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Optimal investment and consumption models with non-linear stock dynamics

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Abstract. We study a generalization of the Merton's original problem of optimal consumption and portfolio choice for a single investor in an intertemporal economy. The agent trades between a bond and a stock account and he may consume out of his bond holdings. The price of the bond is deterministic as opposed to the stock price which is modelled as a diffusion process. The main assumption is that the coefficients of the stock price diffusion are arbitrary nonlinear functions of the underlying process. The investor's goal is to maximize his expected utility from terminal wealth and/or his expected utility of intermediate consumption. The individual preferences are of Constant Relative Risk Aversion (CRRA) type for both the consumption stream and the terminal wealth. Employing a novel transformation, we are able to produce closed form solutions for the value function and the optimal policies. In the absence of intermediate consumption, the value function can be expressed in terms of a power of the solution of a homogeneous linear parabolic equation. When intermediate consumption is allowed, the value function is expressed via the solution of a non-homogeneous linear parabolic equation.

Key words: Portfolio management, Hamilton-Jacobi-Bellman equation, closed form solutions, constrained viscosity solutions

1. Introduction

This paper is a contribution to the theory of optimal portfolio management in intertemporal economies. The underlying task is to specify the maximal

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expected utility and the optimal investment and consumption strategies of a single individual[1] who trades and consumes in the presence of uncertainty for market returns. The market consists of two assets, a bond and a stock with prices already prespecified via equilibrium conditions. The bond price is deterministic and the stock price is modelled as a diffusion process. The objective is to maximize the individual's expected utility which comes from terminal wealth and/or intermediate consumption.

The fundamental stochastic model of optimal investment and consumption, was first introduced by Merton (1969, 1971) who constructed explicit solutions under the assumption that the stock price follows a geometric Brownian motion and the individual preferences are of special type. Specifically, the utilities are either of Constant Relative Risk Aversion (CRRA) type, including the logarithmic case, or of exponential type. There are no binding constraints on the traded portfolios besides the standard no arbitrage constraint of keeping the wealth nonnegative at all times. Merton's pioneering papers initiated a considerable volume of new work on the subject in various directions.

The case of general utilities was analyzed in Karatzas et al. (1986, 1987) who produced the value function in closed form. Models with general utilities and trading constraints were subsequently studied by various authors (see Karatzas et al. (1991), Zariphopoulou (1994), Cvitanic and Karatzas (1995)). Generally speaking, there are two main methodologies for the study of these stochastic optimization problems: one that relies heavily on the theory of nonlinear partial differential equations and the alternative approach that is based on martingale theory. Independently of the specific approach used, the standing assumption in the existing literature is that the asset prices follow a geometric Brownian motion. This special structure enables us to absorb the stock price in the wealth variable – through the budget constraint – and, therefore, to remove one of the state variables. This assumption facilitates the analysis considerably but it does not accommodate a variety of applications like, for example, the case of stochastic volatility. It has been relaxed in a limited way by allowing the linearity coefficients to be deterministic functions of time (see for example, Karatzas et al. (1991)).

In this paper, the above restrictive assumption on the underlying stock price are removed by allowing the coefficients of the price process to be *nonlinear functions* of the current stock level. A special case of this class of prices is the case of stochastic volatility perfectly correlated with the underlying stock. The main contribution herein is the derivation of closed form solutions for the value function and the optimal policies. This is accomplished by exploring the homotheticity properties of the value function combined with a novel transformation. The latter enables us to express the value function in terms of a power of the solution of the so-called "reduced" equation. This representation also facilitates the construction of the optimal portfolio and consumption policies which are given in a simplified feedback form in terms of the solution of the reduced equation and its first derivatives. If the goal of the agent is to maximize his expected utility from terminal wealth, the latter equation turns out to be a homogeneous linear parabolic one. If intermediate consumption is allowed, the reduced equation is also linear parabolic but non-

[1] The investor is assumed to be "small" in the sense that his actions do not affect the equilibrium asset prices.

homogeneous. Nevertheless, in both cases, the reduced equations are similar to the ones arising in bond pricing. An interesting consequence is the representation of their solutions in terms of the expectation of an exponential payoff of a new state process. This process solves the same equation as the stock price but with a *modified drift* which represents the effects of the inherent nonlinearities of the model.

Besides the derivation of new results in the area of optimal portfolio management, this paper offers an exposition – for the technically oriented reader – of the use of a class of weak solutions, namely the *viscosity solutions*, of the relevant Hamilton-Jacobi-Bellman equation. The HJB equation is the offspring of the Dynamic Programming Principle and stochastic analysis and it is expected to be satisfied by the value function. Due to specific degeneracies and other characteristics of the stochastic optimization model, the value function might not satisfy the HJB equation in the classical sense. Such situations are common in finance models with *market imperfections* like trading constraints, transaction costs and stochastic labor income. It is thus useful to relax the notion of solutions to the HJB equation and this has been successfully done in the aforementioned class. Viscosity solutions have become by now an important tool in analyzing stochastic optimization problems that arise in a variety of valuation models in economics, finance and insurance theory.[2]

The paper is organized as follows: in Section 2, we introduce the investment model without intermediate consumption and we state the main results. In Section 3, we derive the Hamilton-Jacobi-Bellman equation and we study its solutions using elements from the theory of viscosity solutions. In Section 4, we derive the closed form solutions and we provide regularity and verification results for the value function and the optimal policies. In Section 5, we study the problem when intermediate consumption is allowed and we discuss possible extensions.

2. The investment model and main results

We consider an optimal investment model of a single agent who manages his portfolio by investing in a bond and a stock account. The price of the bond B_t solves

$$\begin{cases} dB_t = rB_t\,dt \\ B_0 = B \end{cases} \tag{2.1}$$

where $r > 0$ is the interest rate. The price of the stock is modelled as a diffusion process S_t satisfying

$$\begin{cases} dS_t = \mu(S_t)S_t\,dt + \sigma(S_t)S_t\,dW_t \\ S_0 = S \geq 0. \end{cases} \tag{2.2}$$

[2] See for example, Zariphopoulou (1992), Fleming and Zariphopoulou (1991), Fitzpatrick and Fleming (1990), Barles et al. (1993), Scheinkman and Zariphopoulou (1999), Shreve and Soner (1994), Tourin and Zariphopoulou (1994), Young and Zariphopoulou (1999).

The process W_t is a Brownian motion defined on a probability space $(\Omega, \mathscr{F}, P)$. The coefficients μ and σ are functions of the current stock price and they are assumed to satisfy all the required regularity assumptions in order to guarantee that a unique solution to (2.2) exists. These conditions, together with some additional growth assumptions will be introduced later; at this point, we only outlay the underlying structure of the model.

The investor rebalances his portfolio dynamically by choosing at any time s, for $s \in [t, T]$ and $0 \le t \le T$, the amounts π_s^0 and π_s to be invested respectively in the bond and the stock accounts. His total wealth satisfies the budget constraint $X_s = \pi_s^0 + \pi_s$ and the stochastic differential equation

$$\begin{cases} dX_s = rX_s\,dt + (\mu(S_s) - r)\pi_s\,ds + \sigma(S_s)\pi_s\,dW_s \\ X_t = x \ge 0 \quad 0 \le t \le s \le T. \end{cases} \tag{2.3}$$

The above state equation follows from the budget constraint and the dynamics in (2.1) and (2.2). The wealth process must also satisfy the state constraint

$$X_s \ge 0 \quad \text{a.e. } t \le s \le T. \tag{2.4}$$

Remark 2.1: We assume that the coefficients μ and σ do not depend explicitly on time. This is assumed only to ease the presentation since the time-dependent case follows easily from the autonomous one.

The pair of control processes (π_s, C_s) is said to be admissible if it is $\mathscr{F}_s$-progressively measurable, where $\mathscr{F}_s = \sigma(W_u; t \le u \le s)$, satisfies the integrability condition $E\int_t^T \sigma(S_s)^2 \pi_s^2\,ds < +\infty$ and, is such that the above state constraint is satisfied. We denote by $\mathscr{A}$ the set of admissible policies.

The investor's objective is to maximize his expected utility payoff

$$J(x, S, t; \pi) = E[U(X_T)/X_t = x, S_t = S] \tag{2.5}$$

with X_s, S_s given respectively in (2.3) and (2.2).

The value function of the investor is defined as

$$u(x, S, t) = \sup_{\mathscr{A}} J(x, S, t; \pi) \tag{2.6}$$

with the utility function $U : [0, +\infty) \to [0, +\infty)$ being of the form

$$U(x) = \frac{1}{\gamma} x^\gamma. \tag{2.7}$$

The quantity $1 - \gamma$ is known as the *risk aversion coefficient* and it is assumed to satisfy $0 \le 1 - \gamma < 1$. The case $\gamma = 0$ corresponds to logarithmic utilities.

The goal herein is to analyze the value function and to determine the optimal investment strategies.

The special form of the above utilities together with the linearity of the wealth dynamics with respect to the state and control processes (see (2.3)), suggest that the value function may be written in a "separable" form. In other words, the value function may be written as $u(x, S, t) = \frac{x^\gamma}{\gamma} V(S, t)$. To our knowledge, the component V is in general unknown except for some very special cases of the risk aversion parameter $1-\gamma$ and the components of the state dynamics (see Merton (1971)). As a matter of fact, V solves a nonlinear equation for which no closed form solutions are available in general.

The novelty of our results lies in the fact that under a simple power transformation, the factor V can be expressed in terms of the solution of a linear parabolic equation. This representation provides closed form solutions for the value function and the optimal policies which can in turn be used effectively in a more general class of valuation problems with stochastic components. Without stating at this point the necessary technical assumptions and the regularity properties of the solutions, we outline the main results below.

Proposition 2.1:

i) *The value function u is given by*

$$u(x,S,t) = \frac{x^\gamma}{\gamma} v(S,t)^{1-\gamma}$$

where $v : R^+ \times [0,T] \to R^+$ *solves the linear parabolic equation*

$$\begin{cases} v_t + \frac{1}{2}\sigma^2(S,t)S^2 v_{SS} + \left[\mu(S)S + \frac{\gamma(\mu(S)-r)S}{(1-\gamma)}\right] v_S \\ \qquad + \frac{\gamma}{1-\gamma}\left[r + \frac{(\mu(S)-r)^2}{2\sigma^2(S)(1-\gamma)}\right] v = 0 \\ v(S,T) = 1 \quad and \quad v(0,t) = e^{(r\gamma/(1-\gamma))(T-t)}, \quad 0 \le t \le T. \end{cases}$$

ii) *The optimal investment policy* Π_s^* *is given in the feedback form* $\Pi_s^* = \pi^*(X_s^*, S_s, s), t \le s \le T$, *where the function* $\pi^* : R^+ \times R^+ \times [0,T] \to$ *is defined by*

$$\pi^*(x,S,t) = \left[\frac{v_S(S,t)}{v(S,t)} + \frac{1}{1-\gamma}\frac{\mu(S)-r}{\sigma^2(S)}\right] x.$$

We conclude this section by reviewing briefly the celebrated Merton's optimal portfolio management problem (see Merton (1969), (1971) and (1973)). To this end, we consider a market with two securities, a bond whose price solves (2.1) and a stock whose price process satisfies the linear stochastic differential equation

$$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t \tag{2.8}$$

with $S_0 = S > 0$. The market parameters μ and σ are, respectively, the *mean rate of return* and the *volatility*; it is assumed that $\mu > r > 0$ and $\sigma > 0$. The process W_t is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$.

The wealth process satisfies $X_s = \pi_s^0 + \pi_s$ with the amounts π_s^0 and π_s representing the current holdings in the bond and the stock accounts. The state wealth equation (2.3) reduces to

$$dX_s = rX_s\,ds + (\mu - r)\pi_s\,ds + \sigma\pi_s\,dW_s. \tag{2.9}$$

The wealth process must satisfy the state constraint

$$X_s \geq 0 \quad \text{a.e. } t \leq s \leq T. \tag{2.10}$$

The control $\pi_s, t \leq s \leq T$ is admissible if it is $\mathscr{F}_s$-progressively measurable – with $\mathscr{F}_s = \sigma(W_u; t \leq u \leq s)$ – it satisfies $E\int_t^T \pi_s^2\,ds < +\infty$ and, it is such that the state constraint (2.10) is satisfied. We denote the set of admissible policies by $\tilde{\mathscr{A}}$.

The value function is defined as in (2.6), namely

$$\tilde{u}(x,t) = \sup_{\tilde{\mathscr{A}}} E\left[\frac{1}{\gamma}X_T^\gamma / X_t = x\right]. \tag{2.11}$$

Observe that the geometric Brownian motion assumption on prices results in reduction of the number of state variables, from two to one.

A fundamental optimality fact, known as the Dynamic Programming Principle yields that for every stopping time τ,

$$\tilde{u}(x,t) = E[\tilde{u}(X_\tau^*, \tau)/X_t = x] \tag{2.12}$$

with X_τ^* being the optimal wealth at time τ. Using stochastic analysis and under appropriate regularity and growth conditions on the value function, we get that V solves the associated Hamilton-Jacobi-Bellman equation, for $x \geq 0$ and $t \in [0,T)$,

$$\begin{cases} \tilde{u}_t + \max\limits_{\pi}\left[\frac{1}{2}\sigma^2\pi^2\tilde{u}_{xx} + (\mu - r)\pi\tilde{u}_x\right] + rx\tilde{u}_x = 0, & (2.13) \\ \tilde{u}(x,T) = \frac{1}{\gamma}x^\gamma, \quad x \geq 0, & (2.14) \\ \tilde{u}(0,t) = 0, \quad t \in [0,T]. & (2.15) \end{cases}$$

Remark 2.2: The boundary condition $\tilde{u}(0,t)$ is not in general prespecified due to the presence of the state constraint (2.10). The standard by now approach to deal with this issue is to work with the appropriate class of weak solutions, namely the constrained viscosity solutions, and to characterize the value function as the unique solution of the HJB equation in this class. Once this is established, the boundary value may be obtained from the values of the solution in the interior, for $x > 0$, after passing to the limit as $x \to 0$.

The homogeneity of the utility function and the linearity of the state dynamics with respect to both the wealth and the control portfolio process, suggest that the value function must be of the form

$$\tilde{u}(x,t) = \frac{x^\gamma}{\gamma}f(t) \tag{2.16}$$

with $f(T) = 1$. Using the above form in (2.13) and after some cancellations, one gets that f must satisfy the first order equation

$$f'(t) + \lambda f(t) = 0$$

with

$$f(T) = 1$$

where

$$\lambda = r\gamma + \frac{(\mu - r)^2}{2(1-\gamma)\sigma^2}. \tag{2.17}$$

Therefore,

$$\tilde{u}(x, t) = \frac{x^\gamma}{\gamma} e^{\lambda(T-t)}. \tag{2.18}$$

Once the value function is determined, the optimal policy may be obtained in the so-called feedback form as follows: first, we observe that the maximum of the quadratic term appearing in (2.13) is achieved at the point

$$\pi^*(x, t) = -\frac{\mu - r}{\sigma^2} \frac{\tilde{u}_x(x, t)}{\tilde{u}_{xx}(x, t)}$$

or, otherwise,

$$\pi^*(x, t) = \frac{\mu - r}{\sigma^2(1-\gamma)} x$$

where we used (2.18). Next, we recall classical Verification results (see, for example, Chapter VI in the book of Fleming and Soner (1993)) which yield that the candidate solution, given in (2.18) is indeed the value function and that, moreover, the policy

$$\pi_s^* = \frac{\mu - r}{\sigma^2(1-\gamma)} X_s^* \tag{2.19}$$

is the optimal investment strategy.

In other words,

$$\tilde{u}(x, t) = E\left[\frac{(X_T^*)^\gamma}{\gamma} \Big/ X_t^* = x\right]$$

where X_s^* solves

$$dX_s^* = \left(r + \frac{(\mu - r)^2}{\sigma^2(1-\gamma)}\right) X_s^* \, ds + \frac{\mu - r}{\sigma(1-\gamma)} X_s^* \, dW_s.$$

The solution of the optimal state wealth equation is, for $X_t = x$,

$$X_s^* = x \exp\left[\left(r + \frac{(\mu - r)^2}{\sigma^2(1-\gamma)} - \frac{(\mu - r)^2}{2\sigma^2(1-\gamma)^2}\right)(s-t) + \frac{\mu - r}{\sigma(1-\gamma)} W_{s-t}\right].$$

The Merton optimal strategy dictates that it is optimal to keep a *fixed proportion*, namely $\frac{\mu - r}{\sigma^2(1-\gamma)}$, of the current total wealth invested in the stock account. We will refer to this proportionality constant as the *Merton ratio*.

Remark 2.3: It is important to observe that the Merton model uses heavily the assumption that the stock price remains strictly positive even though the stock price does not appear explicitly. One could easily verify this constraint by looking at the actual derivation of the state wealth equation (2.9); we refer the reader to Merton (1969) or Karatzas et al. (1987). Given that the stock price is modelled as a log-normal process, it becomes zero only if it starts at the state 0. In this case, the Merton model degenerates to a deterministic model with no (stochastic) optimization features. In fact, one could easily prove that no investment takes place in the stock account and that the wealth process satisfies the deterministic equation $dX_s = rX_s\, ds$ for $t \le s \le T$. In this case, the value function turns out to be $\tilde{u}(x,t) = \frac{x^\gamma}{\gamma} e^{r\gamma(T-t)}$. We can view this degenerate case as the limiting case of (2.13) as $\mu \to r$ or as $\sigma \to +\infty$. Indeed, if $\mu = r$ or $\sigma = +\infty$, the solution of the HJB equation degenerates to $\tilde{u}(x,t)$ and the optimal policy, given in (2.19), becomes zero.

3. The HJB equation and viscosity solutions

In this section we analyze the associated Hamilton-Jacobi-Bellman (HJB) equation and characterize the value function (2.6) as its solution. Generally speaking, the fact that the value function of a stochastic optimization problem solves, in the classical sense, the relevant HJB equation follows from the optimality principle of Dynamic Programming and stochastic calculus. But it is the case that this can be done only if it is known a priori that the value function has enough regularity. Conversely, classical verification results (see, for example, Fleming and Soner (1993)) yield that if the HJB equation has a unique smooth solution then it coincides with the value function. In the problem at hand, however, it does not follow directly that the value function is smooth. Also the associated HJB equation, see (3.9) below, is a second-order fully nonlinear and possibly degenerate equation and therefore might not have a unique smooth solution. It is thus imperative to relax the notion of solutions to the HJB equation. It turns out that a suitable class of solutions are the so-called viscosity solutions; as a matter of fact, due to the presence of the state constraint (2.4) we will actually work in the class of *constrained viscosity solutions*.

The notion of *viscosity solutions* was introduced by Crandall and Lions (1983) for first-order equations, and by Lions (1983) for second-order equations. For a general overview of the theory we refer to the *User's Guide* by Crandall, Ishii and Lions (1992) and the book by Fleming and Soner (1993).

Next, we recall the notion of *constrained viscosity solutions* which was introduced by Soner (1986) and Capuzzo-Dolcetta and Lions (1990) for first-order equations (see also Ishii and Lions (1990)). To this end, we consider a nonlinear second order partial differential equation of the form

$$F(X, V, DV, D^2V) = 0 \quad \text{in } \Omega \times [0, T] \tag{3.1}$$

where Ω is an open subset of $\mathscr{R}^2$, DV and D^2V denote the gradient vector and the second derivative matrix of V, and the function F is continuous in all its arguments and degenerate elliptic, meaning that

$$F(X, p, q, A + B) \leq F(X, p, q, A) \quad \text{if } B \geq 0. \tag{3.2}$$

Definition 3.1: *A continuous function* $V : \bar{\Omega} \times [0, T] \to R$ *is a constrained viscosity solution of (3.1) if the following two conditions hold:*

i) V is a viscosity subsolution of (3.1) on $\bar{\Omega} \times [0, T]$; *that is, if for any* $\phi \in C^{2,1}(\bar{\Omega} \times [0, T])$ *and any local maximum point* $X_0 \in \bar{\Omega} \times [0, T]$ *of* $V - \phi$,

$$F(X_0, V(X_0), D\phi(X_0), D^2\phi(X_0)) \leq 0, \tag{3.3}$$

ii) V is a viscosity supersolution of (3.1) in $\Omega \times [0, T]$; *that is, if for any* $\phi \in C^{2,1}(\bar{\Omega} \times [0, T])$ *and any local minimum point* $X_0 \in \Omega \times [0, T]$ *of* $V - \phi$,

$$F(X_0, V(X_0), D\phi(X_0), D^2\phi(X_0)) \geq 0. \tag{3.4}$$

Viscosity solutions in stochastic control problems arising in mathematical finance were first introduced by Zariphopoulou (1989) in the context of optimal investment decisions with trading constraints. These solutions were subsequently used in other asset valuation problems in markets with frictions (see, for example, among others, Davis, Panas and Zariphopoulou (1993), Shreve and Soner (1994), Barles and Soner (1999), Duffie et al. (1997), Constantinides and Zariphopoulou (1999a)) and by now they have become a standard tool in studying asset pricing models in general markets.

One of the main advantages to work with viscosity solutions comes from the fact that such a characterization enables us to obtain convergence of a large class of numerical schemes for the maximized utility – value function – as well as the optimal trading strategies. This is desirable given the absence of optimal feedback formulas for the optimal policies due to lack of sufficient regularity of the value function (see, for example, Tourin and Zariphopoulou (1994) and Pichler (1996)). Another contribution of the theory is that, if candidate solutions of the HJB equation can be computed, the characterization of the value function as the unique viscosity solution of the same equation results in identifying it with the candidate solution and, therefore, obtain it in closed form. This is actually the main ingredient that is used herein for the construction of the value function and the optimal policies. Finally, the characterization of the value function as the unique viscosity solution may be used effectively in obtaining estimates when closed form solutions are not available. This methodology has been successfully used in valuation models of derivative pricing and has been rather fruitful (see, for example, Barles et al. (1993), Constantinides and Zariphopoulou (1999a, 1999b)).

In the sequel, we derive the HJB equation and we provide results for its viscosity solutions and the value function. Even though most of the involved argument follow along the lines of existing ones, we choose to present the main steps in order to provide a concise exposition on the use of viscosity solutions for the technically oriented audience[3].

We start with some elementary properties for the value function. We note that *all* the results in this section hold for *general utility functions* U, besides the CRRA ones that were previously introduced (see (2.11)). The only standing assumption about the individual preferences is that $U : [0, +\infty) \rightarrow [0, +\infty)$ is increasing and concave and satisfies the growth condition

$$U(x) \leq K(1+x)^{\gamma}, \quad \forall x \geq 0 \tag{3.5}$$

for some positive constant K and $0 < \gamma < 1$.

Assumption 3.1: The coefficients $\mu : [0, +\infty) \rightarrow [0, +\infty)$ and $\sigma : [0, +\infty) \rightarrow [0, +\infty)$ in (2.2) have the following properties.

i) if f stands for the functions $\mu(S)S$ and $\sigma(S)S$ then, for all $S \geq 0$

$$|f(S) - f(\bar{S})| \leq L|S - \bar{S}| \quad \text{and} \quad f^2(S) \leq L(1+S^2) \tag{3.6}$$

from some generic positive constant L.

ii) The function $\mu : [0, +\infty) \rightarrow [0, +\infty)$ satisfies

$$\mu(S) > r \quad \text{for } S > 0 \quad \text{and} \quad \mu(0) = r. \tag{3.7}$$

The function $\sigma : [0, +\infty) \rightarrow (0, +\infty)$ is bounded away from zero and for all $S \geq 0$,

$$\frac{(\mu(S) - r)^2}{\sigma^2(S)} \leq M \tag{3.8}$$

with M being a large constant.

iii) The coefficients μ and σ are such that $S_s > 0$ a.s. if $S_t > 0$ and $S_s = 0$ if $S_t = 0$, for $t \leq s \leq T$.

The above assumptions were motivated by the following facts. First, stock price processes are expected to stay positive, for positive initial condition, and to remain at zero if they start there. For the necessary technical conditions that ensure (iii), we refer the reader to Gikhman and Skorohod (1972). Secondly, it is natural to assume that the mean rate of return of the risky security dominates the riskless interest rate at all levels of (positive) stock prices. Thirdly, we impose the condition $\mu(0) = r$ motivated by the behavior of the degenerate Merton model for zero stock prices, as it was discussed in Remark 2.3. We will return to this issue at subsequent points in our analysis.

Proposition 3.1: i) *The value function u is non-decreasing and concave with respect to the wealth variable x.*

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[3] We refer the technically non-interested reader to Section 4 which provides the closed form solutions and the related verification results.

ii) *There exists a constant* $\lambda > r\gamma$ *that* $\frac{x^\gamma}{\gamma} e^{r\gamma(T-t)} \leq u(x, S, t) \leq \frac{x^\gamma}{\gamma} e^{\lambda(T-t)}$.

iii) *The value function satisfies* $u(x, 0, t) = \frac{x^\gamma}{\gamma} e^{r\gamma(T-t)}$.

Proof: i) The concavity of u is an immediate consequence of the concavity of the utility function U and the fact that if $\pi^1 \in \mathscr{A}_{(x_1, S)}$, $\pi^2 \in \mathscr{A}_{(x_2, S)}$ and $\lambda \in (0, 1)$ then $(\lambda\pi^1 + (1-\lambda)\pi^2) \in \mathscr{A}_{(\lambda x_1 + (1-\lambda)x_2, S)}$; the latter follows from the linear dependence of the state dynamics (2.4) with respect to the control variables and the state wealth. That u is nondecreasing in x follows from the observation that $\mathscr{A}_{(x_1, S)} \subseteq \mathscr{A}_{(x_2, S)}$ for $x_1 \leq x_2$.

ii) The lower bound for the value function follows directly from the definition of u and the fact that $\pi_s = 0$, $t \leq s \leq T$ is an admissible policy. The upper bound follows from a standard Girsanov transformation, Hölder's inequality and the uniform bound in inequality (3.8). The relevant to the change of measure Randon-Nikodym derivative is given by $Z_T = \exp\{-\int_t^T \theta_s \, dW_s - \frac{1}{2}\int_t^T \theta_s^2 \, dW_s\}$ with $\theta_s = \frac{(\mu(S_s) - r)^2}{\sigma^2(S_s)}$. This kind of analysis is standard in stochastic optimization problems and it is skipped for the sake of the presentation. (We refer the reader to the book of Friedlin (1985) or, in the context of relevant portfolio management problems, to the papers of Huang and Pagès (1992) and Duffie and Zariphopoulou (1993)).

iii) By assumption, $\{0\}$ is an absorbing state and therefore if $S_t = 0$ then $S_s = 0$ for $t \leq s \leq T$. The optimal strategy turns out to be $\pi_s^* = 0$, $t \leq s \leq T$ which in turn yields $X_T^* = xe^{r(T-t)}$. The value $u(x, 0, t)$ then follows from the definition of u and the form of the utility function.

We continue with the main results of this section. We denote the solvency domain by $\bar{D} = [0, +\infty) \times [0, +\infty) \times [0, T]$.

Theorem 3.1: *The value function* $u : \bar{D} \to [0, +\infty)$ *is a constrained viscosity solution on* $\bar{D}$ *of the HJB equation*

$$u_t + \max_\pi \left[\frac{1}{2}\sigma^2(S)\pi^2 u_{xx} + \sigma^2(S)\pi S u_{xS} + (\mu(S) - r)\pi u_x\right] + \frac{1}{2}\sigma^2(S)S^2 u_{SS} + \mu(S)S u_S + rxu_x = 0 \tag{3.9}$$

with

$$u(x, S, T) = \frac{1}{\gamma}x^\gamma \tag{3.10}$$

and

$$u(x, 0, t) = \frac{1}{\gamma}x^\gamma e^{r\gamma(T-t)}. \tag{3.11}$$

The fact that in general, value functions of (stochastic) control problems and differential games turn out to be viscosity solutions of the associated partial differential equations follows directly from the Dynamic Programming Principle, stochastic analysis and the theory of viscosity solutions (see for example Lions (1983), Evans and Souganidis (1984) and Fleming and Souganidis (1989)). The main difficulty with the problem at hand, and in general in all optimization problems of stochastic portfolio management, is that the control processes, representing the risky investments, are not uniformly bounded. In order to overcome this difficulty, we follow the approach of Lions (1983) (see also Krylov (1980)) with which one works with the *normalized HJB equation* which results from a time change. This methodology was introduced in the context of stationary problems of optimal investments with stochastic labor income by Duffie and Zariphopoulou (1993). Even though the main arguments below are modifications of the ones used by Duffie and Zariphopoulou (1993, Theorem 4.1) we present them here for completeness.

Proof: We first show that u is a viscosity supersolution (3.9) in D. Let ϕ be a smooth function on $\bar{D}$ and $(x_0, S_0, t_0) \in D$ be a minimum of $u - \phi$. Without loss of generality we can assume that

$$u(x_0, S_0, t_0) = \phi(x_0, S_0, t_0) \quad \text{and} \quad v \geq \phi \quad \text{in } D. \tag{3.12}$$

We need to show that

$$\phi_t(x_0, S_0, t_0) + \max_{\pi} G(x_0, S_0, t_0; \pi) + \mathscr{L}u(x_0, S_0, t_0) + rx_0\phi_x(x_0, S_0, t_0) \leq 0 \tag{3.13}$$

where

$$\begin{aligned} G(x_0, S_0, t_0; \pi) &\\ = \tfrac{1}{2}\sigma^2(S_0)\pi^2\phi_{xx}(x_0, S_0, t_0) &\\ + \pi\sigma^2(S_0)S_0\phi_{xS}(x_0, S_0, t_0) + (\mu(S_0) - r)\pi\phi_x(x_0, S_0, t_0), & \end{aligned} \tag{3.14}$$

and the operator $\mathscr{L}$ is defined as

$$\mathscr{L}\phi(x_0, S_0, t_0) = \tfrac{1}{2}\sigma^2(S_0)S_0^2\phi_{SS}(x_0, S_0, t_0) + \mu(S_0)S_0\phi_S(x_0, S_0, t_0). \tag{3.15}$$

Next we consider a fixed policy $\pi_s = \pi_0$ for $t \leq s \leq \theta$ where $\theta = \min\{\theta : t \leq \theta \leq T \text{ and } X_\theta^{\pi_0} = 0\}$ with $\tilde{X}_s$ being the solution of (2.3) with the constant policy being used and satisfying $\tilde{X}_{t_0} = x_0$. Since such policies are in general suboptimal, we get that

$$u(x_0, S_0, t_0) \geq E[u(\tilde{X}_\theta, S_\theta, \theta)/\tilde{X}_{t_0} = x_0, \tilde{S}_{t_0} = S_0].$$

On the other hand, applying Itô's lemma to the smooth function $\phi(\tilde{X}_s, S_s, s)$ for $t_0 \leq s \leq \theta$, yields

$$E[\phi(\tilde{X}_\theta, S_\theta, \theta)/\tilde{X}_t = x_0, S_t = S_0]$$

$$= \phi(x_0, S_0, t_0) + E\left[\int_{t_0}^{\theta} \{G(\tilde{X}_s, S_s, s; \pi_0) + \mathscr{L}\phi(\tilde{X}_s, S_s, s)\right.$$

$$\left. + r\tilde{X}_s\phi_x(\tilde{X}_s, S_s, s)\}\, ds/\tilde{X}_{t_0} = x_0, S_{t_0} = S_0\right].$$

Combining the above and using standard estimates from the theory of stochastic differential equations (see Gikhman and Skorohod (1972)), we get

$$E\left[\int_{t_0}^{\theta} \{\phi_t(x_0, S_0, t_0) + G(x_0, S_0, t_0; \pi_0) + \mathscr{L}\phi(x_0, S_0, t_0)\right.$$

$$\left. + rx_0\phi_x(x_0, S_0, t_0)\}\, ds/\tilde{X}_{t_0} = x_0, S_t = S_0\right] + E\int_{t_0}^{\theta} h(s)\, ds \le 0$$

where $h(s) = O(s)$. Dividing both sides by $E(\theta)$ and passing to the limit as $n \to \infty$, inequality (3.13) follows.

We next show that u is a viscosity subsolution of (3.9) on $\bar{D}$. Let ϕ be a smooth solution on $\bar{D}$ and let us assume that $u - \phi$ has a maximum at a point $(x_0, S_0, t_0) \in \bar{D}$. Without loss of generality we may assume that $u(x_0, S_0, t_0) = \phi(x_0, S_0, t_0)$ and $u \le \phi$ otherwise. We need to show that

$$\phi_t(x_0, S_0, t_0) + \max_{\pi} G(x_0, S_0, t_0; \pi) + \mathscr{L}\phi(x_0, S_0, t_0) + rx_0\phi_x(x_0, S_0, t_0) \ge 0. \tag{3.16}$$

In order to show the above inequality, we first recall that the value function u is a viscosity subsolution on $\bar{D}$ of the *normalized HJB equation*

$$\max_{\pi \in \mathscr{R}} \left[\frac{1}{1+\pi^2}\left[\left(u_t + \frac{1}{2}\sigma^2(S)\pi^2 u_{xx} + \sigma^2(S)\pi S u_{xS} + (\mu(S) - r)\pi u_x\right)\right.\right.$$

$$\left.\left. + \mathscr{L}u + rxu_x\right]\right] = 0. \tag{3.17}$$

(For a proof see Lions (1983).) We next look at the following cases.

Case A. $\phi_{xx}(x_0, S_0, t_0) \ge 0$ and $\sigma^2(S_0)S_0\phi_{xS}(x_0, S_0, t_0) + (\mu(S_0) - r)\phi_x(x_0, S_0, t_0) \ne 0$. Then (3.16) is automatically satisfied since the right-hand side of (3.16) is $+\infty$.

Case B. $\phi_{xx}(x_0, S_0, t_0) > 0$ and $\sigma^2(S_0)S_0\phi_{xS}(x_0, S_0, t_0) + (\mu(S_0) - r)\phi_x(x_0, S_0, t_0) \ne 0$. This is the same as the situation in Case A.

Case C. $\phi_{xx}(x_0, S_0, t_0) = 0$ and $\sigma^2(S_0)S_0\phi_{xS}(x_0, S_0, t_0) + (\mu(S_0) - r)\phi_x(x_0, S_0, t_0) \ne 0$. Then (3.16) becomes

$$\phi_t(x_0, S_0, t_0) + \mathscr{L}\phi(x_0, S_0, t_0) + rx_0\phi_x(x_0, S_0, t_0) \ge 0. \tag{3.18}$$

We argue by contradiction. Let us assume that (3.18) is not true. Then define

$$A \equiv -\left[\phi_t(x_0, S_0, t_0) + \mathscr{L}\phi(x_0, S_0, t_0) + rx_0\phi_x(x_0, S_0, t_0)\right] > 0. \tag{3.19}$$

Using the fact that $u - \phi$ has a maximum at (x_0, S_0, t_0) and inequality (3.19), the normalized HJB equation (3.17) yields

$$\max_{\pi \in \mathscr{R}} \frac{A}{1+\pi^2} \leq 0,$$

which is a contradiction because

$$\max_{\pi \in \mathscr{R}} \frac{A}{1+\pi^2} = A > 0.$$

Case D. $\phi_{xx}(x_0, S_0, t_0) < 0$ and $\sigma^2(S_0)S_0\phi_{xS}(x_0, S_0, t_0) + (\mu(S_0) - r)\phi_x(x_0, S_0, t_0) \neq 0$. Then the maximum with respect to π of

$$\tfrac{1}{2}\sigma^2(S_0)\pi^2\phi_{xx}(x_0, S_0, t_0) + \sigma^2(S_0)S_0\pi\phi_{xS}(x_0, S_0, t_0) + (\mu(S_0) - r)\phi_x(x_0, S_0, t_0)$$

occurs at a finite point, denoted π^*. We argue again by contradiction. Let us assume that (3.16) does not hold, that is

$$\begin{aligned} A \equiv -\Big[&\phi_t(x_0, S_0, t_0) + \tfrac{1}{2}\sigma^2(S_0)(\pi^*)^2\phi_{xx}(x_0, S_0, t_0) \\ &+ \sigma^2(S_0)S_0\pi^*\phi_{xS}(x_0, S_0, t_0) \\ &+ \mathscr{L}\phi(x_0, S_0, t_0) + rx_0\phi_x(x_0, S_0, t_0)\Big] > 0. \end{aligned}$$

From the normalized HJB equation, we get $\max_{\pi \in \mathscr{R}} \frac{A}{1+\pi^2} \leq 0$ which again yields a contradiction.

Case E. $\phi_{xx}(x_0, S_0, t_0) < 0$ and $\sigma^2(S_0)S\phi_{xS}(x_0, S_0, t_0) + (\mu(S_0) - r)\phi_x(x_0, S_0, t_0) = 0$. This is the same as the situation in Case C.

We conclude this Section by presenting a uniqueness result for viscosity solutions of the HJB equation. Such a uniqueness results will be used in the sequel to identify the candidate closed form solution with the value function. As it is common in the literature of nonlinear partial differential equations, we present the uniqueness theorem as a comparison result.

Theorem 3.2. *Let u be an upper-semicontinuous concave, with respect to x, viscosity subsolution of (3.9) on $\bar{D}$ and v a supersolution of (3.9) in D that is bounded from below, uniformly continuous on $\bar{D}$ and locally Lipschitz in D, such that $u(x,S,T) = v(x,S,T)$ and $u(x,S,t) + v(x,S,t) \leq O(x^\gamma)$, for x large, uniformly in S and t. Then $u \leq v$ on $\bar{D}$.*

The precise arguments used in the general proof of uniqueness of viscosity solutions for second order non-linear partial differential equations are rather lengthy and can be found in Ishii and Lions (1990) or in the *User's Guide* by Crandall, Ishii and Lions (1992). In the context of HJB equations arising in stochastic optimization problems of portfolio management, such uniqueness proofs can be found in Zariphopoulou (1989), (1994) and Duffie and Zariphopoulou (1993). Below, we only present the main steps of the proof and refer the technically oriented reader to Theorem 4.2 of Duffie and Zariphopoulou (1993).

We consider two functions u and v which have the desired properties and satisfy the appropriate growth conditions as stated in the assumptions of the theorem. The function u is a viscosity subsolution of (3.9) on $\bar{D}$ and v is a viscosity supersolution of (3.9) in D.

Next, we consider an arbitrary constant $m > 0$, we define the function $\Phi^m(y,t) = u(y,t) - v(y,t) - m(T-t)$ and we look at $\sup_{(y,t)\in\bar{D}} \Phi^m(y,t)$. Clearly, if $\sup_{(y,t)\in\bar{D}} \Phi^m(y,t)$ occurs at $t = T$ for all $m > 0$, then the comparison follows from using that $u(y,T) = v(y,T)$ and passing to the limit as $m \downarrow 0$. It remains to investigate if the comparison holds for the other case, i.e. in the case that there exists $\bar{m} > 0$ with $\sup_{(y,t)\in\bar{D}} \Phi^{\bar{m}}(y,t) > 0$ and the maximum occurs at a point $(y_0, t_0) \in \bar{D}$ such that $t_0 < T$. We are going to establish that this case cannot occur. To this end, we consider the function $\tilde{u}(y,t) = u(y,t) - \bar{m}(T-t)$ and we define for $y = (x,S)$, $z = (\bar{x}, \bar{S})$ with $x, \bar{x}, S, \bar{S} \in [0, +\infty)$, the function

$$\phi(y,z,t) = \left|\frac{z-y}{\delta} - 4\eta\right|^4 + \theta(x+S)^\varepsilon + \bar{m}(T-t)$$

where $\eta \in R^2$, $\varepsilon \in (\gamma, 1)$ and θ, δ are positive constants.

From the growth assumptions on u and v, the definition of $\tilde{u}$ and the role of $\bar{m}$, we get that the maximum of $\psi(y,z,t) = \tilde{u}(y,t) - v(z,t) - \phi(y,z,t)$ occurs at a point, say $(\tilde{y}, \tilde{z}, \tilde{t})$ that converges to (y_0, t_0) as $\delta, \theta \downarrow 0$ and $\|\eta\| \downarrow 0$. Using a straightforward variation of the arguments used in Theorem 4.2 of Duffie and Zariphopoulou (1993) we get – after tedious but routine calculations – that $\bar{m} \leq 0$ which is a contradiction. Note that the HJB equation studied in Duffie and Zariphopoulou (1993) has a similar structure to (3.20), (3.21) and (3.22) and the necessary calculations are explicitly outlayed in Theorem 4.2 of their paper.

4. Closed form solutions of the HJB equation

In this section, we derive closed form solutions for the value function and the optimal policies. We establish that the value function can be written as $u(S,y,t) = \frac{1}{\gamma}x^\gamma v(S,t)^\delta$ for $\delta = 1-\gamma$, with v solving a linear parabolic equation. The coefficients of the latter depend on the market coefficients and the risk aversion $1-\gamma$.

In order to demonstrate the key calculations, we start with a formal analysis assuming that all the required derivatives of the relevant solutions exist. The rigorous results together with necessary assumptions on the market coefficients are presented in subsequent theorems.

To this end, we continue with the construction of a candidate solution of the HJB equation (3.9).

As it was discussed earlier, the homogeneity of the utility function together with the fact that the state X_s and the control π_s appear linearly in (2.3), suggest that the value function must be of the form

$$u(x,S,t) = \frac{x^\gamma}{\gamma} V(S,t). \tag{4.1}$$

Direct substitution in the HJB equation (3.9) yields that $V(y,t)$ solves

$$\frac{1}{\gamma}\left[V_t + \frac{1}{2}\sigma^2(S)S^2 V_{SS} + \mu(S)SV_S\right] + rV$$
$$+ \max_{\tilde{\pi}}\left[\frac{1}{2}(\gamma-1)\sigma^2(S)\tilde{\pi}^2 V + \sigma^2(S)S\tilde{\pi}V_S + (\mu(S)-r)\tilde{\pi}V\right] = 0 \tag{4.2}$$

together with the terminal and boundary conditions

$$V(S,T) = 1 \quad \text{and} \quad V(0,t) = e^{r\gamma(T-t)}. \tag{4.3}$$

Note that the control $\tilde{\pi}$ corresponds $\dfrac{\pi}{x}$ with π being the control variable appearing in (3.9).

We now apply, formally, the first order conditions in (4.2). Observe that the maximum over $\tilde{\pi}$ in (4.3) is well defined because $0 < \gamma < 1$ and one has that $V(S,t) > 0$ (see Proposition 3.1). We have that the maximum is achieved at

$$\tilde{\pi}^*(S,t) = \frac{\sigma^2(S)SV_S(S,t) + (\mu(S)-r)V(S,t)}{(1-\gamma)\sigma^2(S)V(S,t)}. \tag{4.4}$$

Substituting the above form of $\tilde{\pi}^*(S,t)$ in (4.2) yields that V must solve

$$\frac{1}{\gamma}\left[V_t + \frac{1}{2}\sigma^2(S)S^2V_{SS} + \mu(S)SV_S\right] + rV + \frac{[(\mu(S)-r)V + \sigma^2(S)SV_S]^2}{2(1-\gamma)\sigma^2(S)V} = 0.$$

Expanding the quadratic term in the above equation and rearranging terms gives

$$V_t + \frac{1}{2}\sigma^2(S)S^2V_{SS} + \left[\mu(S)S + \frac{\gamma(\mu(S)-r)S}{(1-\gamma)}\right]V_S$$
$$+ \left[r\gamma + \frac{\gamma(\mu(S)-r)^2}{2(1-\gamma)\sigma^2(S)}\right]V + \frac{\gamma\sigma^2(S)S^2}{2(1-\gamma)}\frac{V_S^2}{V} = 0. \tag{4.5}$$

We now make the following transformation[4]. We let

$$V(S,t) = v(S,t)^{\delta} \tag{4.6}$$

for a parameter δ to be determined. Differentiating yields

$$V_t = \delta v_t v^{\delta-1}, \quad V_S = \delta v_S v^{\delta-1}, \quad V_{SS} = \delta v_{SS} v^{\delta-1} + \delta(\delta-1) v_S^2 v^{\delta-2}.$$

Substituting the above derivatives in (4.5) gives

$$\begin{aligned} &\delta v_t v^{\delta-1} + \frac{1}{2}\sigma^2(S)S^2 \delta v_{SS} v^{\delta-1} \\ &\quad + \frac{1}{2}\sigma^2(S)S^2\delta(\delta-1)v_S^2 v^{\delta-2} + \left[\mu(S)S + \frac{\gamma(\mu(S)-r)S}{(1-\gamma)}\right]\delta v_S v^{\delta-1} \\ &\quad + \left[r\gamma + \frac{\gamma(\mu(S)-r)^2}{2(1-\gamma)\sigma^2(S)}\right]v^{\delta} + \frac{\gamma\sigma^2(S)S^2\delta^2 v_S^2 v^{2(\delta-1)}}{2(1-\gamma)v^{\delta}} = 0 \end{aligned}$$

which in turn implies that v solves the quasilinear equation

$$\begin{aligned} &v_t + \frac{1}{2}\sigma^2(S)S^2 v_{SS} + \left[\mu(S)S + \frac{\gamma(\mu(S)-r)S}{(1-\gamma)}\right]v_S \\ &\quad + \frac{1}{\delta}\left[r\gamma + \frac{\gamma(\mu(S)-r)^2}{2(1-\gamma)\sigma^2(S)}\right]v + \frac{\sigma^2(S)S^2}{2}\,\frac{v_S^2}{v}\left[(\delta-1) + \frac{\gamma}{1-\gamma}\delta\right] = 0. \end{aligned}$$

The above expression indicates that if we choose the parameter δ to satisfy

$$\delta = 1-\gamma \tag{4.7}$$

then, equation (5) becomes a *linear parabolic differential equation*. In fact, if δ satisfies (4.7) then v solves

$$v_t + \tfrac{1}{2}\sigma^2(S)S^2 v_{SS} + [\mu(S)S + c(S)]v_S + k(S)v = 0 \tag{4.8}$$

with

$$v(S,T) = 1 \quad \text{and} \quad v(0,t) = e^{(r\gamma/(1-\gamma))(T-t)}. \tag{4.9}$$

The functions $c(S)$ and $k(S)$ are given in terms of the market coefficients,

$$c(S) = \frac{\gamma(\mu(S)-r)S}{(1-\gamma)} \tag{4.10}$$

$$k(S) = \frac{\gamma}{1-\gamma}\left[r + \frac{(\mu(S)-r)^2}{2(1-\gamma)\sigma^2(S)}\right]. \tag{4.11}$$

[4] This novel transformation was first introduced in Zariphopoulou (1999a).

We will refer to δ as the *distortion power*.

Observe that, δ coincides with the Risk Aversion coefficient of the utility function. Also, notice that if $0 < \gamma < 1$, then $k(S) > 0$ and under appropriate regularity and growth conditions, one expects, through the Feynman-Kac formula, the stochastic representation

$$v(S,t) = \tilde{E}\left[\exp\int_t^T \frac{\gamma}{1-\gamma}\left[r + \frac{(\mu(\tilde{S}_s) - r)^2}{2\sigma^2(\tilde{S}_s)(1-\gamma)}\right] ds \Big/ \tilde{S}_t = S\right] \tag{4.12}$$

where the process $\tilde{S}_s$, $t \le s \le T$ solves

$$d\tilde{Y}_s = [\mu(\tilde{S}_s)\tilde{S}_s + c(\tilde{S}_s)]\, ds + \sigma(\tilde{S}_s)\tilde{S}_s\, d\tilde{W}_s. \tag{4.13}$$

The process $\tilde{W}_s$ is a standard Brownian motion on a probability space $(\Omega, \mathcal{G}, Q)$ and $\tilde{E}$ is the expectation with respect to Q. Observe that the above stochastic differential equation is similar to (2.2) but with a *modified drift*.

From all the above we see that the value function u is expected to be represented in the form

$$u(x,S,t) = \frac{x^\gamma}{\gamma} v(S,t)^{1-\gamma} \tag{4.14}$$

or, alternatively,

$$u(x,S,t) = \frac{x^\gamma}{\gamma}\left(\tilde{E}\left[\exp\int_t^T \frac{\gamma}{1-\gamma}\left[r + \frac{(\mu(\tilde{S}_s) - r)^2}{2(1-\gamma)\sigma^2(\tilde{S}_s)}\right] ds \Big/ \tilde{S}_t = S\right]\right)^{1-\gamma} \tag{4.15}$$

with $\tilde{S}_s$ solving (4.13).

Using (4.4) and the representation formula (4.14) for the value function, one obtains the following simplified expression for the *optimal feedback portfolio function*

$$\pi^*(x,S,t) = \left[\frac{Sv_S(S,t)}{v(S,t)} + \frac{1}{1-\gamma}\frac{\mu(S) - r}{\sigma^2(S)}\right] x. \tag{4.16}$$

From classical arguments in stochastic control theory, one recovers the optimal portfolio process via $\Pi_s^* = \pi^*(X_s^*, Y_s, s)$ for $t \le s \le T$, where π^* is as in (4.16) and the optimal wealth X_s^* is given by (2.3) with the optimal process Π_s^* being used.

The following theorem provides a verification result for the value function.

Theorem 4.3: *The value function u is given by $u(x,S,t) = \frac{x^\gamma}{\gamma} v(S,t)^{1-\gamma}$ where v is the unique viscosity solution of (4.8) and (4.9).*

Proof: Applying the results of Ishii and Lions (1991) one easily gets that equation (4.8) has a unique viscosity solution satisfying the boundary and terminal conditions (4.9). As a matter of fact, one can show (Lions (1983)) that v admits the stochastic representation (4.12). From (4.12) and the uniform bound on the risk-premium term $\frac{\mu(S) - r}{\sigma^2(S)}$ (see (3.8)) one gets that $v(S,t) \leq e^{\lambda(T-t)}$ with $\lambda = \frac{r\gamma}{1-\gamma} + \frac{\gamma M^2}{2(1-\gamma)^2}$.

Applying once more the definition of viscosity solutions to $F(x,S,t) \equiv \frac{x^\gamma}{\gamma} v(S,t)^{1-\gamma}$, one verifies easily that F is a viscosity solution of the HJB equation for $x > 0$, $S \geq 0$ and $t \in [0,T)$. Moreover, the fact that F is a viscosity subsolution at the boundary points $(0,S,t)$ for $S \geq 0$, $t \in [0,T)$ is a direct consequence of the infinite slope $F_x(x,S,t)$ as $x \to 0$. Therefore F is a constrained viscosity solution of the HJB equation and it also belongs to the appropriate class of solutions in which uniqueness has been established. Because the value function belongs to the same class, we readily conclude that $u(x,S,t) = F(x,S,t) \equiv \frac{x^\gamma}{\gamma} v(S,t)^{1-\gamma}$.

Remark 4.1: Observe that as $S \to 0$, the equation (4.8) "converges" to $v_t + \frac{r\gamma}{1-\gamma} v = 0$ which yields the solution $v(0,t) = e^{(r\gamma/(1-\gamma))(T-t)}$ recovering the boundary condition (4.9).

5. Investment models with intermediate consumption

In this section we analyze the optimal investment model when intermediate consumption is allowed. Models of optimal consumption may allow for either infinite or finite trading horizon. In the case of infinite horizon trading, the investor does not acquire any utility from his wealth holdings in that his utility depends only on the consumption stream. If trading takes place in a finite horizon, the agent maximizes his utility function from intermediate consumption as well as his bequest utility from terminal wealth.

To preserve consistency with the optimal portfolio management model previously studied, we introduce and analyze the investment/consumption model in the case that the trading horizon is finite. We assume that the individual preferences are modelled through a Constant Relative Risk Aversion utility and a bequest function of the same risk aversion. Employing a similar transformation as in the absence of intermediate consumption, we obtain the value function and the optimal policies in closed form. The solutions are provided in terms of the "distorted" solution of an underlying equation as it was the case in (4.8). The fundamental difference when intermediate consumption is allowed is that the underlying equation turns out to be an inhomogeneous as opposed to the homogeneous linear parabolic partial differential equation (4.8).

Below we introduce the optimal investment and consumption model and we derive the closed form solutions. We choose not to present any rigorous results because most of the arguments needed to establish that the value function is the unique constrained viscosity solution of the relevant HJB equation follow along the lines of Theorems 4.1 and 4.2 of Duffie and Zariphopoulou (1993) as well as Theorems 3.1 and 3.2 herein. Once the uniqueness property is established, the verification follows readily if a smooth candidate solution is available.

To this end, we consider an optimal investment/consumption model of a single agent who manages his portfolio by investing in a bond and a stock account. The processes that the prices of the primary securities follow are the same as before solving (2.1) and (2.3).

The investor rebalances his portfolio dynamically by choosing at any time s, for $s \in [t, T]$ and $0 \le t \le T$, the amounts π_s^0 and π_s to be invested respectively in the bond and the stock accounts. He also *consumes* out of his bond holdings at a rate C_s. His total wealth satisfies the budget constraint $X_s = \pi_s^0 + \pi_s$ and the stochastic differential equation

$$\begin{cases} dX_s = rX_s\,dt + (\mu(S_s) - r)\pi_s\,ds - C_s\,ds + \sigma(S_s)\pi_s\,dW_s \\ X_t = x \ge 0 \quad 0 \le t \le s \le T. \end{cases} \tag{5.1}$$

The above state equation follows from the budget constraint and the dynamics in (2.1) and (2.2). The wealth process must also satisfy the state constraint

$$X_s \ge 0 \quad \text{a.e. } t \le s \le T. \tag{5.2}$$

The pair of control processes (π_s, C_s) is said to be admissible if it is $\mathscr{F}_s$-progressively measurable, where $\mathscr{F}_s = \sigma(W_u; t \le u \le s)$, satisfies the integrability conditions $E\int_t^T C_s\,ds < +\infty$ and $E\int_t^T \sigma(S_s)^2\pi_s^2\,ds < +\infty$ and, is such that the above state constraint is satisfied. We denote by $\mathscr{A}$ the set of admissible policies.

The investor's objective is to maximize his expected utility

$$J(x, S, t; \pi, C) = E\left[\int_t^T U(C_s)\,ds + \Phi(X_T)/X_t = x, S_t = S\right], \tag{5.3}$$

with X_s, S_s given in (5.1) and (2.2) respectively.

The *value function* of the investor is

$$v(x, S, t) = \sup_{(\pi, C) \in \mathscr{A}} J(x, y, t; \pi). \tag{5.4}$$

The utility function $U : [0, +\infty) \to [0, +\infty)$ and the *bequest function* Φ : $[0, +\infty) \times R \to [0, +\infty)$ are of the form

$$U(c) = \frac{1}{\gamma}c^\gamma \quad \text{and} \quad \Phi(x) = \frac{1}{\gamma}x^\gamma. \tag{5.5}$$

As it was the case in the model of Section 2, the special form of the above functions together with the fact that the state equation (5.1) is linear with respect to the portfolio and the consumption control processes, suggest that the value function can be written in the "separable form" $v(x, S, t) = \dfrac{x^\gamma}{\gamma}h(S, t)$. The next goal is to determine the function h.

We impose the same assumptions on the coefficients as in the no consumption case. These assumptions imply therefore that if $S_t = 0$, $S_s = 0$ for $t \le s \le T$ which in turn yields the deterministic wealth equation $d\tilde{X}_s = r\tilde{X}_s\,ds - C_s\,ds$. The solution to the deterministic control problem $\Psi(x,t) = \sup_{C\ge 0}\int_t^1 \frac{C_s^\gamma}{\gamma}\,ds$, with the state constraint $\tilde{X}_s \ge 0$, $t \le s \le T$, can be easily computed and it is given by $\Psi(x,t) = \left(\left(1+\frac{1-\gamma}{\gamma}\right)e^{(r\gamma/(1-\gamma))(T-t)} - \frac{1-\gamma}{r\gamma}\right)^{1-\gamma}$. Therefore

$$v(x,0,t) = \frac{x^\gamma}{\gamma}\left[\left(1+\frac{1-\gamma}{\gamma}\right)e^{(r\gamma/(1-\gamma))(T-t)} - \frac{1-\gamma}{r\gamma}\right]^{1-\gamma}.$$

Next, we state the main result and we outline the relevant computations.

Proposition 5.1: i) *The value function v is given by*

$$v(x,S,t) = \frac{x^\gamma}{\gamma}h(S,t) \tag{5.6}$$

where

$$h(S,t) = w(S,t)^{1-\gamma}$$

with $w : R^+ \times [0,T] \to R^+$ *being the solution of*

$$w_t + \frac{1}{2}\sigma^2(S)S^2 w_{ss} + \left[\mu(S)S + \frac{\gamma(\mu(S)-r)S}{(1-\gamma)}\right]w_s$$

$$+\frac{\gamma}{1-\gamma}\left[r + \frac{(\mu(S)-r)^2}{2\sigma^2(S)(1-\gamma)}\right]w + 1 = 0 \tag{5.7}$$

$$w(S,T) = 1 \quad \textit{and} \quad w(0,t) = \left(1+\frac{1-\gamma}{r\gamma}\right)e^{(r\gamma/(1-\gamma))(T-t)} - \frac{1-\gamma}{r\gamma}. \tag{5.8}$$

ii) *The optimal investment policy* Π_s^* *is given in the feedback form* $\Pi_s^* = \pi^*(X_s^*, S_s, s)$, $t \le s \le T$, *where the function* $\pi^* : R^+ \times [0,T] \to R$ *is defined by*

$$\pi^*(x,S,t) = \left[S\frac{w_S(S,t)}{w(S,t)} + \frac{1}{1-\gamma}\,\frac{\mu(S)-r}{\sigma^2(S)}\right]x \tag{5.9}$$

iii) *The optimal consumption policy* C_s^* *is given in the feedback form* $C_s^* = c^*(X_s^*, S_s, s)$, $t \le s \le T$, *where the function* $c^* : R^+ \times [0,T) \to R^+$ *is defined by*

$$c^*(x,S,t) = \frac{x}{w(S,t)}. \tag{5.10}$$

We continue with the formal derivation of (5.6), (5.9) and (5.10).

Following similar arguments as in the proof of Theorem 3.1 we can derive the HJB equation associated with the optimal consumption problem presented below

$$v_t + \max_{\pi}\left[\frac{1}{2}\sigma^2(S)\pi^2 v_{xx} + \sigma^2(S)S\pi v_{xS} + (\mu(S) - r)\pi v_x\right]$$
$$+ \frac{1}{2}\sigma^2(S)S^2 v_{SS} + \mu(S)Sv_S + rxv_x + \max_{C\geq 0}\left[-Cv_x + \frac{1}{\gamma}C^\gamma\right] = 0. \quad (5.11)$$

Direct substitution of a candidate solution $v(x,S,t) = \dfrac{x^\gamma}{\gamma}h(S,t)$ yields

$$\frac{1}{\gamma}\left[h_S + \frac{1}{2}\sigma^2(S)S^2 h_{SS} + \mu(S)Sh_S\right] + rh$$
$$+ \max_{\tilde{\pi}}\left[\frac{1}{2}(\gamma - 1)\sigma^2(S)\tilde{\pi}^2 h + \sigma^2(S)S\pi h_S + (\mu(S) - r)\tilde{\pi} h\right]$$
$$+ \max_{\tilde{C}\geq 0}\left[-\tilde{C}h + \frac{1}{\gamma}\tilde{C}^\gamma\right] = 0. \quad (5.12)$$

Note that the controls $\tilde{\pi}$ and $\tilde{C}$ correspond, respectively, to $\dfrac{\pi}{x}$ and $\dfrac{C}{x}$ with π and C being the control variables appearing in (5.11).

We now apply, formally, the first order conditions. We have that the maximum is achieved at

$$\tilde{\pi}^*(S,t) = \frac{\sigma^2(S)Sh_S(S,t) + (\mu(S) - r)h(S,t)}{(1-\gamma)\sigma^2(S)h(S,t)} \quad (5.13)$$

or, in terms of (x,S,t), at

$$\pi^*(x,S,t) = \left[\frac{\sigma^2(S)Sh_S(S,t) + (\mu(S) - r)h(S,t)}{(1-\gamma)\sigma^2(S)h(S,t)}\right]x. \quad (5.14)$$

Similarly, the maximum over $\tilde{C}$ is achieved at

$$\tilde{C}^*(S,t) = h(S,t)^{1/(\gamma-1)} \quad (5.15)$$

or, in terms of (x,S,t) at

$$C^*(x,S,t) = xh(S,t)^{1/(\gamma-1)}. \quad (5.16)$$

The equalities (5.13) and (5.15) follow from (5.11) and the first order conditions.

Using the form of $\tilde{\pi}^*(S,t)$ and $\tilde{C}^*(S,t)$ in (5.12) yields

$$\frac{1}{\gamma}\left[h_t + \frac{1}{2}\sigma^2(S)S^2 h_{SS} + \mu(S)Sh_S\right] + rh$$
$$+ \frac{[(\mu(S) - r)h + \sigma^2(S)Sh_S]^2}{2(1-\gamma)\sigma^2(S)h} + \frac{1-\gamma}{\gamma}h^{\gamma/(\gamma-1)} = 0.$$

Expanding the quadratic term in the above equation yields

$$h_t + \frac{1}{2}\sigma^2(S)S^2 h_{SS} + \left[\mu(S)S + \frac{\gamma(\mu(S) - r)S}{(1-\gamma)}\right]h_S$$
$$+ \left[r\gamma + \frac{\gamma(\mu(S)-r)^2}{2(1-\gamma)\sigma^2(S)}\right]h + \frac{\gamma\sigma^2(S)S}{2(1-\gamma)}\frac{h_y^2}{h} + (1-\gamma)h^{\gamma/(\gamma-1)} = 0. \quad (5.17)$$

We now make the transformation

$$h(y,t) = w(y,t)^{\delta} \quad (5.18)$$

used in previous sections, with δ being a constant to be determined. Substituting the derivatives of h in (5.17) gives

$$\delta w_t w^{\delta-1} + \frac{1}{2}\sigma^2(S)S^2\delta w_{yy}w^{\delta-1}$$
$$+ \frac{1}{2}\sigma^2(S)S^2\delta(\delta-1)w_y^2 w^{\delta-2} + \left[\mu(S) + \frac{\gamma(\mu(S)-r)}{(1-\gamma)\sigma(S)}\right]\delta w_y w^{\delta-1}$$
$$+ \left[r\gamma + \frac{\gamma(\mu(S)-r)^2}{2(1-\gamma)\sigma^2(S)}\right]w^{\delta} + \frac{\gamma\sigma^2(S)S^2}{2(1-\gamma)}\frac{\delta^2 w_y^2 w^{2(\delta-1)}}{w^{\delta}}$$
$$+ (1-\gamma)w^{\gamma\delta/(\gamma-1)} = 0$$

which in turn implies that v solves the quasilinear equation

$$w_t + \frac{1}{2}\sigma^2(S)S^2 h_{SS} + \left[\mu(S)S + \frac{\gamma(\mu(S)-r)S}{(1-\gamma)}\right]h_S$$
$$+ \frac{1}{\delta}\left[r\gamma + \frac{\gamma(\mu(S)-r)^2}{2(1-\gamma)\sigma^2(S)}\right]w + \frac{\sigma^2(S)S^2}{2}\frac{w_y^2}{w}\left[(\delta-1) + \frac{\gamma}{1-\gamma}\delta\right]$$
$$+ \frac{(1-\gamma)}{\delta}w^{(\delta r/(\gamma-1))-(\delta-1)} = 0. \quad (5.19)$$

The above expression indicates that if we choose the parameter δ to satisfy

$$\delta = 1 - \gamma \quad (5.20)$$

then, equation (5.19) becomes the inhomogeneous linear parabolic equation

$$w_t + \tfrac{1}{2}\sigma^2(S)S^2 w_{SS} + [\mu(S)S + c(S)]w_S + k(S)w + 1 = 0, \tag{5.21}$$

where the coefficients $c(S)$ and $k(S)$ are the same functions as in (4.10) and (4.11), restated below for completeness

$$c(S) = \frac{\gamma(\mu(S) - r)S}{(1-\gamma)}$$

$$k(S) = \frac{\gamma}{1-\gamma}\left[r + \frac{(\mu(S)-r)^2}{2(1-\gamma)\sigma^2(S)}\right].$$

Using (5.15) and (5.16) and the above representation formula, one obtains the following simplified expressions for the optimal feedback portfolio functions

$$\pi^*(x, S, t) = \left[\frac{w_S(S,t)}{w(W,t)} + \frac{1}{1-\gamma}\,\frac{\mu(S)-r}{\sigma^2(S)}\right]x. \tag{5.22}$$

and the optimal feedback consumption rate

$$C^*(x, S, t) = \frac{x}{w(S,t)}. \tag{5.23}$$

From classical arguments in stochastic control, under enough regularity one expects to recover the optimal portfolio and consumption processes via $\Pi_s^* = \pi^*(X_s^*, Y_s, s)$ and $C_s^* = C^*(X_s^*, Y_s, s)$ for $t \le s \le T$, where π^*, C^* and Y_s are as in (5.22), (5.23) and (2.3) respectively and the optimal wealth X_s^* is given by (2.4) with the optimal process Π_s^* being used.

Observe that as $S \to 0$, the equation (5.21) "converges" to $w_t + \dfrac{r\gamma}{1-\gamma}w + 1 = 0$, which yields the solution $w(0,t) = \left(1 + \dfrac{1-\gamma}{r\gamma}\right)e^{r\gamma/(1-\gamma)(T-t)} - \dfrac{1-\gamma}{r\gamma}$ recovering the boundary condition (5.8).

Besides the case of non-linear stock dynamics, one could investigate the optimal consumption/investment model when the stock price is affected by a non-perfectly correlated stochastic factor. Partial results for models without intermediate consumption are presented in Zariphopoulou (1999a). When intermediate consumption is allowed, preliminary work indicates that the "reduced" linear equation (5.8) becomes a *reaction-diffusion equation* (see Zariphopoulou (1999b)) in the case of diffusion price processes and an *integro-differential reaction-diffusion equation* (see Zariphopoulou (1999c)) when asset prices are modelled as jump/diffusions.

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