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Computation of distorted probabilities for diffusion processes via stochastic control methods

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Abstract

We study distorted survival probabilities related to risks in incomplete markets. The risks are modeled as diffusion processes, and the distortions are of general type. We establish a connection between distorted survival probabilities of the original risk process and distortion-free survival probabilities of new pseudo risk diffusions; the latter turns out to be diffusions with killing or splitting rates related, respectively, to concave and convex distortions. The main tools come from the theories of stochastic control, stochastic differential games, and non-linear partial differential equations.

Keywords: Risk diffusion processes; Distorted survival probability functions; Stochastic control; Stochastic differential games; Killing diffusions; Branching diffusions; Viscosity solutions

1. Introduction

Wang et al. (1997) propose axioms for prices (or, more generally, measures of risk) in a competitive insurance market. The market is allowed to behave as if it is risk averse with respect to probability, but required to be risk neutral in wealth. They show that, under their axioms, insurance prices in the market can be represented by the expectation with respect to a distorted probability. Their work is closely related to the work of Yaari (1987) who develops a theory of risk, parallel to expected utility theory, by modifying the independence axiom of Von Neumann and Morgenstern (1944).

In Yaari's theory, attitudes towards risks are characterized by a distortion applied to probability distribution functions, in contrast to expected utility theory in which attitudes towards risks are characterized by a utility function of wealth. A *distorted probability* is a special case of a *non-additive measure* (Denneberg, 1994). One can think of the distorted probability underlying market prices as a 'risk neutral' non-additive probability — a non-additive version of a risk neutral probability in the theory of financial pricing. Thus, one can take a market approach (Wang et al., 1997) or an individual approach (Yaari, 1987) and obtain the same pricing principle.

In work related to that of Wang et al. (1997), Chateauneuf et al. (1996) propose a set of axioms for pricing financial risks. Under their axiomatic system, prices can be represented as the Choquet integral with respect to a non-additive measure. Such a pricing rule can explain violation of put–call parity and the fact that parts of a security

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may sell at a premium to the underlying security. Jouini and Kallal (1995a,b), El Karoui and Quenez (1995), and Artzner et al. (1998) develop similar pricing formulas or risk measurements in *financial markets with frictions*.

Calculating prices of risky prospects as expected values with respect to distorted probabilities, therefore, has a strong theoretical basis and a wide scope of applicability. Given the ever-growing sophistication and complexity of financial claims and insurance plans, there is great need to extend the use of distorted probabilities to *dynamic settings*. This is the task we undertake herein; as a first-step, we concentrate on the case of stochastic risks modeled as diffusion processes. We provide a complete characterization of the associated distorted survival probabilities for general distortion functions. The methodology we develop comes from the theories of stochastic control and non-linear partial differential equations, and it can be applied easily to other risk processes like, for example, diffusion processes with jumps and general Levy processes.

In this work, we do not look yet at the fundamental problem of dynamic pricing rules via distorted probabilities, rather, we concentrate on developing technical tools and alternative characterizations of distorted probabilities. We show that when the distortion is a power function with exponent say γ, then the distorted probability of the original risk process can be interpreted as a non-distorted survival probability of a new underlying pseudo-risk process. In particular, if the power $\gamma \in (0, 1)$ (concave distortions), the new underlying process is a diffusion with killing and if $\gamma > 1$ (convex distortions), the new underlying process is a branching diffusion process. In the case of a general distortion, the pseudo-risk process is a branching diffusion with a combination of 'killing' and 'splitting' characteristics. We expect that these characterizations will shed light on questions related to the valuation of insurance risks in a dynamic framework. Moreover, we expect that this approach will lead to a unified theory of valuation, especially for markets with unhedgeable risks and other frictions. We plan to analyze these valuation problems in subsequent work.

In Section 2, we briefly review the axioms of Wang et al. (1997) and provide a representation theorem for the pricing functional (or risk measure) in the static case. In Section 3.1, we characterize the distorted survival probability of a diffusion process when the distortion is a power function, while in Section 3.2, we consider general distortions. In Section 4, we present examples of a distorted probability when the risk diffusion process follows a geometric Brownian motion and the distortion is either a power function or a piecewise linear function.

2. Axiomatic foundation of measuring risk: the static setting

Fix a probability space $(\Omega, \mathsf{F}, \Pr)$, in which Ω is the space of outcomes, F a σ-algebra of events, and Pr is a probability measure on (Ω, F). A *risk X* is a random variable on $(\Omega, \mathsf{F}, \Pr)$; that is, X is a F-measurable function from the space of outcomes Ω to the real numbers. One can think of $X(\omega)$ as the monetary gain or loss that one incurs if the outcome is $\omega \in \Omega$.

Let X denote the set of risks. One can view a risk measure as a functional RM from X to the extended real numbers $\overline{\mathbf{R}}$:

$$\mathrm{RM} : \mathsf{X} \to \overline{\mathbf{R}} = [-\infty, \infty],$$

in which we allow an insurance risk to have an infinite measure of risk.

Next, we summarize the axioms for a functional RM that imply that RM can be represented as an *expected value with respect to a distorted probability*. See Wang et al. (1997) for arguments that support these axioms.

Axiom 2.1.

(a) *Independence*. The measure of risk depends only on the risk's distribution.

(b) *Monotonicity*. If $X, Y \in \mathsf{X}$, are such that $X(\omega) \leq Y(\omega)$, for all $\omega \in \Omega$, then $\mathrm{RM}[X] \leq \mathrm{RM}[Y]$.

(c) *Comonotonic additivity*. Let $X, Y \in \mathsf{X}$ be comonotonic; then, $\mathrm{RM}[X+Y] = \mathrm{RM}[X] + \mathrm{RM}[Y]$, in which X and Y are said to be comonotonic if there exist non-decreasing real-valued functions f_1 and f_2 and a random variable $Z \in \mathsf{X}$, such that $X = f_1(Z)$ and $Y = f_2(Z)$.

(d) *No unjustified risk loading*. Let 1_Ω represent the degenerate random variable that equals 1 with probability one, then $\mathrm{RM}[1_\Omega] = 1$.
(e) *Continuity*. Let $X \in \mathsf{X}$ and a be a real number; then,

$$\lim_{a\to 0^+} \mathrm{RM}[\max(X-a,0)] = \mathrm{RM}[X] \text{ if } X \geq 0,$$

$$\lim_{a\to \infty} \mathrm{RM}[\min(X,a)] = \mathrm{RM}[X],$$

and

$$\lim_{a\to -\infty} \mathrm{RM}[\max(X,a)] = \mathrm{RM}[X].$$

Notation 2.2. *Denote the decumulative distribution function* (*ddf*) *of X by S_X: $S_X(t) \equiv \Pr(X > t)$, $t \in \mathbf{R}$.*

Wang et al. (1997) prove the following representation theorem for RM, which essentially follows from Yaari's representation theorem applied to not necessarily bounded random variables.

Theorem 2.3. *If the measure of risk* $\mathrm{RM} : \mathsf{X} \to \overline{\mathbf{R}} = [-\infty, \infty]$, *satisfies Axioms* 2.1 (*Independence, Monotonicity, Comonotonic additivity, No unjustified risk loading, and continuity*), *then there exists a non-decreasing function g*: $[0, 1] \to [0, 1]$ *such that* $g(0) = 0$, $g(1) = 1$, *and*

$$\mathrm{RM}[X] = \int X \, \mathrm{d}(g \circ \Pr) = \int_{-\infty}^{0} \{g\,[S_X(t)] - 1\} \, \mathrm{d}t + \int_{0}^{\infty} g\,[S_X(t)] \, \mathrm{d}t. \tag{2.1}$$

Further, if X *contains all the Bernoulli random variables, then g is unique. In this case, for* $p \in [0,1]$, $g(p)$ *is given by the risk measure of a Bernoulli*(p) *risk*.

The integral in (2.1) is a special case of the Choquet integral for non-additive measures. See Denneberg (1994) for further background in non-additive measure theory and Wang (1996) for more details concerning the premium principle given by (2.1).

Definition 2.4. A non-decreasing function g: $[0, 1] \to [0, 1]$ such that $g(0) = 0$ and $g(1) = 1$, is called a distortion and $g \circ \Pr$ is called a distorted probability.

Wang et al. (1997) also argue for a final axiom from which it follows that g is a power function. This final axiom is added in order to avoid a particular arbitrage opportunity.

Axiom 2.5. Let $X = IY$ be a compound Bernoulli random variable, with X, I,and $Y \in \mathsf{X}$, where the Bernoulli random variable I is independent of the random variable $Y = X | X > 0$. Then, $\mathrm{RM}[X] = \mathrm{RM}[I]\mathrm{RM}[Y]$.

Corollary 2.6. *If* RM *is as assumed in Theorem* 2.3 *and if* RM *additionally satisfies Axiom* 2.5, *then there exists* $\gamma > 0$, *such that* $g(p) = p^\gamma$, *for all* $p \in [0, 1]$.

In what follows, we allow X to follow a stochastic process, and we characterize the distorted probability of X at a time T given a value for X at time t.

3. Yaari's risk measures in a stochastic setting

In this section, we look at the case of *dynamic risks* that are modeled as stochastic processes, and we provide a complete *stochastic variational representation* for the associated distorted survival probabilities. The processes that characterize the risks are assumed to be diffusion processes with their generator operator known to us.

We consider the dynamic analog of the static survival probability function that was presented in the previous section. We look at the conditional survival probability function at expiration time T, given the state at initial time t. By using stochastic control methods, we characterize the distorted probability as the conditional expectation of a characteristic function of a new risk process, loaded with a *risk factor*. In the example of Section 4.1 (geometric diffusion with power distortion), we find that this risk factor can be written in terms of the *hazard function* of the normal distribution. The underlying process in the new setting is not the original risk process but a new diffusion risk process with *modified drift*; the latter depends on the shape of the distortion and the original diffusion parameters.

In the case of *concave* power distortion functions, we establish that the distorted survival probability function can be viewed as a distortion-free survival probability function of a new risk process. This new risk process turns out to be a *killed* diffusion whose drift and *killing rate* are influenced by the form of the distortion. When the distortion is a *convex* power function, the new risk process is associated with a *branching* diffusion. Finally, we show that *general distortion* functions correspond to branching diffusions with a combination of 'killing' and 'splitting' characteristics.

This phenomenon is not only observed for the case of diffusion processes. As a matter of fact, when the risks are modeled as jump processes, the authors have established that the new risk process turns out to be a jump process with jumps of modified intensity (Young and Zariphopoulou, 1998).

We start with a brief review of fundamental results from the theory of stochastic processes. In order to make the analysis more tractable, we work with the one-dimensional case. For the case of multi-dimensional problems, the analysis can be carried out with similar arguments, but we choose not to do so here for the sake of presentation.

We consider a diffusion process X_t, which will ultimately be related to the price of a particular asset or liability. We assume that the state process X_t solves the stochastic differential equation

$$\mathrm{d}X_t = b(X_t, t)\,\mathrm{d}t + \sigma(X_t, t)\,\mathrm{d}W_t, \quad X_0 = x,\ x \in \mathbf{R} \tag{3.1}$$

for $t \in [0, T]$. W_t is a standard Brownian motion defined on the probability space $(\Omega, \mathsf{F}, \Pr)$. We denote by F_t the augmentation of the σ-algebra generated by the realizations of the Brownian motion up to time t.

Assume that the *drift* and *volatility* coefficients $b(x, t)$ and $\sigma(x, t)$ satisfy the usual growth and Lipschitz conditions

$$|b(x,t) - b(y,t)| + |\sigma(x,t) - \sigma(y,t)| \le K\,|x - y|\,, \tag{3.2}$$

$$|b(x,t)|^2 + |\sigma(x,t)|^2 \le K(1 + |x|)^2 \tag{3.3}$$

for some positive constant K. These conditions guarantee that a unique solution to Eq. (3.1) exists (Gihman and Skorohod, 1972, Chapter 6). Moreover, we assume that the diffusion process does not 'degenerate', that is, for all $x \in \mathbf{R}$ and for all $t \in [0,T]$, there is an $\varepsilon > 0$, independent of the variables x and t, such that

$$\sigma^2(x, t) > \varepsilon. \tag{3.4}$$

We will use the uniform ellipticity condition (3.4) in subsequent arguments.

Remark 3.1. *Even though some of the above, as well as subsequent conditions, can be refined by the so-called local conditions, we do not try to state the weakest assumptions. We concentrate mostly on the methodology we develop to further explore the nature of distorted probabilities. For a thorough study of diffusion processes and their connection to parabolic partial differential equations, we refer the reader to the books of Friedlin* (1985) *and Fleming and Soner* (1993).

Remark 3.2. *In a wide range of applications, the associated risk processes violate the ellipticity condition* (3.4). *In this case, even though the analysis is more delicate, most results can be modified to incorporate possible degeneracies. The latter require us to relax the notion of related classical solutions and to work with weak ones* (*see Theorem* 3.7).

Next, we consider the conditional survival probability function, also called a transition probability,

$$u(x, t; y, T) = \Pr(X_T > y | X_t = x), \tag{3.5}$$

in which y is a fixed parameter, T is a fixed horizon, and $x \in \mathbf{R}$. Classical elements from the theory of diffusion processes and linear partial differential equations yield the following result.

Theorem 3.3. *Under assumptions* (3.2), (3.3), and (3.4), *we have that* $u \in C^{2,1}(\mathbf{R} \times [0,T))$, $u(x, t) > 0$ for $0 < t < T$, *and u solves the backward* (*in time*) *parabolic problem*

$$u_t + (1/2)\sigma^2(x, t)u_{xx} + b(x, t)u_x = 0, \qquad u(x, T; y, T) = \begin{cases} 1 & \text{if } x > y, \\ 0 & \text{if } x \le y. \end{cases} \tag{3.6}$$

See, for example, Fleming and Soner (1993, Chapter 6).

Example 3.4. In the case of diffusion processes with linear coefficients, the survival probability function can be calculated explicitly. Indeed, suppose that the risk process X_t solves the linear equation $\mathrm{d}X_t = bX_t\,\mathrm{d}t + \sigma X_t\,\mathrm{d}W_t$, with b and σ constants, $\sigma > 0$. Direct calculation yields that

$$u(x, t; y, T) = \Pr(X_T > y \mid X_t = x) = \Phi\left[\frac{-\ln(y/x) + (b - \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}\right]$$

for $x > 0$ and $y > 0$, in which Φ is the cumulative distribution function of the standard normal. Also, one can easily verify that u solves

$$u_t + (1/2)\sigma^2 x^2 u_{xx} + bxu_x = 0.$$

Example 3.5. In an insurance setting, a diffusion process $X_t > 0$ might represent the random loss to an insurance company, while $\int_0^\infty u(x, t; y, T)\,\mathrm{d}y$ is the expected loss at time T, given that the loss is x at time t. Our work in Section 2, argues that $\int_0^\infty g\,[u(x, t; y, T)]\,\mathrm{d}y$, in which g is a concave distortion, is a *measure of the riskiness* of X_T conditional on the information $X_t = x$. Also, see Wang et al. (1997) and Artzner et al. (1998). Also, X_t might represent the value of the assets of an insurance company; in that case, a convex distortion would be more 'conservative' in valuing the asset. Finally, one might consider the stochastic loss random variable of liabilities minus assets; for example, see Norberg (1997). The work in this paper will aid an actuary in calculating these risk measures when the risk process follows a diffusion process. In insurance, the claim process is often modeled as a jump process, and that is the subject of our future research.

Example 3.6. The survival probability function u plays an important role in the valuation theory of financial derivatives. In fact, consider the case of a European call option that is a contingent claim written on an underlying stock with value S_t at time t, expiration at time T, and strike price K. The fundamental pricing problem amounts to determining the *fair price* of the claim, which precludes arbitrage opportunities, as well as to construct the so-called *hedging portfolio*. The main arguments involve transforming the original stock price process to the so-called risk-neutral process, which, in the case of diffusion prices, turns out to be the solution of

$$\mathrm{d}S_t = r(t)S_t\,\mathrm{d}t + z(S_t, t)\,\mathrm{d}W_t. \tag{3.7}$$

The coefficients $r(t)$ and $z(S, t)$ are the riskless interest rate and the volatility function, respectively. It turns out that the stock account component of the dynamic hedging portfolio is related to the so-called delta process ξ_t, given by

$$\mathrm{d}\xi_t = [r(t)\xi_t + z_1(\xi_t, t)z(\xi_t, t)]\,\mathrm{d}t + z(\xi_t, t)\,\mathrm{d}W_t$$

in which $z(\cdot, t)$ is as in (3.7) and $z_1(\cdot, t)$ represents the first partial derivative with respect to the spatial argument. In fact, the number of stock shares used for replication is given by the delta process of the call (Black and Scholes, 1973). The delta of the call represents the 'probability of finishing in the money', or in other words, the survival probability of the process ξ_t to 'finish' above the strike price K (See Grundy and Wiener, 1998):

$$\delta(S,t) = \Pr(\xi_t > K | \xi_t = S).$$

As discussed earlier, a distorted probability function might turn out to be a very important index for valuing dynamic risks in markets with frictions. Before we attack this fundamental pricing problem, we undertake the task of analyzing and characterizing the distorted survival probability functions. Our ultimate goal is to understand how *the distortion of the survival probability affects the stochastic structure of the original underlying risk process*. We will be looking for a 'pseudo' risk process, say, $\tilde{X}_t$, such that the distorted survival probability of the original process X_t can be related to the survival probability of the new process $\tilde{X}_t$, that is,

$$g\left(\Pr\left[X_T > y | X_t = x\right]\right) = \Pr\left[\tilde{X}_T > y | \tilde{X}_t = x\right] = E\left[\mathbf{1}_{\{\tilde{X}_T > y\}} | \tilde{X}_t = x\right], \tag{3.8a}$$

or, in general,

$$g\left(\Pr\left[X_T > y \mid X_t = x\right]\right) = E\left[\mathbf{1}_{\{\tilde{X}_T > y\}} F(\tilde{X}_T) | X_t = x\right], \tag{3.8b}$$

in which $\mathbf{1}_{\{\hat{X}_T > y\}}$ is the indicator function of the set $\{\hat{X}_T > y\}$, E is the expectation operator with respect to the measure Pr, and F is a *stochastic risk load factor*.

We find such a characterization rather useful and interesting for several reasons. From the technical point of view, it shows a direct connection between non-linear probability functionals and linear ones. From the theoretical point of view, this characterization suggests that pricing risks in markets with frictions might possibly be associated with 'friction-free' risks of new pseudo-price processes. Note that there is evidence that distorted probabilities can accommodate unhedgeable risks inherent from incomplete markets, (see for example, Chateauneuf et al., 1996; Wang et al., 1997).

We carry out the analysis by studying the partial differential equation that the distorted probability function solves. By using elements from the theory of stochastic control, we are able to identify the distorted probability as the solution of a stochastic control problem; these solutions are known as *value functions*. We are then able to associate the optimally controlled process with a pseudo-underlying process $\tilde{X}_t$, as denoted in (3.8a) and (3.8b).

We first consider the special case of a *power distortion*. The reason we concentrate on the case of power distortions is twofold. First, this case is easier to analyze and to describe effectively the stochastic control methods used to understand the stochastic nature of the distorted probabilities. Second, Wang et al. (1997) motivate using power distortions from the point of view of conditional risks; see Axiom 2.5. Moreover, see Wang and Young (1998) for a strong link between the power distortion and the Bayes' rule for updating non-additive set functions.

In Section 3.2, we study the case of a general distortion. In this case, the aforementioned stochastic control problem turns out to be a *zero-sum stochastic differential game*. When we restrict ourselves to the class of *concave* (respectively, convex) power distortion functions, the underlying stochastic control problem turns out to be a *minimization* (respectively, maximization) problem of some expected discounted payoff. The intriguing consequence of these results is that, in the case of *concave* (respectively, convex) distortions, the controlled process $\tilde{X}_t$ is related to a *killed* (respectively, branching) diffusion process. Therefore, *the distorted survival probability of the original process can be interpreted as the survival probability of the new killed* (*resp. branching*) *diffusions*.

3.1. Power distortions and their associated HJB equations

3.1.1. The concave case

We consider the distorted probability function

$$v(x,t) = [\Pr(X_T > y \,|X_t = x\,)]^{\gamma} = u(x,t)^{\gamma} \tag{3.9}$$

for some $0 < \gamma < 1$. Direct differentiation and (3.6) yield that v solves

$$v_t + \frac{1}{2}\sigma^2(x,t)v_{xx} + b(x,t)v_x = -\frac{1-\gamma}{\gamma}\sigma^2(x,t)\frac{v_x^2}{2v} \tag{3.10}$$

with terminal condition

$$v(x,\ T) = \begin{cases} 1 & \text{if } x > y, \\ 0 & \text{if } x \le y. \end{cases} \tag{3.11}$$

We observe that the non-linear term $-(v_x^2/2v)$ can be expressed in terms of an auxiliary control variable as follows:

$$-\frac{v_x^2}{2v} = \min_\alpha[(1/2)\alpha^2 v + \alpha v_x].$$

Therefore, Eq. (3.10) becomes

$$v_t + (1/2)\sigma^2(x,t)v_{xx} + b(x,t)v_x = \min_\alpha[(1/2)\delta\alpha^2\sigma^2(x,t)v + \delta\alpha\sigma^2(x,t)v_x] \tag{3.12}$$

with $\delta = (1-\gamma)/\gamma$.

We continue with a formal discussion regarding the interpretation of the distorted probability as a solution of Eq. (3.12). First observe that Eq. (3.12), together with the appropriate terminal condition as in (3.11), is the Hamilton–Jacobi–Bellman (HJB) equation of a stochastic control problem described below.

Consider the controlled diffusion process $\hat{X}_s$ that solves the stochastic differential equation

$$\mathrm{d}\hat{X}_s = \left[b(\hat{X}_s, s) - \delta\alpha_s(\hat{X}_s)\sigma^2(\hat{X}_s, s)\right]\mathrm{d}s + \sigma(\hat{X}_s, s)\,\mathrm{d}W_s \tag{3.13}$$

with $\hat{X}_t = x,\ t \le s \le T$. The process W_s and the coefficients $b(x, s)$ and $\sigma(x, s)$ are the same as in (3.1). The process α_s will play the role of an admissible control process that is assumed to be F_s-progressively measurable, where F_s is the complete filtration generated by the Brownian motion, and to satisfy the integrability condition $E[\int_t^T |\alpha_s|\ \mathrm{d}s] < \infty$. We denote the set of admissible policies α_s by A. In addition to affecting the drift of the state process $\hat{X}_s$, α_s affects the discount factor of the expected cost functional given by

$$J(x,t;\alpha) = E\left[\mathbf{1}_{\{\hat{X}_T > y\}}\exp\left(-\int_t^T \frac{1}{2}\delta\sigma^2(\hat{X}_s, s)\alpha_s^2\,\mathrm{d}s\right)\middle|\hat{X}_t = x\right], \tag{3.14}$$

in which E is the expectation with respect to the original measure Pr.

Next, we define the value function

$$V(x,t) = \inf_{\alpha\in\mathsf{A}} J(x,\ t;\alpha), \tag{3.15}$$

for $x \in \mathbf{R}$, and $t \ge 0$. By using standard results from the theory of stochastic control, we see that the above value function solves the same equation as the distorted survival probability function $v(x, t)$, namely Eq. (3.10), together with the terminal condition (3.11). Therefore, if we know that there exists a *unique* solution of (3.10) and (3.11), then it would follow that the distorted survival probability *coincides* with the value function of the above stochastic control problem. Recent advances in the theory of non-linear partial differential equations enable us to provide a unique characterization of the solution of (3.10) and (3.11) in a very rich class of *weak* solutions, namely the *viscosity solutions*. This unique characterization will be used in turn to identify the value function with the distorted probability as we demonstrate in Theorem 3.7 below.

Viscosity solutions were introduced by Crandall and Lions (1983) for first-order non-linear partial differential equations and were later generalized for the second-order case by Lions (1983a,b); see also Ishii and Lions (1990).

Viscosity solutions of non-linear partial differential equations are now routinely used in non-linear problems related to stochastic control, and in particular, in a wide range of applications of financial models with unhedgeable risks, which is the case in markets with transaction costs, stochastic income, or trading constraints. For these models, viscosity solutions were first employed by the second author in Zariphopoulou (1992, 1994) and Davis et al. (1993). We provide the definition of viscosity solutions in Appendix A.

Theorem 3.7. *The value function V, given in* (3.15), *is the unique bounded viscosity solution of the HJB equation* (3.12) *satisfying the terminal condition* (3.11).

The proof of Theorem 3.7 follows from standard arguments well known in viscosity theory, and therefore, we omit it.

We are now ready for one of our main results.

Theorem 3.8.
1. *The distorted survival probability function $v(x, t; y, T)$ identically equals the value function V.*
2. *The optimal policy α_s^* is given in feedback form by*

$$\alpha_s^* = -\frac{V_x(X_s^*, s)}{V(X_s^*, s)} = -\frac{v_x(X_s^*, s)}{v(X_s^*, s)} = -\gamma\frac{u_x(X_s^*, s)}{u(X_s^*, s)} \tag{3.16}$$

for $t \le s \le T$, in which X_s^* is the optimally controlled risk process solving

$$dX_s^* = \left[b(X_s^*, s) + (1-\gamma)\sigma^2(X_s^*, s)\frac{u_x(X_s^*, s)}{u(X_s^*, s)}\right] ds + \sigma(X_s^*, s)\, dW_s, \tag{3.17}$$

and V, u, b, and σ are given in (3.15), (3.5), (3.1), *and* (3.1), *respectively*.

Proof.
1. From Theorem 3.7, we have that the value function V is the unique bounded viscosity solution of (3.12) that also satisfies (3.11). On the other hand, the distorted survival probability function is a smooth function ($C^{2,1}(\mathbf{R} \times [0,T))$) that satisfies (3.10); therefore, v is automatically a viscosity solution of (3.12) satisfying the terminal condition (3.11). Thus, v coincides with V by the uniqueness of viscosity solutions.
2. The first-order conditions in the HJB equation (3.12) yield that

$$-\frac{V_x(X_s, s)}{V(X_s, s)} = -\frac{v_x(X_s, s)}{v(X_s, s)} = \arg\min_{\alpha}\left\{\frac{1}{2}\alpha^2 V + \alpha V_x\right\}.$$

Also, observe that $v_x/v = \gamma(u_x/u)$. By using the regularity of V together with a standard verification (for example, Fleming and Soner, 1993, Chapter 4), we establish the optimality of the process α_s^*, $t \le s \le T$. □

The above theorem yields a very useful representation of the value function in terms of the original survival probability and the distortion factor γ. In fact, part (2) of Theorem 3.8 says that

$$V(x,t) = E\left[\mathbf{1}_{\{X_T^* > y\}} \exp\left(-\int_t^T k(X_s^*, s)\, ds\right) | X_t^* = x\right] \tag{3.18}$$

with the 'risk load' factor $k(x,t) \ge 0$ given in functional form by

$$k(x,t) = \frac{1}{2}\gamma(1-\gamma)\sigma^2(x,t)\left(\frac{u_x(x,t)}{u(x,t)}\right)^2. \tag{3.19}$$

On the other hand, the HJB equation (3.12), together with (3.9) and (3.16), becomes the linear parabolic equation

$$V_t + \frac{1}{2}\sigma^2(x,t)V_{xx} + \left[b(x,t) + (1-\gamma)\sigma^2(x,t)\frac{u_x(x,t)}{u(x,t)}\right]V_x = k(x,t)V \tag{3.20}$$

with terminal condition as in (3.11) and the risk load factor as in (3.19). Note that if $k \equiv 0$, the above equation gives us the survival probability function of the optimally controlled process X_t^*. The case for which k is not identically zero corresponds to a new diffusion process with killing, as the next result shows.

Proposition 3.9. *Consider the process* X_s^* *solving* (3.17) *with* $X_t^* = x$, $t \le s \le T$, *and the factor* k (x, t) *as in* (3.19). *Let Z be an independent and exponentially distributed random variable with mean one. Define the random time by*

$$\tau_1(x,t) = \inf\left\{s \ge t : \int_t^s k(X_u^*, u)\,\mathrm{d}u \ge Z\right\}, \tag{3.21}$$

and define the process M_s *by*

$$M_s = \begin{cases} X_s^*, & t \le s < \tau_1(x,t), \\ A, & s > \tau_1(x,t), \end{cases} \tag{3.22}$$

in which A is an isolated state in **R**. *Then,*

$$V(x,t) = \Pr[M_T > y | M_s = x]. \tag{3.23}$$

Proof. See, for example, Durrett (1996). □

Remark 3.10. *Intuitively, the load factor k in* (3.18) *can be thought of as a variable force of interest* (*Kellison*, 1991) *that depends on time s and on the optimally controlled diffusion process* X_s^*, $t \le s \le T$.

3.1.2. The convex case

The case of a distorted survival probability function of the form

$$v(x,t) = [\Pr(X_T > y \,|X_t = x)]^\gamma = u(x,t)^\gamma$$

for $\gamma > 1$ can be treated similarly as the case for $\gamma \in (0,1)$. We first observe that the equality $v = u^\gamma$, together with Eq. (3.6) that u solves, yields

$$v_t + \frac{1}{2}\sigma^2(x,t)v_{xx} + b(x,t)v_x = \frac{\gamma-1}{\gamma}\sigma^2(x,t)\frac{v_x^2}{2v} \tag{3.24}$$

with terminal condition

$$v(x,T) = \begin{cases} 1 & \text{if } x > y, \\ 0 & \text{if } x \le y. \end{cases} \tag{3.25}$$

Next, it is immediate that the non-linear term in (3.24) can be expressed in terms of an auxiliary control variable as follows:

$$\frac{v_x^2}{2v} = \max_\alpha\left[-\frac{1}{2}\alpha^2 v + \alpha v_x\right].$$

Therefore, Eq. (3.24) becomes

$$v_t + (1/2)\sigma^2(x,t)v_{xx} + b(x,t)v_x = \max_{\alpha}[-(1/2)\theta\alpha^2\sigma^2(x,t)v + \theta\alpha\sigma^2(x,t)v_x] \tag{3.26}$$

with $\theta = (\gamma - 1)/\gamma$.

Proceeding as in the case of a concave power distortion, we can interpret the above equation as the HJB equation of an underlying stochastic control problem. Subsequently, we can show that its value function actually coincides with the distorted survival probability. Since the arguments are a straight modification of the ones used previously, we only state the relevant results.

We consider the controlled diffusion process $\hat{X}_s$ that solves the stochastic differential equation

$$\mathrm{d}\hat{X}_s = \left[b(\hat{X}_s, s) - \alpha_s\theta\sigma^2(\hat{X}_s, s)\right]\mathrm{d}s + \sigma(\hat{X}_s, s)\,\mathrm{d}W_s \tag{3.27}$$

with $\hat{X}_t = x$, $t \le s \le T$. The process α_s is a control policy that is assumed to belong to the set of admissible policies A, as defined in Section 3.1.1. We define the value function

$$V(x,t) = \sup_{\alpha\in\mathsf{A}} E\left[\mathbf{1}_{\{\hat{X}_T > y\}} \exp\left(\int_t^T \frac{1}{2}\theta\sigma^2(\hat{X}_s, s)\alpha_s^2\,\mathrm{d}s\right)\Big|\hat{X}_t = x\right]. \tag{3.28}$$

As in the case of a concave power distortion, the goal is first to identify the above value function as the distorted survival probability. The next theorem summarizes the relevant results.

Theorem 3.11.

1. *The distorted survival probability function v(x, t; y, T) identically equals the value function V.*
2. *The optimal policy* α_s^* *is given in feedback form by*

$$\alpha_s^* = \frac{V_x(X_s^*, s)}{V(X_s^*, s)} = \frac{v_x(X_s^*, s)}{v(X_s^*, s)} = \gamma\frac{u_x(X_s^*, s)}{u(X_s^*, s)}, \tag{3.29}$$

for $t \le s \le T$, *in which* X_s^* *is the optimally controlled risk process solving*

$$\mathrm{d}X_s^* = \left[b(X_s^*, s) - (\gamma - 1)\sigma^2(X_s^*, s)\frac{u_x(X_s^*, s)}{u(X_s^*, s)}\right]\mathrm{d}s + \sigma(X_s^*, s)\,\mathrm{d}W_s. \tag{3.30}$$

This theorem yields the following stochastic representation of the distorted survival probability.

$$V(x,t) = E\left[\mathbf{1}_{\{X_T^* > y\}} \exp\left(\int_t^T l(X_s^*, s)\,\mathrm{d}s\right)|X_t^* = x\right] \tag{3.31}$$

with the load factor $l(x,t) \ge 0$ given in functional form by

$$l(x,t) = \frac{1}{2}\gamma(\gamma - 1)\sigma^2(x,t)\left(\frac{u_x(x,t)}{u(x,t)}\right)^2, \tag{3.32}$$

in which u is the original survival probability function.

By using classical results from the theory of branching diffusions, we can easily relate (3.31) to the expectation of a branching process as follows.

Proposition 3.12. *Consider a branching diffusion* Y_s *that starts at a single particle at location x at time t, with particles splitting at a rate* $l(Y_s, s)$ *with the two 'offspring' born at the location of the parent and then independently diffusing according to the diffusion operator as in* (3.27). *If* $N(\tau)$ *is the number of particles alive at* *time τ, then*

$$V(x,t) = E\left[\mathbf{1}_{\{Y_T>y\}}\exp\left(\int_t^T l(Y_s,s)\,\mathrm{d}s\right)|Y_t=x\right] = E\left[\sum_{i=1}^{N(T)}\mathbf{1}_{\{Y_i(T)>y\}}|N(t)=1, Y_1(t)=x\right]$$
$$= \sum_{i=1}^{N(T)}[\Pr\{Y_i(T)>y\}\,|N(t)=1,\ Y_1(t)=x]. \tag{3.33}$$

Proof. See, for example, Durrett (1996). □

The above result enables us to identify the distorted survival probability of the original diffusion risk process with the distortion-free survival probability of a new underlying risk process. The latter is a branching diffusion process Y_s whose splitting rate, given in (3.32), is state dependent with coefficients depending on the form of the distortion. Moreover, Y_s diffuses according to a new diffusion operator, say L given by

$$\mathsf{L} = \frac{1}{2}\sigma^2(x,t)\frac{\partial^2}{\partial x^2} + \left[b(x,t) - (\gamma-1)\sigma^2(x,t)\frac{u_x(x,t)}{u(x,t)}\right]\frac{\partial}{\partial x}$$

that differs from the diffusion operator of the original risk process X_s (3.1) by the drift term $-(\gamma-1)\sigma^2(x,t)\,[u_x(x,t)/u(x,t)]$. In conclusion, the convex power of the survival probability of X_s can be interpreted as the *distortion-free survival probability of a branching diffusion* Y_s whose diffusion operator is similar to the one of X_s modulus a change of drift and with a splitting rate depending on the distortion power.

3.2. General distorted probabilities and stochastic differential games

We now look at the case of a general distortion function g: $[0,1]\to[0,1]$ applied to the original survival probability function. We assume that g is strictly increasing so that it has an inverse. We will demonstrate that the distorted survival probability can be represented as the value of a stochastic differential game and, ultimately, as the distortion-free survival probability of a new diffusion process with a risk load factor. The latter has similar characteristics to the diffusions with killing and splitting that we discussed in the previous section.

We pursue along the same lines as before. By using the form of the distorted probability and the fundamental equation (3.6) satisfied by the original survival probability, we derive a non-linear equation that turns out to be a *Bellman–Isaac* (BI) *equation*. This equation is the analogue of the HJB equation for a wider class of stochastic optimization problems, namely the (*zero-sum*) *stochastic differential games*.

To this end, we set

$$F(x,t) = g(u(x,t)), \tag{3.34}$$

which easily yields that F solves

$$F_t + \frac{1}{2}\sigma^2(x,t)F_{xx} + b(x,t)F_x = \frac{1}{2}\sigma^2(x,t)F_x^2\frac{g''(u)}{[g'(u)]^2}. \tag{3.35}$$

Equivalently,

$$F_t + (1/2)\sigma^2(x,t)F_{xx} + b(x,t)F_x = (1/2)\sigma^2(x,t)F_x^2 G(F), \tag{3.36}$$

in which

$$G(F) = \frac{g''(g^{-1}(F))}{[g'(g^{-1}(F))]^2}. \tag{3.37}$$

Note that the ratio in Eq. (3.37) is actually the *Arrow-Pratt*, or *absolute*, *risk aversion coefficient* of the inverse distortion function g^{-1}. Indeed, by differentiating the identity $u = g^{-1}(g(u))$ twice, we obtain

$$g''(u)[g^{-1}(g(u))]' = -[g'(u)]^2[g^{-1}(g(u))]'',$$

which, in turn, implies that

$$G(F) = \frac{g''(g^{-1}(F))}{[g'(g^{-1}(F))]^2} = -\frac{[g^{-1}(F)]''}{[g^{-1}(F)]'}, \tag{3.38}$$

where we used $u = g^{-1}(F)$.

For the analysis to follow, we will need the following assumption.

Assumption 3.13. The distortion function g: [0, 1] → [0, 1] is invertible, twice continuously differentiable in (0, 1), and such that the reciprocal of the absolute risk aversion coefficient of g^{-1} is a Lipschitz function in [0, 1].

Example 3.14. In addition to the family of power distortions, the *dual* power distortions satisfy Assumption 3.13, in which a dual power distortion is of the form $g(p) = 1-(1-p)^\gamma$, $\gamma > 0$. Moreover, $g(p) = (a^p-1)/(a-1)$, $a > 1$, defines a family of distortions which also satisfies the above assumption.

Next, we put Eq. (3.36) in the form of the BI equation associated with a stochastic differential game. At this point, we proceed formally assuming that all the necessary operations are valid. We first observe that

$$p^2 = \max_l \left[lp - \frac{1}{4}l^2 \right]. \tag{3.39}$$

Thus

$$p^2G(q) = \max_l \left[lpG(q) - \frac{1}{4}l^2G(q) \right] = \max_l \left[(lG(q))p - \frac{1}{4}\frac{(lG(q))^2}{G(q)} \right] = \max_{\alpha_1} \left[\alpha_1 p - \frac{1}{4}\alpha_1^2 H(q) \right], \tag{3.40}$$

in which $\alpha_1 = \alpha G(q)$, and $H(q) = 1/G(q)$.

By Assumption 3.13, the function H is Lipschitz with Lipschitz constant, say L. Then, H can be represented as

$$H(q) = \min_{\alpha_2} \left[H(\alpha_2) + L\,|q - \alpha_2| \right]. \tag{3.41}$$

Moreover, it is well known that

$$|q - \alpha_2| = \max_{|z|\le 1} (\alpha_2 - q)z,$$

which together with Eq. (3.41), yields

$$H(q) = \min_{\alpha_2} \max_{|z|\le 1} [H(\alpha_2) - Lqz + L\alpha_2 z]. \tag{3.42}$$

Therefore,

$$\begin{aligned} p^2G(q) &= \max_{\alpha_1} \left[\alpha_1 p - \frac{1}{4}\alpha_1^2 \min_{\alpha_2} \max_{|z|\le 1} \left[H(\alpha_2) - Lqz + L\alpha_2 z \right] \right] \\ &= \max_{\alpha=(\alpha_1,\,\alpha_2)} \min_{|z|\le 1} \left[\alpha_1 p + \frac{1}{4}\alpha_1^2 Lqz - \frac{1}{4}\alpha_1^2 (H(\alpha_2) + L\alpha_2 z) \right]. \end{aligned} \tag{3.43}$$

By evaluating the above expression at $p = F_x$ and $q = F$, and by using Eq. (3.36), we obtain

$$F_t + \frac{1}{2}\sigma^2(x,t)F_{xx} + b(x,\ t)F_x + \frac{1}{2}\sigma^2(x,\ t)\min_{\alpha=(\alpha_1,\ \alpha_2)}\max_{|z|\le 1}\left[-\alpha_1 F_x - \frac{1}{4}\alpha_1^2 zLF + \frac{1}{4}\alpha_1^2(H(\alpha_2) + L\alpha_2 z)\right] = 0. \tag{3.44}$$

Next, we review fundamental elements of the theory of *stochastic differential games*, and we recall the notion of their solution. For the sake of presentation, we start with generic coefficients, and we specify them according to (3.44) later on.

Consider the state diffusion risk process $\hat{X}_s$ that solves the stochastic differential equation

$$\mathrm{d}\hat{X}_s = f(\hat{X}_s, \alpha_s, z_s, s)\,\mathrm{d}s + \sigma(\hat{X}_s, s)\,\mathrm{d}W_s \tag{3.45}$$

with initial condition $\hat{X}_t = x$, with $x \in \mathbf{R}$, and the criterion

$$J(x,t;\alpha,z) = E\left[\int_t^T h(\hat{X}_s, \alpha_s, z_s, u_s)\exp\left(-\int_t^s c(\hat{X}_\tau, \alpha_\tau, z_\tau, u_\tau)\,\mathrm{d}\tau\right)\mathrm{d}s + \Phi(\hat{X}_T)\right]. \tag{3.46}$$

As before, W_s is a Brownian motion on the underlying probability space $(\Omega, \mathsf{F}, \Pr)$; the processes α_s and z_s are the control processes assumed to belong to appropriately defined sets of admissible policies A and Z. The functions $c(X, \alpha, z, u)$, $h(X, \alpha, z, u)$, and $\Phi(x)$ represent the discount factor, the running cost (or payoff), and the terminal penalty (or bequest) function, respectively.

The intuitive idea is that the processes α_s and z_s represent the actions of two players, I and II, respectively, in the following way. Player I controls α_s and wishes to maximize J over all choices of z_s. On the other hand, Player II controls z_s and wishes to minimize J over all choices of α_s. The main difficulty in the study of such games lies in the fact that, although at any time $s \in [t, T]$ both players know the states $\hat{X}_s$, W_s, α_s, and z_s, instantaneous switches of α_s and z_s are possible in continuous time.

This fundamental point was addressed by introducing two approximate games, namely the *lower* and the *upper* game. In the lower game, Player II is allowed to know α_s before choosing z_s, while in the upper game, Player I chooses α_s knowing z_s. Each game has a value, the lower value $\underline{V}$ and the upper value $\overline{V}$ that turn out to be (viscosity) solutions of the BI equations

$$\underline{V}_t + \frac{1}{2}\sigma^2(x,t)\underline{V}_{xx} + \mathsf{H}^-(\underline{V}_x, \underline{V}, x, t) = 0, \tag{3.47}$$

and

$$\overline{V}_t + \frac{1}{2}\sigma^2(x,t)\overline{V}_{xx} + \mathsf{H}^+(\overline{V}_x, \overline{V}, x, t) = 0 \tag{3.48}$$

with the same terminal condition, say

$$\underline{V}(x,T) = \overline{V}(x,T) = \Phi(x). \tag{3.49}$$

The expressions H^- and H^+ are the so-called *Hamiltonians* and are given by

$$\mathsf{H}^-(p,q,x,t) = \max_\alpha \min_z [f(x,\alpha,z,t)p + h(x,\alpha,z,t) - c(x,\alpha,z,t)q],$$

and

$$\mathsf{H}^+(p,q,x,t) = \min_z \max_\alpha [f(x,\alpha,z,t)p + h(x,\alpha,z,t) - c(x,\alpha,z,t)q].$$

We say that the *Isaac's condition* is satisfied if for all (p, q, x, t),

$$\mathsf{H}^-(p,q,x,t) = \mathsf{H}^+(p,q,x,t). \tag{3.50}$$

The main objective is to establish that the lower and the upper values of the game are equal, in which case we say that the *differential game has a value*. The following theorem gives some of the most general results and was proved by Fleming and Souganidis (1989).

Theorem 3.15 (FS). *Assume that* A *and* Z *are compact metric spaces, that the functions f,* σ, *h, and c are bounded, uniformly continuous, and Lipschitz continuous with respect to* (x, t), *uniformly with respect to* (α, z) in A *and* Z, *and that the function* Φ *is bounded. Then, the lower value* $\underline{V}$ (*respectively, the upper value* $\overline{V}$) *of the stochastic differential game* (*SDG*) *with state* $\hat{X}_s$ *and payoff J, given by* (3.45) *and* (3.46), *is the unique viscosity solution of the lower* (*respectively, upper*) BI equation (3.47) (*respectively*, (3.48)). *Moreover, if the Isaac's condition* (3.50) *holds, then the SDG has a value* $V = \underline{V} = \overline{V}$ *given by*

$$V(x,t) = \inf_{\mathsf{A}} \sup_{\mathsf{Z}} J(x,t;\alpha,z) = \sup_{\mathsf{Z}} \inf_{\mathsf{A}} J(x,t;\alpha,z). \tag{3.51}$$

Remark 3.16. *As it is stated in Fleming and Souganidis* (1989), *the above conditions are by no means the weakest possible. We do not attempt to look for the weakest assumptions because our main goal is to demonstrate a potentially very useful connection between general distorted probability functions and stochastic differential games.*

We are now ready to discuss the representation of the distorted survival probability as the value of a stochastic differential game. To this end, we first observe that Eq. (3.44) can be viewed as the upper BI equation (3.48) with coefficients

$$f(x,\alpha,z,t) = b(x,t) - (1/2)\sigma^2(x,t)\alpha_1, \tag{3.52}$$

$$h(x,\alpha,z,t) = (1/8)\sigma^2(x,t)\alpha_1^2(H(\alpha_2) + L\alpha_2 z), \tag{3.53}$$

$$c(x,\alpha,z,t) = (1/8)\sigma^2(x,t)\alpha_1^2 zL, \tag{3.54}$$

$$\Phi(x) = \mathbf{1}_{\{x>y\}}. \tag{3.55}$$

Proposition 3.17. *Assume that the conditions of Theorem* 3.15 (*FS*) *hold for the functions f, h, and c defined in* (3.52), (3.53) and (3.54). *Then, the distorted survival probability F*(x, t) *is the value of a stochastic differential game with state* $\hat{X}_s$ *and criterion J, given by* (3.45) and (3.46), *with* Φ *given in* (3.55).

4. Examples: geometric Brownian motion

In this section, we present two examples in which the diffusion process is a geometric Brownian motion. First, we consider the case of a power distortion, as in Section 3.1. Then, we analyze the case of a piecewise linear distortion. The work in Section 3 is not directly applicable to the latter case because the distortion is not differentiable, but the observations in Section 2 still apply. Specifically, one can use the expectation with respect to a probability distorted by a piecewise linear distortion as a measure of risk. In the case of a piecewise linear distortion, we provide explicit formulae for the distorted survival probability, and we analyze their behavior in terms of the various risk parameters.

To this end, let the state process X_s, $t \le s \le T$, follow the diffusion process

$$\mathrm{d}X_s = bX_s\,\mathrm{d}s + \sigma X_s\,\mathrm{d}W_s, \qquad X_t = x, x \in \mathbf{R},$$

in which b and σ are real constants, with $\sigma > 0$. In this case, we can write X_s explicitly:

$$X_s = x \exp\left[\left(b - \frac{\sigma^2}{2}\right)(s-t) + \sigma W_{s-t}\right], \quad t \le s \le T.$$

Because the state $x = 0$ is an absorbing state (see Gihman and Skorohod, 1972), if $x > 0$, the survival probability $u(x, t; y, T)$ equals 1, for $y \leq 0$; and

$$u(x,t;y,T) = \Phi\left[\left(b - \frac{\sigma^2}{2}\right)\frac{\sqrt{T-t}}{\sigma} - \frac{1}{\sigma\sqrt{T-t}}\ln\left(\frac{y}{x}\right)\right] \quad \text{for } y > 0,$$

in which Φ is the cumulative distribution function of the standard normal random variable. Similarly, if $x < 0$, the survival probability for $y < 0$ is given by

$$u(x,t;y,T) = \Phi\left[-\left(b - \frac{\sigma^2}{2}\right)\frac{\sqrt{T-t}}{\sigma} + \frac{1}{\sigma\sqrt{T-t}}\ln\left(\frac{y}{x}\right)\right],$$

and it equals 0, for $y \geq 0$.

4.1. Power distortion functions

Let g be the power distortion given by $g(p) = p^\gamma$. For concreteness, we assume that g is concave $(0 < \gamma < 1)$ and that $x > 0$ and $y > 0$. The other cases follow similarly and are not treated here. From Theorem 3.8, the distorted probability $v(x,t) = u(x,t)^\gamma$ is given by the expectation in (3.18)

$$\mathrm{E}\left[\mathbf{1}_{\{X_T^* > y\}} \exp\left(-\int_t^T k(X_s^*, s)\,\mathrm{d}s\right) | X_t^* = x\right],$$

in which X_s^* is the optimally controlled risk process solving

$$\mathrm{d}X_s^* = \left[bX_s^* + (1-\gamma)\sigma^2 (X_s^*)^2 \frac{u_x(X_s^*, s)}{u(X_s^*, s)}\right]\mathrm{d}s + \sigma X_s^*\,\mathrm{d}W_s,$$

and $k(x, t)$ is the 'load' factor

$$k(x,t) = \frac{1}{2}\gamma(1-\gamma)\sigma^2 x^2 \left(\frac{u_x(x,t)}{u(x,t)}\right)^2.$$

In this case,

$$\frac{u_x(x,t)}{u(x,t)} = \frac{1}{\sigma x\sqrt{T-t}}\lambda(z(x,t;y)),$$

in which $z(x,t;y) = [\ln(\frac{y}{x}) - (b - \frac{\sigma^2}{2})(T-t)]/\sigma\sqrt{T-t}$, and $\lambda(z)$ is the *hazard function* of the standard normal distribution.

In general, the *hazard function* of a continuous random variable X is given by $f(x)/S(x)$, in which f is the probability density function of X and S is the decumulative distribution function of X.

It follows that X_s^* solves

$$\mathrm{d}X_s^* = \left[bX_s^* + \frac{(1-\gamma)\sigma X_s^*}{T-t}\lambda(z(X_s^*, s; y))\right]\mathrm{d}s + \sigma X_s^*\,\mathrm{d}W_s,$$

and the load factor $k(x, t)$ equals

$$k(x,t) = \frac{\gamma(1-\gamma)}{2(T-t)}[\lambda(z(x,t;y))]^2.$$

4.2. Piecewise linear distortion

Consider the concave piecewise linear distortion given by

$$g_{a,c}(p) = \begin{cases} \dfrac{c}{a} p, & 0 \le p \le a, \\ \dfrac{(c-a)+(1-c)p}{(1-a)}, & a < p \le 1, \end{cases}$$

in which $0 \le a \le c \le 1$. In this case, we can explicitly calculate the risk measure from Section 2, Eq. (2.1). The risk measure of the geometric Brownian motion X_T, given that $X_t = x$, determined by $g_{a,c}$ is

$$\mathrm{RM}[X_T; x, t; a, c] = \int X_T \,\mathrm{d}(g_{a,c} \circ \Pr_{|X_t=x}) = \frac{x\,\mathrm{e}^{b(T-t)}}{1-a}\left[(1-c) + \frac{c-a}{a}\Phi(z_a + \sigma\sqrt{T-t})\right]$$

for $x > 0$, in which z_a is the 100a-th percentile of the standard normal. If $x \ne 0$, then the risk measure of X_T, given that $X_t = x$, is

$$\mathrm{RM}[X_T; x, t; a, c] = \int X_T \,\mathrm{d}(g_{a,\,c} \circ \Pr_{|X_t=x}) = \frac{x\,\mathrm{e}^{b(T-t)}}{1-a}\left[(1-c) + \frac{c-a}{a}\Phi(z_a + \sigma\sqrt{T-t})\right].$$

When $x > 0$, it is straightforward to demonstrate that $\mathrm{RM}[X_T; x, t; a, c]$ satisfies the properties listed below; note that similar properties hold when $x < 0$.

- $\mathrm{RM}[X_T; x, t; a, c] \ge x\,\mathrm{e}^{b(T-t)} = \mathrm{E}[X_T | X_t = x]$. This result follows from the fact that $g_{a,c}(p) \ge p$ for all $p \in [0, 1]$.
- For fixed $c \ge a$, $\mathrm{RM}[X_T; x, t; a, c]$ decreases as a increases. Intuitively, as $g_{a,c}$ becomes less concave, the risk measure of X_T decreases. When $a = c$, note that $\mathrm{RM}[X_T; x, t; a, c] = x\,\mathrm{e}^{b(T-t)} = \mathrm{E}[X_T | X_t = x]$, as expected because in that case, there is no distortion.
- For fixed $a \le c$, $\mathrm{RM}[X_T; x, t; a, c]$ increases as c increases. Intuitively, as $g_{a,c}$ becomes more concave, the risk measure of X_T increases.

5. Summary

In this paper, we studied the distorted survival probability functions of risk processes which are modeled as diffusion processes. We established that these probabilities can be related, under general weak assumptions, to distortion-free survival probabilities of new ‘pseudo’ risk processes with modified drift. This representation is important because it relates distorted survival probabilities — associated with risks in incomplete markets — with distortion-free probabilities of new risk processes. This relation is expected to facilitate the pricing of risks in the presence of frictions via the relevant values for the friction-free case. This is the subject of our future research.

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Appendix A

In this appendix, we present the definition of viscosity solutions of non-linear parabolic partial differential equations. The notion of *viscosity solutions* was introduced by Crandall and Lions (1983) for first-order equations and by Lions (1983a,b) for second-order equations. For a general overview of the theory, see Fleming and Soner (1993).

Consider a non-linear second-order partial differential equation of the form

$$F(X, V, DV, D^2V) = 0 \text{ in } \Omega \times [0, T], \tag{A.1}$$

in which $\Omega \subseteq \mathbf{R}^2$, DV and D^2V denote the gradient vector and the second-derivative matrix of V, and the function F is continuous in all its arguments and degenerate elliptic, meaning that

$$F(X, p, q, A + B) \le F(X, p, q, A) \text{ if } B \ge 0.$$

Definition A.1. A continuous function $V : \overline{\Omega} \times [0, T] \to \mathbf{R}$ is a viscosity solution of Eq. (A.1) if the following two conditions hold:

1. V is a viscosity subsolution of (A.1) on $\overline{\Omega} \times [0, T]$; that is, for any $\phi \in C^{2,1}(\overline{\Omega} \times [0, T])$ and any local maximum point $X_0 \in \overline{\Omega} \times [0, T]$ of $V-\phi$,

$$F(X_0, V(X_0), D\phi(X_0), D^2\phi(X_0)) \le 0.$$

2. V is a viscosity supersolution of (A.1) on $\overline{\Omega} \times [0, T]$; that is, if for any $\phi \in C^{2,1}(\Omega \times [0, T])$ and any local minimum point $X_0 \in \overline{\Omega} \times [0, T]$ of $V-\phi$,

$$F(X_0, V(X_0), D\phi(X_0), D^2\phi(X_0)) \ge 0.$$

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