

AN OPTIMAL INVESTMENT/CONSUMPTION MODEL WITH BORROWING*

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This paper considers a consumption and investment decision problem for a single agent. Wealth is divided between a riskless asset and a risky asset with logarithmic Brownian motion price fluctuations. Short-selling is not allowed, but borrowing is allowed at rate exceeding the rate of return on the riskless asset. An explicit solution of the dynamic programming differential equation for the maximum total discounted expected utility function U is available only in the HARA case. However, using viscosity solution methods the asymptotic behavior of the value function $v(x)$ is found for small wealth x and for large wealth x .

1. Introduction. This paper considers a consumption and investment decision problem for a single agent. The investor consumes his current wealth X_t at some nonnegative rate C_t and he distributes it between two assets continuously in time. One asset is a riskless security (e.g. a bond) with instantaneous rate of return r . The other asset is risky (e.g. a stock) with price fluctuating according to a logarithmic Brownian motion with mean rate of return $b > r$. The objective is to maximize the total expected discounted utility of consumption over an infinite time interval, where utility is measured by some increasing, concave function $U(c)$. In our version of the model, short-selling of stock is not allowed. However, stock purchases may be financed by borrowing at an interest rate $R \geq r$.

We treat the problem by the usual method of dynamic programming, which leads to the nonlinear second order differential equation (2.8) for the maximum expected discounted utility $v(x)$ as a function of the investor's wealth x .

When $R = r$ and short-selling is allowed a rather explicit solution to the problem is known. For the special case of HARA (hyperbolic absolute risk aversion) utility function, i.e. $U(c) = c^k/p$, $p < 1$, the solution is completely explicit as shown by Merton [7] in his pioneering paper on this topic. For general utility function the dynamic programming equation can be linearized by a change of variables and the value function can be expressed in terms of certain integrations. (See Karatzas, Lehoczky, Sethi and Shreve [5].)

We consider the case $R > r$, where no explicit solution is available. Our aim is to obtain a qualitative understanding of optimal investment policies for a range of parameter values and for various choices of utility functions $U(c)$. This is done via analytical study of the value function v including its asymptotic behavior as $x \rightarrow 0$ and $x \rightarrow \infty$ (§4). The theory of viscosity solutions to differential equations (ordinary or partial) provides a convenient tool for our study of the asymptotics of v .

The paper is organized as follows. In §2 we describe the market and the rules according to which the investor trades. We also state a theorem in which we see that

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the value function is a smooth solution of the dynamic programming equation. This enables us to find the optimal policies which are given in a feedback form as functions of the current wealth of the investor. More precisely we show that the investor borrows money only if his wealth drops below a certain level which depends on the market parameters. In §3 we examine the case of HARA utility function and we give a complete description of the optimal investment. The last section is devoted to the asymptotic behavior of the value function and the optimal investment. We consider a general utility function which behaves like a HARA one for very small or very large values of the consumption. We show that as the wealth of the investor approaches zero or infinity, the value function and the optimal investment look like the ones of a model with HARA utility function (see Table 3.1).

2. The financial model. We consider a market with two assets: a bond and a stock. The price P_t^0 of the bond evolves according to the equation

$$(2.1) \quad \begin{cases} dP_t^0 = rP_t^0 dt, \\ P_0^0 = p^0. \end{cases}$$

The price of the stock satisfies the stochastic differential equation

$$(2.2) \quad \begin{cases} dP_t = bP_t dt + \sigma P_t dw_t, \\ P_0 = p. \end{cases}$$

The process w_t is a standard Brownian motion through which randomness in stock price is assumed to enter. It is assumed that $\sigma > 0$ and $0 < r \leq R \leq b$. The investor consumes, trades and borrows continuously in time. Let π_t^0, π_t be respectively the amount of money invested in bond and stock and ϕ_t be the amount of money borrowed. The net amount of current wealth X_t has the role of the state variable and is given by

$$(2.3) \quad X_t = \pi_t^0 + \pi_t - \phi_t.$$

Taking into account the price equations (2.1), (2.2) and (2.3) we derive (see [9], [5]) the state equation

$$(2.4) \quad \begin{cases} dX_t = rX_t dt + [(b-r)\pi_t - (R-r)\phi_t] dt - C_t dt + \sigma\pi_t dw_t, \\ X_0 = x. \quad (x \geq 0), \end{cases}$$

where x is the initial endowment of the investor.

The control variables are π_t, C_t, ϕ_t with control constraints $C_t \geq 0, \pi_t \geq 0$ and $\phi_t \geq 0$ a.e. $\forall t \geq 0$. Moreover, the wealth of the investor has to stay nonnegative, i.e. $X_t \geq 0$ a.e. $\forall t \geq 0$. The controls are stochastic processes and they are called *admissible* if they satisfy the above constraints. In the Appendix we describe the set \mathcal{A} of admissible controls.

The objective of the investor is to maximize, over all admissible controls, the total expected discounted utility coming from consumption. We define the *value function*

$$(2.5) \quad v(x) = \sup_{\mathcal{A}} E \int_0^{+\infty} e^{-\beta t} U(C_t) dt$$

where U is the *utility function* and $\beta > 0$ is the *discount factor*. The utility function

$U: [0, +\infty) \rightarrow [0, +\infty)$ is assumed to have the following properties:

$$\left\{ \begin{array}{l} U \text{ is a strictly increasing, strictly concave, } C[0, +\infty) \text{ and } C^3(0, +\infty) \text{ function,} \\ \lim_{c \rightarrow 0} U'(c) = +\infty, \quad \lim_{c \rightarrow \infty} U'(c) = 0 \quad \text{and} \quad U(0) = 0. \end{array} \right.$$

We also assume that

$$(2.6) \quad \int_c^{+\infty} \frac{d\theta}{(U'(\theta))^{\rho_-}} < +\infty, \quad \forall c > 0,$$

where ρ_- is the negative root of the equation $\gamma\rho^2 - (r - \beta - \gamma)\rho - r = 0$, $\gamma = (b - r)^2/2\sigma^2$. Condition (2.6) is sufficient for the value function to be finite. In fact, if (2.6) holds, the value function of the unconstrained problem (i.e. $R = r$ and short-selling being allowed) is finite (see [5]). Moreover, the latter is always bigger or equal to v . In the case that $U(c) = c^p$, $0 < p < 1$, condition (2.6) is equivalent to

$$(2.7) \quad 1 + \rho_-(1 - p) < 0$$

and also to

$$(2.8) \quad \beta > rp + \frac{\gamma p}{1 - p}.$$

In the sequel we will need the quantity

$$(2.9) \quad F(y) = \max_{c \geq 0} [-cy + U(c)], \quad y \geq 0,$$

which is a convex function (Legendre transform) dual to the concave function U . Let I be defined by $I(y) = -F'(y)$, $y \geq 0$. Then I is convex, decreasing with $\lim_{y \rightarrow \infty} I(y) = 0$. We define

$$g(p, q, \pi, \phi) = \frac{1}{2}\sigma^2\pi^2q + (b - r)\pi p - (R - r)\phi p,$$

$p > 0$, $q < 0$ and $(\pi, \phi) \in \Delta(x)$ where (see Figure 1)

$$\Delta(x) = \{(\pi, \phi) : \pi \geq 0, \phi \geq 0, x - \pi + \phi \geq 0\}.$$

Also, let

$$(2.10) \quad G(x, p, q) = \max_{(\pi, \phi) \in \Delta(x)} g(p, q, \pi, \phi).$$

In the rest of the section we talk about the value function and the optimal policies.

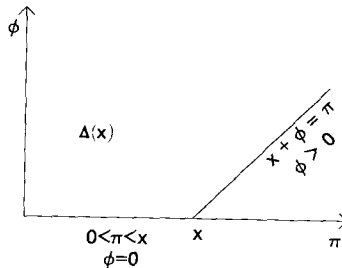


FIGURE 1

PROPOSITION 1.1. *The value function is concave, strictly increasing and continuous on $[0, +\infty)$. Moreover $v(0) = 0$ and if U is bounded then*

$$\lim_{x \rightarrow \infty} v(x) = \frac{\|U\|_{\infty}}{\beta}.$$

For a proof see [9].

THEOREM 2.1. *The value function is the unique increasing, concave, twice continuously differentiable in $(0, +\infty)$ and continuous on $[0, +\infty)$ solution of the Bellman equation*

$$(2.11) \quad \begin{cases} \beta v = G(x, v_x, v_{xx}) + F(v_x) + rxv_x, & \forall x > 0, \\ v(0) = 0. \end{cases}$$

The idea of the proof is as follows. From the form of G we see that (2.11) may be degenerate unless we prove that $|v_{xx}|$ is bounded from above. We regularize equation (2.11) by adding a term $\frac{1}{2}\epsilon^2\sigma^2v_{xx}$ and we show that if v^ϵ is the classical solution of the regularized equation, then in any interval $[x_1, x_2]$ with $0 < x_1 < x_2 < \infty$, we can find a constant $C = C([x_1, x_2])$ such that $|v_{xx}^\epsilon| \leq C, \forall x \in [x_1, x_2]$ (see [9]). Then we send ϵ to zero and we get that the limit (along subsequences) of v^ϵ is a viscosity solution of (2.11). (For the definition of viscosity solution see Definition 4.1.) Moreover equation (2.11) has a unique viscosity solution in the class of nondecreasing and concave functions which is indeed the value function. For uniqueness of viscosity solution of the Bellman equation in the case of a bounded utility function see [9] and [4]. If the utility function is unbounded but remains below Kc^γ for some $K > 0, \gamma < 1$ and $\gamma \neq 0$, we first need an a priori sublinear growth condition of v (see Theorem 4.1).

THEOREM 2.2. *The optimal investment policy of the investor is as follows:*

(i) *If*

$$x > -\frac{b-r}{\sigma^2} \frac{v_x}{v_{xx}},$$

then he does not borrow and he invests

$$-\frac{b-r}{\sigma^2} \frac{v_x}{v_{xx}}$$

amount of money in stock and

$$x + \frac{b-r}{\sigma^2} \frac{v_x}{v_{xx}}$$

in bond; i.e.

$$\pi_0^*(x) = x - \pi^*(x), \quad \pi^*(x) = -\frac{b-r}{\sigma^2} \frac{v_x}{v_{xx}} \quad \text{and} \quad \phi^*(x) = 0.$$

(ii) *If*

$$-\frac{b-R}{\sigma^2} \frac{v_x}{v_{xx}} \leq x \leq -\frac{b-r}{\sigma^2} \frac{v_x}{v_{xx}},$$

then he does not borrow and he invests all his wealth in stock; i.e.,

$$\pi_0^*(x) = 0, \quad \pi^*(x) = x \quad \text{and} \quad \phi^*(x) = 0.$$

(iii) If

$$x < -\frac{b-R}{\sigma^2} \frac{v_x}{v_{xx}},$$

then he invests the amount of

$$-\frac{b-R}{\sigma^2} \frac{v_x}{v_{xx}}$$

in stock and he borrows

$$-\frac{b-R}{\sigma^2} \frac{v_x}{v_{xx}} - x$$

amount; i.e.

$$\pi_0^*(x) = 0, \quad \pi^*(x) = -\frac{b-R}{\sigma^2} \frac{v_x}{v_{xx}} \quad \text{and} \quad \phi^*(x) = -\frac{b-R}{\sigma^2} \frac{v_x}{v_{xx}} - x.$$

(iv) The optimal consumption policy is given by $C^*(x) = I(v_x(x))$.

Moreover π^* , ϕ^* and C^* are locally Lipschitz functions of x for $0 < x < +\infty$.

PROOF. Claims (i)–(iii) are an immediate consequence of the following lemma.

LEMMA 2.1. For $p > 0$ and $q < 0$, the maximum in (2.10) is obtained at (π^*, ϕ^*) where

$$(i) \quad \text{if } x \geq -\frac{b-r}{\sigma^2} \frac{p}{q}, \quad \text{then } \pi^* = -\frac{(b-r)}{\sigma^2} \frac{p}{q} \quad \text{and} \quad \phi^* = 0,$$

$$(ii) \quad \text{if } -\frac{b-R}{\sigma^2} \frac{p}{q} \leq x \leq -\frac{b-r}{\sigma^2} \frac{p}{q}, \quad \text{then } \pi^* = x \quad \text{and} \quad \phi^* = 0,$$

$$(iii) \quad \text{if } x \leq -\frac{b-R}{\sigma^2} \frac{p}{q}, \quad \text{then } \pi^* = -\frac{b-R}{\sigma^2} \frac{p}{q} \quad \text{and} \quad \phi^* = \pi - x.$$

PROOF. The maximum in (2.10) occurs either at $\phi = 0$, $\pi \leq x$ or $\phi = \pi - x$, $\pi > x$.

Case (i): $x > -((b-r)/\sigma^2)(p/q)$. If $\phi^* > 0$ then $\phi^* = \pi^* - x$. Maximizing over π we get

$$\pi^* = -\frac{b-R}{\sigma^2} \frac{p}{q} < -\frac{b-r}{\sigma^2} \frac{p}{q}$$

which contradicts the fact that $\phi^* > 0$. Therefore, $\phi^* = 0$ and maximizing over π we get that

$$\pi^* = -\frac{b-r}{\sigma^2} \frac{p}{q}.$$

Case (ii): $((b - R)/\sigma^2)(p/q) \leq x \leq -((b - r)/\sigma^2)(p/q)$. If $\phi^* = 0$ then maximization over π gives that

$$(2.12) \quad \begin{cases} \text{(i)} & \pi^* = -\frac{b - r}{\sigma^2} \frac{p}{q} \quad \text{or} \\ \text{(ii)} & \pi^* = x. \end{cases}$$

If π^* is given by (2.12) (i) then it is not admissible because it gives $x - \pi^* < 0$. If $\phi^* > 0$ then maximization over π gives that

$$\pi^* = -\frac{b - R}{\sigma^2} \frac{p}{q}$$

which is not admissible because it gives $\pi^* - x < 0$. Therefore, $\phi^* = 0$ and $\pi^* = x$.

Case (iii): $x < -((b - R)/\sigma^2)(p/q)$. If $\phi^* = 0$ then maximization over π gives that

$$(2.13) \quad \begin{cases} \text{(i)} & \pi^* = -\frac{b - r}{\sigma^2} \frac{p}{q} \quad \text{or} \\ \text{(ii)} & \pi^*(x) = x. \end{cases}$$

If $\phi^* = \pi^* - x$ then

$$\pi^* = -\frac{b - R}{\sigma^2} \frac{p}{q}.$$

But (2.13)(i) implies $\pi^* > x$ which violates the constraint $x - \pi + \phi \leq 0$. Moreover,

$$g(p, q, x, 0) < g\left(p, q, \frac{b - R}{\sigma^2} \frac{p}{|q|}, \frac{b - R}{\sigma^2} \frac{p}{|q|} - x\right)$$

and therefore

$$\pi^* = -\frac{b - R}{\sigma^2} \frac{p}{q} \quad \text{and} \quad \phi^* = \pi^* - x.$$

We continue the proof of Theorem 2.2. A straightforward maximization in (2.9) gives that $c^* = I(y)$. Therefore, $C^*(x) = I(v_x(x))$. Finally, we show that the optimal policies are locally Lipschitz functions of x . From Theorem 2.5 of [9] we have that in any interval $[a, b]$ with $a > 0$, there exist positive constants $C_1 = C_1([a, b])$ and $C_2 = C_2([a, b])$ such that $0 < v_x(x) \leq C_1$ and $|v_{xx}(x)| \leq C_2 \forall x \in (a, b)$. Therefore v_x is locally Lipschitz. Moreover, using the values of the optimal policies we get that

$$(2.14) \quad G(x, v_x, v_{xx}) = \begin{cases} -\frac{(b - r)^2}{2\sigma^2} \frac{v_x^2}{v_{xx}}, & x > -\frac{b - r}{\sigma^2} \frac{v_x}{v_{xx}}, \\ \frac{1}{2}\sigma^2 x^2 v_{xx} + (b - r)xv_x, & -\frac{b - R}{\sigma^2} \frac{v_x}{v_{xx}} \leq x \leq -\frac{b - r}{\sigma^2} \frac{v_x}{v_{xx}}, \\ -\frac{1}{2} \frac{(b - R)^2}{\sigma^2} \frac{v_x^2}{v_{xx}} + (R - r)xv_x, & x < -\frac{b - R}{\sigma^2} \frac{v_x}{v_{xx}}. \end{cases}$$

From the Bellman equation we have that

$$(2.15) \quad \beta v - F(v_x) - rxv_x = G(x, v_x, v_{xx}),$$

and the right side is positive by (2.10).

Since $\beta v - F(v_x) - rxv_x > 0$, using (2.14) and (2.15) we can find a function $H = H(x, v, v_x)$ such that $v_{xx} = H(x, v, v_x)$, $\forall x \in (a, b)$. Clearly H is locally Lipschitz in x and therefore v_{xx} is too. Thus all optimal policies are locally Lipschitz functions of x , for $0 < x < \infty$. If we put in (2.4)

$$\pi_t = \pi^*(X_t), \quad \phi_t = \phi^*(X_t), \quad C_t = C^*(X_t),$$

then (2.4) has a unique strong solution defined on a maximal interval $0 \leq t < \tau$. (See [2].) If $\tau < \infty$ then τ can be regarded as the time of either bankruptcy or explosion of wealth to infinity. Under the general assumptions made on $U(c)$ above, we have not shown that $\tau = +\infty$ with probability 1. However, this is true under the additional assumptions made in §4, and follows from Theorems 4.2, 4.4 and 4.6.

3. The HARA utility functions. In this section we look at the class of utility functions which have Hyperbolic Absolute Risk Aversion. In other words, the utility function is of the form

$$(3.1) \quad U(c) = \frac{1}{\gamma} c^\gamma, \quad \gamma < 1 \quad \text{and} \quad \gamma \neq 0.$$

We observe that the value function must satisfy

$$(3.2) \quad v(kx) = k^\gamma v(x), \quad k > 0.$$

This follows from (3.1) and the fact that if the control (π_t, ϕ_t, C_t) is admissible for initial wealth x then $(k\pi_t, k\phi_t, kC_t)$ is also admissible at the point kx . Relation (3.2) gives

$$(3.3) \quad v(x) = Kx^\gamma, \quad \forall x \geq 0,$$

where $K = v(1)$ and it is determined by the dynamic programming equation (2.11). (See Table 3.1.)

Since $v_x(x)/v_{xx}(x) = x/(\gamma - 1)$ we can derive the optimal policies from Theorem 2.3. These optimal policies are linear in x and therefore the corresponding solution X_t^* to (2.4) is a logarithmic brownian motion with drift. In particular $0 < X_t^* < \infty$ for all $t \geq 0$ (no bankruptcy or explosion of wealth).

4. Asymptotic behavior of the value function and the optimal policies. In this section we discuss the behavior of the value function and the optimal policies when the wealth of the investor is either very small or very large. The asymptotic behavior depends on the choice of the utility function which is assumed to be either bounded or unbounded. In the sequel, although we state the properties of the utility function we will use for convenience the equivalent properties of the dual function.

I. Asymptotic analysis near zero. In this section we examine the asymptotic behavior of the value function and the optimal policies when the wealth is very small under the assumption that the utility function behaves like a HARA function near

TABLE 3.1

Market	K	Optimal Portfolio		Amount borrowed ϕ^*	Optimal consumption C^*
		$\pi_0^*(x)$	$\pi^*(x)$		
$(b-r) < \sigma^2(1-\gamma)$	$\left(\frac{\delta}{\beta - \frac{\gamma}{1-\gamma} \frac{(b-r)^2}{2\sigma^2} - r\gamma} \right)^{1-\gamma}$	$x - \pi^*(x)$	$\frac{(b-r)x}{\sigma^2(1-\gamma)}$	0	$(K\gamma)^{1/(\gamma-1)}x$
$b-R \leq \sigma^2(1-\gamma)$ $\leq b-r$	$\left(\frac{\delta}{\beta + \frac{\sigma^2\gamma(1-\gamma)}{2} - b\gamma} \right)^{1-\gamma}$	0	x	0	$(K\gamma)^{1/(\gamma-1)}x$
$\sigma^2(1-\gamma) < b-R$	$\left(\frac{\delta}{\beta - \frac{\gamma}{1-\gamma} \frac{(b-R)^2}{2\sigma^2} - R\gamma} \right)^{1-\gamma}$	0	$\frac{(b-R)x}{\sigma^2(1-\gamma)}$	$\pi^*(x) - x$	$(K\gamma)^{1/(\gamma-1)}x$

Note that because of (2.8) K is well defined. Also

$$\delta = \frac{1-\gamma}{\gamma} \gamma^{\gamma/(\gamma-1)}.$$

zero i.e.,

$$\lim_{c \rightarrow 0} \frac{U(c)}{c^p} = \frac{1}{p}, \quad 0 < p < 1.$$

For convenience we introduce the equivalent assumption

$$(4.1) \quad \lim_{c \rightarrow 0} \frac{U'(c)}{c^{p-1}} = 1.$$

Then the dual function satisfies (see Appendix)

$$(4.2) \quad \lim_{y \rightarrow \infty} \frac{F(y)}{y^{p/(p-1)}} = \frac{1-p}{p}.$$

THEOREM 4.1. *If (4.2) holds then*

$$\lim_{x \rightarrow 0} \frac{v(x)}{x^p} = K$$

where K is given in Table 3.1.

PROOF. Let $u^\lambda(x) = \lambda^p v(x/\lambda)$, $\forall x \geq 0$, $\lambda > 0$. Then from (2.11) we can see that u^λ satisfies

$$(4.3) \quad \begin{cases} \beta u^\lambda(x) = G(x, u_x^\lambda, u_{xx}^\lambda) + \max_{c \geq 0} [-c\lambda u_x^\lambda + \lambda^p U(c)] + rxu_x^\lambda, \\ u^\lambda(0) = 0. \end{cases}$$

We are interested in the behavior of u^λ as λ goes to infinity. We must first prove that the u^λ are bounded uniformly in λ .

LEMMA 4.1. $u^\lambda(\xi) \leq m\xi^p, 0 \leq \xi \leq \lambda x_1$ for some $x_1, m > 0$.

PROOF. Let $v^{0,r}$ be the solution of the “unconstrained problem” i.e., when borrowing and short-selling are allowed and $R = r$ (see [5]). Then for $R > r, v^{0,R} \leq v^{0,r}$ and since $v \leq v^{0,R}$ we get that

$$(4.4) \quad v(x) \leq v^{0,r}(x), \quad \forall x \geq 0.$$

We now show that for some x_1 small

$$v^{0,r}(x) \leq mx^p, \quad \forall x \leq x_1, m > 0.$$

To this end, since $v^{0,r}(0) = 0$ (see [5]), it suffices to show that $v_x^{0,r}(x) \leq pmx^{p-1}, \forall x \leq x_1$, for some x_1 small or equivalently $h(y) \leq Ay^{1/(p-1)}$ for $y \geq y_1$ for some y_1 large, $h = (v_x^{0,r})^{-1}$ where $()^{-1}$ denotes the inverse function and $A = (pm)^{1/(1-p)}$. From [5, §6] we have that

$$(4.5)$$

$$h(y) = \frac{I(y)}{r} - \frac{1}{\gamma(\rho_+ - \rho_-)} \left\{ \frac{y^{\rho_+}}{\rho_+} \int_0^{I(y)} \frac{d\theta}{U'(\theta)^{\rho_+}} + \frac{y^{\rho_-}}{\rho_-} \int_{I(y)}^{+\infty} \frac{d\theta}{U'(\theta)^{\rho_-}} \right\}$$

where $I = (U')^{-1}, \gamma = (b - r)^2 / 2\sigma^2$ and ρ_-, ρ_+ are respectively the negative and positive roots of $\gamma\rho^2 - (r - \beta - \gamma)\rho - r = 0$. Since $cU'(c) \leq U(c)$ and

$$\lim_{c \rightarrow 0} \frac{U(c)}{c^p} = \frac{1}{p}$$

we get that for c small and some $L > 1$

$$(4.6) \quad U'(c) \leq Lc^{p-1}.$$

This also gives

$$(4.7) \quad I(y) \leq ly^{1/p-1}, \quad \forall y \geq y_1$$

where $L = l^{1-p}$.

Using (4.5), (4.6) and (2.7) we get

$$\int_{I(y)}^{+\infty} \frac{d\theta}{U'(\theta)^{\rho_-}} \leq \frac{1}{L^{\rho_-}(\rho_-(p-1) - 1)} I(y)^{1-\rho_-(p-1)}.$$

Therefore,

$$h(y) \leq \frac{I(y)}{r} + k_1 y^{\rho_-} I(y)^{1-\rho_-(p-1)} \quad \text{where}$$

$$k_1 = \frac{1}{\gamma(\rho_+ - \rho_-)|\rho_-|L^{\rho_-}(\rho_-(p-1) - 1)}.$$

Using (4.7) once more we obtain

$$h(y) \leq Ay^{1/(p-1)}, \quad \forall y \leq y_1$$

with $A = l/r + k_1 l^{1-\rho_-(p-1)}$. Therefore $v(x) \leq mx^p, \forall x \leq x_1$ which implies that, for $\xi = \lambda x, u^\lambda(\xi) \leq m\xi^p, 0 \leq \xi \leq \lambda x_1$.

REMARK 4.1. From the previous lemma we have that the u^λ are bounded locally uniformly. Moreover, by concavity, u_x^λ are also bounded locally uniformly and therefore u^λ converge along subsequences locally uniformly.

We wish to show that u^λ converge to w where $w(x) = Kx^p$, $K > 0$ is a classical solution of the "limiting" Bellman equation (4.8).

Before we give the next lemma we present the following definition which was introduced by M. G. Crandall and P.-L. Lions [1].

DEFINITION 4.1. A continuous function $u: [0, +\infty) \rightarrow \mathfrak{R}$ is a viscosity solution of $\beta u = G(x, u_x, u_{xx}) - Bu_x^q + rxu_x$ if and only if for all $\phi \in C^2[0, \infty)$:

if $x_0 > 0$ is a local maximum point of $u - \phi$, then

$$\beta u(x_0) \leq G(x_0, \phi_x(x_0), \phi_{xx}(x_0)) - B\phi_x^q(x_0) + rx_0\phi_x(x_0) \quad \text{and}$$

if $x_0 > 0$ is a local minimum point of $u - \phi$, then

$$\beta u(x_0) \geq G(x_0, \phi_x(x_0), \phi_{xx}(x_0)) - B\phi_x^q(x_0) + rx_0\phi_x(x_0).$$

LEMMA 4.2. Let $\lambda_n \rightarrow 0$ and $u^{\lambda_n} \rightarrow \bar{u}$. If (4.2) holds then \bar{u} is a viscosity solution of

$$(4.8) \quad \begin{cases} \beta u = G(x, u_x, u_{xx}) + ku_x^{p/(p-1)} + rxu_x, \\ u(0) = 0, \end{cases}$$

where $k = (1/p)^{p/(p-1)} - (1/p)^{1/(p-1)}$.

PROOF. The proof follows along the lines of Proposition I.3 of [6] and therefore we omit it.

We now continue the proof of Theorem 4.1. We first show that \bar{u} coincides with $w(x) = Kx^p$ which is a classical solution of (4.8). Let us assume that $\sup_{R^+} [\bar{u} - w] > 0$. Let $\Phi(x) = \bar{u}(x) - w(x) - \epsilon x^{p+\delta}$ with $\delta > 0$ and $p + \delta < 1$. Since $\Phi(0) = 0$ and $\lim_{x \rightarrow \infty} \Phi(x) = -\infty$, Φ attains a maximum at some point x_0 . We examine the following cases:

Case 1: $x_0 = 0$. Then $\Phi(x) \leq 0, \forall x \geq 0$.

Case 2: $x_0 > 0$ and $\Phi(x_0) \leq 0$. Then $\Phi(x) \leq 0, \forall x \geq 0$.

Case 3: $x_0 > 0$ and $\Phi(x_0) > 0$. We show that this case cannot happen.

Let $\psi(x) = w(x) + \epsilon x^{\gamma+\delta}$. Since \bar{u} is a viscosity solution of (4.8) and $\bar{u} - \psi$ has a maximum at x_0 we have that

$$(4.9) \quad \beta \bar{u}(x_0) \leq G(x_0, t + \hat{t}, q + \hat{q}) + k(t + \hat{t})^{p/(p-1)} + rx_0(t + \hat{t}) \quad \text{where}$$

$$(4.10) \quad \begin{cases} t = w_x(x_0), & \hat{t} = \epsilon(\gamma + \delta)x_0^{\gamma+\delta-1} \quad \text{and} \\ q = w_{xx}(x_0), & \hat{q} = \epsilon(\gamma + \delta)(\gamma + \delta - 1)x_0^{\gamma+\delta-2}. \end{cases}$$

Note that $G_q \geq 0$ and $\hat{q} < 0$. Moreover, $G_i(x_0, t + \lambda\hat{t}, q + \lambda\hat{q}) \leq Kx$ for some positive constant K . To show this let $t_\lambda = t + \lambda\hat{t}$ and $q_\lambda = q + \lambda\hat{q}$. We examine the following cases:

Case (i): $x_0 \geq ((b-r)/\sigma^2)(t_\lambda/|q_\lambda|)$. Then

$$G(x_0, t_\lambda, q_\lambda) = \frac{(b-r)^2}{2\sigma^2} \frac{t_\lambda^2}{|q_\lambda|}$$

(see Lemma 2.1) and

$$G_t(x_0, t_\lambda, q_\lambda) = \frac{(b-r)^2 t_\lambda}{\sigma^2 |q_\lambda|} = \frac{(b-r)^2}{\sigma^2} \frac{xt_\lambda}{at + (\alpha - \delta)\lambda\hat{t}}$$

where $\alpha = 1 - p$. Therefore

$$G_t(x_0, t_\lambda, q_\lambda) \leq \frac{(b-r)^2}{\sigma^2} \frac{x_0}{\alpha - \delta}.$$

Case (ii): $((b-R)/\sigma^2)(t_\lambda/|q_\lambda|) \leq x \leq ((b-r)/\sigma^2)(t_\lambda/|q_\lambda|)$. Then $G(x_0, t_\lambda, q_\lambda) = \frac{1}{2}\sigma^2 x_0^2 q_\lambda + (b-r)x_0 t_\lambda$ and $G_t(x_0, t_\lambda, q_\lambda) = (b-r)x_0$.

Case (iii): $x \leq ((b-R)/\sigma^2)(t_\lambda/|q_\lambda|)$. Then

$$G(x_0, t_\lambda, q_\lambda) = \frac{(b-R)^2}{\sigma^2} \frac{t_\lambda}{|q_\lambda|}$$

and from Case (i) we get

$$G_t(x_0, t_\lambda, q_\lambda) \leq \frac{(b-R)^2}{\sigma^2} \frac{x_0}{\alpha - \delta}.$$

Let

$$K = (b-r) \max\left[\frac{b-r}{\sigma^2(1-p-\delta)}, 1\right].$$

Using the mean value theorem we get

$$G(x_0, t + \hat{t}, q + \hat{q}) - G(x_0, \hat{t}, q) = \int_0^1 [G_t(x_0, t_\lambda, q_\lambda)\hat{t} + G_q(x_0, t_\lambda, q_\lambda)\hat{q}] d\lambda.$$

Note that $G_q \geq 0$ and $\hat{q} < 0$. Therefore

$$(4.11) \quad G(x_0, t + \hat{t}, q + \hat{q}) \leq G(x_0, t, q) + Kx_0\hat{t}.$$

From (4.9) and (4.11) we get

$$\beta\bar{u}(x_0) \leq G(x_0, t, q) + k(t + \hat{t})^{p/(p-1)} + rx_0(t + \hat{t}) + Kx_0\hat{t}.$$

Since w solves (4.8) we have

$$\beta\bar{u}(x_0) \leq \beta w(x_0) + k\left[(t + \hat{t})^{p/(p-1)} - t^{p/(p-1)}\right] + (K+r)x_0\hat{t}$$

and using that $t^{p/(p-1)}$ is a decreasing function we get

$$\beta(\bar{u}(x_0) - w(x_0)) \leq \epsilon(K+r)(p+\delta)x_0^{p+\delta-1}$$

which contradicts Case 3.

Sending ϵ to zero we get that $\bar{u} \leq w, \forall x \geq 0$.

We now assume that $\inf_{R^+} [\bar{u} - w] < 0$. Let $\Phi(x) = \bar{u}(x) - w(x) + \epsilon x^{p+\delta}$ with $\delta > 0$ and $p + \delta < 1$. Then Φ attains a minimum at some point $x_0 \geq 0$. We examine the following cases:

Case 1: $x_0 = 0$. Then $\Phi(x) \geq 0, \forall x \geq 0$.

Case 2: $x_0 > 0$ and $\Phi(x_0) \geq 0$. Then $\Phi(x) \geq 0, \forall x \geq 0$.

Case 3: $x_0 > 0$ and $\Phi(x_0) < 0$.

Let $\phi(x) = w(x) - \epsilon x^{p+\delta}$. Since \bar{u} is a viscosity supersolution of (4.8) and $\bar{u} - \phi$ has a minimum at x_0 , we have that

$$(4.12) \quad \beta \bar{u}(x_0) \geq G(x_0, t - \hat{t}, q - \hat{q}) + k(t - \hat{t})^{p/(p-1)} + rx_0(t - \hat{t})$$

where t, \hat{t}, q and \hat{q} are given by (4.10).

Working similarly we can show that there exists a positive constant K such that

$$(4.13) \quad G(x_0, t - \hat{t}, q - \hat{q}) \geq G(x_0, t, q) - Kx_0 \hat{t}.$$

From (4.12), (4.13) and using that w solves (4.12) we get

$$\beta(\bar{u}(x_0) - w(x_0)) \geq -\epsilon(K + r)(p + \delta)x_0^{p+\delta-1}$$

which contradicts Case 3. Sending ϵ to zero we get that $\bar{u} \geq w, \forall x \geq 0$. Therefore $\bar{u} = w \forall x \geq 0$.

We conclude now as follows. Let $x = \xi/\lambda$. Since $\lim_{\lambda \rightarrow \infty} u^\lambda(\xi) = K\xi^p$ and $u^\lambda(\xi) = v(\xi/\lambda)$ we get that $\lim_{x \rightarrow 0} [v(x)/x^p] = K$.

THEOREM 4.2. *Under Assumption 4.2 the optimal policies satisfy*

$$(i) \quad \lim_{x \rightarrow 0} \frac{C^*(x)}{x} = (Kp)^{1/(p-1)}.$$

$$(ii) \quad \lim_{x \rightarrow 0} \frac{\pi^*(x)}{x} = \begin{cases} \frac{b-r}{\sigma^2(1-p)} & \text{if } (b-r) < \sigma^2(1-p), \\ 1 & \text{if } b-R \leq \sigma^2(1-p) \leq b-r, \\ \frac{b-R}{\sigma^2(1-p)} & \text{if } \sigma^2(1-p) < b-R. \end{cases}$$

$$(iii) \quad \lim_{x \rightarrow 0} \frac{\phi^*(x)}{x} = \begin{cases} 0 & \text{if } b-R \leq \sigma^2(1-p), \\ \frac{b-R}{\sigma^2(1-p)} - 1 & \text{if } \sigma^2(1-p) < b-R. \end{cases}$$

PROOF. We first show that for each $x > 0$

$$(4.14) \quad \lim_{\lambda \rightarrow \infty} u_x^\lambda(x) = Kpx^{p-1} \quad \text{and}$$

$$(4.15) \quad \lim_{\lambda \rightarrow \infty} u_{xx}^\lambda(x) = Kp(p-1)x^{p-2}.$$

From Lemma 4.1 we have that u^λ are bounded locally uniformly in λ . Moreover, since u^λ are concave we get that u_x^λ are also bounded locally uniformly. Using that $\lim_{\lambda \rightarrow \infty} u^\lambda(x) = Kx^p$ we get that, for each x , $\lim_{\lambda \rightarrow \infty} u_x^\lambda(x) = Kpx^{p-1}$ and (4.14)

follows. We now observe that because u^λ are concave the above convergence is uniform on $[x_1, x_2]$ for $0 < x_1 < x_2 < \infty$. Finally we show that $\lim_{\lambda \rightarrow 0} u_{xx}^\lambda(x) = Kp(p-1)x^{p-2}$ for each x . From the Bellman equation we have that

$$\beta u^\lambda - \max_{c \geq 0} [-c\lambda u_x^\lambda + \lambda^p U(c)] - rxu_x^\lambda = G(x, u_x^\lambda, u_{xx}^\lambda).$$

Let $A_\lambda(x) = \beta u^\lambda - \max_{c \geq 0} [-c\lambda u_x^\lambda + \lambda^p U(c)] - rxu_x^\lambda$.

Since $\lim_{\lambda \rightarrow 0} A_\lambda(x) = A(x) = \beta v - \max_{c \geq 0} [-cu_x + U(c)] - rxv_x$ and $A(x) > 0$ we get that, for sufficiently small λ , $A_\lambda(x) > 0$. Moreover from (2.14) and using that $A_\lambda(x) > 0$ we can find a function H such that $u_{xx}^\lambda(x) = H(x, u^\lambda, u_x^\lambda)$. We let $\lambda \rightarrow \infty$ and (4.15) follows.

We now conclude the proof of (i)–(iii) as follows.

(i) Since $C^*(x) = I(v_x(x))$ we need to show that

$$\lim_{x \rightarrow 0} \frac{I(v_x(x))}{x} = (Kp)^{1/(p-1)}$$

or equivalently

$$\lim_{\lambda \rightarrow \infty} \lambda I\left(v_x\left(\frac{1}{\lambda}\right)\right) = (Kp)^{1/(p-1)}.$$

Using that $v_x(1/\lambda) = \lambda^{1-p}u_x^\lambda(1)$ it suffices to show that

$$\lim_{\lambda \rightarrow \infty} \lambda I(\lambda^{1-p}u_x^\lambda(1)) = (Kp)^{1/(p-1)}.$$

From (4.2) and the fact that $I(y) = -F'(y)$ we have that (see Appendix)

$$(4.16) \quad \lim_{y \rightarrow 0} I(y)y^{1/(1-p)} = 1.$$

Using that $\lim_{\lambda \rightarrow 0} u_x^\lambda(1) = Kp$ we get

$$\lim_{\lambda \rightarrow \infty} \lambda I(\lambda^{1-p}u_x^\lambda(1)) = \lim_{\lambda \rightarrow \infty} \frac{[\lambda^{1-p}u_x^\lambda(1)]^{1/(1-p)}}{[u_x^\lambda(1)]^{1/(1-p)}} I(\lambda^{1-p}u_x^\lambda(1)) = (Kp)^{1/(p-1)}.$$

(ii) It suffices to show that

$$\lim_{x \rightarrow 0} -\frac{v_x(x)}{xv_{xx}(x)} = \frac{1}{1-p}$$

or equivalently

$$\lim_{\lambda \rightarrow \infty} \frac{\lambda v_x(1/\lambda)}{v_{xx}(1/\lambda)} = -\frac{1}{1-p}.$$

Using that $v_x(1/\lambda) = \lambda^{1-p}u_x^\lambda(1)$, $v_{xx}(1/\lambda) = \lambda^{2-p}u_{xx}^\lambda(1)$ it suffices to show that

$$\lim_{\lambda \rightarrow \infty} \frac{u_x^\lambda(1)}{u_{xx}^\lambda(1)} = -\frac{1}{1-p}$$

which follows from (4.14) and (4.15).

(iii) It follows as (ii).

Note that the three possibilities shown in (ii) are the same for HARA utility functions in Table 3.1.

II. Asymptotic analysis near infinity.

IIa. Bounded utility function.

We consider a utility function which satisfies

$$(4.17) \quad \lim_{c \rightarrow \infty} [U(c) - M]c^\alpha = -A$$

where α , A and M are positive constants. Then the dual function has the property (see Appendix)

$$(4.18) \quad \lim_{y \rightarrow 0} \frac{F(y) - M}{y^q} = -B$$

where $q = \alpha/(1 + \alpha)$ and $B = A^{1/(1+\alpha)}[1/\alpha^q + 1/\alpha^{q-1}]$.

THEOREM 4.3. *If the dual function satisfies (4.18) then*

$$\lim_{x \rightarrow \infty} \left[v(x) - \frac{M}{\beta} \right] x^\alpha = -K$$

for some $K > 0$.

PROOF. Let $v^\lambda(x) = \lambda^{-\alpha}[v(x/\lambda) - M/\beta]$, $\forall x \geq 0$ and $\lambda > 0$. Since v is a solution of (2.11) we get that v^λ satisfies

$$(4.19) \quad \begin{cases} \beta v^\lambda = G(x, v_x^\lambda, v_{xx}^\lambda) + \lambda^{-\alpha}[F(\lambda^{1+\alpha}v_x^\lambda) - M] + rxv_x^\lambda, & \forall x > 0, \\ v^\lambda(0) = -\frac{M}{\beta}\lambda^{-\alpha}, & \lim_{x \rightarrow \infty} v^\lambda(x) = 0. \end{cases}$$

Our goal is to examine the behavior of v^λ as λ goes to zero. In the sequel we will need the following lemmas.

LEMMA 4.3. (i) $v^\lambda \geq w$, $\forall x \geq 0$, where w is a classical solution of (4.20). In particular $w(x) = -mx^{-\alpha}$ with $m > 0$ and determined by direct substitution from (4.20)

$$(4.20) \quad \begin{cases} \beta w = G(x, w_x, w_{xx}) - lw_x^q + rxw_x, \\ \lim_{x \rightarrow 0} w(x) = -\infty, & \lim_{x \rightarrow \infty} w(x) = 0 \end{cases} \text{ where}$$

$$\inf_{y > 0} \frac{F(y) - M}{y^q} = -l.$$

(ii) $v^\lambda \leq -k(x+h)^{-\alpha}$ for λ sufficiently small, where $h > 0$ and k are such that

$$(4.21) \quad k^{(2-q)/(1-q)} < \frac{B\left(\frac{q}{1-q}\right)^{1/(q-1)}}{\beta + \frac{(b-r)^2}{\sigma^2}q + r\frac{q}{1-q}}.$$

PROOF. (i) Note that $-\infty < l < 0$ because

$$\lim_{y \rightarrow 0} \frac{F(y) - M}{y^q} = -B \quad \text{and} \quad \lim_{y \rightarrow \infty} \frac{F(y) - M}{y^q} = 0.$$

We now observe that

$$v^\lambda(0) = -\frac{M}{\beta} \lambda^{-\alpha} \quad \text{and} \quad \lim_{x \rightarrow \infty} v^\lambda(x) = 0.$$

Since w is a solution of (4.20) with

$$\lim_{x \rightarrow 0} w(x) < v^\lambda(0) \quad \text{and} \quad \lim_{x \rightarrow \infty} w(x) = \lim_{x \rightarrow \infty} v^\lambda(x)$$

we get from the Maximum Principle for elliptic equations (see [8]) that it suffices to show that v^λ is a supersolution of (4.20). In other words, we need to show

$$\beta v^\lambda \geq G(x, v_x^\lambda, v_{xx}^\lambda) - l(v_x^\lambda)^q + r x v_x^\lambda.$$

Using that v^λ solves (4.19) we need to prove that

$$\lambda^{-\alpha} [F(\lambda^{1+\alpha} v_x^\lambda) - M] \geq -l(v_x^\lambda)^q.$$

Since $\alpha = q/(1 - q)$ the above condition is equivalent to

$$\frac{F(\lambda^{1/(1-q)} v_x^\lambda) - M}{(\lambda^{1/(1-q)} v_x^\lambda)^q} \geq -l$$

and using the definition of l we conclude.

(ii) Fix $h > 0$ and let $z(x) = -k(x + h)^{-\alpha}$, $\forall x \geq 0$ where k satisfies (4.21). We observe that $z(0) = -kh^{-\alpha} > v^\lambda(0)$ for λ sufficiently small and $\lim_{x \rightarrow \infty} z(x) = \lim_{x \rightarrow \infty} v^\lambda(x)$. Using again the Maximum Principle we need to show that z is a supersolution of (4.19). In other words we need to show

$$\beta z \geq G(x, z, z_{xx}) + \lambda^{-\alpha} [F(\lambda^{1+\alpha} z_x) - M] + r x z_x, \quad \forall x > 0.$$

Moreover,

$$G(x, z_x, z_{xx}) \leq \max_{x \geq \pi} \left[\frac{1}{2} \sigma^2 \pi^2 z_{xx} + (b - r) \pi z_x \right] \leq -\frac{(b - r)^2}{\sigma^2} \frac{z_x^2}{z_{xx}}.$$

Therefore we need to show that

$$(4.22) \quad \beta z \geq -\frac{(b - r)^2}{\sigma^2} \frac{z_x^2}{z_{xx}} + \lambda^{-\alpha} [F(\lambda^{1+\alpha} z_x) - M] + r(x + h) z_x.$$

Since $z_x(x) = k\alpha(x+h)^{-(1+\alpha)}$ and $z_{xx}(x) = -k\alpha(1+\alpha)(x+h)^{-(2+\alpha)}$, (4.22) is equivalent to

$$-k \left[\beta + \frac{(b-r)^2}{\sigma^2} \frac{\alpha}{1+\alpha} + r\alpha \right] \geq \frac{F \left[k\alpha \left(\frac{x+h}{\lambda} \right)^{-(1+\alpha)} \right] - M}{\left(\frac{x+h}{\lambda} \right)^{-\alpha}}.$$

Using that $\alpha = q/(1-q)$ the above condition is equivalent to

$$-\frac{k \left[\beta + \frac{(b-r)^2}{\sigma^2} q + \frac{rq}{1-q} \right]}{\left(\frac{kq}{1-q} \right)^{1/q-1}} \geq \frac{F(y_\lambda^h) - M}{(y_\lambda^h)^q} \quad \text{where}$$

$$y_\lambda^h = \frac{kq}{1-q} \left(\frac{x+h}{\lambda} \right)^{1/(q-1)}.$$

We now observe that y_λ^h is close to zero for λ sufficiently small. Using the fact that

$$-B = \lim_{y \rightarrow 0} \frac{F(y) - M}{y^q}$$

and that k satisfies (4.21) we conclude.

REMARK 4.2. From the previous lemma we have that, in any interval $[x_0, +\infty)$ with $x_0 > 0$, $-mx^{-\alpha} \leq v^\lambda < 0$. Moreover since v^λ are increasing and concave they form a family of equicontinuous functions and by Ascoli's theorem they converge along subsequences. Moreover, by concavity, the convergence is locally uniform.

REMARK 4.3. Let \bar{v} be the limit of a subsequence v_{λ_n} with $\lambda_n \rightarrow 0$. From Lemma 4.3(i) and (ii) respectively we have that $\lim_{x \rightarrow \infty} \bar{v}(x) = 0$ and $\lim_{x \rightarrow 0} \bar{v}(x) = -\infty$.

In the sequel we show that the limit function is unique and it coincides with $w = -kx^{-\alpha}$, $k > 0$ where w is a classical solution of

$$(4.23) \quad \begin{cases} \beta v = G(x, v_x, v_{xx}) - Bv^q + rxv_x, \\ \lim_{x \rightarrow 0} v(x) = -\infty, \quad \lim_{x \rightarrow \infty} v(x) = 0. \end{cases}$$

First we show that \bar{v} is a viscosity solution of (4.23). Second we show that (4.23) has a unique viscosity solution in the class of nondecreasing concave functions which actually coincides with the classical solution w of the form $w(x) = -Kx^p$. The positive constant K can be computed from (4.19) by direct substitution and it is given in Table 3.1.

LEMMA 4.4. *Let $\lambda_n \rightarrow 0$ and $v_{\lambda_n} \rightarrow \bar{v}$. Then \bar{v} is a viscosity solution of (4.23).*

PROOF. The proof follows along the lines of Proposition I.3 of [6] and therefore we omit it.

LEMMA 4.5. *The boundary-value problem (4.23) has a unique viscosity solution in the class of nondecreasing concave functions.*

PROOF. We show that if f and g are viscosity solutions of (4.23), then, for fixed $h > 0$, $f_h \geq g$ where $f_h(x) = f(x+h)$, $\forall x > 0$. We observe that $\lim_{x \rightarrow \infty} f_h(x) =$

$\lim_{x \rightarrow \infty} g(x) = 0$ and $f_h(0) = f(h) > g(0)$, since $g(0) = -\infty$. Therefore, using a comparison theorem for viscosity solutions (see [4]), it suffices to show that f_h is a viscosity supersolution of (4.23). In other words, we need to show that if ψ is smooth and $f_h - \psi$ has a minimum at $x_0 > 0$ then

$$\beta f_h(x_0) \geq G(x_0, \psi_x(x_0), \psi_{xx}(x_0)) - B\psi_x^q(x_0) + rx_0\psi_x(x_0).$$

Since f is a viscosity solution of (4.23), f_h is a viscosity solution of

$$\beta f_h = G_h(x, f_h, f_{h,xx}) - Bf_{h,x}^q + r(x+h)f_{h,x} \quad \text{where}$$

$$G_h(x, p, q) = \max_{\substack{x+h \geq 0, \phi > 0 \\ x+h-\pi+\phi \geq 0}} \left[\frac{1}{2}\sigma^2\pi^2q + (b-r)\pi p - (R-r)\phi p \right].$$

Therefore we have to prove that

$$\begin{aligned} &G(x_0, \psi_x(x_0), \psi_{xx}(x_0)) + rx_0\psi_x(x_0) \\ &\leq G_h(x_0, \psi_x(x_0), \psi_{xx}(x_0)) + r(x_0+h)\psi_x(x_0) \end{aligned}$$

which follows since G is increasing in x and $\psi_x(x_0) \geq 0$. The latter follows from the fact that f is nondecreasing.

Let us now suppose that (4.23) has two nondecreasing concave solutions f and g . Then for $h > 0$ we have $f_h \geq g$ and $g_h \geq f$. By sending h to zero we get that $f = g$.

We now conclude the proof of Theorem 4.3. By direct substitution we can show that $w = -Kx^{q/(q-1)}$ is a classical nondecreasing concave solution of (4.23). (The constant K is given in Table 3.1.) Therefore it is the unique limit of v^λ . In other words,

$$(4.24) \quad \lim_{\lambda \rightarrow 0} v^\lambda(x) = \lim_{\lambda \rightarrow 0} \lambda^{q/(q-1)} \left[v\left(\frac{x}{\lambda}\right) - \frac{M}{\beta} \right] = -Kx^{q/(q-1)}.$$

Let $\xi = x/\lambda$, $x > 0$. Then (4.24) gives

$$\lim_{\xi \rightarrow \infty} \xi^{-\alpha} \left[u(\xi) - \frac{M}{\beta} \right] = -K$$

where $\alpha = q/(q-1)$.

THEOREM 4.4. *Under assumption (4.17) the optimal policies satisfy*

- (i) $\lim_{x \rightarrow 0} \frac{C^*(x)}{x} = qB \left(\frac{Kq}{1-q} \right)^{1-q};$
- (ii) $\lim_{x \rightarrow 0} \frac{\pi^*(x)}{x} = \begin{cases} \frac{b-r}{\sigma^2(1-q)} & \text{if } b-r < \sigma^2(1-q), \\ 1 & \text{if } b-R \leq \sigma^2(1-q) \leq b-r, \\ \frac{b-R}{\sigma^2(1-q)} & \text{if } \sigma^2(1-q) < b-R; \end{cases}$
- (iii) $\lim_{x \rightarrow 0} \frac{\phi^*(x)}{x} = \begin{cases} 0 & \text{if } b-R \leq \sigma^2(1-q), \\ \frac{b-R}{\sigma^2(1-q)} - 1 & \text{if } \sigma^2(1-q) < b-R. \end{cases}$

PROOF. The proof follows along the lines of Theorem 4.2 and therefore we omit it.

I**b**. *Unbounded utility function.* We consider a utility function which satisfies

$$(4.25) \quad \lim_{c \rightarrow \infty} \frac{U(c)}{c^p} = \frac{1}{p}, \quad 0 < p < 1,$$

or equivalently,

$$(4.26) \quad \lim_{c \rightarrow \infty} \frac{U'(c)}{c^{p-1}} = 1.$$

Then the dual function satisfies (see Appendix)

$$(4.27) \quad \lim_{y \rightarrow 0} \frac{F(y)}{y^{p/(p-1)}} = \frac{1-p}{p}.$$

THEOREM 4.5. *If the dual function satisfies (4.27) then*

$$\lim_{x \rightarrow \infty} \frac{v(x)}{x^p} = K$$

where K is given in Table 3.1.

PROOF. Let $w^\lambda(x) = \lambda^p v(x/\lambda)$, $\forall x \geq 0$. Then w^λ satisfies equation (4.3). We are interested in the behavior of w^λ as λ goes to zero. We need the following lemma and the rest of the proof follows along the lines of the proof of the previous theorem.

LEMMA 4.6. $w^\lambda(x) \leq Mx^p + \delta/\beta$, $\forall x \geq 0$ for some $M, \delta > 0$ and λ sufficiently small.

PROOF. Let $v^{0,r}$ be the solution of the “unconstrained” problem (i.e. short-selling and borrowing are allowed) with $R = r$ and utility function \bar{U} such that $\bar{U}(c) = \gamma c^p + \delta$ with $\gamma > 1$, $\delta > 0$. Then (see [5]), $v^{0,r}(x) = Mx^p + \delta/\beta$ for some $M > 0$. We can choose δ sufficiently large such that $U(c) \leq \bar{U}(c)$, $\forall c \geq 0$. Since $U \leq \bar{U}$ we get that $v(x) \leq v^{0,r}(x)$ for all x . Therefore $w^\lambda(x) \leq Mx^p + \delta\lambda^p/\beta$ and for λ small $w^\lambda(x) \leq Mx^p + \delta/\beta$.

THEOREM 4.6. *Under assumption (4.25) the optimal policies satisfy*

$$(i) \quad \lim_{x \rightarrow \infty} \frac{C^*(x)}{x} = (Kp)^{1/(p-1)};$$

$$(ii) \quad \lim_{x \rightarrow \infty} \frac{\pi^*(x)}{x} = \begin{cases} \frac{b-r}{\sigma^2(1-p)} & \text{if } b-r < \sigma^2(1-p), \\ 1 & \text{if } b-R \leq \sigma^2(1-p) \leq b-r, \\ \frac{b-R}{\sigma^2(1-p)} & \text{if } \sigma^2(1-p) < b-R; \end{cases}$$

$$(iii) \quad \lim_{x \rightarrow \infty} \frac{\phi^*(x)}{x} = \begin{cases} 0 & \text{if } b-R \leq \sigma^2(1-p), \\ \frac{b-R}{\sigma^2(1-p)} - 1 & \text{if } \sigma^2(1-p) < b-R. \end{cases}$$

PROOF. The proof follows along the lines of Theorem 4.2 and therefore we omit it.

Concluding remarks. We have obtained the asymptotic behavior of the value function v as a function of the initial wealth x and of the optimal investment and consumption policies for small x and for large x . The parameters $\alpha > 0$ in (4.17) and $0 < p < 1$ in (4.1), (4.25) are arbitrary. Moreover, the p in (4.1) may differ from the q in (4.25). Table 3.1 shows three possible investment strategies in the HARA case: partly in stock and no borrowing, all in stock and no borrowing, or borrowing. From Theorems 4.2, 4.4 and 4.6 each of these various possibilities can occur for small or large initial wealth, depending on the behavior of the utility function $U(c)$ for small c and for large c .

Since no explicit formula for v is available when the borrowing rate R exceeds the rate of return r on the riskless asset, the dynamic programming equation must be solved numerically to find the behavior of optimal investment and consumption policies for intermediate values of x .

5. Appendix.

5.1. *Admissible controls.* Let (Ω, P, F) be the probability space where the Brownian motion w_t in equation (2.2) is defined. Let also F_t to be the σ -algebra generated by the process w . The controls π_t, C_t, ϕ_t are called admissible if:

- (i) they are F_t -progressively measurable real valued processes;
- (ii) they satisfy the constraints $\pi_t \geq 0, C_t \geq 0,$ and $\phi_t \geq 0$ a.e. $\forall t \geq 0$;
- (iii) the state equation (2.4) has a strong solution X_t which satisfies $0 < X_t < \infty$ a.s. $\forall t \geq 0$.

We also allow feedback controls $\pi(t), \phi(t), C(t)$ where:

$$\pi(t) = \pi(X_t), \quad \phi(t) = \phi(X_t) \quad \text{and} \quad C(t) = C(X_t)$$

with π, ϕ and C locally Lipschitz functions from R into R^+ .

5.2. *Asymptotic behavior of the dual function.*

5.2.a. *Asymptotic behavior near zero.* We assume that the utility function satisfies (4.1). Let $c = I(y)$ where $I = (U')^{-1}$. Then (4.1) is equivalent to

$$\lim_{y \rightarrow \infty} \frac{y}{I(y)^{p-1}} = 1$$

which gives

$$\lim_{y \rightarrow \infty} \frac{I(y)}{y^{1/(p-1)}} = 1.$$

Therefore, (4.2) holds.

By L'Hôpital's rule we have that

$$\lim_{y \rightarrow \infty} \frac{F(y)}{y^{p/(p-1)}} = \lim_{y \rightarrow \infty} \frac{F'(y)}{\frac{p}{p-1} y^{1/(p-1)}} = \lim_{y \rightarrow \infty} \frac{-F'(y)}{y^{1/(p-1)}} \frac{p-1}{p}.$$

Using that $I(y) = -F'(y)$ and (4.16) we get (4.2).

5.2.b. *Asymptotic behavior near infinity.* We show (4.18) under the assumption (4.17). From the definition of F we get that $F(y) = -yI(y) + U(I(y))$. Therefore, to prove (4.18) it suffices to prove

$$(5.1) \quad \lim_{y \rightarrow 0} \frac{-yI(y) + U(I(y)) - M}{y^{\alpha/(1+\alpha)}} = A^{1/(1+\alpha)} \left[\frac{1}{\alpha^q} + \frac{1}{\alpha^{q-1}} \right].$$

From (4.17) we get

$$\lim_{c \rightarrow \infty} \frac{U'(c)}{\alpha c^{-(\alpha+1)}} = A.$$

Therefore

$$(5.2) \quad \lim_{c \rightarrow \infty} \left[\frac{U'(c)}{\alpha c^{-(1+\alpha)}} \right]^{1/(1+\alpha)} = A^{1/(1+\alpha)}.$$

Let $c = I(y)$. Then (5.2) becomes

$$\lim_{y \rightarrow 0} \left[\frac{y}{\alpha I(y)^{-(1+\alpha)}} \right]^{1/(1+\alpha)} = A^{1/(1+\alpha)}$$

or equivalently

$$(5.3) \quad \lim_{y \rightarrow 0} I(y) y^{1/(1+\alpha)} = \alpha^{1/(1+\alpha)} A^{1/(1+\alpha)}.$$

Moreover,

$$(5.4) \quad \lim_{y \rightarrow 0} \frac{U(I(y)) - M}{y^{\alpha/(1+\alpha)}} = \lim_{c \rightarrow \infty} \frac{(U(c) - M)c^\alpha}{[U'(c)]^{\alpha/(1+\alpha)} c^\alpha}.$$

From (5.2) we have that

$$(5.5) \quad \lim_{c \rightarrow \infty} c^\alpha [U'(c)]^{\alpha/(1+\alpha)} = \alpha^{\alpha/(1+\alpha)} A^{\alpha/(1+\alpha)}.$$

Combining (5.3), (5.4) and (5.5) we obtain

$$\begin{aligned} \lim_{y \rightarrow 0} \left[-\frac{yI(y)}{y^{\alpha/(1+\alpha)}} + \frac{U(I(y)) - M}{y^{\alpha/(1+\alpha)}} \right] &= -\alpha^{1/(1+\alpha)} A^{1/(1+\alpha)} - \frac{A}{\alpha^{\alpha/(1+\alpha)} A^{\alpha/(1+\alpha)}} \\ &= -A^{1/(1+\alpha)} [\alpha^{1/(1+\alpha)} + \alpha^{-\alpha/(1+\alpha)}]. \end{aligned}$$

Using that $1/(1+\alpha) = 1 - q$ and $\alpha/(1+\alpha) = q$ we get (5.1).

Finally, equation (4.27) is obtained the same way as (4.2).

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