J^\epsilon$ are respectively viscosity sub- and super-solutions of*

$$\begin{aligned} &\widetilde{J}_\tau = \max_\pi \left[\tfrac{1}{2}\sigma^2\pi^2\widetilde{J}_{xx} + (\mu - r)\pi\widetilde{J}_x\right] + rx\widetilde{J}_x = -\frac{(\mu-r)^2}{2\sigma^2}\frac{\widetilde{J}x^2}{\widetilde{J}_{xx}} + rx\widetilde{J}_x \\ &\widetilde{J}(x,0) = \tfrac{1}{\gamma}x^\gamma, \\ &\widetilde{J}(0,\tau) = 0. \end{aligned} \tag{3.13}$$

For the proof see Ishii and Lions (1990).

Lemma 3.3: *The initial value problem (3.13) has a unique viscosity solution in the class of functions that are increasing and concave in x. Moreover this solution is given by*

$$\widetilde{J}(x,\tau) = \frac{e^{\lambda\tau}}{\gamma}x^\gamma,$$

where

$$\lambda \equiv \frac{\gamma(\mu-r)^2}{2\sigma^2(1-\gamma)} + \gamma r.$$

Proof. For the uniqueness part, see Theorem 4.1 in Zariphopoulou (1994). Also, observe that the function $\widetilde{J}$ is smooth and therefore, it is a viscosity solution of (3.13). Moreover, $\widetilde{J}$ is concave and increasing in x and we easily conclude that it must coincide with the unique viscosity solution of (3.13).

Proposition 3.2: *Suppose that Assumptions 2.1 and 3.1 hold. Then*

$$\lim_{\epsilon\to 0} J^\epsilon(x,\tau) = \frac{e^{\lambda\tau}x^\gamma}{\gamma} \tag{3.14}$$

and therefore

$$\lim_{x\to\infty} \frac{J(x,\tau)}{e^{\lambda\tau}x^\gamma} = \frac{1}{\gamma}. \tag{3.15}$$

where J solves (2.3).

Proof. Using (3.12) and direct differentiation give

$$\begin{aligned} J_x^\epsilon(x,\tau) &= \epsilon^{\gamma-1}J_x\left(\tfrac{x}{\epsilon},\tau\right), \\ J_{xx}^\epsilon(x,\tau) &= \epsilon^{\gamma-2}J_{xx}\left(\tfrac{x}{\epsilon},\tau\right), \\ J_\tau^\epsilon(x,t) &= \epsilon^{\gamma}J_\tau\left(\tfrac{x}{\epsilon},\tau\right). \end{aligned} \tag{3.16}$$

Evaluating (2.3) and (2.4) at the point $\left(\frac{x}{\epsilon},\tau\right)$ yields

$$J_\tau\left(\frac{x}{\epsilon},\tau\right) = -\frac{(\mu-r)^2}{2\sigma^2}\frac{J_x^2\left(\frac{x}{\epsilon},\tau\right)}{J_{xx}\left(\frac{x}{\epsilon},\tau\right)} + r\frac{x}{\epsilon}J_x\left(\frac{x}{\epsilon},\tau\right)$$

and

$$J\left(\frac{x}{\epsilon},0\right) = U\left(\frac{x}{\epsilon}\right).$$

These imply, using (3.16), that

$$\begin{aligned} J_\tau^\epsilon(x,\tau) &= -\tfrac{(\mu-r)^2}{2\sigma^2}\tfrac{(J_x^\epsilon(x,\tau))^2}{J_{xx}^\epsilon(x,\tau)} + rxJ_x^\epsilon(x,\tau), \\ J^\epsilon(x,0) &= \epsilon^\gamma U\left(\tfrac{x}{\epsilon}\right). \end{aligned} \tag{3.17}$$

We next show that the $J^\epsilon(x,\tau)$'s are locally bounded in $\Re_+ \times [0,T]$ uniformly in ϵ. To this end, let $w(x,\tau)$ be the indirect utility function of the utility maximization problem given by (2.2), with utility function

$$\hat{U}(x) = K_1 x^\gamma + K_2,$$

where $K_1 > \frac{1}{\gamma}$ and $K_2 > 0$ is a sufficiently large (local) constant. Given that $\lim_{x\to\infty}\frac{U(x)}{x^\gamma} = \frac{1}{\gamma}$, one sees that for sufficiently large K_2, $\hat{U}(x) > U(x)$, $\forall x \geq 0$.

Let $\hat{w}$ be defined as

$$\hat{w}(x,\tau) \equiv K_1 e^{\lambda\tau}x^\gamma + K_2 e^{\lambda\tau}.$$

Direct calculation shows that $\hat{w}$ is a (viscosity) supersolution of

$$\begin{cases} w_\tau(x,\tau) = -\frac{(\mu-r)^2}{2\sigma^2}\frac{w_x^2(x,\tau)}{w_{xx}(x,\tau)} + rxw_x(x,\tau), \\ w(x,0) = K_1x^\gamma + K_2. \end{cases} \tag{3.18}$$

Since by Proposition 2.1 equation (3.18) admits a unique concave solution, the supersolution $\hat{w}$ must lie above the solution w. Moreover w dominates J because $\hat{U} > U$. Therefore

$$\hat{w}(x,\tau) \geq w(x,\tau) \geq J(x,\tau),$$

which in turn yields,

$$\begin{aligned} J^\epsilon(x,\tau) &\leq \epsilon^\gamma\hat{w}\left(\tfrac{x}{\epsilon},\tau\right) = \epsilon^\gamma\left[K_1e^{\lambda\tau}\left(\tfrac{x}{\epsilon}\right)^\gamma + K_2e^{\lambda\tau}\right] \\ &= K_1e^{\lambda\tau}x^\gamma + K_2e^{\lambda\tau}\epsilon^\gamma \\ &< e^{\lambda\tau}\left(K_1x^\gamma + K_2\right). \end{aligned}$$

Applying Lemma 3.2 with $w = e^{\lambda\tau}(K_1x^\gamma + K_2)$, we get that J^* are J_* are viscosity sub- and super-solutions of (3.13). On the other hand, by Lemma 3.3, the initial value problem (3.13) has a unique viscosity solution which implies that

$$J^* \leq J_*.$$

Moreover, by construction, $J^* \geq J_*$ which combined with the above inequality yields

$$J^*(x,\tau) = J_*(x,\tau) = \widetilde{J}(x,\tau)$$

where $\widetilde{J}$ is the unique visicosity solution of (3.13) given in Lemma 3.3.

Therefore,

$$\lim_{\epsilon\to 0} J^\epsilon(x,\tau) = \frac{e^{\lambda\tau}x^\gamma}{\gamma}.$$

Equivalently,

$$\lim_{\epsilon\to 0} \frac{\epsilon^\gamma J\left(\frac{x}{\epsilon},\tau\right)}{e^{\lambda\tau}x^\gamma} = \lim_{\epsilon\to 0} \frac{J\left(\frac{x}{\epsilon},\tau\right)}{e^{\lambda\tau}\left(\frac{x}{\epsilon}\right)^\gamma} = \frac{1}{\gamma},$$

which in turn implies

$$\lim_{x\to\infty} \frac{J(x,\tau)}{e^{\lambda\tau}x^\gamma} = \frac{1}{\gamma}.$$

□

Remark 3.1. For the case $\gamma = 0$, consider $\hat{U}(x) = M_1 x^\delta + M_2$ for some $\delta \in (0,1)$ and M_2 sufficiently large.

An immediate corollary of the above proposition is that $J_x(x,\tau)$ also behaves like a power function for large x.

Corollary 3.1: *The function J_x satisfies*

$$\lim_{x\to\infty} \frac{J_x(x,\tau)}{e^{\lambda\tau}x^{\gamma-1}} = 1.$$

The proof is similar to the proof of part (iii) of Lemma 3.1 and therefore we present only the main steps.

Proof. The concavity of J yields

$$J(x(1+\delta),\tau) \leq J(x,\tau) + \delta x J'(x,\tau)$$

which, in turn, implies $\forall\delta > 0$, $\tau \in [0,T]$

$$\frac{1}{\delta\gamma}\left\{\frac{J((1+\delta)x,\tau)}{x^\gamma} - \frac{J(x,\tau)}{x^\gamma}\right\} \leq \frac{J_x(x,\tau)}{\gamma x^{\gamma-1}}.$$

Sending $x \to \infty$ and $\delta \to 0$, yields

$$\varliminf_{x\to\infty} \frac{J_x(x,\tau)}{x^{\gamma-1}} \geq 1.$$

We can get similarly that

$$\varlimsup_{x\to\infty} \frac{J_x(x,\tau)}{x^{\gamma-1}} \leq 1$$

and the assertion follows from the latter inequalities.

From the above result, we see J and J_x behave like power functions when x is sufficiently large. It turns out that these properties of J and J_x imply that $A(x,\tau)$ behaves like a *linear function when x is sufficiently large for all trading horizons*. This is the subject of the main theorem of this subsection.

Theorem 3.2: *Under Assumptions 3.1 and 3.2,*

$$\lim_{x\to\infty} \frac{A(x,\tau)}{x} = \frac{1}{1-\gamma}\frac{\mu - r}{\sigma^2} \quad \forall \tau \in [0,T]. \tag{3.19}$$

Proof. First observe that by (2.9) and (3.7)

$$\lim_{x\to\infty} \frac{A(x,0)}{x} = \frac{1}{1-\gamma}\frac{\mu - r}{\sigma^2}.$$

We are left to prove that the assertion holds for any $\tau > 0$.

From (2.5) and Corollary 3.1 it follows that (3.19) holds if we establish that for every $\tau \geq 0$

$$\lim_{x\to\infty} \frac{J_{xx}(x,\tau)}{(\gamma-1)e^{\lambda\tau}x^{\gamma-2}} = 0. \tag{3.20}$$

To prove (3.20), it suffices to show that the function $J_x(x,\tau)$ is convex in x.

In fact, using the convexity of J_x together with its asymptotic behavior, stated in Corollary 3.1 we can show (3.20). The arguments to show (3.20) are similar to the ones used in Lemma 3.1 (part (iii)) and therefore are omitted.

To prove the convexity of $J_x(x,\tau)$ we use a transformation employed in Karatzas et al. (1987). To this end, let $f : \Re_+ \times [0,T] \to \Re_+$ be such that

$$J_x(f(y,\tau),\tau) = y. \tag{3.21}$$

Differentiating the HJB equation (2.3) with respect to x and using the definition of f, yields that f solves the initial value problem

$$\text{(IVP)} \qquad \begin{cases} f_\tau + rf = \frac{1}{2}\frac{(\mu-r)^2}{\sigma^2}y^2 f_{yy} + \left(\frac{(\mu-r)^2}{\sigma^2} - r\right) y f_y \\ f(y,0) = (U')^{-1}(y) \end{cases}$$

with $(U')^{-1}$ being the inverse function of U'.

On the other hand, the above (IVP) has a unique smooth solution (see, for example Krylov 1987) which has the probabilistic representation

$$f(y,\tau) = E\left[e^{-r\tau}(U')^{-1}(y_\tau); y_0 = y\right] \tag{3.22}$$

where the process $y(x)$, $0 \leq s \leq \tau$ solves the linear stochastic differential equation

$$dy(s) = \left[\frac{(\mu-r)^2}{\sigma^2} - r\right] y(s)ds + \frac{\mu-r}{\sigma} y(s)dw(s). \tag{3.23}$$

Using (3.22), the convexity of $(U')^{-1}$ and the linear dynamics of (3.23) we get that f is convex in y. Finally, differentiating (3.21) with respect to y, yields

$$J_{xx}(x,\tau) = \frac{1}{f_y(y,\tau)}$$

and

$$J_{xxx}(x,\tau) = -\frac{f_{yy}(y,\tau)}{f_y^3} \tag{3.24}$$

where $x = f(y,\tau)$.

Using now the concavity of J which yields that $f_y < 0$, and the convexity of f, (3.24) implies $J_{xxx} > 0$ which in turn yields that J_x is convex in x.

3.3 Asymptotic properties of $A(x,\tau)$, as $\tau \to \infty$, for a class of utility functions

In the previous subsection, Theorem 3.2 establishes that the optimal dollar amount invested in the risky asset is approximately linear in wealth, for any investment horizon when the wealth is high enough. We will now establish that this implies that the optimal dollar amount invested in the risky asset is, in the limit, *linear for all levels of wealth* at the beginning of a very long investment horizon.

We begin by showing that $A(x,\tau)$ must lie in a certain region, denoted by $\mathscr{W}$, for all $x \geq 0$ and all $\tau \geq 0$. Note that this "turnpike window" $\mathscr{W}$ is *independent* of time τ.

Proposition 3.3: *Suppose that Assumptions 2.1 and 3.1 hold. Then for every $\theta > 0$ there exist $x_\theta > 0$ and $y_\theta > 0$ so that for all $(x,\tau) \in \Re_+ \times \Re_+$,*

$$A(x,\tau) \leq f_1(x), \tag{3.25}$$

$$A(x,\tau) \geq f_2(x), \tag{3.26}$$

where

$$f_1(x) = \left(\frac{\mu - r}{\sigma^2(1-\gamma)} + \theta\right) x + y_\theta, \tag{3.27}$$

$$f_2(x) = \max\left[\left(\frac{\mu - r}{\sigma^2(1-\gamma)} - \theta\right)(x - x_\theta), 0\right]. \tag{3.28}$$

Proof. We will only prove (3.25) as (3.26) follows from similar arguments. By Theorem 3.2,

$$\lim_{x\to\infty} \frac{A(x,0)}{x} = \frac{\mu - r}{\sigma^2(1-\gamma)}$$

for every $\theta > 0$. Therefore, there exists y_θ large enough, such that

$$A(x,0) \leq f_1(x) \equiv \left(\frac{\mu - r}{\sigma^2(1-\gamma)} + \theta\right) x + y_\theta, \quad \forall x \geq 0. \tag{3.29}$$

We will next show that (3.29) holds for all $\tau > 0$. To this end, we first observe that $f_1(x)$ is a viscosity supersolution of (2.8); to see this, we note that

$$rf_1(x) = r\left(\frac{\mu - r}{\sigma^2(1-\gamma)} + \theta\right) x + ry_\theta > rxf_1'(x) = r\left(\frac{\mu - r}{\sigma^2(1-\gamma)} + \theta\right) x, \quad \forall x \geq 0.$$

We now prove a comparison result between the viscosity solution $A(x,\tau)$ and the viscosity supersolution f_1 of (2.8). To see this, we argue by contradiction. For fixed T, assume

$$\sup_{(x,\tau)\in\Re_+\times[0,T]} [A(x,\tau) - f_1(x)] > 0. \tag{3.30}$$

Given the asymptotic behavior of $A(x,\tau)$ and the form of f_1, it follows that the supremum in (3.30) should occur at a point (x_0,τ_0) with $x_0 < \infty$, and, by assumption,

$$A(x_0,\tau_0) > f_1(x_0). \tag{3.31}$$

Since both A and f_1 are twice continuously differentiable at the point (x_0,τ_0),

$$\begin{aligned} A_x(x_0,\tau_0) &= f_1'(x_0) = \frac{\mu - r}{\sigma^2(1-\gamma)} + \theta, \\ A_\tau(x_0,\tau_0) &= 0, \\ A_{xx}(x_0,\tau_0) &\le 0. \end{aligned}$$

Using the above relations and (2.8) we get

$$rA(x_0,v_0) \le r\left(\frac{\mu - r}{\sigma^2(1-\gamma)} + \theta\right) x_0,$$

which contradicts (3.31). Thus $A(x,\tau) \le f_1(x)$ for all $\tau \in [0,T]$.

Relation (3.28) is proved the same way. □

Proposition 3.3 shows that for any $(x,\tau) \in \Re_+ \times \Re_+$, $A(x,\tau)$ lies in a time-independent “turnpike window”, $\mathscr{W}$, depicted in Fig. 1.

Now we are ready for the main theorem.

Theorem 3.3: *Suppose that the utility function U satisfies Assumptions 2.1 and 3.1 hold. Then*

$$A^*(x) = A_*(x) = \frac{\mu - r}{\sigma^2(1-\gamma)} x, \quad x \ge 0.$$

Proof. Our proof is organized as follows. We first show that for every $\theta > 0$ and for all $x \ge 0$,

$$f_2(x) \le A^*(x) \le \left(\frac{\mu - r}{\sigma^2(1-\gamma)} + \theta\right) x; \tag{3.32}$$

$$\left(\frac{\mu - r}{\sigma^2(1-\gamma)} - \theta\right) x \le A_*(x) \le f_1(x), \tag{3.33}$$

that is, A_* and A^* lie, respectively, in part I and II of $\mathscr{W}$ (see Fig. 1). Consequently,

$$\left(\frac{\mu - r}{\sigma^2(1-\gamma)} - \theta\right) x \le A_*(x) \le A^*(x) \le \left(\frac{\mu - r}{\sigma^2(1-\gamma)} + \theta\right) x$$

and sending $\theta \to 0$ we conclude.

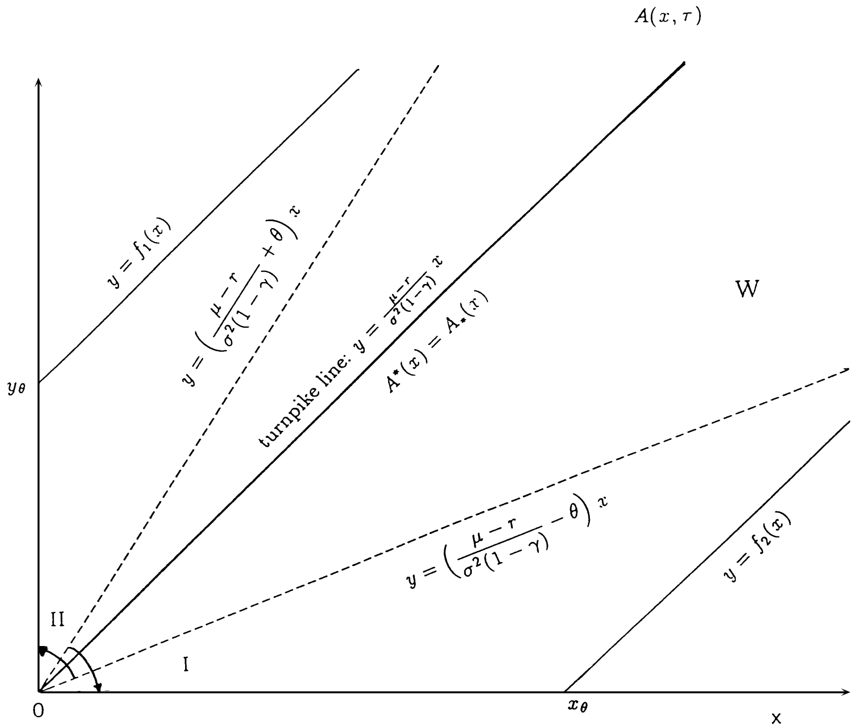

Fig. 1. Time-independent window for $A(x, \tau)$

We start with (3.32). Note that (3.25) and (3.26) imply

$$f_2(x) \le A^*(x) \le f_1(x), \quad \forall x \in \Re_+. \tag{3.34}$$

We claim that the upper bound of A^* can be refined to

$$A^*(x) \le \left(\frac{\mu - r}{\sigma^2(1-\gamma)} + \theta\right) x. \tag{3.35}$$

To see this, we first observe that at $x = 0$, $A^*(0) = 0$ by Theorem 3.1. Thus (3.35) holds at $x = 0$. For $x > 0$, we argue by contradiction. We look at the following cases.

Case 1: Suppose that A^* is strictly above $\left(\frac{\mu-r}{\sigma^2(1-\gamma)} + \theta\right)x$ at some point and then goes back down to be equal to or less than $\left(\frac{\mu-r}{\sigma^2(1-\gamma)} + \theta\right)x$. Formally, assume for some $\hat{x} > 0$, $A^*(\hat{x}) > \left(\frac{\mu-r}{\sigma^2(1-\gamma)} + \theta\right)\hat{x}$ and for some $\overline{x} > \hat{x}$, $A^*(\overline{x}) \le \left(\frac{\mu-r}{\sigma^2(1-\gamma)} + \theta\right)\overline{x}$. Then by the upper semicontinuity of A^* (see the first assertion of Theorem 3.1, $A^*(x) - \left(\frac{\mu-r}{\sigma^2(1-\gamma)} + \theta\right)x$ achieves a local maximum at some $x^0 \in (0, \overline{x})$ with $A^*(x^0) > \left(\frac{\mu-r}{\sigma^2(1-\gamma)} + \theta\right)x^0$. Since $f(x) = \left(\frac{\mu-r}{\sigma^2(1-\gamma)} + \theta\right)x$ is smooth and A^* is a viscosity subsolution (for the latter see Theorem 3.1), it follows that

$$rx^0\left(\frac{\mu-r}{\sigma^2(1-\gamma)}+\theta\right)\geq rA^*(x^0),$$

which is a contradiction. Thus A^* cannot be strictly above $\left(\frac{\mu-r}{\sigma^2(1-\gamma)}+\theta\right)x$ and then go back down to be equal to or less than $\left(\frac{\mu-r}{\sigma^2(1-\gamma)}+\theta\right)x$.

Case 2: Suppose that A^* lies strictly above $\left(\frac{\mu-r}{\sigma^2(1-\gamma)}+\theta\right)x$ for all $x>\bar{x}$ for some $\bar{x}\geq 0$; that is,

$$\left(\frac{\mu-r}{\sigma^2(1-\gamma)}+\theta\right)x<A^*(x)\leq\left(\frac{\mu-r}{\sigma^2(1-\gamma)}+\theta\right)x+y_\theta,\quad \forall x>\bar{x}. \tag{3.36}$$

To prove (3.35) we argue by contradiction. Assume that

$$\sup_{x\geq 0}\left[A^*(x)-\left(\frac{\mu-r}{\sigma^2(1-\gamma)}+\theta\right)x\right]>0. \tag{3.36}$$

It then follows that for $\delta\in(0,1)$ and $\epsilon>0$ sufficiently small,

$$\sup_{x\geq 0}\left[A^*(x)-\left(\frac{\mu-r}{\sigma^2(1-\gamma)}+\theta\right)x-\epsilon x^\delta\right]>0. \tag{3.37}$$

Let $\varphi(x)=\left(\frac{\mu-r}{\sigma^2(1-\gamma)}+\theta\right)x+\epsilon x^\delta$. Since $A^*(0)=\varphi(0)=0$, (3.37) yields that the above supremum is achieved at a point, say $x_0>0$. Using that A^* is a subsolution of (3.1) and the form of φ we get

$$\frac{1}{2}\sigma^2[A^*(x_0)]^2\epsilon\delta(\delta-1)x^{\delta-2}+rx_0\left[\left(\frac{\mu-r}{\sigma^2(1-\gamma)}+\theta\right)+\epsilon\delta x_0^{\delta-1}\right]\geq rA^*(x_0).$$

Given that $\delta\in(0,1)$, the above relation implies

$$r\varphi(x_0)>r\left[\left(\frac{\mu-r}{\sigma^2(1-\gamma)}+\theta\right)x_0+\epsilon\delta x_0^\delta\right]\geq rA^*(x_0)$$

which contradicts (3.37).

The inequality (3.33) can be proved along the same line of arguments used to prove (3.32). □

4 Concluding remarks

In this paper we provided further results on the turnpike theory in a continuous time framework. We employed recent advances of the theory of viscosity solutions which enabled us to obtain asymptotic results via a delicate interplay between the Bellman equation (2.7) and the equation (2.8) that the optimal policy solves. Although we restricted ourselves to the case of state dynamics with linear coefficients (see, equation (2.1)) the methodology developed herein could be applied to the general case at the expense of tedious calculations and long arguments. Moreover, the same analysis could be applied in cases where trading constraints are binding, e.g. limited or not at all shortselling/borrowing etc. Although the above cases would give rise to more complicate Bellman equations, and subsequently to more complex equations for the optimal policies, the theory of viscosity solutions, and in particular their *stability properties* would be valid even for the "more nonlinear" situations.

Endnotes

[1] The function $J(x,\tau)$ is said to be $C^{2,1}(\Re_+,\Re_+)$ if it is twice continuously differentiable in x and continuously differentiable in t, for (x,t) in $(\Re_+,\Re_+)$.

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Appendix

In this Appendix, we derive (2.8) when $r = 0$. The calculations for $r \neq 0$ are similar but much more tedious. (The motivated reader could look at He and Huang (1933).) First, we observe that combining (2.5) and (2.7) yields

$$J_\tau = \frac{\mu}{2} A(x,\tau) J_x . \tag{A.1}$$

Moreover, differentiating equality (2.5) with respect to τ and rearranging terms gives

$$A_\tau = -\frac{\mu}{\sigma^2}\frac{J_{xt}}{J_{xx}} + \frac{\mu}{\sigma^2}\frac{J_x J_{xxt}}{J_{xx}^2}. \tag{A.2}$$

Differentiating (A.1) with respect to x yields

$$2J_{\tau x} = \mu A_x J_x + \mu A J_{xx} \tag{A.3}$$

and

$$2J_{\tau xx} = \mu A_{xx} J_x + 2\mu A_x J_{xx} + \mu A J_{xxx} . \tag{A.4}$$

We claim that

$$A_\tau = \tfrac{1}{2}\sigma^2 A^2 A_{xx}$$

or, equivalently

$$\begin{aligned} -\frac{\mu}{\sigma^2}\frac{J_{xt}J_{xx} - J_x J_{xx\tau}}{J_{xx}^2} &= \tfrac{1}{2}\sigma^2\Big(-\frac{\mu}{\sigma}\frac{J_x}{J_{xx}}\Big)^2 A_{xx} \\ \Leftrightarrow 2J_x J_{xx\tau} - 2J_{x\tau}J_{xx} &= \mu J_x^2 A_{xx} . \end{aligned} \tag{A.5}$$

Using (A.3) and (A.4), (A.5) becomes

$$\begin{aligned} &J_x(\mu A_{xxx} J_x + 2\mu A_x J_{xx} + \mu A J_{xxx}) \\ &\quad -J_{xx}(\mu A_x J_x + \mu A J_{xx}) = \mu A_{xx} J_x^2 \\ &\quad \Leftrightarrow A_x J_x J_{xx} = A(J_{xx}^2 - J_x J_{xxx}) \\ &\quad \Leftrightarrow A_x = -\frac{\mu}{\sigma^2}\frac{1}{J_{xx}^2}\left(J_{xx}^2 - J_x J_{xxx}\right) \\ &\quad \Leftrightarrow A_x = -\frac{\mu}{\sigma^2} + \frac{\mu}{\sigma^2}\frac{J_x J_{xxx}}{J_{xx}^2} \end{aligned}$$

which follows from the form of A.