Finance Stochast. 3, 345–369 (1999)

Finance and Stochastics

Bounds on prices of contingent claims in an intertemporal economy with proportional transaction costs and general preferences

George M. Constantinides[1,2,⋆], **Thaleia Zariphopoulou**[3,⋆⋆]

[1] Graduate School of Business, The University of Chicago, 1101 East 58th Street, Chicago, IL 60637, USA (e-mail: gmc@gsb.chicago.edu)
[2] The National Bureau of Economic Research, 1050 Massachusetts Avenue, Cambridge, MA 02138, USA
[3] School of Business and Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA

Abstract. Analytic bounds on the reservation write price of European-style contingent claims are derived in the presence of proportional transaction costs in a model which allows for intermediate trading. The option prices are obtained via a utility maximization approach by comparing the maximized utilities with and without the contingent claim. The mathematical tools come mainly from the theories of singular stochastic control and viscosity solutions of nonlinear partial differential equations.

Key words: Contingent claim prices, bounds on prices, transaction costs, viscosity solutions

JEL classification: C6, D9, G1

Mathematics Subject Classification (1991): 93E20, 60G40

1. Introduction

In a frictionless market Black and Scholes (1973) and Merton (1973) relied on an ingenious no-arbitrage argument to price an option on a stock when the interest rate is constant and the stock price follows a geometric Brownian process. They presented a self-financing, dynamic trading policy between the bond and stock accounts which replicates the payoff of the option. They then argued that absence of arbitrage dictates that the option price is equal to the cost of setting up the

⋆ George Constantinides acknowledges partial financial support from the Center for Research in Security Prices, the University of Chicago.
⋆⋆ Thaleia Zariphopoulou acknowledges partial support from the Alfred P. Sloan Foundation, the H. and M. Laun Foundation and the Graduate School of the University of Wisconsin, Madison.
Manuscript received: October 1997; final version received: August 1998

replicating portfolio. The appeal of the argument lies in its reliance on the absence of arbitrage alone and is independent of other aspects of the equilibrium, such as a particular asset pricing model.

The Achilles' heel of the argument is that the frictionless market assumption must be taken literally. The dynamic replication policy incurs an infinite volume of transactions over any finite trading interval, given the fact that the Brownian process which drives the stock price has infinite variation. In a market with proportional transaction costs, the dynamic replication policy incurs infinite transaction costs over any finite trading interval and cannot be self-financing, no matter how small the finite transaction costs rate is.

Merton (1990, Chapter 14) maintained the goal of a dynamic trading policy as that of replicating the option payoff and modeled the path of the stock price as a two-period binomial process. The initial cost of the replication policy is finite and serves as an upper bound to the write price of a call which is arbitrage-free. Shen (1990) and Boyle and Vorst (1992) extended Merton's model to a multiperiod binomial process for the stock price and provided numerical solutions to the initial cost of the replicating portfolio. As the number of periods increases within the given lifetime of a call option, the initial cost of the replicating portfolio tends to infinity.

Bensaid et al. (1992) and Edirisinghe et al. (1993) noted that a tighter upper bound on the write price of a call option is obtained by replacing the goal of replicating the payoff of the option with the goal of dominating the payoff. For example, the payoff of a share of stock dominates the payoff of a call option and, therefore, the cost of initially buying one share provides an upper bound to the cost of a minimum-cost dominating policy as the number of periods increases within the given lifetime of the option.

Davis and Clark (1993) conjectured and Soner et al. (1995) proved that the cost of initially buying one share of stock is indeed the cost of the cheapest dominating policy in the presence of finite proportional transaction costs, and concluded that this bound is of little economic interest.

Leland (1985) introduced a class of imperfectly replicating policies in the presence of proportional transaction costs. He calculated the total cost, including transaction costs, of an imperfectly replicating policy and the "tracking error", that is the standard deviation of the difference between the payoff of the option and the payoff of the imperfectly replicating policy. Imperfectly replicating policies were further studied by Figlewski (1989), Flesaker and Hughston (1994), Grannan and Swindle (1996), Henrotte (1993), Hoggard et al. (1994) and Toft (1996). Avellaneda and Paras (1994) extended the notion of imperfectly replicating policies to that of imperfectly dominating policies.

An alternative approach, initiated by Hodges and Neuberger (1989) and developed further by Davis et al. (1993), is to consider an investor endowed with bonds, stocks and an option and to derive the investor's optimal trading policy in the stock and bond accounts which maximizes the investor's expected utility in the presence of proportional transaction costs. The optimal trading policy is solved numerically for the case of exponential utility by approximating the stock price process by a multiperiod binomial process. One may then compute the investor's reservation purchase price and reservation write price of the option.

The setup in our paper is similar to that of the above two papers in that we consider an investor's intertemporal consumption and investment problem in the presence of proportional transaction costs with and without the opportunity to write a call option. The bond is riskless and the stock price is a geometric Brownian Motion. The investors' preferences are modeled by an increasing and concave utility function. Unlike the above two papers, our goal is to derive in closed form an upper bound to the reservation write price of a call option, thereby bypassing the need for a numerical solution. Indeed we derive such a bound in closed form as a function of the initial conditions and the model parameters.

We motivate our paper by considering in Sect. 2 a simple one-period model, where the end of the period coincides with the expiration date of the option. We modify the stochastic dominance arguments of Perrakis and Ryan (1984), Levy (1985) and Ritchken (1985) to account for proportional transaction costs and derive bounds on the reservation purchase price and reservation write price of a call option which apply to any concave utility function. We then explain why the stochastic dominance argument breaks down when intermediate trading is allowed. The seemingly innocuous generalization of the model to allow for intermediate trading makes the problem far more difficult.

In Sect. 3 we set up the model with intermediate trading and state some preliminary technical results. The main result is derived and stated in Sect. 4 as Theorem 4.1.

2. Bounds on option prices in a single-period model

We consider an economy with two securities, a riskless bond and a risky stock. We denote by B and S the bond and the stock prices, respectively, at the beginning of the (single) period and by B_T and S_T the prices at the end of the period which is assumed to have length T.

Trading in the bond and the stock accounts occurs only at the beginning and end of the period and is subject to transaction costs. Specifically, β dollars of the bond may be converted into one dollar of the stock; and, one dollar of the stock may be converted into α dollars of the bond. We assume that the constants α and β satisfy $0 < \alpha < 1 < \beta$.

The important simplifying assumption is that no trading may occur at intermediate times. This assumption is relaxed in the next section and the implications are fully explored therein.

The investor's pre-trade endowment consists of x_0 dollars in the bond account and y_0 dollars in the stock account. The investor trades at the beginning of the period incurring transaction costs and attains a post-trade endowment of x dollars in the bond account and y dollars in the stock account.

We assume that $y > \frac{S}{\alpha}$ that is the investor invests in at least $\frac{1}{\alpha}$ shares of the stock. At the end of the period, the investor converts the stock account into the bond account and consumes $c(S_T) = xR_F + y\frac{S_T}{S}$, where $R_F = \frac{B_T}{B}$.

We assume that the investor's expected utility is the expectation of $u(c(S_T))$, where $u : \mathbb{R} \to \mathbb{R}$ is increasing and concave. In the absence of the opportunity to invest in an option, the investor chooses (x, y) to maximize the expected utility.

Given (x, y), we now present the investor with the opportunity to write one cash-settled, European-style call option with expiration at the end of the period and strike price K. Let C denote the post-transaction-cost price at which the investor may write the call: if the investor writes the call, the bond account increases by C dollars at the beginning with the period and decreases by $[S_T - K]^+$ dollars at the end of the period.

To provide an upper bound to the reservation write price of a call, we adopt the stochastic dominance arguments of Perrakis and Ryan (1984), Levy (1985) and Ritchken (1985), modified to account for transaction costs.

Consider the zero-net-cost portfolio which consists of a short position in one call and a long position in $\frac{C}{\beta S}$ shares of stock. The net payoff in the bond account at the end of the period is $z(S_T)$, where $z(S_T) = \frac{\alpha C S_T}{\beta S} - [S_T - K]^+$. Note that $z(S_T) \gtreqless 0$ as $S_T \lesseqgtr \hat{S}$ where $\hat{S}$ is defined by $\frac{\alpha C \hat{S}}{\beta S} - S_T + K = 0$.

The investor has post-trade endowment (x, y) and contemplates whether to write the call. If the investor writes the call and invests the proceeds in the stock, the expected utility is

$$E[u(c(S_T) + z(S_T))] \geq E[u(c(S_T))] + E[z(S_T)u'(c(S_T) + z(S_T))]$$

(by the concavity of u)

$$\geq E[u(c(S_T))] + E[z(S_T)u'(c(\hat{S}) + z(\hat{S}))]$$

(since $z(S_T) \gtreqless 0$ and $u'(c(s_T) + z(S_T)) \gtreqless u'(c(\hat{S}) + z(\hat{S}))$ as $S_T \lesseqgtr \hat{S}$)

$$\geq E[u(c(S_T))] + u'(c(\hat{S}) + z(\hat{S}))E[z(S_T)]$$

and exceeds the expected utility from refraining to write the call, unless $E[z(S_T)] < 0$, i.e.

$$(\alpha/\beta)CE[S_T/S] - E[[S_T - K]^+] < 0.$$

Therefore,

$$C < \beta E[[S_T - K]^+]/\alpha E[S_T/S_0] \equiv \bar{C}_1 \tag{2.1}$$

and $\bar{C}_1$ is an upper bound to the reservation write price of a call option.

We consider next a different zero-net-cost portfolio which consists of a short position in one call and a long position in C dollars in the bond. Proceeding as before, we conclude that the expected utility in writing the call exceeds the expected utility in not writing the call, unless

$$C < E[[S_T - K]^+]/R_F \equiv \bar{C}_2. \tag{2.2}$$

We combine equations (2.1) and (2.2) and conclude that $\bar{C}$ is an upper bound to the reservation write price of a call option, where

$$\bar{C} = E[[S_T - K]^+] \min \left[R_F^{-1}, \frac{\beta/\alpha}{E[S_T/S]} \right]. \tag{2.3}$$

To derive a lower bound to the reservation purchase price of a call option, let C denote the post-transaction-cost price at which the investor may purchase the call. Consider the zero-net-cost portfolio which consists of

(a) a long position in one call;
(b) a short position in $1/\beta$ shares of stock; and
(c) investment of $(\alpha S_T/\beta) - C$ dollars in the bond account.

Denote by $z(S_T)$ the net payoff in the bond account at the end of the period, where $z(S_T) = [S_T - K]^+ - S_T + \left\{ \alpha \frac{S}{\beta} - C \right\} R_F$. Repeating the earlier argument, we conclude that the expected utility in purchasing the call exceeds the expected utility in refraining from purchasing the call, unless $E[z(S_T)] < 0$, which yields $\underline{C}$ as a lower bound to the reservation purchase price of a call, where

$$\underline{C} = \frac{E[[S_T - K]^+]}{R_F} - \frac{E[S_T]}{R_F} + \frac{\alpha S}{\beta}. \tag{2.4}$$

It is easily shown that $\underline{C} \leq \bar{C}$. In equilibrium, transaction prices of a call option must lie in the region $[\underline{C}, \bar{C}]$. For, if a transaction occurs at a price $C < \underline{C}$, then the writer is acting suboptimally as the writer could have found a willing buyer of the call at a price as high as $\underline{C}$. Likewise, if a transaction occurs at a price $C > \bar{C}$, then the buyer of the call is acting suboptimally as the buyer could have found a willing writer of the call at a price as low as $\bar{C}$.

The stochastic dominance bounds are appealing in that they apply for any increasing and concave utility function. It turns out, however, that the derivation of these bounds breaks down when intermediate trading is permitted in the open interval $(0, T)$.

Let us reconsider the stochastic dominance argument for the reservation write price of a call. The plausible assumption was made that the investor's endowment satisfies the condition $y > \frac{S}{\alpha}$. Without intermediate trading, the consumption at the end of the period is $c(S_T)$ and has two crucial properties:

(1) it is monotone increasing in S_T with slope greater than one; and
(2) given S_T, $c(S_T)$ is independent of the stock price path ω_T over $(0, T)$.

The first property is crucial in the proof in that it implies that $c(S_T) + z(S_T)$ is increasing in S_T and therefore $u'(c(S_T) + z(S_T))$ is decreasing in S_T. The second property is crucial in the step which allowed us to take $u'(c(S_T) + z(S_T))$ outside the expectation: if c is a function of the price path ω_T, $u'(c(\omega_T) + z(S_T)) \mid S_T$ is a random variable and cannot be taken outside the expectation. Another problem is that, in the presence of intermediate trading, $c(\omega_T) + z(S_T)$ is not even bounded from below and expected utility is undefined for utility functions which are only defined for consumption bounded from below. Similar problems arise in attempting to generalize the stochastic dominance argument in the derivation of a lower bound to the reservation purchase price when intermediate trading is allowed.

In the next section we address the problem of the derivation of bounds when intermediate trading is allowed. It turns out that the seemingly innocuous generalization of intermediate trading results in considerable weakening of the bounds.

3. The continuous time model

We consider an economy with two securities, a bond with price $B(t)$ and a stock with price $S(t)$ at date $t \geq 0$. Prices are denominated in units of a consumption good, say dollars.

The bond pays no coupons, is default free and has price dynamics

$$B_t = e^{rt} B_0, \qquad t \geq 0 \tag{3.1}$$

where r is the *constant rate of interest*.

We denote by $W(t)$ a one-dimensional standard Brownian motion which generates the filtration $\mathscr{F}_t$ on a fixed, complete probability space $(\Omega, \mathscr{F}, P)$. The stock price is the diffusion process

$$S_t = S_0 \exp\left\{ \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right\}, \tag{3.2}$$

where the *mean rate of return* μ and the *volatility* σ are constants such that $\mu > r$ and $\sigma \neq 0$.

The investor holds x_t dollars of the bond and y_t dollars of the stock at date t, and consumes at the rate c_t dollars out of the bond account. We consider a pair of right-continuous with left limits (CADLAG), non-decreasing processes (L_t, M_t) such that L_t represents the cumulative dollar amount transferred into the stock account and M_t the cumulative dollar amount transferred out of the stock account. By convention, L_0=M_0=0. The stock account process, starting with $y_0 = y$, is

$$y_t = y + \int_0^t \mu y_\tau d\tau + \int_0^t \sigma y_\tau dW_\tau + L_t - M_t. \tag{3.3}$$

Transfers between the stock and the bond accounts incur *proportional transaction costs*. In particular, the cumulative transfer L_t into the stock account reduces the bond account by βL_t and the cumulative transfer M_t out of the stock account increases the bond account by αM_t, where $0 < \alpha < 1 < \beta$.

The bond account process, starting with $x_0 = x$, is

$$x_t = x + \int_0^t \{r x_\tau - c_\tau\} d\tau - \beta L_t + \alpha M_t\,. \tag{3.4}$$

The integral represents the accumulation of interest and the drain due to consumption. The last two terms represent the cumulative transfers between the stock and bond accounts, net of transaction costs.

A policy is a $\mathscr{F}_t$-progressively measurable triple (c_t, L_t, M_t). We restrict our attention to the set of admissible policies $\mathscr{A}$ such that, a.s. for $t \geq 0$,

$$c_t \geq 0,\ E\int_0^t c_\tau d\tau < \infty, \quad \text{and} \quad w_t = x_t + \binom{\alpha}{\beta} y_t \geq 0 \tag{3.5}$$

where we adopt the notation

$$\binom{\alpha}{\beta} z = \begin{cases} \alpha z & \text{if } z \geq 0 \\ \\ \beta z & \text{if } z < 0. \end{cases} \tag{3.6}$$

We refer to w_t as the *net worth*. It represents the investor's bond holdings, if the investor were to transfer the holdings from the stock account into the bond account, incurring in the process the transaction costs.

The investor has von Neumann-Morgenstern preferences $E\left[\int_0^{+\infty} e^{-\rho t} U(c_t) dt\right]$ over the consumption stream $\{c_t, t \geq 0\}$, where ρ is the *subjective discount rate* and the *felicity function* $U : R_0^+ \to R_0^+$ is assumed to have the following properties:

i) $U \in C([0, +\infty)) \cap C^1((0, +\infty))$ is increasing and concave.
ii) There exist λ_1 and λ_2 positive constants and γ, with $0 < \gamma < 1$, such that $\lambda_1 c^\gamma \leq U(c) \leq \lambda_2 c^\gamma$.
iii) The function $\dfrac{U(c)}{c^\gamma}$ is non-decreasing.

Examples of felicity functions that satisfy the above assumptions are, among others,

i) the CRRA (constant relative risk aversion) utility function $U(c) = \frac{1}{\gamma} c^\gamma$, with $0 < \gamma < 1$, and
ii) concave functions of the form $U(c) = f(c) c^\gamma$ with $0 < \gamma < 1$ and f nondecreasing such that $\lambda_1 \leq f(c) \leq \lambda_2$.

Given the initial endowment (x, y) in $D = \left\{(x, y) \in \mathbb{R}^2 : x + \binom{\alpha}{\beta} y \geq 0\right\}$, we define the *value function* V as

$$V(x, y) = \sup_{(c,L,M) \in \mathscr{A}} E\left[\int_0^{+\infty} e^{-\rho t} U(c_t) dt \mid x_0 = x, y_0 = y\right]. \tag{3.7}$$

To guarantee that the value function is well defined, we assume either as in Davis and Norman (1990) that

$$\rho > r\gamma + \frac{\gamma(\mu - r)^2}{2\sigma^2(1 - \gamma)}, \tag{3.8}$$

or as in Shreve and Soner (1994) that

$$\rho > r\gamma + \gamma^2(\mu - r)^2 / 2\sigma^2(1 - \gamma)^2, \tag{3.9}$$

without an associated upper bound on $\mu - r$. Either set of conditions (3.8) and (3.9) yield that the value function corresponding to $\alpha = \beta = 1$ and $U(c) = \gamma^{-1}\lambda_2 c^\gamma$ is finite and, therefore, all functions with $0 \leq \alpha < 1 \leq \beta$ are finite. We also assume that $\rho \geq \mu$.

A straightforward argument along the lines of Constantinides (1979) shows that the value function is increasing and jointly concave in (x, y). It can also be shown that it is uniformly continuous on D (see Tourin and Zariphopoulou 1994). Furthermore, the value function is expected to solve the Hamilton-Jacobi-Bellman equation (HJB) associated with the stochastic control problem (3.7). The HJB equation turns out to be the following Variational Inequality with gradient constraints

$$\min\left[\mathscr{L}V, \beta\frac{\partial V}{\partial x} - \frac{\partial V}{\partial y}, -\alpha\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}\right] = 0 \tag{3.10}$$

where the differential operator $\mathscr{L}$ is

$$\mathscr{L}V = \rho V - \tfrac{1}{2}\sigma^2 y^2 \frac{\partial^2 V}{\partial y^2} - \mu y \frac{\partial V}{\partial y} - rx\frac{\partial V}{\partial x} - \max_{c \geq 0}\left\{ -c\frac{\partial V}{\partial x} + U(c)\right\}. \tag{3.11}$$

In the special case of power utility functions, Davis and Norman (1990) obtained a closed form expression for the value function employing the special homogeneity of the problem. They also showed that the optimal policy confines the investor's portfolio to a certain wedge-shaped region in the wealth plane and they provided an algorithm and numerical computations for the optimal investment rules. The same class of utility functions was later further explored by Shreve and Soner (1994) who relaxed some of the technical assumptions on the market parameters of Davis and Norman (1990) related to the finiteness of the value function and the nature of the optimal policies. Shreve and Soner (1994) also provided results related to the regularity of the value function and the location of the exercise boundaries.

In the case of general utility functions that we study herein, the value function is not necessarily smooth and, therefore, it might not satisfy the HJB equation in the classical (strong) sense. It turns out that the appropriate class of weak solutions are the so-called (constrained) *viscosity solutions* and this is the class of solutions we will be using throughout the paper. (See Appendix A for their definition.) The characterization of V as a constrained solution is natural because of the presence of state constraints given by (3.5). In models with transaction costs, this class of solutions was first employed by Zariphopoulou (1992) and, subsequently, among others by Davis et al. (1993), Tourin and Zariphopoulou (1994), Shreve and Soner (1994), and Barles and Soner (1995). The following theorem is proved in Tourin and Zariphopoulou (1994) and Shreve and Soner(1994):

Theorem 3.1. *The value function V is the unique constrained viscosity solution of (3.10) on D, in the class of uniformly continuous, concave and increasing functions.*

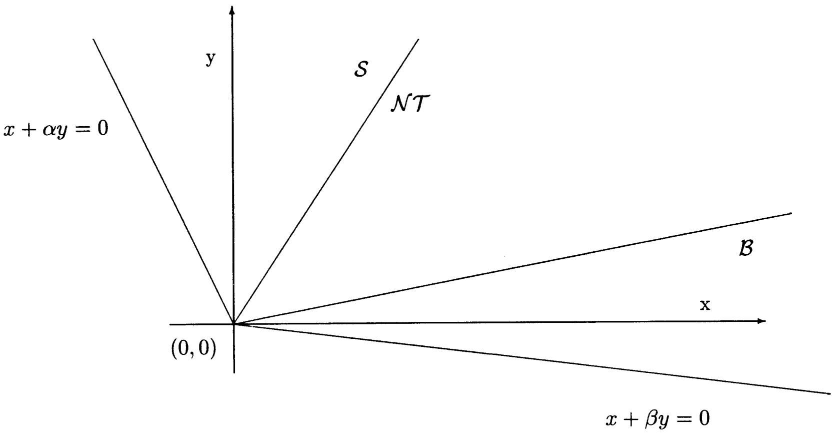

Fig. 1.

For the special case of power utility function, Davis and Norman (1990) and Shreve and Soner (1994) showed that the state space depletes into three regions, namely the $\mathscr{NT}$ region (no transactions), the $\mathscr{B}$ region (buy stock shares) and the $\mathscr{S}$ region (sell stock shares). The homogeneity properties of the utility function yield that the boundaries of the $\mathscr{NT}$ with the $\mathscr{B}$ and the $\mathscr{S}$ regions are straight lines that pass through the origin (see Fig. 1); moreover, for a big set of parameters, the wedge $\mathscr{NT}$ is a subset of the first quadrant, i.e. $\mathscr{NT} \subset \{(x,y) : x \geq 0, y \geq 0\}$. For the case of more general utilities that we examine herein, there are no analytic results to date for the location and the regularity of the transaction boundaries. On the other hand, numerical results obtained first by Tourin and Zariphopoulou (1994) and more recently by Pichler (1996), indicate that the $\mathscr{NT}$ is a wedge-shaped region, located between the $\mathscr{B}$ and $\mathscr{S}$ regions, which, for a wide range of parameters, belongs to the first quadrant as well.

To make the analysis in this paper more tractable, we are going to make the following assumption (see Fig. 2).

Assumption 3.1. *An optimal policy exists such that the* $\mathscr{NT} \subset \{(x,y) : Ax \leq y \leq Bx, x \geq 0\}$ *for some constants A and B with* $A > 0$.

In the next section, it will be apparent how this assumption is used when we construct the candidate price bound. For a general overview of existence of optimal policies in (singular) stochastic control problems with constraints, we refer the reader to Kurtz (1991) and Zhu (1991).

We now introduce a third asset, a cash settled European-style contingent claim with expiration at date T and payoff $g(S_T)$ at expiration. If the investor writes the claim at date t with $0 \leq t \leq T$, the bond account is credited with an amount, say C dollars, which represents the price of the claim, and is debited $g(S_T)$ dollars at the expiration date T. To keep the problem tractable we assume that the investor may not trade the claim in the open interval $(0, T)$.

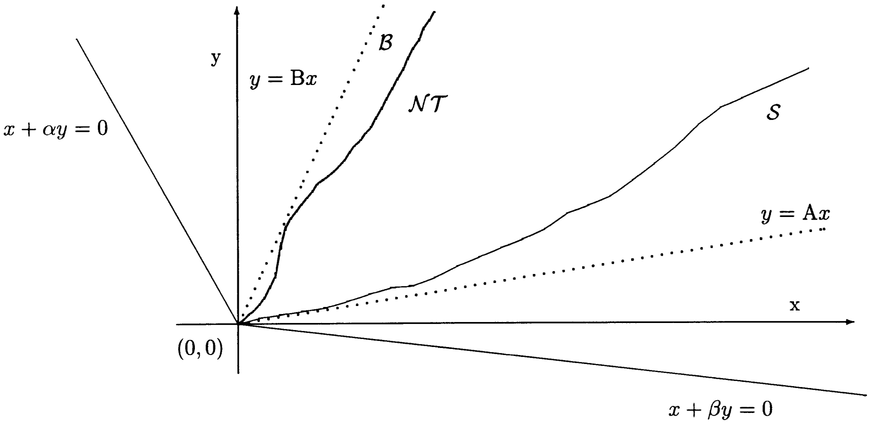

Fig. 2.

Let x_t and y_t be the initial endowment at time t *after* the bond account has been credited with the proceeds from writing the claim. Once the claim is written, the writer's objective is to maximize his expected utility from consumption, as before with the extra obligation to surrender to the buyer $g(S_T)$ dollars at time T. If V is defined in (3.7) and S_t is given by (3.2), the utility payoff of the writer is

$$E\left[\int_t^T e^{-\rho(s-t)}U(c_s)ds + e^{-\rho(T-t)}V(x_T - g(S_T), y_T) \mid x_t = x, y_t = y, S_t = S\right].$$

The value function of the writer is

$$\begin{aligned} J(x, y, S, t) \quad = \quad & \sup_{\mathscr{A}_1} E\left[\int_t^T e^{-\rho(s-t)}U(c_s)ds\right. \\ & \left. + e^{-\rho(T-t)}V(x_T - g(S_T), y_T) \mid x_t = x, y_t = y, S_t = S\right] \end{aligned} \tag{3.12}$$

where $\mathscr{A}_1$ is the set of admissible policies defined below.

We assume that the payoff g satisfies the following:

$$g : [0, +\infty) \to [0, +\infty) \quad \text{is convex,} \quad g(0) = 0 \quad \text{and} \quad \lim_{S\to\infty} \frac{g(S)}{S} = 1. \tag{3.13}$$

It immediately follows that $0 \leq g(S) \leq S$ and $0 \leq g_s \leq 1$.

To motivate the definition of $\mathscr{A}_1$, we state a proposition which follows directly from the results of Soner et al. (1995) as generalized by Leventhal and Skorohod (1997):

Proposition 3.1. *Let S_t, x_t and y_t be given by (3.2), (3.4) and (3.3). Then, in order to have at $t = T$*

$$x_T + \binom{\alpha}{\beta} y_T \geq g(S_T) \ \ a.e. \tag{3.14}$$

the following constraint must hold for all t, $0 \leq t < T$

$$x_t + \binom{\alpha}{\beta}\left(y_t - \frac{S_t}{\alpha}\right) \geq 0 \quad a.e. \tag{3.15}$$

Thus the cheapest (super) replicating strategy is the trivial one, to hold one share of the stock.

We define the set $\mathscr{A}_1$ of admissible policies of the investor who has written a contingent claim, as the set of $\mathscr{F}_t$-progressively measurable processes (c_t, L_t, M_t), with L_t and M_t being CADLAG which also satisfy, a.s. for $0 \leq t \leq T$, the conditions

$$c_t \geq 0,\ E\int_0^t c_\tau d\tau < \infty, \quad \text{and} \quad w_t = x_t + \binom{\alpha}{\beta}\left(y_t - \frac{S_t}{\alpha}\right) \geq 0\,. \tag{3.16}$$

We also define the set of admissible policies $\{c_t, L_t, M_t; t > T\}$ of the investor who has written a claim by $\mathscr{A}$, as given in (3.5). Note that for $t > T$, the option has expired and settled and the investor's problem is indistinguishable from that of an investor who has not written the claim. Thus it is natural to define the set of admissible policies for $t > T$ as $\mathscr{A}$. The set $\mathscr{A}_1$ is a subset of $\mathscr{A}$ for $0 \leq t \leq T$ in the sense that the second restriction ensures that the investor will have non-negative net worth upon closing up the short position in the call option and, therefore, that it is feasible to write a call option in the first place. The results of Soner et al. (1995) (for $g(S) = (S-K)^+$) and Leventhal and Skorohod (1997) (for general g) state that the set of policies in $\mathscr{A}_1$ is not overly restrictive given the goal of ensuring that it is feasible to write the claim option.

The value function $J(x,y,S,t)$ is given by (3.12) and is defined for $(x,y,S) \in D_1 = \left\{(x,y,S) : x + \binom{\alpha}{\beta}\left(y - \frac{S}{\alpha}\right) \geq 0,\ S \geq 0\right\}$.

A generalization of Theorem 4 in Tourin and Zariphopoulou (1994) yields the following result. The proof is not presented here due to the tedious albeit standard arguments.

Theorem 3.2. *The value function is a constrained viscosity solution on $D_1 \times [0,T)$ of the Variational Inequality*

$$\min\left[\mathscr{L}J - \bar{\mathscr{L}}J, \beta\frac{\partial J}{\partial x} - \frac{\partial J}{\partial y}, -\alpha\frac{\partial J}{\partial x} + \frac{\partial J}{\partial y}\right] = 0 \tag{3.17}$$

with

$$J(x,y,S,T) = V(x - g(S), y) \tag{3.18}$$

where the operator $\mathscr{L}$ is given in (3.11) and the operator $\bar{\mathscr{L}}$ is

$$\bar{\mathscr{L}}J = \frac{\partial J}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 J}{\partial S^2} + \sigma^2 yS\frac{\partial^2 J}{\partial y \partial S} + \mu S\frac{\partial J}{\partial S}. \tag{3.19}$$

Moreover, J is the unique constrained viscosity solution of (3.17) in the class of uniformly continuous and concave functions, with respect to the state variables (x,y,S).

Consider now the writer with endowment $(x, y) \in D$ at time t before writing the claim. If the writer chooses to write the claim at price C, the endowment becomes $(x + C, y)$ and by Proposition 3.1, the price C must be such that $(x + C, y, S) \in D_1$. In the case of *zero-transaction costs*, the function $C = C(S, t)$ is determined as the price that makes the writer *indifferent* between writing the claim or refraining from writing it, i.e.

$$V(x, y) = J(x + C(S, t), y, S, t) .$$

In the special case $g(S) = (S - K)^+$, one can show that $C(S, t)$ is the Black and Scholes price which is of course independent of the current portfolio holdings (x, y) and the utility function. Moreover, because of the absence of transaction costs, perfect replication is possible and the constraints (3.16) are not binding.

In the case of *non-zero transaction costs*, the above equality *is not feasible for all* $(x, y, S) \in D_1$, if C is allowed to depend *only* on (S, t). This motivates the following definitions.

Definition 3.1. *The reservation write price* $C(x, y, S, t)$*, for initial endowment* (x, y)*, is defined as the minimum value at which the investor is willing to write the claim. Therefore,* C *satisfies for* $(x + C(x, y, S, t), y, S) \in D_1$

$$V(x, y) = J(x + C(x, y, S, t), y, S, t) . \tag{3.20}$$

Definition 3.2. *The write price* $\overline{C}(S, t)$ *is defined as the maximum of reservation write prices across all admissible states* (x, y, S)*. Therefore,* $\overline{C}$ *satisfies*

$$V(x, y) \leq J(x + \overline{C}(S, t), y, S, t) . \tag{3.21}$$

Inequality (3.21) guarantees that the writer will be willing to write the option at any price higher than $\overline{C}(S, t)$, independently of his current portfolio position.

Our goal is to derive an upper bound $h = h(S, t)$ for the write price; the upper bound will satisfy (3.21) on D_1. The construction and characterization of the upper bound is worked out in the next section.

4. Bounds on prices of contingent claims

In this section we derive *analytic* bounds for the write price of a European-type contingent claim. The underlying idea is to construct suitable subsolutions of the Bellman equations (3.10) and (3.17) in order to use a comparison result to establish inequality (3.21). The main difficulty stems from the fact that the value functions V and J are defined on different domains and that there are no explicit solutions of the two associated free-boundary problems (3.7) and (3.12).

We start with a formal discussion in order to motivate the construction of the analytic bound. To ease the presentation, we recall that the value functions V and J solve, respectively, (3.10) in $D = \left\{(x, y) : x + \binom{\alpha}{\beta} y \geq 0, \right\}$ and (3.17) in $D_1 \times [0, T]$, where $D_1 = \left\{(x, y, S) : x + \binom{\alpha}{\beta}\left(y - \frac{S}{\alpha}\right) \geq 0,\ S \geq 0\right\}$. The domains D and D_1 are illustrated in Figs. 2 and 3.

The goal is to construct a function $h = h(S, t)$, *independent* of (x, y), such that, for $(x + h, y, S) \in D_1$

$$V(x, y) \leq J(x + h(S, t), y, S, t) \tag{4.1}$$

Using the suboptimality inequality

$$J(x + h(S, t), y, S, t) \geq J\left(x, y + \frac{h(S, t)}{\beta}, S, t\right) \tag{4.2}$$

and a simple transformation, we observe that (4.1) follows if we find an h such that

$$V\left(x, y - \frac{h(S, t)}{\beta}\right) \leq J(x, y, S, t) \tag{4.3}$$

for $(x, y, S) \in D_1$.

We start with a formal construction of a candidate solution and then we establish its existence and validity. The underlying idea in the choice of the candidate bound is first to find a price that satisfies (4.3) in the case that $(x, y, S) \in \partial D_1$, i.e. when the writer holds the *minimal allowed position* which amounts to the value of one stock share, taking into account the transaction costs. We then need to show that this price works for all wealth levels greater than the minimal one.

To this end, we start with the following lemma which gives us information about the value function J on $\partial D_1 = \left\{(x, y, S) : x + \binom{\alpha}{\beta}\left(y - \frac{S}{\alpha}\right) = 0,\ S \geq 0\right\}$.

Lemma 4.1. *For $(x, y, S) \in \partial D_1$, the value function J is given by*

$$J(x, y, S, t) = E\left[e^{-\rho(T-t)} V\left(-g(S_T), \frac{S_T}{\alpha}\right) \,\middle|\, S_t = S\right]. \tag{4.4}$$

Proof. The proof follows directly from the fact that the only admissible policy for the boundary points (x, y, S) is to move instantaneously at time t to the point $(0, \frac{S}{\alpha}, S)$ and remain there until time T.

The next result will give us the main ingredient for the construction of the candidate solution.

Lemma 4.2. *If $h_{\hat\rho} = h_{\hat\rho}(S, t)$ is such that*

$$V\left(\frac{\beta}{\alpha}S, -\frac{h_{\hat\rho}}{\beta}\right) = E\left[e^{-\hat\rho(T-t)} V(S_T - g(S_T), 0) \mid S_t = S\right]. \tag{4.5}$$

with $0 \leq h_{\hat\rho} \leq \frac{\beta}{\alpha}S$ and $\hat\rho \geq \rho$, then (4.3) holds for $(x, y, S) \in \partial D_1$.

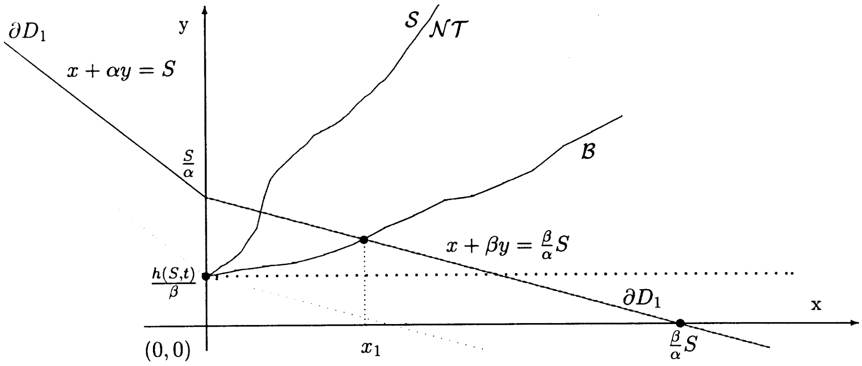

Fig. 3.

Proof. We first observe that, by suboptimality, for $\hat{\rho} \geq \rho$,

$$E\left[e^{-\rho(T-t)}V\left(-g(S_T), \frac{S_T}{\alpha}\right) \,\Big|\, S_t = S\right] \geq E\left[e^{-\hat{\rho}(T-t)}V(S_T - g(S_T), 0) \mid S_t = S\right].$$

Therefore, for $(x, y, S) \in \partial D_1$, (4.4) yields

$$J(x, y, S, t) \geq V\left(\frac{\beta}{\alpha}S, -\frac{h_{\hat{\rho}}}{\beta}\right). \tag{4.6}$$

We next claim that for $(x, y, S) \in \partial D_1$

$$V\left(x, y - \frac{h_{\hat{\rho}}(S,t)}{\beta}\right) \leq V\left(\frac{\beta}{\alpha}S, -\frac{h_{\hat{\rho}}(S,t)}{\beta}\right). \tag{4.7}$$

The above inequality follows easily once we understand the monotonicity of $V\left(x, y - \frac{h_{\hat{\rho}}}{\beta}\right)$ along the set ∂D_1. We refer the reader to Fig. 3 to facilitate the exposition; we also eliminate the $\hat{\rho}$-notation for convenience. To this end, first observe that by Assumption 3.1 the $\mathscr{NT}$ region is a subset of the first quadrant (appropriately translated for $x \geq 0$, $y \geq \frac{h(S,t)}{\beta}$) and, by assumption, $0 \leq \frac{h}{\beta} \leq \frac{S}{\alpha}$. Therefore, $V\left(x, y - \frac{h(S,t)}{\beta}\right)$ is constant along the line segment $x + \alpha y = S$ for $x \leq 0$, stays non-decreasing on $x + \beta y = \frac{\beta}{\alpha}S$ for $0 \leq x \leq x_1$ and remains constant afterwards. Inequality (4.7) follows from the following relations:

(i) $V\left(x, y - \frac{h(S,t)}{\beta}\right) = V\left(0, \frac{S}{\alpha} - \frac{h(S,t)}{\beta}\right)$ on $\{x + \alpha y = S,\ x \leq 0\}$,

(ii) $V\left(0, \frac{S}{\alpha} - \frac{h(S,t)}{\beta}\right) \leq V\left(x_1, \frac{S}{\alpha} - \frac{x_1}{\beta} - \frac{h(S,t)}{\beta}\right)$ on $\{x + \beta y = \frac{\beta}{\alpha}S,\ 0 \leq x \leq x_1\}$,

(iii) $V\left(x, y - \frac{h(S,t)}{\beta}\right) = V\left(\frac{\beta}{\alpha}S, -\frac{h(S,t)}{\beta}\right)$ on $\{x + \beta y = \frac{\beta}{\alpha}S,\ x \geq x_1\}$.

We next observe that the points $\left(\frac{\beta}{\alpha}S, 0\right)$ and $(S - g(S), 0)$ belong to the $\mathscr{B}$ region (see Fig. 3) because of the standing Assumption 3.1. But since in the $\mathscr{B}$

region the value function V satisfies $\beta V_x = V_y$, there exists a function G such that, for $(x_0, y_0) \in \mathscr{B}$, $V(x_0, y_0) = G(x_0 + \beta y_0)$. Therefore,

$$V\left(\frac{\beta}{\alpha}S, -\frac{h(S,t)}{\beta}\right) = G\left(\frac{\beta}{\alpha}S - h(S,t)\right) \tag{4.8}$$

and

$$V(S_T - g(S_T), 0) = G(S_T - g(S_T)) \tag{4.9}$$

Combining the above equalities and (4.5) yields

$$G\left(\frac{\beta}{\alpha}S - h(S,t)\right) = E\left[e^{-\hat{\rho}(T-t)}G(S_T - g(S_T)) \mid S_t = S\right].$$

It follows easily from the monotonicity properties of the value function V (see, for example, Tourin and Zariphopoulou 1994) that G is strictly increasing and therefore invertible. Hence the function h is well defined and

$$h(S,t) = \frac{\beta}{\alpha}S - G^{-1}\left(E\left[e^{-\hat{\rho}(T-t)}G(S_T - g(S_T)) \mid S_t = S\right]\right). \tag{4.10}$$

Our goal now is to show that the above function is a candidate upper bound for the write price. We start with some elementary properties of h.

Proposition 4.1. *The function h satisfies*
(i) $0 \le h(S,t) \le \frac{\beta}{\alpha}S$, *and (ii)* $h_s \le \frac{\beta}{\alpha}$. *(4.11)*

Proof. i) The fact that $h(S,t) \le \frac{\beta}{\alpha}S$ follows from the definition of h and the fact that G is continuous with $G(0) = 0$ and $G \ge 0$. To show that $h(S,t) \ge 0$, we first observe that

$$h(0,t) = 0 \quad \text{for} \quad 0 \le t \le T, \quad \text{and} \quad h(S,T) = \frac{\beta - \alpha}{\alpha}S + g(S) \ge 0. \tag{4.12}$$

We next claim that

$$\mu h - h_t - \frac{1}{2}\sigma^2 S^2 h_{ss} - \mu S h_s \ge 0. \tag{4.13}$$

To this end define

$$f(S,t) = E[e^{-\hat{\rho}(T-t)}G(S_T - g(S_T)) \mid S_t = S]. \tag{4.14}$$

The Feynman-Kac formula implies that f solves the terminal-value problem

$$\begin{cases} \hat{\rho}f = f_t + \frac{1}{2}\sigma^2 S^2 f_{ss} + \mu S f_s \\ f(S,t) = G(S - g(S)). \end{cases} \tag{4.15}$$

Using that

$$h(S,t) = \frac{\beta}{\alpha}S - G^{-1}(f(S,t)) \tag{4.16}$$

yields

$$\mu h - h_t - \frac{1}{2}\sigma^2 S^2 h_{ss} - \mu S h_s$$
$$\geq \quad (G^{-1}(f))'\left[f_t + \frac{1}{2}\sigma^2 S^2 f_{ss} + \mu S f_s - \hat{\rho} f\right] + [(G^{-1}(f))'\hat{\rho} f - \mu G^{-1}(f)] \quad (4.17)$$

where we used that $\hat{\rho} \geq \rho \geq \mu$ and the convexity of G^{-1}. Note that the latter property follows from the fact that V is concave and $V(x,y) = G(x + \beta y)$ for $(x,y) \in \mathscr{B}$. Moreover the fact that $G^{-1}(0) = 0$ and the convexity and monotonicity of G^{-1} yield $f(G^{-1}(f))' \geq G^{-1}(f)$, which combined with (4.15) gives (4.13). Inequality (4.11) then follows from classical results from the theory of linear parabolic differential equations.

ii) It follows from (4.16) that we have $h_s = \frac{\beta}{\alpha} - (G^{-1}(f))' f_s(S,t)$. To conclude, it suffices to show that $(G^{-1}(f))' \geq 0$ and that $f_s(S,t) \geq 0$. The first inequality follows from the fact that G is nondecreasing. Given that f solves the linear parabolic equation (4.15), to show that $f_s \geq 0$, it suffices to establish that $f_s(S,T) \geq 0$. Since $f_s(S,T) = (1 - g_s(S))G'(S - g(S))$, we may conclude easily using the properties of g.

Proposition 4.2. *Let $\hat{\rho}$ be a discount factor in (4.10) given by*

$$\hat{\rho} = \max\left[\rho, \mu + \frac{m\sigma^2(1-\beta A)}{2\beta A}\right] \quad (4.18)$$

where m is a constant given in Lemma 4.3 below. Then, if the candidate price h, as in (4.10) with $\hat{\rho}$ as above, satisfies $Sh_s - h \geq 0$ for $S \geq 0$, the function $F : D_1 \times [0,T] \to [0,+\infty)$ given by

$$F(x,y,S,t) = V\left(x, y - \frac{h(S,t)}{\beta}\right) \quad (4.19)$$

is a viscosity subsolution of the HJB equation (3.17).

Proof. The main ingredients of the proof are the fact that V solves (3.10) and the special choice of h.

Below, we first show the above under the assumption that the function V has all the necessary derivatives. It should be noted here that the strength as well as the beauty of viscosity solutions is that they eventually reduce all the calculations to the ones in the case of smooth solutions. After we show the claim under the regularity assumptions on V, we will indicate briefly how it can be relaxed.

First, we observe that if $S = 0$ in (4.10), then $h(S,t) = 0$ and hence $F(x,y,0,t) = V(x,y)$ and the assertion follows immediately. We next consider the case $S > 0$. By inspecting the two (HJB) equations, (3.10) and (3.17), and using the fact that V solves (3.10), we see that it suffices to establish

$$\rho F \leq F_t + \frac{1}{2}\sigma^2 S^2 F_{ss} + \sigma^2 S y F_{sy} + \frac{1}{2}\sigma^2 y^2 F_{yy} + \mu S F_s + \mu y F_y + H(F_x) + r x F_x \quad (4.20)$$

with $H(p) = \max_{c\geq 0}\{-cp + U(c)\}$, only if the point $(x, y - \frac{h}{\beta})$ belongs to the $\mathscr{NT}$ region, i.e. when

$$\begin{aligned}\rho V\left(x, y - \frac{h}{\beta}\right) &= \frac{1}{2}\sigma^2\left(y - \frac{h}{\beta}\right)^2 V_{yy}\left(x, y - \frac{h}{\beta}\right) \\ &+\mu\left(y - \frac{h}{\beta}\right) V_y\left(x, y - \frac{h}{\beta}\right) \\ &+H\left(V_x\left(x, y - \frac{h}{\beta}\right)\right) + rxV_x\left(x, y - \frac{h}{\beta}\right). \quad (4.21)\end{aligned}$$

Using the definition of F, inequality (4.20) becomes

$$\begin{aligned}\rho V \leq\ & -\frac{h_t}{\beta}V_y + \frac{1}{2}\sigma^2 S^2(-\frac{h_{ss}}{\beta}V_y + \frac{h_s^2}{\beta^2}V_{yy}) - \frac{\sigma^2}{\beta}ySh_s V_{yy} \\ & \frac{1}{2}\sigma^2 y^2 V_{yy} - \mu S\frac{h_s}{\beta}V_y + \mu y V_y + H(V_x) + rxV_x\end{aligned}$$

where all the above derivatives of V are evaluated at the point $(x, y - \frac{h}{\beta})$. Using (4.21) and rearranging terms the above inequality reduces to

$$\frac{V_y}{\beta}(\mu h - h_t - \frac{1}{2}\sigma^2 S^2 h_{ss} - \mu S h_s) + \frac{\sigma^2}{2\beta}V_{yy}[(Sh_s - h)\cdot(\frac{Sh_s + h}{\beta} - 2y)] \geq 0 \quad (4.22)$$

We now look at the following cases:

Case i: $y \geq \dfrac{S}{\alpha}$. Because $V_y \geq 0$, (4.13) yields that the first term in (4.22) is nonnegative. Since V is concave, to show that the second term is nonnegative as well, it suffices to prove that

$$(Sh_s - h)\Big(\frac{Sh_s + h}{\beta} - 2y\Big) \leq 0. \quad (4.23)$$

Observe that from (4.11) we have

$$Sh_s + h \leq \frac{2\beta}{\alpha}S$$

and, in turn,

$$Sh_s + h \leq 2\beta y$$

where we used that $y \geq \frac{S}{\alpha}$. Inequality (4.23) then follows from the above inequality and the assumption that $Sh_s - h \geq 0$.

Case ii: $\dfrac{h}{\beta} \leq y < \dfrac{S}{\alpha}$. Note that the reason we do not look at the case $\beta y < h$ is because we only need to establish (4.22) when the point $(x, y - \frac{h}{\beta})$ belongs to the region $\mathscr{NT}$ which has been assumed to be a subset of the first quadrant. As a matter of fact, recall that the region $\mathscr{NT}$ is assumed to belong to the cone $\left\{(x, y) : Ax \leq y - \frac{h}{\beta} \leq Bx,\ x \geq 0\right\}$ (see Fig. 3).

First observe that if $Sh_s = h$, then (4.22) follows directly. Next we assume that $Sh_s > h$.

If $\beta A \geq 1$, we claim that

$$\frac{Sh_s + h}{\beta} \leq 2y \tag{4.24}$$

We argue by contradiction. Suppose that the opposite inequality holds; then,

$$\frac{Sh_s + h}{\beta} > 2y \geq \frac{2}{1+\beta A}\Big(A\frac{\beta}{\alpha}S + \frac{h}{\beta}\Big) \tag{4.25}$$

where the right inequality follows from the fact that $x + \beta y \geq \frac{\beta}{\alpha}S$ and $y - \frac{h}{\beta} \geq Ax$. Combining inequality (4.25) with (4.11), yields $(\beta A - 1)h \geq (\beta A - 1)Sh_s$ which contradicts the assumption that $Sh_s > h$. The rest of the arguments follow as in Case i.

If $\beta A < 1$ and $Sh_s + h \leq 2\beta y$, we can argue as in Case i. It remains to establish (4.22) when $Sh_s + h > 2\beta y$.

To this end, we claim the following result which is proved in Appendix B.

Lemma 4.3. *The value function* V *satisfies, for* $(x, y) \in \mathcal{NT}$,

$$yV_{yy}(x, y) \geq -mV_y(x, y), \tag{4.26}$$

where m *is a constant depending on the market parameters and is given by*

$$m = \Big[\Big(\frac{\lambda_2\big(\beta + \frac{1}{A}\big)}{\lambda_1\big(\alpha + \frac{1}{B}\big)^{\gamma}}\Big)^{\frac{1}{1-\gamma}}\frac{\rho - r\gamma}{\alpha\gamma} + \Big(\mu + \frac{r}{\alpha A}\Big) - \frac{\rho}{2}\Big(1 + \frac{1}{\beta B}\Big)\Big]\sigma^{-2}.$$

We proceed now with the proof of the proposition. Observing first that (4.16), (4.17) and fact that $\widehat{\rho} \geq \mu$ yield

$$\mu h - h_t - \frac{1}{2}\sigma^2 S^2 h_{ss} - \mu Sh_s \geq (\widehat{\rho} - \mu)\Big(\frac{\beta}{\alpha}S - h\Big) \geq 0.$$

Combining this last inequality and (4.26) – applied to the point $\Big(x, y - \frac{h}{\beta}\Big)$ – reduces the desired inequality (4.22) to the inequality

$$\frac{V_y}{y - \frac{h}{\beta}}\left[(\hat{\rho} - \mu)(\frac{\beta}{\alpha}S - h)(y - \frac{h}{\beta}) - \frac{1}{2}\sigma^2 m(Sh_s - h)(\frac{Sh_s + h}{\beta} - 2y)\right] \geq 0. \tag{4.27}$$

Next observe that the constraint $x \geq \frac{\beta}{\alpha}S - \beta y$ and the assumption that $\Big(x, y - \frac{h}{\beta}\Big) \in \mathcal{NT}$ yield

$$y - \frac{h}{\beta} \geq \frac{\beta A}{\alpha(1+\beta A)}(S - \frac{\alpha}{\beta}h), \tag{4.28}$$

and in turn

$$\frac{Sh_s + h}{\beta} - 2y \le -\frac{2\beta A}{\alpha(1+\beta A)}(S - \frac{\alpha}{\beta}h) + \frac{Sh_s - h}{\beta}.$$

Multiplying by $-\frac{1}{2}\sigma^2 m(Sh_s - h)$ and using (4.11) gives

$$\begin{aligned}
&-\frac{1}{2}\sigma^2 m(Sh_s - h)(\frac{Sh_s + h}{\beta} - 2y) \\
&\ge -\frac{1}{2}\sigma^2 m\frac{(Sh_s - h)^2}{\beta} + \sigma^2 m\frac{\beta A}{\alpha(1+\beta A)}(S - \frac{\alpha}{\beta}h)(Sh_s - h) \\
&\ge -\frac{1}{2}\sigma^2 m\frac{(Sh_s - h)^2}{\beta} + \sigma^2 m\frac{A}{1+\beta A}(Sh_s - h)^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{V_y}{y - \frac{h}{\beta}}\big[(\hat{\rho} - \mu)(\frac{\beta}{\alpha}S - h)\frac{\beta A}{\alpha(1+\beta A)}(S - \frac{\alpha}{\beta}h) \\
&-\frac{1}{2}\sigma^2 m\frac{(Sh_s - h)^2}{\beta} + \sigma^2 m\frac{A}{1+\beta A}(Sh_s - h)^2\big] \\
&\ge \frac{V_y}{y - \frac{h}{\beta}}\big[(\hat{\rho} - \mu)\frac{A}{1+\beta A} - \frac{\sigma^2 m}{2\beta} + \frac{\sigma^2 mA}{1+\beta A}\big](Sh_s - h)^2 \ge 0
\end{aligned}$$

which holds if $\hat{\rho} \ge \mu + \frac{m\sigma^2(1-\beta A)}{2\beta A}$.

The rigorous argument amounts to repeating the proof of the subsolution property for the function $F^\theta(x, y, S, t) = V^\theta\left(x, y - \frac{h(S,t)}{\beta}\right)$, where V^θ is the classical sup-convolution regularization of V given by

$$V^\theta(x, y) = \sup_{\substack{\bar{x}\to x \\ \bar{y}\to y}}\left\{V(\bar{x}, \bar{y}) - \frac{|x - \bar{x}|^2}{2\theta} - \frac{|y - \bar{y}|^2}{2\theta}\right\}.$$

It follows that $V^\theta \to V$, locally uniformly as $\theta \to 0$. Moreover, the regularity properties of V^θ together with the arguments used in the comparison principle, allow us to work with points $(\bar{x}, \bar{y})$ at which $D^2_y V^\theta$, and therefore $D^2_{y,y}F^\theta$, $D^2_{y,s}F^\theta$ and $D^2_{s,s}F^\theta$ exist. Finally, it can be shown that the function F^θ is a subsolution of the appropriately modified equation for V. All the above can be made precise, but the arguments, which are tedious and also routine in the theory of viscosity solutions are rather long, and beyond the interests of the readership.

Remark 4.1. If $\beta A > 1$, it suffices to choose $\widehat{\rho} = \rho$ in view of Proposition 4.2 and the assumption that $\rho \ge \mu$.

Remark 4.2. The assumption that $A > 0$ was motivated by the fact that, as $A \to 0$, we have $\lim_{A\to 0}\frac{m\sigma^2(1-\beta A)}{2\beta A} = \infty$. This yields $\widehat{\rho} = +\infty$, in which case $h_{\widehat{\rho}}$ degenerates to the trivial upper bound $\frac{\beta}{\alpha}S$.

Before we present the main result we introduce the following assumption.

Assumption 4.1. *The function h given by (4.10), where $\widehat{\rho}$ is defined as in (4.18), satisfies $Sh_s - h \geq 0$.*

Remark 4.3. The above assumption on the candidate solution h played an important role in proving Proposition 4.2. Although we were not able to remove this assumption, we were able to provide examples in which h has the desired properties. These examples are presented after the main theorem.

Theorem 4.1. *Let h be given by (4.10), i.e.*

$$h(S,t) = \frac{\beta}{\alpha}S - G^{-1}\left(E\left[e^{-\widehat{\rho}(T-t)}G(S_T - g(S_T)) \mid S_t = S\right]\right)$$

where $\widehat{\rho}$ is defined in (4.18). Under Assumptions 3.1 and 4.1, the function $h(S,t)$ is an upper bound to the reservation write price.

Proof. In order to prove the theorem, it suffices to show that inequality (4.3) holds on $D_1 \times [0,T]$, i.e. that $V\left(x, y - \frac{h(S,t)}{\beta}\right) \leq J(x,y,S,t)$.

First, we observe that at $t = T$ the above inequality holds. In fact, from (3.18) and (4.10), the above inequality reduces to

$$V\left(x, y - \frac{\beta-\alpha}{\alpha\beta}S + \frac{g(S)}{\beta}\right) \leq V(x - g(S), y),$$

which holds by suboptimality, the monotonicity of V and the fact that $g \geq 0$.

Next, recall that by the special choice of h, (see Lemma 4.2), the desired inequality holds for $(x,y,S) \in \partial D_1$. Moreover, from Proposition 4.2 we have that the function $V\left(x, y - \frac{h(S,t)}{\beta}\right)$ is a viscosity subsolution of the (HJB) equation (3.17) whose unique solution is the value function J. Finally, routine arguments can be used to show that the rest of the conditions for the comparison results for solutions of (3.17) hold. We therefore conclude that the subsolution $V(x, y - \frac{h(S,t)}{\beta})$ is dominated by the solution $J(x,y,S,t)$ and the validity of the price bound h is established.

Remark 4.4. Note that when the utility function is of the form $U(c) = \frac{1}{\gamma}c^\gamma$ with $0 < \gamma < 1$, the value function V turns out to be homogeneous of degree γ. Many steps of the main proofs can then be considerably simplified; see Constantinides and Zariphopoulou (1997) for details and numerous illustrations.

We conclude this section by presenting some examples of value functions that satisfy Assumption 4.1.

Example 1. $G(z) = kz^\gamma$ for some constant $k > 0$.

This is the case when $U(c) = \frac{1}{\gamma}c^\gamma$ with $0 < \gamma < 1$ (see Davis and Norman 1987). Then,

$$Sh_s - h = G^{-1}(f) - Sf_s(G^{-1}(f))' = k^{-\frac{1}{\gamma}} \frac{f^{\frac{1-\gamma}{\gamma}}}{\gamma}[\gamma f - Sf_s].$$

Direct calculations show that the function $v = \gamma f - Sf_s$ solves the same linear equation as f. Therefore, in order to show that $Sh_s - h \geq 0$, or equivalently, that $v = \gamma f - Sf_s \geq 0$, it suffices, using classical results from the theory of linear parabolic equations, to show, that $v(S, T) \geq 0$. (Note also that $v(0, t) = 0$.) Indeed, at $t = T$, $\gamma f - Sf_s = \gamma k(S - g(S))^{\gamma-1}[Sg_s - g(S)] \geq 0$ which follows from the properties of the payoff function g.

Example 2. The function G has nondecreasing relative risk aversion coefficient, i.e. $\left(-\frac{zG''}{G'}\right)' \geq 0$.

Differentiating (4.16) yields $Sh_s - h = (G^{-1})'(f)\left[\frac{G^{-1}(f)}{(G^{-1})'(f)} - Sf_s\right]$. Let $K(f) = \frac{G^{-1}(f)}{(G^{-1})'(f)}$. Since G^{-1} is increasing, we have $\operatorname{sgn}(Sh_s - h) = \operatorname{sgn}[K(f) - Sf_s]$. Therefore, we need to show that

$$w(S, t) = K(f(S, t)) - Sf_s(S, t) \geq 0. \tag{4.29}$$

Straightforward calculations and equation (4.15) yield that w solves

$$\hat{\rho}w - w_t - \frac{1}{2}\sigma^2 S^2 w_{ss} - \mu S w_s = \hat{\rho}(K(f) - fK'(f)) - \frac{1}{2}\sigma^2 S^2 f_s^2 K''(f). \tag{4.30}$$

It follows easily that $w(S, T) \geq 0$ and $w(0, t) = 0$. Therefore for inequality (4.29) to hold, it suffices to find conditions which will ensure that

$$\hat{\rho}w - w_t - \tfrac{1}{2}\sigma^2 S^2 w_{ss} - \mu S w_s \geq 0. \tag{4.31}$$

To this end, observe that $K \geq 0$ and , if $z = G^{-1}(f)$, then $K(G(z)) = zG'(z)$. Differentiating once more we get

$K'(G(z)) = 1 + \frac{zG''(z)}{G'(z)}$ and $G'(z)K''(G(z)) = \left(\frac{zG''(z)}{G'(z)}\right)'$.

Given that G is increasing and concave, the above equality yields that K is concave, if G has nondecreasing relative risk aversion coefficient. Moreover, the utility function satisfies $U(0) = 0$ which in turn implies that $G(0) = 0$. The latter together with the concavity of G imply that $\lim_{z\to 0} zG'(z) = 0$. It follows then easily that $K(0) = 0$ which, together with the concavity of K, hence, yields $K(f) - fK'(f) \geq 0$ and (4.31).

Remark 4.5. Although Example 1 comes from the case of power utilities, $U(c) = \gamma^{-1}c^{\gamma}$, it is not clear what class of utility functions generate value functions with nondecreasing relative risk aversion coefficient like the ones discussed in Example 2. Despite the fact that, in the absence of transaction costs, such characteristics are inherited from the utility functions to the value functions, it remains an open and interesting question to study the case when transaction costs are paid. As a matter of fact, such questions are rather challenging on their own right independently of the problem of derivative pricing with transaction costs.

Appendix A

The notion of *viscosity solutions* was introduced by Crandall and Lions (1983) for first-order equations, and by Lions (1983) for second-order equations. For a general overview of the theory we refer to the *User's Guide* by Crandall, Ishii and Lions (1992) and the book by Fleming and Soner (1993). Next, we recall the notion of *constrained viscosity solutions* which was introduced by Soner (1986) and Capuzzo-Dolcetta and Lions (1987) for first-order equations (see also Ishii and Lions 1990). To this end, we consider a nonlinear second order partial differential equation of the form

$$F(X, V, DV, D^2V) = 0 \quad \text{in} \quad \Omega \times [0, T] \tag{A.1}$$

where $\Omega \subseteq \mathbb{R}^2$, DV and D^2V denote the gradient vector and the second derivative matrix of V, and the function F is continuous in all its arguments and degenerate elliptic, meaning that

$$F(X, p, q, A + B) \leq F(X, p, q, A) \ \text{ if } B \geq 0. \tag{A.2}$$

Definition A.1. *A continuous function $V : \overline{\Omega} \times [0, T] \to \mathbb{R}$ is a constrained viscosity solution of (A.1) if the following two conditions hold:*

i) V is a viscosity subsolution of (A.1) on $\overline{\Omega} \times [0, T]$; that is, if for any $\phi \in C^{1,2}(\overline{\Omega} \times [0, T])$ and any local maximum point $X_0 \in \overline{\Omega} \times [0, T]$ of $V - \phi$,

$$F(X_0, V(X_0), D\phi(X_0), D^2\phi(X_0)) \leq 0\,.$$

ii) V is a viscosity supersolution of (A.1) in $\Omega \times [0, T]$; that is, if for any $\phi \in C^{1,2}(\Omega \times [0, T])$ and any local minimum point $X_0 \in \Omega \times [0, T]$ of $V - \phi$,

$$F(X_0, V(X_0), D\phi(X_0), D^2\phi(X_0)) \geq 0.$$

Appendix B

Proof of Lemma 4.3.
We first recall that in the region $\mathscr{NT}$ region, the value function satisfies

$$\rho V = \frac{1}{2}\sigma^2 y^2 V_{yy} + \mu y V_y + r x V_x + \max_{c \geq 0}\{-cV_x + U(c)\}. \tag{B.1}$$

The monotonicity and concavity of the value function V yields that $2V \geq xV_x + yV_y$; this combined with

$$\alpha V_x \leq V_y \leq \beta V_x \tag{B.2}$$

and the inequality (valid by Assumption 3.1)

$$Ax \leq y \leq Bx \tag{B.3}$$

implies

$$2V \geq yV_y\left(1+\frac{1}{\beta B}\right). \tag{B.4}$$

Combining (B.1), (B.2), (B.3) and (B.4) yields

$$\frac{1}{2}\sigma^2 y^2 V_{yy} + \max_{c\geq 0}\{-cV_x + U(c)\} \geq yV_y\left[\frac{\rho}{2}\left(1+\frac{1}{\beta B}\right) - \left(\mu + \frac{r}{\alpha A}\right)\right]. \tag{B.5}$$

We next claim that there exist a constant δ such that

$$\max_{c\geq 0}\{-cV_x + U(c)\} \leq \delta y V_y. \tag{B.6}$$

In fact, because $U(c) \leq \lambda_2 \frac{c^\gamma}{\gamma}$ for some $\lambda_2 > 0$ and $0 < \gamma < 1$ and (B.2), we get

$$\max_{c\geq 0}\{-cV_x + U(c)\} \leq \frac{1-\gamma}{\gamma}\lambda_2^{\frac{1}{1-\gamma}}\frac{V_y}{\alpha}V_x^{\frac{1}{\gamma-1}}. \tag{B.7}$$

By the monotonicity of $\frac{U(c)}{c^\gamma}$ we have, for $\lambda > 1$, $U(\lambda c) \geq \lambda^\gamma U(c)$; this together with the linearity of the state dynamics x_t and y_t, gives

$$V(\lambda x, \lambda y) \geq \lambda^\gamma V(x, y). \tag{B.8}$$

Next, using the above inequality and following similar arguments as in Section 3 of Shreve and Soner (1994) we get

$$V_x(x, y) \geq \frac{\gamma}{x+\beta y}V(x, y) \tag{B.9}$$

and

$$V(x, y) \geq \lambda_1\left[\frac{1}{\gamma}\frac{(\rho-\gamma r)^{\gamma-1}}{1-\gamma}(x+\alpha y)^\gamma\right] \tag{B.10}$$

Combining (B.9) and (B.10) yields

$$V_x^{\frac{1}{\gamma-1}} \leq \lambda_1^{\frac{1}{\gamma-1}}\frac{\rho-\gamma r}{1-\gamma}\left[\frac{(\alpha+\frac{1}{B})^\gamma}{\beta+\frac{1}{A}}\right]^{\frac{1}{\gamma-1}}$$

and, from (B.7), $\max_{c\geq 0}\{-cV_x + U(c)\} \leq \delta y V_y$ for

$$\delta = \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1}{1-\gamma}}\frac{\rho-\gamma r}{\alpha\gamma}\left[\frac{(\alpha+\frac{1}{B})^\gamma}{\beta+\frac{1}{A}}\right]^{\frac{1}{\gamma-1}}. \tag{B.10}$$

Finally, (B.5) and the above inequality yield $yV_{yy} \geq -mV_y$ for

$$m = \frac{\left[\delta + \left(\mu+\frac{r}{\alpha A}\right) - \frac{\rho}{2}\left(1+\frac{1}{\beta B}\right)\right]}{\sigma^2}.$$

References

1. Avellaneda, M., Paras, A.: Optimal hedging portfolios for derivative securities in the presence of large transaction costs. Applied Mathematical Finance **1**, 165–193 (1994)
2. Barles G., Soner, H. M.: Option pricing with transaction costs and a non-linear Black-Scholes equation. Finance and Stochastics **2**, 369–397 (1998)
3. Bensaid, B., Lesne, J.-P., Pages, H., Scheinkman, J.: Derivative asset pricing with transaction costs. Mathematical Finance **2**, 63–86 (1992)
4. Black, F., Scholes, M. S.: The pricing of options and corporate liabilities. Journal of Political Economy **81**, 637–654 (1973)
5. Boyle, P. P., Vorst, T.: Option replication in discrete time with transaction costs. Journal of Finance **47**, 271–293 (1992)
6. Capuzzo-Dolcetta, I., Lions, P.-L.: Viscosity solutions of Hamilton-Jacobi equations. Transactions of the American Mathematical Society (to appear)
7. Constantinides, G. M.: Multiperiod consumption and investment behavior with convex transactions costs. Management Science **25**, 1127–1137 (1979)
8. Constantinides, G. M.: Option pricing bounds based on stochastic dominance. Working paper, University of Chicago (1997)
9. Constantinides, G. M., Zariphopoulou, T.: Bounds on option prices in an intertemporal setting with proportional transaction costs and multiple securities. CRSP working paper, No. 442, University of Chicago (1997)
10. Crandall, M., Lions, P.-L.: Viscosity solutions of Hamilton-Jacobi equations. Transactions of the American Mathematical Society **277**, 1–42 (1983)
11. Crandall, M., Ishii, H., Lions, P.-L.: User's guide to viscosity solutions of second order partial differential equations. Bulletin of the American Mathematical Society **27**, 1–67 (1992)
12. Davis, M. H. A., Norman, A. R.: Portfolio selection with transaction costs. Mathematics of Operations Research **15**, 676–713 (1990)
13. Davis, M. H. A., Clark, J. M. C.: Analysis of financial models including transaction costs. Royal Society Discussion Meeting on Mathematical Models in Finance, London, November 10–11 (1993)
14. Davis, M. H. A., Panas, V. G., Zariphopoulou, T.: European option pricing with transaction costs. SIAM Journal of Control and Optimization **31**, 470–493 (1993)
15. Edirisinghe, C., Naik, V., Uppal, R.: Optimal replication of options with transactions costs and trading restrictions. Journal of Financial and Quantitative Analysis **28**, 117–138 (1993)
16. Figlewski, S.: Options arbitrage in imperfect markets. Journal of Finance **44**, 1289–1311 (1989)
17. Fleming, W., Soner, H. M.: Controlled Markov processes and viscosity solutions. Berlin Heidelberg New York: Springer 1993
18. Flesaker, B., Hughston, L. P.: Contingent claim replication in continuous time with transaction costs. Working paper, Merrill Lynch International (1994)
19. Grannan, E. R., Swindle, G. H.: Minimizing transaction costs of option hedging strategies. Mathematical Finance **6**, 341–364 (1996)
20. Henrotte, P.: Transactions costs and duplication strategies. Working paper, Stanford University (1993)
21. Hodges, S. D., Neuberger, A.: Optimal replication of contingent claims under transaction costs. Review of Futures Markets **8**, 222–239 (1989)
22. Hoggard, T., Whalley, A. E., Wilmott, P.: Hedging option portfolios in the presence of transaction costs. Advances Futures Operations Research **7**, 21 (1994)
23. Ishii, H., Lions, P.-L.: Viscosity solutions of fully nonlinear second-order elliptic partial differential equations. Journal of Differential Equations **83**, 26–78 (1990)
24. Kurtz, T.: A control formulation for constrained Markov processes. Mathematics of Random Media (Lectures in Applied Mathematics, **27**) AMS, Providence, 1991, pp. 139–150
25. Leland, H. E.: Option pricing and replication with transactions costs. Journal of Finance **40**, 1283–1301 (1985)
26. Leventhal, S., Skorohod, A. V.: On the possibility of hedging options in the presence of transaction costs. Annals of Applied Probability **7**, 410–443 (1997)
27. Levy, H.: Upper and lower bounds of put and call option value: Stochastic dominance approach. Journal of Finance **40**, 1197–1217 (1985)

28. Lions, P.-L.: Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. Part 1: The dynamic programming principle and applications. Part 2: Viscosity solutions and uniqueness. Communications in PDE **8**, 1101–1174 and 1229–1276 (1983)
29. Merton, R. C.: Theory of rational option pricing. Bell Journal of Economics and Management Science **4**, 141–183 (1973)
30. Merton, R. C.: Continuous-time finance. Oxford: Basil Blackwell 1990
31. Perrakis, S., Ryan, P. J.: Option pricing bounds in discrete time. Journal of Finance **39**, 519–525 (1984)
32. Pichler, A.: On transaction costs and HJB equations. Preprint (1996)
33. Ritchken, P. H.: On option pricing bounds. Journal of Finance **40**, 1219–1233 (1985)
34. Shen, Q.: Bid-Ask prices for call options with transaction costs. Working paper, University of Pennsylvania (1990)
35. Shreve, S. E., Soner, H. M.: Optimal investment and consumption with transaction costs. Annals of Applied Probability **4(3)**, 609–692 (1994)
36. Soner, H. M.: Optimal control with state space constraints. SIAM Journal of Control and Optimization **24**, 552–562, 1110–1122 (1986)
37. Soner, H. M., Shreve, S. E., Cvitanic, J.: There is no nontrivial hedging portfolio for option pricing with transaction costs. Annals of Applied Probability **5(2)**, 327–355 (1995)
38. Toft, K. B.: On the mean-variance tradeoff in option replication with transaction costs. Journal of Financial and Quantitative Analysis **31**, 233–263 (1996)
39. Tourin, A., Zariphopoulou, T.: Numerical schemes for investment models with singular transactions. Computational Economics **7**, 287–307 (1994)
40. Zariphopoulou, T.: Investment-consumption models with transaction fees and Markov-chain parameters. SIAM Journal Control Optimization **30**, 613–636 (1992)
41. Zhu, H.: Characterization of variational inequalities in singular control. Ph.D. Thesis, Brown University, Providence, RI (1991)