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ON LEVEL CURVES OF VALUE FUNCTIONS IN OPTIMIZATION MODELS OF EXPECTED UTILITY

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We study the level sets of value functions in expected utility stochastic optimization models. We consider optimal portfolio management models in complete markets with lognormally distributed prices as well as asset prices modeled as diffusion processes with nonlinear dynamics. Besides the complete market cases, we analyze models in markets with frictions like correlated nontraded assets and diffusion stochastic volatilities. We derive, for all the above models, equations that their level curves solve and we relate their evolution to power transformations of derivative prices. We also study models with proportional transaction costs in a finite horizon setting and we derive their level curve equation; the latter turns out to be a Variational Inequality with mixed gradient and obstacle constraints.

Key Words: portfolio optimization, transaction costs, level sets, derivative prices

1. INTRODUCTION

In this paper, we initiate a study of the level sets of the value functions of stochastic optimization problems that arise in utility maximization models. Level sets are sets on which the value function is constant and, as the examples below indicate, they might have a natural connection with derivative prices. The utility maximization models are the cornerstone in both areas of portfolio management and derivative security pricing, especially in incomplete markets. In fact in the latter case, such models arise in the hedging of contingent claims (see Example 1.1) as well as in the pricing of claims via utility methods. Even though when perfect replication is feasible the utility formulation is clearly redundant, this method has produced fruitful results in the presence of frictions that prohibit exact replication.

The study of the level curves has always been of central interest in nonlinear evolution problems. Problems of this nature also arise in a variety of mathematical finance models but the level curves of their solutions have not been analyzed yet. Besides studying these curves for their own sake, there is concrete evidence that they may also contain valuable information for asset valuation as the following examples indicate.

EXAMPLE 1.1. It is well known that in the presence of transaction costs perfect replication of contingent claim payoffs is not feasible. Thus one needs to relax the notion of exact replication in order to be able to price derivatives with transaction costs. Among the various methodologies proposed—the utility maximization approach, the imperfect hedging technique, and the superreplication method—the latter produces, from the practical

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point of view, the least interesting results. In fact, as Davis and Clark (1994) conjectured and Soner, Shreve, and Cvitanić (1995) established, the cheapest superreplication strategy is to buy and hold one share of the underlying security. This result was subsequently generalized by Leventhal and Skorohod (1997) who showed that if the derivative payoff $g(S_T)$ satisfies $g(S) \sim \ell S$ for large S, then in order to have exact superreplication at expiration, the least expensive strategy is to hold ℓ shares of the underlying security. Because these constraints are rather stringent and produce prices of little practical significance, it is imperative to relax the requirement of exact superreplication by allowing for a "small slippage." In other words, one may replace the almost surely superreplication requirement by the condition that the candidate (super) hedging portfolio dominates the security payoff with probability $\epsilon \in (0, 1)$ only.

A convenient way to study such questions is to formulate the problem as a singular stochastic control one and identify its value function with the maximal probability of hedging

$$V(x, y, S, t) = \sup_{(L,M)} E\left[\mathbf{1}_{\left\{x_T + \binom{\alpha}{\beta}(y_T - g(S_T)) \geq 0\right\}} / x_t = x, y_t = y, S_t = S\right]. \tag{1.1}$$

The constants α and β are related to the proportional transaction costs (see (3.13) for the definition of $\binom{\alpha}{\beta} z$) and the controlled processes $x_s, y_s, t \leq s \leq T$ represent the current size of the bond and the stock accounts. The optimization is over the set of admissible (super) hedging strategies (L, M) and the value function gives the probability of (super) hedging. It is then immediate that given a slippage threshold corresponding to superhedging probability $\epsilon \in (0, 1)$, we can determine the new price by studying the ϵ-*level sets* of V.

EXAMPLE 1.2. The utility maximization approach has been proven to be a powerful method in obtaining the so-called reservation derivative prices in the presence of market frictions. The prices are determined by comparing the maximal utility of the derivative holder/buyer to the value function without the opportunity to trade the derivative (see Hodges and Neuberger 1989; Davis, Panas, and Zariphopoulou 1993; Constantinides and Zariphopoulou 1999b). Generally speaking and with a slight abuse of notation, for a European-type derivative of payoff $g(S_T)$, the buyer's value function is

$$u(x, S, t) = \sup_{\text{Å}} \left[E \int_t^T U(C_s)\, ds + V(x_T + g(S_T), T)/X_t = x, S_t = S\right],$$

where

$$V(x, t) = \sup_{\text{Å}_0} E\left[\int_t^{T_1} U(C_s)\, ds + \Phi(X_{T_1})/X_t = x\right].$$

The processes X_s and S_s represent, respectively, the wealth and the primitive asset price; the functions U and Φ are the utility functions of intermediate consumption and terminal wealth, satisfying $U(0) = \Phi(0) = 0$; the trading horizon T_1 is taken to dominate the expiration time T. The sets of admissible policies Å and Å_0 are appropriately defined to guarantee that the necessary nonnegativity wealth constraints are met.

In the frictionless case, the price of the derivative is the unique function $h \equiv h(S,t)$ such that for *all* (x, S, t)

$$V(x,t) = u(x - h(S,t), S, t).$$

One may easily show, after some tedious but routine calculations, that $h(S,t)$ solves the Black and Scholes equation and that the *zero-level sets* of u are described by the derivative price.

EXAMPLE 1.3. Recently Carr, Tari, and Zariphopoulou (1999) showed that in the absence of arbitrage, the so-called absolute volatility function $a(S_s, s)$, $t \le s \le T$, of the underlying stock price process S_s, must satisfy the nonlinear parabolic problem

$$\begin{cases} a_t + \frac{1}{2}a^2 a_{yy} + k(t) y a_y = q(t)a \\ a(0,t) = 0, \qquad a(y,T) = \psi(y), \qquad (y,t) \in \mathcal{R}^+ \times [0,T]. \end{cases} \tag{1.2}$$

The functions $k(t)$ and $q(t)$ depend on the interest rate and the dividends. The terminal condition $\psi(y)$ represents the volatility data for a given "smile." As we show in Section 2, the *slope* $f(x,t)$ of the *level curves* of the value function of the classical Merton problem (see Merton 1969, 1971), is given by $f(x,t) = \delta\pi(x,t) + rx$. The coefficients δ and r are positive constants and π solves a problem similar to (1.2) (see equation (2.18) later).

Motivated by the examples above, we start herein a systematic, albeit preliminary, study of the level sets that arise in various utility maximization problems. The basic analysis is carried out through the properties of the relevant Hamilton–Jacobi–Bellman (HJB) equation that their value function is expected to solve. We analyze the level curves of the Merton problem for lognormally distributed prices as well as for the case of nonlinear price dynamics. In the first case, the slope of the level curves solves a terminal value problem similar to (1.2) and in the second case, under constant relative risk aversion (CRRA) references, the level curves are expressed directly as powers of a derivative price.

In Section 3, we study the portfolio optimization problems with stochastic volatility, when the latter is modeled as a diffusion correlated with the underlying stock price, and with transaction costs.

2. MODELS WITH NO FRICTIONS

We study the level curves of the value function of the classical optimal portfolio management model with general preferences. This model was introduced by Merton (1969, 1971) for the case of hyperbolic absolute risk aversion (HARA) utility functions and lognormally distributed stock prices, and subsequently was generalized by various authors (see, among others, Karatzas et al. 1987; Grossman and Zhou 1993; Cvitanić and Karatzas 1996; Vila and Zariphopoulou 1997; and Karatzas 1997).

We show that for general preferences the slope of the level curves is proportional to the optimal feedback portfolio rule. Moreover, we prove that it solves a nonlinear partial differential equation for which we establish uniqueness of solutions. A by-product of the latter fact is a comparison result for the optimal feedback portfolio policies in terms of the individual's absolute risk aversion coefficient.

2.1. Models with Lognormal Stock Prices

We start with a brief review of the Merton model assuming general utility functions and market completeness. To this end, we consider an economy with two securities, a bond and a stock. The bond's price B_s is deterministic and evolves, for $0 \le t \le s \le T$, according to

$$\begin{cases} dB_s = rB_s ds \\ B_t = B > 0 \end{cases} \tag{2.1}$$

with r being the *interest rate*. The stock price is modeled as a diffusion process S_s solving for $0 \le t \le s \le T$, the stochastic differential equation

$$\begin{cases} dS_s = \mu S_s\, ds + \sigma S_s\, dW_s \\ S_t = S > 0. \end{cases} \tag{2.2}$$

The market parameters μ and σ are respectively the *mean rate of return* and the *volatility*; it is assumed that $\mu > r > 0$ and $\sigma > 0$. The process W_s is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$.

Trading takes place between the bond and the stock accounts continuously in time, in the trading horizon $[0, T]$. The *wealth process* satisfies $X_s = \pi_s^0 + \pi_s$ with π_s^0 and π_s representing the current holdings in the bond and the stock accounts.

Using the price equations (2.1) and (2.2) one may easily derive the equation for the state process

$$dX_s = rX_s\, ds + (\mu - r)\pi_s\, ds + \sigma\pi_s dW_s. \tag{2.3}$$

The wealth process must also satisfy the state constraint

$$X_s \ge 0 \text{ a.e.} \qquad t \le s \le T. \tag{2.4}$$

The control $\pi_s, t \le s \le T$ is admissible if it is $\mathcal{F}_s$-progressively measurable, with $\mathcal{F}_s = \sigma(W_u; t \le u \le s)$, if it satisfies $E\int_t^T \pi_s^2\, ds < +\infty$ and, if it is such that the state constraint (2.4) is satisfied. We denote the set of admissible policies by Å.

The value function is defined as

$$u(x, t) = \sup_{\text{Å}} E\big[U(X_T)/X_t = x\big], \tag{2.5}$$

where $U = R^+ \to R^+$ is the utility function modeling the individual preferences.

ASSUMPTION 2.1. The utility function $U \in (C^1[0, +\infty) \cap C^2(0, +\infty))$ is increasing, concave, and satisfies the growth condition $U(x) \le K(1 + x)^\gamma$ for some positive constants K and γ, with $\gamma \in (0, 1)$. Moreover, $U(0) = 0$ and $-U'(x)/U''(x) = O(x)$ for large x.

The following result was proved in Karatzas et al. (1987).

PROPOSITION 2.1.

(i) *The value function* $u \in C^{2,1}((0, +\infty), [0, T])$ *is the unique increasing and concave solution of the Hamilton–Jacobi–Bellman equation*

$$\begin{cases} u_t + \max_{\pi} \left[\frac{1}{2}\sigma^2\pi^2 u_{xx} + (\mu - r)\pi u_x \right] + rxu_x = 0 & (2.6) \\ u(x, T) = U(x) \quad \textit{and} \quad u(0, t) = 0, \qquad t \in [0, T]. & (2.7) \end{cases}$$

(ii) *The optimal policy* π_s^*, $t \le s \le T$ *is given in the feedback form* $\pi_s^* = \hat{\pi}(X_s^*, s)$ *where* $\hat{\pi} : R^+ \times [0, T] \to R^+$ *is*

$$\hat{\pi}(x, t) = -\frac{\mu - r}{\sigma^2} \frac{u_x(x, t)}{u_{xx}(x, t)} \tag{2.8}$$

and X_s^* *is the solution of (2.3) with the policy* π_s^* *being used.*

We now explore the HJB equation (2.6) from a different point of view. First, we evaluate it at the optimum point (2.8), which yields

$$u_t - \frac{\mu - r}{2\sigma^2} \frac{u_x^2}{u_{xx}} + rxu_x = 0.$$

Therefore, one may interpret the HJB equation (2.6) as the *first-order wave equation*

$$\begin{cases} u_t + f(x, t)u_x = 0 \\ u(x, T) = U(x) \quad \text{and} \quad u(0, t) = 0, \end{cases} \tag{2.9}$$

where

$$f(x, t) = \frac{\mu - r}{2} \hat{\pi}(x, t) + rx. \tag{2.10}$$

The above equation is known as the *traveling wave equation of first order* (see, e.g., Zauderer 1983). It is well known for this class of equations that the solution u of (2.9) is constant along the *characteristic curves*, denoted herein by $\tilde{x}(s), t \le s \le T$. For a given positive constant c, the characteristic curve, say $\tilde{x}^c(s)$, is defined as the set $\tilde{x}^c(s)$ on which the value function remains constant; that is,

$$u(\tilde{x}^c(s), s) = c. \tag{2.11}$$

It is then immediate, in view of (2.9), that the characteristic curves of (2.6) have slope

$$\frac{d\tilde{x}^c(s)}{ds} = f(\tilde{x}^c(s), s) = \frac{\mu - r}{2} \hat{\pi}(\tilde{x}^c(s), s) + r\tilde{x}^c(s) \tag{2.12}$$

and satisfy at $t = T$,

$$\tilde{x}^c(T) = U^{-1}(c). \tag{2.13}$$

The goal for the rest of this section is to study the evolution of the level curves $\tilde{x}^c(s)$. We accomplish this by studying an autonomous equation that their slope f solves. To this end, we show that f solves a nonlinear equation, see (2.15), and that, under mild growth and regularity conditions, f is in fact its unique solution.

PROPOSITION 2.2. *The slope of the characteristic curves $f(x,t)$ given in (2.10) satisfies, for $x > 0$,*

$$f(x,t) > rx, \tag{2.14}$$

and it solves the nonlinear parabolic problem

$$\begin{cases} f_t + \dfrac{2\sigma^2}{(\mu - r)^2}(f - rx)^2 f_{xx} + rxf_x = rf, & (2.15)\\ f(x,T) = -\dfrac{(\mu - r)^2}{2\sigma^2}\dfrac{U'(x)}{U''(x)} + rx, \qquad \forall x \geq 0, & (2.16)\\ f(0,t) = 0, \qquad 0 \leq t \leq T. & (2.17)\end{cases}$$

Proof. First, we recall that the value function u is concave and strictly increasing for $x > 0$ (see Karatzas 1997). Therefore, $\hat{\pi}(x,t) > 0$, which in view of (2.10) yields (2.14). To derive equation (2.15), we first use that under Assumption 2.1, the optimal portfolio feedback function $\hat{\pi}(x,t)$ solves

$$\hat{\pi}_t + \frac{1}{2}\sigma^2\hat{\pi}^2\hat{\pi}_{xx} + rx\hat{\pi}_x = r\hat{\pi} \tag{2.18}$$

with

$$\hat{\pi}(x,T) = -\frac{\mu - r}{\sigma^2}\frac{U'(x)}{U''(x)} \quad \text{and} \quad \hat{\pi}(0,t) = 0. \tag{2.19}$$

The above equalities follow respectively from (2.8) and (2.9) and, the state constraint (2.4). Equation (2.18) was derived by He and Huang (1994) and it was further studied by Huang and Zariphopoulou (1999). The arguments used for its derivation are rather technical and tedious and we do not present them here; instead, we refer the technically oriented reader to the above references.

Equation (2.15) and the terminal and boundary conditions (2.16) and (2.17) are then a direct consequence of (2.18), (2.19) and the definition of f in (2.10). □

The following theorem provides a uniqueness result for the solutions of the fully nonlinear equation (2.15).[1]

THEOREM 2.1. *Let $f : \mathcal{R}^+ \times [0,T] \to \mathcal{R}^+$ be a solution of (2.15)–(2.17) satisfying the terminal condition $\phi(x) \equiv f(x,T)$ with $\phi \in C^2[0,+\infty)$ and $\phi(x) \sim O(x)$ for x large. Then f is the unique solution of (2.15)–(2.17) in the class of functions satisfying $f(x,t) \sim O(x)$ for x large and $|(f^2(x,t))_{xx}| \leq C$ for $(x,t) \in \mathcal{R}^+ \times [0,T]$ and some given constant C.*

Proof. The uniqueness result will follow once we establish that $\hat{\pi}(x,t)$ is the unique solution of (2.18) and (2.19). To simplify the presentation we assume that all coefficients

[1] Similar results have been recently used by Carr et al. (1999) to establish the unique characterization of volatility surfaces given a specified "volatility smile" at the expiration time of European derivatives.

appearing in (2.15)–(2.17) are equal to one and we denote its solution by $a(x,t)$; in other words, with a slight abuse of notation we define,

$$a(x,t) = \hat{\pi}(x,t;\sigma = 1, \mu - r = 1, r = 1)$$

to be a solution of

$$\begin{cases} a_t + \frac{1}{2}a^2 a_{xx} + x a_x = a & (2.20) \\ a(x,T) = -\frac{U'(x)}{U''(x)} \quad \text{and} \quad a(x,t) = 0. & (2.21) \end{cases}$$

First, we observe that if $\tilde{a}(x,t)$ satisfies (2.21) and solves the nonlinear problem

$$\tilde{a}_t + \frac{1}{2}\tilde{a}^2\tilde{a}_{xx} = 0, \tag{2.22}$$

then the function

$$a(x,t) = e^{-(T-t)}\tilde{a}(xe^{(T-t)}, t)$$

solves (2.20) and (2.21); this can be easily verified by direct differentiation.

Given the above, it suffices to establish uniqueness for the solutions of (2.21) and (2.22). To this end, we define $F : \mathcal{R}^+ \times [0,T] \to \mathcal{R}^+$ to be

$$F(x,t) = \tilde{a}^2(x,t). \tag{2.23}$$

Direct calculations yield that F solves

$$\begin{cases} F_t(x,t) + \frac{1}{2}F(x,t)F_{xx}(x,t) = F_x^2(x,t) & (2.24) \\ F(x,T) = \left(-\frac{U'(x)}{U''(x)}\right)^2 \quad \text{and} \quad F(0,t) = 0, \qquad 0 \le t \le T. & (2.25) \end{cases}$$

From the assumptions on $f(x,t)$ and therefore on $\tilde{\pi}(x,t)$ and, in turn, on $\tilde{a}(x,t)$, we get that $F(x,t) \sim O(x^2)$ for x large and that $F(x,t)_{xx} \le C$ for $(x,t) \in \mathcal{R}^+ \times [0,T)$. Using a variation of the results of Fukuda, Ishii, and Tsutsumi (1993) we get that (2.24), (2.25) has a unique solution.

Therefore, if $a_1(x,t)$ and $a_2(x,t)$ are two solutions of (2.22), satisfying also (2.21), the above uniqueness result yields that

$$a_1^2(x,t) = a_2^2(x,t). \tag{2.26}$$

Next, we look at the difference $G(x,t) = a_1(x,t) - a_2(x,t)$. Differentiation and use of (2.21) yield that G solves

$$\begin{cases} G_t(x,t) + \frac{1}{2}a_1^2(x,t)G_{xx}(x,t) = 0 \\ G(0,t) = 0 \quad \text{and} \quad G(x,T) = 0, \qquad 0 \le t \le T. \end{cases} \tag{2.27}$$

Working as above for $\widehat{G}(x,t) = a_2(x,t) - a_1(x,t)$ yields that $\widehat{G}$ solves

$$\widehat{G}_t(x,t) + \frac{1}{2}a_2^2(x,t)\widehat{G}_{xx}(x,t) = 0, \tag{2.28}$$

which, in view of (2.26), coincides with (2.27). Moreover, $\widehat{G}(0,T) = 0$ and $\widehat{G}(x,T) = 0$. We can easily verify that equation (2.27) (or (2.28)) admits a comparison principle and we readily conclude that $G(x,t) \equiv 0$ and therefore, $a_1(x,t) = a_2(x,t)$ for $(x,t) \in \mathcal{R}^+ \times [0,T]$. □

The following result is an interesting consequence of the uniqueness of solutions of the autonomous portfolio equation (2.18). It shows that two investors with absolute risk aversion coefficients, say $R_1(x)$ and $R_2(x)$ satisfying $R_1(x) \le R_2(x)$, always choose their optimal portfolio policies $\hat{\pi}_1(x,t)$ and $\hat{\pi}_2(x,t)$, such that $\hat{\pi}_1(x,t) \ge \hat{\pi}_2(x,t)$. Therefore, it is only the *terminal* ordering in the optimal portfolios, via the absolute risk aversion coefficient, that determines the dynamic ordering of all trading times. Even though this result follows easily in the case of constant relative risk aversion (CRRA) and exponential utilities, to our knowledge, this is the first time that this monotonic behavior is established for dynamic trading models with general individual preferences.

PROPOSITION 2.3. *Assume that utilities U_1 and U_2 have absolute risk aversion coefficients $\mathcal{R}_1$ and $\mathcal{R}_2$ satisfying $\mathcal{R}_1(x) \le \mathcal{R}_2(x)$; that is,*

$$-\frac{U_1''(x)}{U_1'(x)} \le -\frac{U_2''(x)}{U_2'(x)} \tag{2.29}$$

and $U_1(0) = U_2(0) = 0$. *Consider the relevant utility maximization problems (2.6) and (2.7) for utilities U_1 and U_2 and denote by $\pi_1^*(x,t)$ and $\pi_2^*(x,t)$ respectively their optimal feedback portfolio rules. Assume that π_1^* and π_2^* satisfy the growth and regularity conditions $\pi_i^*(x,t) \sim O(x)$ and $|(\pi_i^*)^2_{xx}| \le C$, for a large constant C and $i = 1,2$. Then*

$$\pi_1^*(x,t) \ge \pi_2^*(x,t), \qquad 0 \le t \le T. \tag{2.30}$$

2.2. Models with Nonlinear Stock Dynamics

We consider the generalization of the Merton model in a market with two securities, a deterministic bond and a stock. We allow for the stock price to follow a diffusion process with nonlinear dynamics. In this setting, the portfolio optimization problem becomes two-dimensional and closed-form solutions are not generally available. The case of CRRA functions was recently studied by Zariphopoulou (1999a) who produced the solutions in a reduced form (see Proposition 2.3 below).

We represent the stock price as the solution of

$$dS_s = \mu(S_s)S_s\,ds + \sigma(S_s)S_s\,dW_s. \tag{2.31}$$

The process W_s is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ and the coefficients μ, σ are given functions of the current stock price. They are assumed to satisfy, respectively, the global Lipschitz and linear growth conditions $|f(y) - f(\bar{y})| \le k|y - \bar{y}|$ and $f^2(y) \le k^2(1 + y^2)$ for $y \ge 0$, k being a positive constant and f standing for μ and σ. Moreover there exist positive constants ℓ_1 and ℓ_2 such that, for $y \ge 0$, $\sigma(y) \ge \ell_1$ and $((\mu(y) - r)^2)/\sigma^2(y) \le \ell_2$.

With the above nonlinear stock price dynamics, the wealth state equation becomes

$$\begin{cases} dX_s = rX_s\,ds + (\mu(S_s) - r)\pi_s\,ds + \sigma(S_s)\pi_s\,dW_s \\ X_t = x \ge 0, \qquad 0 \le t \le s \le T \end{cases} \tag{2.32}$$

with X_s being the current wealth satisfying the state constraint $X_s \ge 0$ a.s., $t \le s \le T$.

The utility function is of CRRA type

$$U(x) = \frac{1}{\gamma} x^{\gamma} \tag{2.33}$$

with $\gamma \in (0, 1)$.

The value function is

$$u(x, S, t) = \sup_{\text{Å}} E\left[U(X_T)/X_t = x, S_t = S\right]$$

with Å being the set of admissible portfolios.

The proof of the following result is in Zariphopoulou (1999a).

PROPOSITION 2.4.

(i) *The value function u is given by*

$$u(x, S, t) = \frac{x^{\gamma}}{\gamma} V(S, t)^{1-\gamma}, \tag{2.34}$$

where $V : R^+ \times [0, T] \to R^+$ solves the linear parabolic equation

$$\begin{cases} V_t + \frac{1}{2}\sigma^2(S)S^2 V_S + \left[\mu(S)S + \frac{\gamma(\mu(S) - r)S}{(1-\gamma)}\right] V_S \\ \qquad + \frac{\gamma}{1-\gamma}\left[r + \frac{(\mu(S) - r)^2}{2\sigma^2(S)(1-\gamma)}\right] V = 0 \\ V(S, T) = 1 \quad \text{and} \quad V(0, t) = e^{\frac{r\gamma}{1-\gamma}(T-t)}, \qquad 0 \le t \le T. \end{cases} \tag{2.35}$$

(ii) *The optimal portfolio policy π_s^* is given in the feedback form $\pi_s^* = \tilde{\pi}_s(X_s^*, S_s, s)$ where*

$$\tilde{\pi}(x, S, t) = \left[\frac{SV_S}{V} + \frac{\mu(S) - r}{(1-\gamma)\sigma^2(S)}\right] x.$$

Using the above representation, one may obtain the level sets of u in a simplified form. In fact, given $c > 0$ and $x^c(S, t)$ being such that

$$u(x^c(S, t), S, t) = c,$$

the representation (2.34) yields

$$x^c(S, t) = (c\gamma)^{\frac{1}{\gamma}} [V(S, t)]^{(\gamma-1)/\gamma}, \tag{2.36}$$

with V solving the linear equation (2.35).

So we see that in the case of complete markets with stocks modeled as diffusion prices but with nonlinear dynamics the level sets are represented as powers of solutions of linear parabolic equations. Since such equations are directly related to prices of European type derivative securities, we observe an interesting connection between level sets and derivative prices.

3. MODELS WITH FRICTIONS

In this section we derive the level sets of two fundamental models of optimal portfolio management in markets with frictions.

3.1. Models with Nontraded Assets

These models are similar to the ones we studied in the previous section but we allow for a *nontraded asset* in the market environment. This asset affects the returns of the underlying security and it is in general correlated with it. A special case is when the volatility is stochastic and it is modeled as a correlated diffusion process. Of course, since the volatility is usually unobservable the model might not be very realistic albeit useful for certain approximations.

We assume that trading takes place between a bond account (with the bond price given by (2.1)) and a stock account with the stock price S_s solving

$$dS_s = \mu S_s\, ds + \sigma(Y_s) S_s\, dW_s^1, \tag{3.1}$$

where $\mu > r > 0$ and Y_s is given by

$$dY_s = b(Y_s, s)\, ds + a(Y_s, s)\, dW_s^2. \tag{3.2}$$

The processes W_s^1 and W_s^2 are standard Brownian motions on a probability space $(\Omega, \mathcal{F}, P)$ correlated with *correlation coefficient* $\rho \in (-1, 1)$. The coefficients $\sigma : \mathcal{R} \to \mathcal{R}^+$ and $b, \alpha : \mathcal{R} \times [0, T] \to \mathcal{R}$ satisfy the global Lipschitz and linear growth conditions $|f(y, t) - f(\bar{y}, t)| \le K|y - \bar{y}|$ and $f^2(y, t) \le k^2(1 + y^2)$, for every $t \in [0, T]$, $y, \bar{y} \in \mathcal{R}$, K being a positive constant and f standing for σ, b, and a. Moreover, uniformly in $y \in \mathcal{R}$ and $t \in [0, T]$, there is a positive constant ℓ such that for $y \in \mathcal{R}$ and $t \in [0, T]$, $\sigma(y) \ge \ell$.

The value function w is

$$w(x, y, t) = \sup_{Å_1} E\left(\frac{1}{\gamma} X_T^\gamma \Big/ X_t = x, Y_t = y\right). \tag{3.3}$$

Here $Å_1$ is the set of admissible policies π_s which are $\mathcal{F}_s$-progressively measurable processes, with $\mathcal{F}_s = \sigma((W_u^1, W_u^2); t \le u \le s)$, which satisfy the integrability condition

$$E \int_t^T (\sigma(Y_s)^2) \pi_s^2\, ds < +\infty,$$

and which are such that the state wealth X_s satisfies $X_s \ge 0$ a.e., $t \le s \le T$.

Using the state equations (2.1), (3.1), and (3.2), one easily derives the stochastic differential equation for X_s, namely

$$dX_s = rX_s\, ds + (\mu - r)\pi_s\, ds + \sigma(Y_s)\pi_s\, dW_s^1. \tag{3.4}$$

This generalization of the Merton problem was recently solved by the author (see Zariphopoulou 1999b). Using the apparent homogeneity of the problem and a convenient power transformation, one may obtain the value function in a reduced form. For the proof of the following result we refer the reader to Theorem 3.3 of Zariphopoulou (1999b).

THEOREM 3.1. *The value function w is given by*

$$w(x, y, t) = \frac{x^\gamma}{\gamma} H(y, t)^{(1-\gamma)/(1-\gamma+\rho^2\gamma)}, \tag{3.5}$$

where $H : R \times [0, T] \to R^+$ solves the linear parabolic problem

$$H_t + \frac{1}{2} a^2(y, t) H_{yy} + \left[b(y, t) + \rho \frac{\gamma(\mu - r) a(y, t)}{(1 - \gamma)\sigma(y)} \right] H_y + \frac{\gamma(1 - \gamma + \rho^2\gamma)}{1 - \gamma} \left[r + \frac{(\mu - r)^2}{2\sigma^2(y)(1 - \gamma)} \right] H = 0 \tag{3.6}$$

$$H(y, T) = 1. \tag{3.7}$$

The following result is a direct consequence of the representation formula (3.5) for the value function.

PROPOSITION 3.1. *The curve $x^c(y, t)$ on which the value function satisfies $w(x^c(y, t), y, t) = c$ is given by*

$$x^c(y, t) = (c\gamma)^{1/\gamma} H(y, t)^{(1-\gamma)/(\gamma(1-\gamma+\rho^2\gamma))} \tag{3.8}$$

with H solving (3.6) and (3.7).

3.2. Models with Transaction Costs

Transaction costs have always been present in financial transactions and their role in asset pricing has long been of central interest, especially when the financial assets involved have different liquidity.

The stochastic control problems that arise in models with transaction costs are of singular type and their HJB equation becomes a Variational Inequality with gradient constraints. The majority of existing work on the subject deals with infinite horizon problems of optimal consumption; see, the pioneering paper of Magill and Constantinides (1976) and the seminal paper of Davis and Norman (1990). Given that a considerable number of applications deal with dynamic trading in a finite horizon, it is highly desirable to study the finite horizon case as well. Important optimization problems in which the finiteness of the horizon is crucial arise in models of derivative pricing with transaction costs via the utility maximization approach. These stochastic portfolio optimization problems consider the optimal policies of the writer and/or the buyer of the derivative security, which in turn yield useful bounds on the selling and the buying price (see, e.g., Davis et al. 1993; Davis and Zariphopoulou 1995; Barles and Soner 1998; Constantinides and Zariphopoulou 1999a, 1999b).

In the sequel we review briefly the underlying finite horizon model and we proceed with the derivation of the equation of the level curves. To this end, we consider a market with two securities, a bond and a stock whose prices solve (2.1) and (2.2) respectively. Trading takes place between the bond the stock accounts and there is no intermediate consumption. The amounts x_s and y_s invested, respectively, in the bond and the stock account evolve according to the controlled state equations

$$\begin{cases} dx_s = rx_s\,ds - (1+\lambda)\,dL_s + (1-\mu)\,dM_s \\ x_t = x, \qquad 0 \le t \le s \le T, \end{cases} \tag{3.9}$$

and

$$\begin{cases} dy_s = \mu y_s\,ds + \sigma y_s\,dW_s + dL_s - dM_s \\ y_t = y, \qquad 0 \le t \le s \le T. \end{cases} \tag{3.10}$$

The control processes L_s and M_s represent the cumulative purchases and sales of stock. The pair (L_s, M_s) is admissible if the processes L_s and M_s are $\mathcal{F}_s$-progressively measurable, right continuous with left limits, and the state constraint

$$x_s + \binom{\alpha}{\beta} y_s \ge 0 \text{ a.e.} \qquad t \le s \le T \tag{3.11}$$

is satisfied, where

$$\alpha = 1 - \mu \quad \text{and} \quad \beta = 1 + \lambda. \tag{3.12}$$

For the rest of the paper, to ease the presentation we adopt the notation

$$\binom{\alpha}{\beta} z = \begin{cases} \alpha z & \text{if } z \ge 0 \\ \beta z & \text{if } z < 0. \end{cases} \tag{3.13}$$

We denote the set of admissible policies by $\mathring{A}_2$. The value function is defined as

$$V(x, y, t) = \sup_{\mathring{A}_2} E\left[\frac{1}{\gamma}\left(x_T + \binom{\alpha}{\beta} y_T\right)^{\gamma} \Big/ x_t = x, y_t = y\right], \tag{3.14}$$

where

$$(x, y) \in \overline{D} = \left\{(x, y) \in \mathcal{R} : x + \binom{\alpha}{\beta} y \ge 0\right\}.$$

Following arguments similar to the ones used in Constantinides and Zariphopoulou (1999b) yields the following result.

THEOREM 3.2. *The value function is the unique concave and increasing in x and y, constrained viscosity solution on $\overline{D} \times [0, T]$ of the Variational Inequality*

$$\min\left\{-V_t - \frac{1}{2}\sigma^2 y^2 V_{yy} - \mu y V_y - rxV_x, \beta V_x - V_y, -\alpha V_x + V_y\right\} = 0 \tag{3.15}$$

satisfying

$$V(x, y, T) = \frac{1}{\gamma}\left(x + \binom{\alpha}{\beta} y\right)^{\gamma}. \tag{3.16}$$

The fact that one needs to relax the notion of solutions to the Hamilton–Jacobi–Bellman equation of stochastic control problems involving models with frictions is by now well established. For the use of viscosity solutions in models with transaction costs, we refer the technically interested reader to the review article by Zariphopoulou (1999c).

We are now ready to derive the equation which the level curves of V satisfy. Note that up-to-date complete results on the regularity of the value function are generally not available and the calculations below are formal.

To this end, we consider a constant $c > 0$ and we look for the function $g : \mathcal{R} \times [0, T] \to \mathcal{R}$ such that

$$V(x, g(x,t), t) = c. \tag{3.17}$$

We recall that V is jointly homogeneous of degree γ which yields

$$xV_x(x, g(x,t), t) + g(x,t)V_y(x, g(x,t), t) = \gamma V(x, g(x,t), t) \tag{3.18}$$

and, in turn, that

$$\begin{aligned} &xV_{xx}\big(x, g(x,t), t) + g(x,t)V_{xy}(x, g(x,t), t\big) \\ &\qquad = (1-\gamma)g_x(x,t)V_y(x, g(x,t), t). \end{aligned} \tag{3.19}$$

Differentiating twice, (3.17) with respect to x yields

$$\begin{aligned} &V_{xx}(x, g(x,t), t) + 2g_x(x,t)V_{yy}(x, g(x,t), t) \\ &\qquad + g_{xx}(x,t)V_y(x, g(x,t), t) + g_x^2(x,t)V_{yy}(x, g(x,t), t) = 0. \end{aligned} \tag{3.20}$$

Combining (3.19) and (3.20) gives

$$V_{xy} = \frac{\big[(1-\gamma)g_x + xg_{xx}\big]V_y + xg_x^2 V_{yy}}{g - 2xg_x} \tag{3.21}$$

with all the above derivatives of V being evaluated at the point $(x, g(x,t), t)$.

Using again the homogeneity of V implies

$$xV_{xy}(x, g(x,t), t) + g(x,t)V_{yy}(x, g(x,t), t) = -(1-\gamma)V_y(x, g(x,t), t),$$

which together with (3.21) results in

$$\frac{V_{yy}(x, g(x,t), t)}{V_y(x, g(x,t), t)} = -\frac{1-\gamma}{g(x,t) - xg_x(x,t)} - \frac{x^2 g_{xx}(x,t)}{\big(g(x,t) - xg_x(x,t)\big)^2}. \tag{3.22}$$

Differentiating (3.17) with respect to time and x respectively, implies

$$V_t(x, g(x,t), t) = -g_t(x,t)V_y(x, g(x,t), t) \tag{3.23}$$

and

$$V_x(x, g(x,t), t) = -g_x(x,t)V_y(x, g(x,t), t). \tag{3.24}$$

Combining (3.22), (3.23), and (3.24) yields that the second-order operator appearing in (3.15), namely

$$\mathcal{L}V = -\left\{V_t + \tfrac{1}{2}\sigma^2 y^2 V_{yy} + \mu y V_y + rxV_x\right\}, \tag{3.25}$$

when evaluated at $(x, g(x,t), t)$ becomes

$$\begin{aligned}\mathcal{L}V(x, g(x,t), t) &= V_y(x, g(x,t), t)\\ &\times\left[g_t(x,t) + \frac{1}{2}\sigma^2 g^2(x,t)\left(\frac{1-\gamma}{g(x,t) - xg_x(x,t)}\right.\right.\\ &\qquad\left.\left. + \frac{x^2 g_{xx}(x,t)}{(g(x,t) - xg_x(x,t))^2}\right) - \mu g(x,t)\right].\end{aligned} \tag{3.26}$$

From (3.24) we get that the gradient terms

$$\mathcal{L}_1 V = \beta V_x - V_y \quad \text{and} \quad \mathcal{L}_2 V = -\alpha V_x + V_y$$

evaluated at $(x, g(x,t), t)$ become

$$\mathcal{L}_1 V(x, g(x,t), t) = -V_y(x, g(x,t), t)(\beta g_x(x,t) + 1) \tag{3.27}$$

and

$$\mathcal{L}_2 V(x, g(x,t), t) = V_y(x, g(x,t), t)(\alpha g_x(x,t) + 1). \tag{3.28}$$

Combining (3.26)–(3.28) and canceling the common term V_y gives the equation that $g(x,t)$ satisfies. The latter turns out to be the Variational Inequality

$$\min\left\{g_t + \frac{1}{2}\sigma^2 g^2\left[\frac{1-\gamma}{g - xg_x} + \frac{x^2 g_{xx}}{(g - xg_x)^2}\right] - \mu g, -(\beta g_x + 1), \alpha g_x + 1\right\} = 0. \tag{3.29}$$

The terminal condition $g(x,T)$ is recovered easily from (3.16) and it is given by

$$g(x,T) = \begin{cases} \dfrac{c^{\frac{1}{\gamma}} - x}{\beta} & \text{if } x \geq c^{1/\gamma} \\ \dfrac{c^{\frac{1}{\gamma}} - x}{\alpha} & \text{if } x < c^{1/\gamma}. \end{cases} \tag{3.30}$$

Next we make the following transformations.

REMARK 3.1. One may further simplify the second-order part in (3.29) using a number of transformations. In fact, if $k : R \times [0, T] \to R$ is such that $k(x,t) = e^{-\mu t} e^{-\frac{x}{\gamma}} g\big(e^{\frac{x}{\gamma}}, t\big)$, $0 \leq t \leq T$ and $p : \mathcal{R} \times [0, T] \to \mathcal{R}$ is given by $p(x,t) = k\big(x, \frac{2}{\sigma^2} t\big)$ for $0 \leq t \leq \overline{T}$ with

$\overline{T} = \sigma^2 T/2$, after lengthy arguments, one can argue that there is a well-defined function $q(x,t)$ such that $p(q(x,t),t) = x$. Defining

$$S(x,t) = \exp\left\{-q(e^x,t) + \frac{x}{2} + \frac{t}{4}\right\}$$

one gets, after tedious but routine calculations, that S solves

$$\min\left\{S_t + S_{xx},\ \alpha e^{\frac{2\mu}{\sigma^2}t}\left(-\frac{S_x}{S} + \gamma + \frac{1}{2}\right) + 1,\ -\beta e^{\frac{2\mu}{\sigma^2}t}\left(-\frac{S_x}{S} + \gamma + \frac{1}{2}\right) - 1\right\} = 0$$

with terminal condition

$$S(x,\overline{T}) = e^{-\frac{x}{2}-\frac{\overline{T}}{4}}\left[\frac{\alpha\mathbf{1}_{\{x<0\}} + \beta\mathbf{1}_{\{x\geq 0\}}}{e^{\mu\overline{T}}}e^x + 1\right]^\gamma.$$

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