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Numerical Schemes for Variational Inequalities Arising in International Asset Pricing

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Abstract. This paper introduces a valuation model of international pricing in the presence of political risk. Shipments between countries are charged with shipping costs and the country specific production processes are modelled as diffusion processes. The political risk is modelled as a continous time jump process that affects the drift of the returns in the politically unstable countries. The valuation model gives rise to a singular stochastic control problem that is analyzed numerically. The fundamental tools come from the theory of viscosity solutions of the associated Hamilton–Jacobi–Bellman equation which turns out to be a system of integral-differential Variational Inequalities with gradient constraints.

Key words: international asset pricing, political risk, shipping costs, variational inequalities, gradient constraints, viscosity solutions

1. Introduction

In this paper, we develop a continuous time model of international asset pricing in a two-country framework with political risks. Political risk is an important aspect of international investment decisions and is discussed in international finance texts such as Sercu and Uppal (1995). There are organizations which rate countries regarding political risk, and there have been empirical studies of international asset returns which consider political risk as an explanatory variable. Erb, Harvey and Viskanta (1996) is an example of such a study, which also lists several commerical services that provide political risk assessments. However, there has been a notable lack of work on equilibrium asset pricing models which consider such risks. We view this paper as an early step in the development of such models.

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Fundamentally, political risk represents uncertainty about future government actions which may impact the value of firms and/or the welfare of individuals. If we focus simply on the value of firms, there are still a host of government actions which can affect firm profitability and the values of securities it issues. Changes in the tax code (or its implementation), price ceilings, local content requirements, quotas on imported inputs, labor law provisions, and numerous other areas of government regulation can affect firm profitability and/or security values. One could argue that all governments exhibit political risk in that there is some uncertainty about their future actions. However, the degree of risk tends to vary dramatically with some governments (countries) viewed as politically stable and others as quite risky.

A typical characteristic of political risk is discrete changes in government regulations or policies. To some extent, these changes can be forecasted; and there may well be a partial adjustment of market prices to reflect the anticipated change. However, there is often a substantial amount of uncertainty as to whether the anticipated change will actually take place. When this uncertainty is resolved (e.g., via an announcement), market prices and other economic variables affected by the change are likely to move by (potentially large) discrete amounts. Consequently, it seems appropriate to model political risk using a stochastic process with jumps.

In our model, political risk enters via uncertainty in the drift of the stochastic production process in the politically risky country. For simplicity, we treat one of the countries as exhibiting no political risk. That is, the drift is known for the production process of assets located in that country. However, there is still uncertainty about the value of those assets located in that country due to market forces, technology, weather, etc. We model such uncertainty via a (country specific) Brownian motion, with the drift of the process being known for the politically stable country. In the politically risky (unstable) country, we assume that the expected productivity of assets is not constant but it can take a range of values. In effect, the return process jumps between the allowed levels which correspond to the possible states of a suitable stochastic jump process. The lower states can be interpreted as representing the local government's ability to impose a tax or regulation on firms producing in that country which negatively impacts their profitability. Symmetrically, the high states can be interpreted as either a lower tax rate or even a subsidy for local production, perhaps in an indirect form via changing a restrictive regulation.

In practice, government actions can have positive as well as negative effects on firm value. Furthermore, a negative action can be followed by a positive one and vice versa. Recently, we have seen international asset prices and exchange rates (a special type of asset price) yoyoing up and down in response to sequences of government actions, as well as conjectures about future actions. In a rather simplified manner, we are attempting to capture this sort of phenomenon by having the drift in the risky country determined by a continuous-time jump process which can take several possible values. Consequently, the extent of political risk in our model is determined by both the spread between the different states of the jump process and by the relevant transition probabilities.

We formulate this model as a *singular stochastic control problem* whose states describe the production technology processes in both countries. The collective utility is the value function of this optimization problem and it is characterized as the unique (weak) solution of the associated Hamilton–Jacobi–Bellman (HJB) equation. Because of the presence of shipping costs and the effects of the jump Markov process, the HJB equation actually turns out to be an integral-differential Variational Inequality with gradient constraints. Such problems typically result in a depletion of the state space into regions of idleness and regions where singular controls are exercised. In the context of the model we are developing herein, the singular policies correspond to 'lump-sum' shipments from one country to the other.

Related problems with singular policies arise in a wide range of models in the areas of asset and derivative pricing. They are essentially linked to the fundamental issue of *irreversibility* of financial decisions in markets with frictions such as transaction or shipping costs, or an irreversible loss of an investment opportunity related to *unhedgeable risks*. Unfortunately, such problems do not have in general smooth solutions, let alone closed form ones. It is therefore necessary to analyze these problems numerically by building accurate schemes for the value function as well as the free boundaries which characterize the singular investment policies. Our problem is further complicated by the random drift coefficient used to model political risks.

To deal with this situation, we construct a family of numerical schemes for calculating the collective utilities and the equilibrium prices. These schemes have all the desired properties for convergence, stability, monotonicity and consistency. They belong to the class of the so-called 'time-splitting' schemes which approximate separately – in each half-time iteration – the first- and the second-order derivatives. This class of schemes is known to be very suitable for approximating solutions of second-order nonlinear partial differential equations similar to the ones arising in our model.

Although it is highly simplified, the proposed model captures some of the flavor of an international environment where assets may be exposed to substantially different risks because they are located within the jurisdictions of different governments. In effect, they are different assets and will generally exhibit different prices because of their location. As we shall see, political risk not only influences asset values but also consumption patterns. Furthermore, if we interpret the ratio of the output prices in the two economies as a real exchange rate, then that exchange rate will exhibit sustained deviations from its Purchasing Power Parity (PPP) value.

The paper is organized as follows. In Section 2, we describe the basic model and we provide analytic results for the value function. In Section 3, we construct the numerical schemes for the value function and the trading policies. In Section 4, we interpret the numerical results, and we provide some conclusions and suggestions for extensions of this work.

2. The Asset Pricing Model and the Associated Variational Inequalities

We concentrate on a simplified two-country model where capital markets are fully integrated in the sense that individuals from each country can own claims to assets located in either country. We denote the two countries by $\mathcal{X}$ and $\mathcal{Y}$. The consumption good for residents of country $\mathcal{X}$ is denoted by $\mathbf{X}$ and its level at time t is x_t. $\mathbf{Y}$ and y_t are similarly defined for the country $\mathcal{Y}$. These consumption goods are homogeneous except for location and also serve as production inputs. Hence, x_t and y_t also represent the capital stocks at time t in countries $\mathcal{X}$ and $\mathcal{Y}$, respectively. These production assets/consumption goods can be shipped between countries; however, this will incur shipping costs that will be defined shortly.

Consumption in country $\mathcal{X}$ is denoted by C^x, which includes both consumption of local output and of imports from country $\mathcal{Y}$. Consumption in country $\mathcal{Y}$ is defined in an analogous manner and is denoted by C^y. Cumulative shipments, as of time t, from country $\mathcal{X}$ to country $\mathcal{Y}$ are denoted by L_t; such shipments (exports from country $\mathcal{X}$) incur proportional shipping costs at a rate λ. In a similar manner, cumulative shipments from country $\mathcal{Y}$ (imports by country $\mathcal{X}$), denoted by M_t incur proportional shipping costs at a rate μ. Without loss of generality, we assume that country $\mathcal{X}$ is charged with the shipping costs.

The production process in country $\mathcal{Y}$ is modeled as a diffusion with a constant positive drift coefficient b and a volatility parameter σ_2. The constant drift coefficient is our representation of a *stable political situation* in country $\mathcal{Y}$ in the sense that the expected productivity of the capital stock is known. In contrast, we model country $\mathcal{X}$ as exhibiting political risk via a random drift parameter for its production process. That is, the production process in country $\mathcal{X}$ is a diffusion with volatility parameter σ_1 and a drift parameter which follows a jump Markov process z_t.

The process z_t is a continuous-time jump Markov process with values in a compact state space Z. The intensity with which jumps occur from a state, say $z \in Z$ is denoted by $\theta(z)$. The distribution of the post-jump (from z) location ξ is given by $\pi(z, \cdot)$. Then the generator operator of the jump process is given by

$$\mathcal{B} f(z) = \theta(z) \int_Z [f(\xi) - f(z)]\pi(z, d\xi). \tag{1}$$

(See, for example, Fleming and Soner (1993).)

As discussed above, the low states are associated with unfavorable political solutions (from the perspective of the production process owners) as opposed to the high states which represent the favorable political states in country $\mathcal{X}$.

Recalling that goods can be shipped between countries as well as consumed locally, we can write the state processes for the capital stocks in the two countries as

$$dx_t = z_t x_t dt - C_t^x dt + \sigma_1 x_t dW_t^1 - (1+\lambda)dL_t + (1-\mu)dM_t \tag{2}$$

$$dy_t = by_t dt - C_t^y dt + \sigma_2 y_t dW_t^2 + dL_t - dM_t \tag{3}$$

with W_t^1 and W_t^2 being Brownian motions on a probability space $(\Omega, \mathcal{F}, P)$ with correlation $\delta \in [-1, 1]$; for this we can take $W_t^2 = \delta W_t^1 + \sqrt{1-\delta^2} B_t$ with B_t being a Brownian motion independent of W_t^1. The constants σ_1, σ_2 are assumed to be positive.

Interpreting Equation (2), we see that the capital stock in country $\mathcal{X}$ changes due to production $z_t x_t dt + \sigma_1 x_t dW_t^1$ and consumption C_t^x, as well as, imports dM_t and exports dL_t. The proportional shipping charge on exports λ is born by country $\mathcal{X}$ and has a dissipative effect on its capital stock. The proportional shipping charge on imports μ has a similar effect. Equation (3) describes the process followed by the capital stock in country $\mathcal{Y}$. It is similar to the process for country $\mathcal{X}$ except that country $\mathcal{Y}$ has a known drift coefficient b (due to political stability) and there are no shipping cost parameters (since these costs were charged to country $\mathcal{X}$).

We assume that consumers in both countries seek to maximize expected discounted utility of consumption over an infinite horizon. Using E to denote the expectation operator with $U(C^x, C^y)$ as the joint utility consumption and ρ as the rate of time preferences, we represent the collective (or integrated) utility payoff for consumers of both countries as

$$E \int_0^{+\infty} e^{-\rho t} U(C_t^x, C_t^y) dt .$$

A policy (C_t^x, C_t^y, L_t, M_t) is admissible if it satisfies the following conditions:

(i) it is $\mathcal{F}_t$-progressively measurable, where $\mathcal{F}_t$ is the σ-algebra generated by (W_s^1, B_s, z_s) for $0 \leq s \leq t$.

(ii) L_t and M_t as defined above, are nondecreasing CADLAG[1] processes such that $C_t^x \geq 0$, $C_t^y \geq 0$ a.s. $\forall t \geq 0$.

(iii) $E \int_0^t e^{-\rho s}(C_s^x + C_s^y) ds < +\infty, \qquad \forall t \geq 0.$

(iv) The following *state constraints* are satisfied for $\forall t \geq 0$,

$$x_t \geq 0 \text{ and } y_t \geq 0 \text{ a.e.} \tag{4}$$

Note that these constraints on admissible policies allow for instantaneous shipments (imports or exports) which are not infinitesimal. That is, singular policies with *discrete (lumpy)* shipments are allowed. As we shall see later, these policies turn out to be singular ones and the valuation model (5) gives rise to a *singular* *stochastic control* problem. (For the technically oriented reader, we refer to the book of Fleming and Soner (1993), Chapter VI on singular stochastic control theory).

We assume that international capital markets are integrated with no restrictions or frictions that inhibit individuals from each country buying or selling claims to assets located in either country. Thus, the capital stock of country $\mathcal{X}$ may be owned by individuals from both countries and similarly for the capital stock of country $\mathcal{Y}$. The value of these claims depends on the current level of the capital stocks (denoted by x and y) as well as the current political state (denoted by z) in country $\mathcal{X}$. Furthermore, the current value of these claims also depends on the policy regarding consumption and shipping which is expected to be followed in the future. Note that z influences the consumption choice via its influence on the productivity of the capital stock in country $\mathcal{X}$.

Let $\mathcal{A}_z$ denote the set of admissible policies (C_t^x, C_t^y, L_t, M_t) which satisfy the above measurability and integrability conditions plus the state constraints (4). Then we can define a *collective (across-countries) value function* $V(x, y; z)$ as

$$V(x, y; z) = \sup_{\mathcal{A}_z} E \int_0^{+\infty} e^{-\rho t} U(C_t^x, C_t^y) dt \,. \tag{5}$$

The collective consumer utility function $U : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is assumed to be increasing and concave in both arguments. Also, $U(0, 0) = 0$ and U is bounded above in the sense that

$$U(C^x, C^y) \leq M\Big(\frac{1}{1+\lambda} C^x + (1-\mu) C^y\Big)^\gamma$$

for some constants $M > 0$ and $0 < \gamma < 1$.

In order to guarantee that V is well defined for all $x \geq 0$, $y \geq 0$ and $z \in Z$, we impose the following technical restrictions on the market parameters.

First, we assume that the maximum element of the state space of the z_t process is $\widehat{z}$. Next, we define σ, k_i, $\hat{\rho}_i$, $i = 1, 2$ as follows:

$$\sigma = \sqrt{\frac{1}{2}\sigma_1^2 - \delta\sigma_1\sigma_2 + \frac{1}{2}\sigma_2^2} \,.$$

If $b \geqq \widehat{z}$,

$$\begin{cases} k_1 = \sigma_1^2(1-\gamma) - (1-\gamma)\delta\sigma_1\sigma_2 + (b - \hat{z}) \\ \hat{\rho}_1 = \rho + \frac{1}{2}\gamma(1-\gamma)\sigma_1^2 - \gamma\hat{z} \end{cases}$$

and if $b < \widehat{z}$,

$$\begin{cases} k_2 = \sigma_2^2(1-\gamma) - (1-\gamma)\delta\sigma_1\sigma_2 + (\hat{z} - b) \\ \hat{\rho}_2 = \rho + \frac{1}{2}\gamma(1-\gamma)\sigma_2^2 - \gamma b \,. \end{cases}$$

We assume that at least one of the following sets of inequalities holds

$$\{b \geq \hat{z}, k_1 > 0, \hat{\rho}_1 > 0\} \quad \text{or} \quad \{b < \hat{z}, k_2 > 0, \hat{\rho}_2 > 0\}, \tag{6a}$$

together with the additional related conditions

$$\hat{\rho}_1 > \hat{z}(\gamma - 1) + \frac{\gamma k_1^2}{2\sigma^2(1-\gamma)}, \quad \text{if} \quad b \geq \hat{z}$$

or (6b)

$$\hat{\rho}_2 > b(\gamma - 1) + \frac{\gamma k_2^2}{2\sigma^2(1-\gamma)}, \quad \text{if} \quad b < \hat{z}\,.$$

We continue with some elementary properties of the value function whose proofs appear in Appendix B.

PROPOSITION 1. *The value function V is increasing and jointly concave in the spatial arguments (x, y). Moreover, for fixed z, V is uniformly continuous on* $[0, +\infty) \times [0, +\infty)$.

PROPOSITION 2. *Under the growth conditions (6a) and (6b) and the properties of the utility function U, the value function is well defined on* $[0, +\infty) \times [0, +\infty)$ *for $z \in Z$.*

REMARK 1. Even though we use linear coefficients in the state equations (2) and (3), this assumption is by no means restrictive. In fact, all the arguments presented herein can be easily generalized to the case of general coefficients $\sigma_1(X_t)$, $b(Y_t)$ and $\sigma_2(Y_t)$ as long as $\sigma_1, b, \sigma_2 : [0, +\infty) \to [0, +\infty)$ are Lipschitz and concave functions of their argument with $\sigma_1(0) = b(0) = \sigma_2(0)$ and, at least one of the σ_i's, $i = 1, 2$ satisfies $\sigma_i(w) > mw$, for $w \geq 0$ and $m > 0$.[2] The motivation behind the choice of linear coefficients is for the sake of simplicity since the methodology is easier to present and the numerical schemes have been validated for such coefficients.

The classical way to attack problems of stochastic control is to analyze the relevant equation that the value function is expected to solve, namely the *Hamilton–Jacobi–Bellman equation*. This HJB equation is the offspring of the Dynamic Programming Principle and stochastic analysis. When singular policies (lump adjustments of controls) are allowed, the HJB equation becomes a Variational Inequality with gradient constraints. These constraints are associated to the 'optimal direction' of instantaneously moving the optimally controlled state processes. In the context of optimal consumption and investment problems, such situations arise when transaction fees are paid (see, for example, Zariphopoulou (1992), Tourin and Zariphopoulou (1995)). In the problem we study herein, the analysis is more complicated because the drift of the state process x_t is influenced by the fluctuations of the jump Markov process z_t. This feature, together with the presence of singular policies results in an HJB equation which is actually *an integral-differential Variational Inequality with gradient constraints* (see Equation (7)).

If it is known a priori that the value function is smooth, then standard verification results guarantee that the value function is the unique smooth solution of the HJB equation. Moreover, if first order conditions for optimality apply, then they are sufficient to determine the optimal policies in the so-called feedback formula. (See, for example, Fleming and Soner (1993)). Unfortunately, this is rarely the case. In our problem, the value function might not be smooth and therefore it is necessary to relax the notion of solutions of the (HJB) equation. The class of solutions we will be using in this paper are weak solutions or the so-called (constrained) *viscosity solutions*. In models of optimal investment and consumption with transaction costs, this class of solutions was first employed by Zariphopoulou (1992). Subsequently this class of solutions was used among others by Davis, Panas and Zariphopoulou (1993), Zariphopoulou (1994), Davis and Zariphopoulou (1995), Tourin and Zariphopoulou (1994), Shreve and Soner (1994) and Barles and Soner (1998). Such investment models were developed in a single-currency (one-country) context; however, they are related to the current paper's model in that shipping costs between countries have similar effects to transaction costs for adjusting an investment portfolio's composition. However, the problem addressed by the current paper is substantially more complicated due to the modeling of political risk via a stochastic drift coefficient. Finally, the characterization of V as a constrained solution is natural because of the presence of state constraints given by (4).

The notion of viscosity solutions was introduced by Crandall and Lions (1983) for first-order equations, and by Lions (1983) for second-order equations.

Constrained viscosity solutions were introduced by Soner (1986) and Capuzzo–Dolcetta and Lions (1990) for first-order equations (see also Ishii and Lions (1990)). For a general overview of the theory we refer to the *User's Guide* by Crandall, Ishii and Lions (1992) and to the book by Fleming and Soner (1993). We provide the definition of constrained viscosity solutions in Appendix A.

The following theorem provides a unique characterization of the value function. Its proof is discussed in Appendix B.

THEOREM 1. *The value function is the unique constrained viscosity solution on* $[0, +\infty) \times [0, +\infty)$, *of the integral-differential Variational Inequality*

$$\begin{aligned} \min \big\{ &\rho V(x, y; z) - \mathcal{B} V(x, y; z) - \mathcal{L} V(x, y; z) - \mathcal{H}(V_x(x, y; z),\ V_y(x, y; z)) \\ &- z x V_x(x, y; z) - b y V_y(x, y; z),\ (1+\lambda) V_x(x, y; z) - V_y(x, y; z), \\ &-(1-\mu) V_x(x, y; z) + V_y(x, y; z) \big\} = 0 \end{aligned} \tag{7}$$

in the class of concave and increasing functions with respect to the spatial argument (x, y). *Here* $\mathcal{L}$ *is the differential operator*

$$\mathcal{L} V = \frac{1}{2}\sigma_1^2 x^2 V_{xx} + \delta \sigma_1 \sigma_2 x y V_{xy} + \frac{1}{2}\sigma_2^2 y^2 V_{yy}\,, \tag{8}$$

and

$$\mathcal{H}(q_1, q_2) = \max_{C^x \geq 0, C^y \geq 0} \left\{ -q_1 C^x - q_2 C^y + U(C^x, C^y) \right\}. \tag{9}$$

The generator operator $\mathcal{B}V(x, y; z)$ is given in (1).

As it was mentioned earlier, the presence of singular policies leads to a depletion of the state space into regions of three types, namely the 'Import to country $\mathcal{Y}$' ($I_{\mathcal{Y}}$), 'Export from country $\mathcal{Y}$' ($\mathcal{E}_{\mathcal{Y}}$) and 'No shipping' ($\mathcal{NS}$) regions. The choice for the country $\mathcal{Y}$ as the baseline for describing the optimal trading rules is arbitrary and does not change the nature of the results. The regions ($I_{\mathcal{Y}}$), ($\mathcal{E}_{\mathcal{Y}}$) and ($\mathcal{NS}$) are related to the optimal shipping rules as follows:

(i) if at time t the production technology state (x_t, y_t) belongs to ($\mathcal{NS}$) region, only the consumption and production processes are used and no shipments take place from one country to the other.

(ii) If the state (x_t, y_t) belongs to the ($I_{\mathcal{Y}}$) (respectively, $\mathcal{E}_{\mathcal{Y}}$) region, it is beneficial to import (respectively, export) a shipment from country $\mathcal{X}$ to country $\mathcal{Y}$.

In regions $I_{\mathcal{Y}}$ and $\mathcal{E}_{\mathcal{Y}}$ where shipments are optimal, a singular policy – which represents the 'lump-sum' shipment – is used to move to a new state, say (x_{t^+}, y_{t^+}) which belongs to the boundary of the ($\mathcal{NS}$) and the ($I_{\mathcal{Y}}$) (respectively, ($\mathcal{E}_{\mathcal{Y}}$)) regions.

No closed-form solutions exist to date for the free boundaries of the aforementioned ($I_{\mathcal{Y}}$), ($\mathcal{E}_{\mathcal{Y}}$) and ($\mathcal{NS}$) regions. Therefore, it is highly desirable to analyze these boundaries as well as other related quantities, numerically. This is the task we undertake in the next section.

REMARK 2. In the special case of a collective utility function of the CRRA type, $U(C^x, C^y) = [(C^x)^\gamma + (C^y)^\gamma]$ for $0 < \gamma < 1$, one can show that the value function is homogeneous of degree γ. This fact provides valuable information about the free boundaries which turn out to be straight lines passing through the origin.

We continue this section by presenting some results related to analytic bounds of the value function as well as alternative characterizations of it in terms of a class of 'pseudo-collective' value functions. The latter results are expected to enhance our intuition for the economic significance of the proposed pricing model. We only present the main steps of the proofs of these results; the underlying idea is to use the HJB equation (7) and interpret it as the HJB equation of new pseudo-utility problems. The comparison between the new 'pseudo-value functions' and the original value function stems from the uniqueness result in Theorem 1 as well as the fact that the pseudo-value functions are viscosity solutions of the associated HJB equations.

To this end, consider the following pairs $(\underline{x}_t, \underline{y}_t)$ and $(\bar{x}_t, \bar{y}_t)$ of state dynamics where $\underline{x}_t$, $\underline{y}_t$, $\bar{x}_t$ and $\bar{y}_t$ solve respectively

$$d\underline{x}_t = \check{z}\underline{x}_t dt - \underline{C}_t^x dt + \sigma_1 \underline{x}_t dW_t^1 - (1+\lambda) d\underline{L}_t + (1-\mu) d\underline{M}_t \tag{10}$$

$$d\underline{y}_t = b\underline{y}_t dt - \underline{C}_t^y dt + \sigma_2 \underline{y}_t dW_t^2 + d\underline{L}_t - d\underline{M}_t \tag{11}$$

and

$$d\bar{x}_t = \hat{z}\bar{x}_t dt - \bar{C}_t^{\bar{x}} dt + \sigma_1 \bar{x}_t dW_t^1 - (1+\lambda) d\bar{L}_t + (1-\mu) d\bar{M}_t \tag{12}$$

$$d\bar{y}_t = b\bar{y}_t dt - \bar{C}_t^{\bar{y}} dt + \sigma_2 \bar{y}_t dW_t^2 + d\bar{L}_t - d\bar{M}_t\,, \tag{13}$$

where $\check{z}$, $\hat{z}$ are, respectively, the minimal and maximal elements of the state space Z and $\underline{x}_0 = \overline{x}^0 = x$, $\underline{y}_0 = \overline{y}^0 = y$. The above dynamics correspond to the case of deterministic drifts with no effect from the jump Markov process.

We define for (10), (11) and (12), (13) the sets of admissible policies $\mathcal{A}_{\check{z}}$ and $\mathcal{A}_{\hat{z}}$ along the same lines as before. That is, $\mathcal{A}_{\check{z}}$ (respectively, $A_{\hat{z}}$) is the set of admissible policies given that we are currently in state $\check{z}$ (respectively, $\hat{z}$). Note that the definition of $A_{\check{z}}$ is consistent both with a model structure where the drift coefficient is fixed at $\check{z}$ and with one where it follows a stochastic process z_t whose current value is $\check{z}$.

The following result shows that the original value function V is bounded between $\underline{v}$ and $\bar{v}$, the value functions of two international asset pricing models without political risk. More precisely, $\underline{v}$ (respectively $\bar{v}$) is the collective value function for countries $\underline{\mathcal{X}}$ and $\underline{\mathcal{Y}}$ (respectively $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$) with $\underline{\mathcal{X}}$ (respectively $\overline{\mathcal{X}}$) not exhibiting political instability but with a fixed drift coefficient of $\check{z}$ (respectively, $\hat{z}$). Models of this type were studied by Dumas (1992) in the case of CRRA utilities.

PROPOSITION 3. *Consider the value functions* $\underline{v}$, $\bar{v} : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$,

$$\underline{v}(x, y) = \sup_{\mathcal{A}_{\check{z}}} E \int_0^{+\infty} e^{-\rho t} U(\underline{C}_t^x, \underline{C}_t^y) dt \tag{14}$$

and

$$\bar{v}(x, y) = \sup_{\mathcal{A}_{\hat{z}}} E \int_0^{+\infty} e^{-\rho t} U(\bar{C}_t^{\bar{x}}, \bar{C}_t^{\bar{y}}) dt\,. \tag{15}$$

Then

$$\underline{v}(x, y) \le V(x, y; z) \le \bar{v}(x, y)$$

for $(x, y) \in [0, +\infty] \times [0, +\infty)$ *and* $z \in Z$.

In addition to understanding the behavior of the optimal shipping policies, we are particularly interested in specifying the equilibrium price behavior of goods **X** and **Y**. Note that we can interpret the ratio of the partial derivatives of $V(x, y; z)$ as the relative price of good **X** in terms of good **Y**. Then, the equilibrium price, for the state z, is given by

$$P_i(x, y; z_i) = \frac{V_x(x, y; z_i)}{V_y(x, y; z_i)}\,. \tag{16}$$

As mentioned earlier, there are no closed form solutions for the value functions and the optimal policies in the type of problem we are addressing, mainly because the HJB equation corresponds to a free boundary problem. Consequently, we use a numerical approach which is described in the next section. After obtaining the numerical results, we will return to the above price equation (16) and discuss the influence of shipping costs on the equilibrium prices.

3. Numerical Schemes

This section is devoted to the construction of numerical schemes for the solution of the Variational Inequality (7). To make the approximations tractable, we assume that the jump Markov process is actually a continuous-time Markov chain which can take two values, say z_1 and z_2 for which we have $z_1 < z_2$. The level z_1 (respectively z_2) represents the unfavorable (respectively, favorable) political state for the unstable country $\mathcal{X}$. The transitional probabilities are denoted by p_{ij}, $i, j = 1, 2$. For such a process, the generator operator $\widetilde{\mathcal{B}}$ is given by

$$\widetilde{\mathcal{B}} f(z_i) = p_{ij}(f(z_j) - f(z_i)) \quad \text{for } i \neq j,\ \ j = 1, 2\,.$$

It also follows that the Variational Inequality (7) becomes a system of two differential inequalities coupled through the zeroth order terms, namely

$$\begin{aligned}\min\bigl\{&\rho V(x, y; z_1) - \mathcal{L}V(x, y; z_1) - \mathcal{H}(V_x(x, y; z_1), V_y(x, y; z_1))\\ &- p_{12}(V(x, y; z_2) - V(x, y; z_1)) - z_1 x V_x(x, y; z_1)\\ &- byV_y(x, y; z_1), (1+\lambda)V_x(x, y; z_1) - V_y(x, y; z_1),\\ &- (1-\mu)V_x(x, y; z_1) + V_y(x, y; z_1)\bigr\} = 0\end{aligned} \tag{17a}$$

and

$$\begin{aligned}\min\bigl\{&\rho V(x, y; z_2) - \mathcal{L}V(x, y; z_2) - \mathcal{H}(V_x(x, y; z_2), V_y(x, y; z_2))\\ &- p_{21}(V(x, y; z_1) - V(x, y; z_2)) - z_2 x V_x(x, y; z_2)\\ &- byV_y(x, y; z_2), (1+\lambda)V_x(x, y; z_2) - V_y(x, y; z_2),\\ &- (1-\mu)V_x(x, y; z_2) + V_y(x, y; z_2)\bigr\} = 0\end{aligned} \tag{17b}$$

where the operator $\mathcal{L}$ and the function $\mathcal{H}$ are given respectively by (8) and (9). For the rest of the section, we will be working with the Variational Inequalities (17a) and (17b) instead of (7).

The first goal in choosing the appropriate class of schemes is to find a scheme with three key properties: *consistency, monotonicity* and *stability*. We define these properties below and we use a generic notation for our equation in order to simplify the presentation.

To this end, we consider a nonlinear equation $F(w, u(w), Du(w), D^2u(w)) = 0$ for $w \in \overline{\Omega}$, Ω is an open subset of $\mathbb{R}^N$ and, where Du and D^2u denote respectively the gradient and the second order derivative matrix of the solution u; F is continuous in all its arguments and the equation is degenerate elliptic, meaning that $F(w, p, q, A + B) \leq F(w, p, q, A)$ if $B \geq 0$.

DEFINITION 1. *We consider a sequence of approximations* $S : \mathbb{R}^+ \times \overline{\Omega} \times \mathbb{R} \times \mathcal{B}_{\text{loc}}(\overline{\Omega}) \to \mathbb{R}$ *where* $S = S(\theta, w, u^\theta(w), u^\theta)$ *and* $\mathcal{B}_{\text{loc}}(\overline{\Omega})$ *is the space of locally bounded functions on* $\overline{\Omega}$.

We say that S is:
monotone if

$$S(\theta, w, t, u) \leq S(\theta, w, t, v) \quad \text{for } u \geq v,$$

consistent if

$$\limsup_{(\theta,y,\xi)\to(0,w,0)} \frac{S(\theta, w, \phi(w) + \xi, \phi + \xi)}{\theta} =$$

$$\liminf_{(\theta,y,\xi)\to(0,w,0)} \frac{S(\theta, w, \phi(w) + \xi, \phi + \xi)}{\theta} = F(w, \phi(w), D\phi(w), D^2\phi(w)),$$

stable if

$\forall \theta > 0$, there exists a solution $u^\theta \in \mathcal{B}_{\text{loc}}(\overline{\Omega})$ of $S(\theta, w, u^\theta(w), u^\theta) = 0$ and its (local) bound is independent of θ.

The motivation to use such schemes for our model comes from the fact that they exhibit excellent convergence properties to the (viscosity) solution of fully nonlinear degenerate elliptic partial differential equations as long as the latter have a unique solution. This result was established by Barles and Souganidis (1991)[3] and it is stated below for completeness.

THEOREM 2 (Barles and Souganidis). *Assume that the equation* $F(w, u, Du, D^2u) = 0$ *admits the strong uniqueness property, i.e. if u (resp. v) is a viscosity subsolution (resp. supersolution) of* $F = 0$, *then* $u \leq v$. *If the approximation sequence* $\{S^\theta\}$ *satisfies the monotonicity, consistency and stability properties then the solution* u^θ *of* $S(\theta, w, u^\theta(w), u^\theta) = 0$ *converges locally uniformly to the unique viscosity solution of* $F(w, u, Du, D^2u) = 0$.

Besides, it is well known that the rate of convergence for such a scheme is of order $\sqrt{\theta}$ in the L_∞ norm as θ tends to 0, even if in practice generally one observes a first-order accuracy (see for example, Crandall and Lions (1983)).

We continue with the description of our scheme. To this end, we first write (17a) and (17b) in the concise form

$$\min\{\rho V - \mathcal{M}V - \mathcal{L}_0 V,\ \mathcal{L}_1 V,\ \mathcal{L}_2 V\} = 0\,, \tag{18}$$

where for $i = 1, 2$ at (x, y, z_i)

$$\begin{aligned}\mathcal{M}V(x, y; z_i) &= \mathcal{L}V(x, y; z_i) + z_i x V_x + byV_y \\ &\quad + \mathcal{H}(V_x(x, y; z_i), V_y(x, y; z_i))\end{aligned}$$

with $\mathcal{L}$ given in (8) and,

$$\begin{aligned}\mathcal{L}_0 V(x, y; z_1) &= p_{12}(V(x, y; z_2) - V(x, y; z_1)),\\ \mathcal{L}_0 V(x, y; z_2) &= p_{21}(V(x, y; z_1) - V(x, y; z_2)),\\ \mathcal{L}_1 V(x, y; z_i) &= (1 + \lambda)V_x(x, y; z_i) - V_y(x, y; z_i),\\ \mathcal{L}_2 V(x, y; z_i) &= -(1 - \mu)V_x(x, y; z_i) + V_y(x, y; z_i).\end{aligned}$$

The first step consists of approximating the equation in the whole space by an equation set in a bounded domain $\mathcal{B}_R = [0, R] \times [0, R]$ and proving the existence of a solution V_R of the Variational Inequalities in B_R and the convergence of V_R to V as R tends to the infinity. As there is no natural condition satisfied at infinity by $V(x, y; z_1)$ and $V(x, y; z_2)$, we have to decide what kind of condition we impose on ∂B_R. Barles, Daher and Romano (1995) answered this question and exhibited an exponential rate of convergence for the heat equation complemented either with Dirichlet or Neumann conditions. The generalization of their result to more general parabolic equations is straightfoward (for more details, see Barles, Daher and Romano (1995)). In the degenerate elliptic case, there is no natural choice for the Dirichlet or Neumann boundary value.

We impose here a simple, arbitrary Neumann condition $\dfrac{\partial V_R}{\partial n}(x, y; z) = K$ where n is the outer unit vector and K is a preassigned positive constant. Note that this condition must be taken in the viscosity sense and that the corners of $\mathcal{B}_R$ require a specific treatment.

The second step is the approximation to the solution of the equation set in the above bounded domain. We denote by Δx and Δy, respectively, the mesh sizes in the x and y directions. Moreover, for $i = 0..N$, and $j = 0..M$, $x_i = i\Delta x$, $y_j = j\Delta y$ are the grid points (with $\Delta x = \frac{R}{N}$ and $\Delta y = \frac{R}{M}$) and $V^1_{i,j}$ (resp. $V^2_{i,j}$) are the approximations for the value function $V(x, y; z_1)$ (resp. $V(x, y; z_2)$) at the grid point (x_i, y_j). We then propose an iterative algorithm to compute $V^1_{i,j}$ and $V^2_{i,j}$. For this purpose, we introduce a time step Δt and the approximation for $V^1_{i,j}$ (resp. $V^2_{i,j}$) at step n will be denoted by $V^{1,n}_{i,j}$ (resp. $V^{2,n}_{i,j}$). If $(V^{1,n}_{i,j}, V^{2,n}_{i,j})$ is known at step

n, the monotone scheme which allows us to compute at step $n+1$, $(V_{i,j}^{1,n+1} V_{i,j}^{2,n+1})$ may be ultimately written as

$$S^1(\Delta t, \Delta x, \Delta y, n\Delta t, x_i, y_j, V_{i,j}^{1,n+1}, V_{i,j}^{2,n}, V_{i,j}^{1,n}, V_{i-1,j-1}^{1,n}, V_{i-1,j-1}^{1,n+1}, V_{i-1,j}^{1,n}, V_{i-1,j}^{1,n+1}, V_{i-1,j+1}^{1,n}, V_{i-1,j+1}^{1,n+1}, V_{i,j-1}^{1,n}, V_{i,j-1}^{1,n+1}, V_{i,j+1}^{1,n}, V_{i,j+1}^{1,n+1}, V_{i+1,j-1}^{1,n}, V_{i+1,j-1}^{1,n+1}, V_{i+1,j}^{1,n}, V_{i+1,j}^{1,n+1}, V_{i+1,j+1}^{1,n}, V_{i+1,j+1}^{1,n+1}) = 0,$$

and

$$S^2(\Delta t, \Delta x, \Delta y, n\Delta t, x_i, y_j, V_{i,j}^{2,n+1}, V_{i,j}^{1,n}, V_{i,j}^{2,n}, V_{i-1,j-1}^{2,n}, V_{i-1,j-1}^{2,n+1}, V_{i-1j}^{2,n}, V_{i-1,j}^{2,n+1}, V_{i-1,j+1}^{2,n}, V_{i-1j+1}^{2,n+1}, V_{i,j-1}^{2,n}, V_{i,j-1}^{2,n+1}, V_{i,j+1}^{2,n}, V_{i,j+1}^{2,n+1}, V_{i+1,j-1}^{2,n}, V_{i+1,j-1}^{2,n+1}, V_{i+1,j}^{2,n}, V_{i+1,j}^{2,n+1}, V_{i+1,j+1}^{2,n}, V_{i+1,j+1}^{2,n+1}) = 0.$$

Both S^1 and S^2 are consistent with (17a), (17b) as Δt, Δx, Δy converge to 0 and $n\Delta t$ converges to $+\infty$. Moreover, one easily establishes that both S^1 and S^2 satisfy the monotonicity and stability properties as stated in Definition 1.

Note that $(\Delta t, \Delta x, \Delta y, n\Delta t)$ correspond to the variable θ in Definition 1, whereas (x_i, y_j) stands for w and $V_{i,j}^{1,n+1}$ in S^1 (resp. $V_{i,j}^{2,n+1}$ in S^2) represents $u^\theta(w)$. Finally, the role of the variable u^θ is played here for S^1 by the vector

$$(V_{i,j}^{2,n}, V_{i,j}^{1,n}, V_{i-1,j-1}^{1,n}, V_{i-1,j-1}^{1,n+1}, V_{i-1,j}^{1,n}, V_{i-1,j}^{1,n+1}, V_{i-1,j+1}^{1,n}, V_{i-1,j+1}^{1,n+1}, V_{i,j-1}^{1,n}, V_{i,j-1}^{1,n+1}, V_{i,j+1}^{1,n}, V_{i,j+1}^{1,n+1}, V_{i+1,j-1}^{1,n}, V_{i+1,j-1}^{1,n+1}, V_{i+1,j}^{1,n}, V_{i+1,j}^{1,n+1}, V_{i+1,j+1}^{1,n}, V_{i+1,j+1}^{1,n+1}).$$

We are now ready to start the construction of the scheme. First, we define the following explicit approximation to the gradient operators $\mathcal{L}_1 V$ and $\mathcal{L}_2 V$

$$\frac{V_{i,j}^{n+1} - V_{i,j}^n}{\Delta t} = -(1+\lambda)\frac{V_{i,j}^n - V_{i-1,j}^n}{\Delta x} + \frac{V_{i,j+1}^n - V_{i,j}^n}{\Delta y}. \tag{19}$$

$$\frac{V_{i,j}^{n+1} - V_{i,j}^n}{\Delta t} = (1-\mu)\frac{V_{i+1,j}^n - V_{i,j}^n}{\Delta x} - \frac{V_{i,j}^n - V_{i,j-1}^n}{\Delta y}, \tag{20}$$

where $V_{i,j}^n$ stands for both $V_{i,j}^{1,n}$ and $V_{i,j}^{2,n}$. It is easy to verify that these approximations are monotone as long as $\Delta t \le \dfrac{\min(\Delta x, \Delta y)}{2+\lambda}$.

For the elliptic operator $\mathcal{L}$, we use a *time-splitting method* in order to approximate separately the first-order derivatives in a first-half iteration and the second-order ones in the second-half iteration.

For the first-half iteration, we consider the *first-order operator* $\tilde{\mathcal{L}}$ obtained by eliminating the second-order terms in $\mathcal{M}$, i.e., for $i = 1, 2$,

$$\tilde{\mathcal{L}}V(x, y; z_i) = z_i x V_x(x, y; z_i) + byV_y(x, y; z_i) + \mathcal{H}(V_x(x, y; z_i), V_y(x, y; z_i)).$$

Observe that the above operator corresponds to two new state processes, say $\widetilde{x}_t$ and $\widetilde{y}_t$, which do not have any diffusion component. The randomness comes only through the process z_t, affecting the drift of $\widetilde{x}_t$. In other words, $\widetilde{x}_t$ and $\widetilde{y}_t$ solve

$$\begin{cases} d\tilde{x}_t = z_t\tilde{x}_t dt - \tilde{C}_t^x dt - (1+\lambda)d\tilde{L}_t + (1-\mu)d\tilde{M}_t \\ d\tilde{y}_t = b\tilde{y}_t dt - \tilde{C}_t^y dt + d\tilde{L}_t - d\tilde{M}_t \\ \tilde{x}_0 = x, \quad \tilde{y}_0 = y \end{cases}$$

with z_t being the two-state Markov chain.

The 'first-order' analogues of (17a) and (17b) are, respectively,

$$\begin{aligned} &\min\{\rho\widetilde{V}(x,y;z_1) - \widetilde{\mathcal{L}}\widetilde{V}(x,y;z_1) - p_{12}(\widetilde{V}(x,y;z_2) - \widetilde{V}(x,y;z_1)), \\ &\quad (1+\lambda)\widetilde{V}_x(x,y;z_1) - V_y(x,y;z_1), -(1-\mu)\widetilde{V}_x(x,y;z_1) + \widetilde{V}_y(x,y;z_1)\} = 0 \end{aligned}$$

and

$$\begin{aligned} &\min\{\rho\widetilde{V}(x,y;z_2) - \widetilde{\mathcal{L}}\widetilde{V}(x,y;z_2) - p_{21}(\widetilde{V}(x,y;z_1) - \widetilde{V}(x,y;z_2)), \\ &\quad (1+\lambda)\widetilde{V}_x(x,y;z_2) - \widetilde{V}_y(x,y;z_2), -(1-\mu)\widetilde{V}_x(x,y;z_2) + \widetilde{V}_y(x,y;z_2)\big\} = 0. \end{aligned}$$

Following routine arguments from the theory of singular stochastic control one can get that the solution $\widetilde{V}$ of these two Variational Inequalities coincides with the value function of the 'first-order' problem

$$\widetilde{V}(x,y;z_i) = \sup_{\widetilde{\mathcal{A}}} \widetilde{E}\Big\{\int_0^{+\infty} e^{-\rho t}U(\widetilde{C}_t^x, \widetilde{C}_t^y)dt/\widetilde{x}_0 = x, \widetilde{y}_0 = y\Big\}$$

with $\widetilde{x}_t$ and $\widetilde{y}_t$ defined above and, $\widetilde{\mathcal{A}}$ being the set of admissible policies defined along the same lines as $\mathcal{A}$. (For this characterization, we refer the reader to Zariphopoulou (1992).)

We apply the Dynamic Programming Principle to the above control problem and we discretize it, that is, for Δt positive and sufficiently small, we choose a constant approximation to each consumption rate on the time interval $[0, \Delta t]$.

In order to be able to work with a closed-form expression, we concentrate on the class of utility functions of Constant Relative Risk Aversion, i.e., $U(C^x, C^y) = 2\big[(C^x)^{1/2} + (C^y)^{1/2}\big]$. We obtain the following numerical scheme for the operator $\rho V - \tilde{\mathcal{L}}V - \mathcal{L}_0$; this scheme is monotone for Δt sufficiently small:

$$\begin{aligned} \frac{V_{i,j}^{1,n+1/2} - V_{i,j}^{1,n}}{\Delta t} &= p_{12}(V_{i,j}^{2,n} - V_{i,j}^{1,n}) - \rho V_{i,j}^{1,n} \\ &+ h_1(\Delta x, x_i, V_{i-1,j}^{1,n}, V_{i,j}^{1,n}, V_{i+1,j}^{1,n}) + h_2(\Delta y, y_j, V_{i,j-1}^{1,n}, V_{i,j}^{1,n}, V_{i,j+1}^{1,n}). \end{aligned} \tag{21}$$

The coefficient h_1 above is defined by:

(i) $$h_1(\Delta x, x_i, V_{i-1,j}^{1,n}, V_{i,j}^{1,n}, V_{i+1,j}^{1,n}) = \frac{\Delta x}{V_{i+1,j}^{1,n} - V_{i,j}^{1,n}} + z_1 x_i \frac{V_{i+1,j}^{1,n} - V_{i,j}^{1,n}}{\Delta x}$$

$$\text{if } z_1 x_i \geq \frac{\Delta x}{(V^{1,n}_{i+1,j} - V^{1,n}_{i,j})^2} \text{ and } z_1 x_i \geq \frac{\Delta x}{(V^{1,n}_{i,j} - V^{1,n}_{i-1,j})^2},$$

$$\text{(ii) } h_1(\Delta x, x_i, V^{1,n}_{i-1,j}, V^{1,n}_{i,j}, V^{1,n}_{i+1,j}) = \frac{\Delta x}{V^{1,n}_{i,j} - V^{1,n}_{i-1,j}} + z_1 x_i \frac{V^{1,n}_{i,j} - V^{1,n}_{i-1,j}}{\Delta x}$$

$$\text{if } \left(z_1 x_i \geq \frac{\Delta x}{(V^{1,n}_{i+1,j} - V^{1,n}_{i,j})^2} \text{ and } z_1 x_i < \frac{\Delta x}{(V^{1,n}_{i,j} - V^{1,n}_{i-1,j})^2} \right) \text{ or}$$

$$\left(z_1 x_i < \frac{\Delta x}{(V^{1,n}_{i+1,j} - V^{1,n}_{i,j})^2} \text{ and } z_1 x_i < \frac{\Delta x}{(V^{1,n}_{i,j} - V^{1,n}_{i-1,j})^2} \right)$$

and

$$\text{(iii) } h_1(\Delta x, x_i, V^{1,n}_{i-1j}, V^{1,n}_{i,j}, V^{1,n}_{i+1,j}) = 2\sqrt{z_1 x_i},$$

$$\text{if } z_1 x_i < \frac{\Delta x}{(V^{1,n}_{i+1,j} - V^{1,n}_{i,j})^2} \text{ and } z_1 x_i \geq \frac{\Delta x}{(V^{1,n}_{i,j} - V^{1,n}_{i-1,j})^2}.$$

Symmetrically h_2 is deduced from h_1 by replacing Δx, $z_1 x_i$,$V^{1,n}_{i-1,j}$ and $V^{1,n}_{i+1,j}$ respectively by Δy, by_j, $V^{1,n}_{i,j-1}$ and $V^{1,n}_{i,j+1}$ and the approximation $V^{2,n}_{i,j}$ is obtained similarly.

A simple sufficient condition for the *monotonicity* of the previous approximation is provided by the following upper bound on the time-step

$$\Delta t \leq \min_{k\in\{1,2\},i,j} \left\{ \frac{1}{z_2 i + bj + \max(p_{12}, p_{21}) + \rho + \frac{\Delta x}{(V^{k,n}_{i,j} - V^{k,n}_{i-1,j})^2} + \frac{\Delta y}{(V^{k,n}_{i,j} - V^{k,n}_{i,j-1})^2}} \right\}.$$

The *second order degenerate elliptic term* is then approximated by the well-known Crank-Nicolson scheme with a parameter θ equal to 0.5. To simplify the presentation, we chose the following approximation for the second-order derivatives which in fact is not monotone but the replacement by a monotone approximation is routine and this modification does not affect the convergence of the scheme. As before, we use the notation $V_{i,j}^n$ for both $V_{i,j}^{1,n}$ and $V_{i,j}^{2,n}$.

$$
\begin{aligned}
\frac{V_{i,j}^{n+1}-V_{i,j}^{n+1/2}}{\Delta t} &= \frac{1}{2}\sigma_1^2 x_i^2 \left[\frac{1}{2}\frac{(V_{i+1,j}^{n+1/2}+V_{i-1,j}^{n+1/2}-2V_{i,j}^{n+1/2})}{\Delta x^2}\right. \\
&\quad \left. + \frac{1}{2}\frac{(V_{i+1,j}^{n+1}+V_{i-1,j}^{n+1}-2V_{i,j}^{n+1})}{\Delta x^2}\right] + \frac{1}{2}\sigma_2^2 y_j^2 \left[\frac{1}{2}\frac{(V_{i,j+1}^{n+1/2}+V_{i,j-1}^{n+1/2}-2V_{i,j}^{n+1/2})}{\Delta y^2}\right. \\
&\quad \left. + \frac{1}{2}\frac{(V_{i,j+1}^{n+1}+V_{i,j-1}^{n+1}-2V_{i,j}^{n+1})}{\Delta y^2}\right] \\
&\quad + \delta\sigma_1\sigma_2 x_i y_j \left[\frac{1}{2}\left(\frac{V_{i+1,j+1}^{n+1/2}+V_{i-1,j-1}^{n+1/2}-V_{i-1,j+1}^{n+1/2}-V_{i+1,j-1}^{n+1/2}}{4\Delta x\Delta y}\right)\right. \\
&\quad \left. + \frac{1}{2}\left(\frac{V_{i+1,j+1}^{n+1}+V_{i-1,j-1}^{n+1}-V_{i+1,j-1}^{n+1}-V_{i-1,j+1}^{n+1}}{4\Delta x\Delta y}\right)\right].
\end{aligned} \tag{22}
$$

On the x-axis, we impose for $V_{i,j}^1$ and $V_{i,j}^2$ the gradient constraint in the following format

$$
\frac{V_{i,j}^{n+1}-V_{i,j}^{n}}{\Delta t} = -(1+\lambda)\frac{V_{i,j}^{n}-V_{i-1,j}^{n}}{\Delta x} + \frac{V_{i,j+1}^{n}-V_{i,j}^{n}}{\Delta y}. \tag{23}
$$

Similarly, on the y-axis, we impose

$$
\frac{V_{i,j}^{n+1}-V_{i,j}^{n}}{\Delta t} = (1-\mu)\frac{V_{i+1,j}^{n}-V_{i,j}^{n}}{\Delta x} - \frac{V_{i,j}^{n}-V_{i,j-1}^{n}}{\Delta y}. \tag{24}
$$

(Here we assume that the no shipping ($\mathcal{NS}$) region is strictly included in the first quadrant; from the numerical experiments, it turns out that this hypothesis is satisfied for a large set of parameters).

The Crank–Nicolson scheme requires boundary conditions that are chosen as follows: on the x and y-axis we impose Dirichlet conditions whose values are provided by the above approximations ((23) and (24)) and we use the Neumann condition $\frac{\partial V_R}{\partial n}(x, y; z) = K$ on ∂B_R. Thus the second half-iteration consists of inverting a block matrix using a standard iterative Jacobi procedure performed by block.

At each iteration, we choose the following adaptive time-step which actually is not far from being constant but may evolve a little during the convergence:

$$\Delta t = \min\left\{ \min_{k\in\{1,2\},i,j} \left\{ \frac{1}{z_2 i + bj + \max(p_{12}, p_{21}) + \rho + \frac{\Delta x}{(V^{k,n}_{i,j} - V^{k,n}_{i-1j})^2} + \frac{\Delta y}{(V^{k,n}_{i,j} - V^{k,n}_{ij-1})^2}} \right\}, \frac{\min(\Delta x, \Delta y)}{2+\lambda} \right\}. \tag{25}$$

Given the approximations to the elliptic and the gradient operators, $V^{1,n}_{i,j}$ is then set to the maximal value over these three ones. Futhermore, we let the algorithm converge until the conditions $\sup_{i,j} |V^{1,n}_{i,j} - V^{1,n-1}_{i,j}| < \epsilon$ and $\sup_{i,j} |V^{2,n}_{i,j} - V^{2,n-1}_{i,j}| < \epsilon$ are reached, ϵ being a preassigned small positive constant. After the last iteration, we compute the equilibrium prices by using centered finite differences and finally, the no shipping region is defined as the set of the points where the approximation to the value function at the last step comes from the discretization of the elliptic operator.

We recapitulate by the following description of the algorithm we implemented on a Hewlett Packard workstation in Fortran with double precision:

ALGORITHM

1st Step:

$$V^{1,1}_{i,j} = V^{2,1}_{i,j} = x_i + y_j$$

K given

$(n+1)$*st step:* $V^{1,n}_{i,j}$, $V^{2,n}_{i,j}$ are given

1. Approximations to the gradient constraints:

Use (19) and (20) as follows: Given $V^{1,n}_{i,j}$, deduce $u^{1,n+1}_{i,j}$ from (19)

$$\frac{u^{1,n+1}_{i,j} - V^{1,n}_{i,j}}{\Delta t} = -(1+\lambda)\frac{V^{1,n}_{i,j} - V^{1,n}_{i-1,j}}{\Delta x} + \frac{V^{1,n}_{i,j+1} - V^{1,n}_{i,j}}{\Delta y}$$

and $w^{1,n+1}_{i,j}$ from (20).

$$\frac{w^{1,n+1}_{i,j} - V^{1,n}_{i,j}}{\Delta t} = (1-\mu)\frac{V^{1,n}_{i+1,j} - V^{1,n}_{i,j}}{\Delta x} - \frac{V^{1,n}_{i,j} - V^{1,n}_{i,j-1}}{\Delta y}.$$

Then, from $V^{2,n}_{i,j}$, compute similarly $u^{2,n+1}_{i,j}$ and $w^{2,n+1}_{i,j}$.

2. Construction of $V_{i,j}^{1,n+1}$, $V_{i,j}^{2,n+1}$ *on the x- and y-axis:*

Apply (23) along the x-axis and (24) along the y-axis.

3. Approximation to the second-order operator:

(a) *First half-iteration in* B_R*:*
Choose the time step as in (25) and apply the scheme (25) in order to compute $V_{i,j}^{1,n+1/2}$ from $V_{i,j}^{1,n}$ and $V_{i,j}^{2,n}$. Then $V_{i,j}^{2,n+1/2}$ is obtained similarly.

(b) *Second half-iteration in* B_R*:* Given the values $V_{i,j}^{1,n+1/2}$ in B_R and $V_{i,j}^{1,n+1}$ along the x-and y-axis, together with

$$V_{N,j}^{1,n+1} = V_{N-1,j}^{1,n+1} + K\Delta x$$

and

$$V_{i,M}^{1,n+1} = V_{i,M-1}^{1,n+1} + K\Delta y \quad \text{for } i = 1, \dots, N,\ j = 1, \dots, M-1,$$

perform a Jacobi iterative procedure by block to solve (22). The outcome is denoted by $v_{i,j}^{1,n+1}$.
Compute in the same way, $v_{i,j}^{2,n+1}$ from $V_{i,j}^{2,n+1/2}$.

4. Construction of $V_{i,j}^{1,n+1}$, $V_{i,j}^{2,n+1}$*:*

(a) in B_R, set

$$\begin{aligned} V_{ij}^{1,n+1} &= \max\left(v_{i,j}^{1,n+1}, u_{i,j}^{1,n+1}, w_{i,j}^{1,n+1}\right), \\ V_{ij}^{2,n+1} &= \max\left(v_{i,j}^{2,n+1}, u_{i,j}^{2,n+1}, w_{i,j}^{2,n+1}\right). \end{aligned}$$

(b) on ∂B_R, set

$$\begin{aligned} &V_{N,j}^{1,n+1} = V_{N-1,j}^{1,n+1} + K\Delta x \text{ and } V_{N,j}^{2,n+1} = V_{N-1,j}^{2,n+1} + K\Delta x \\ &\text{for } j = 1, \dots, M-1, \\ &V_{i,M}^{1,n+1} = V_{i,M-1}^{1,n+1} + K\Delta x \text{ and } V_{i,M}^{2,n+1} = V_{i,M-1}^{2,n+1} + K\Delta y \\ &\text{for } i = 1, \dots, N. \end{aligned}$$

If $\sup_{ij} |V_{ij}^{1,n} - V_{ij}^{1,n+1}| < \epsilon$ and $\sup_{ij} |V_{ij}^{2,n} - V_{ij}^{2,n+1}| < \epsilon$ then stop. Here ϵ is a *tolerance bound* prescribed by the user. The approximations after the last step are denoted by $V_{i,j}^1$ and $V_{i,j}^2$.

5. *After the convergence is established:*

(a) Compute the equilibrium prices

$$P_1(x_i, y_j, z_1) = \left\{ \frac{V^1_{i+1,j} - V^1_{i-1,j}}{V^1_{i,j+1} - V^1_{i,j-1}} \right\} \times \frac{\Delta y}{\Delta x} ,$$

and

$$P_2(x_i, y_j, z_2) = \left\{ \frac{V^2_{i+1,j} - V^2_{i-1,j}}{V^2_{i,j+1} - V^2_{i,j-1}} \right\} \times \frac{\Delta y}{\Delta x} .$$

(b) Find the no shipping ($\mathcal{NS}$) region for z_k, $k = 1, 2$

$$(\mathcal{NS}) = \{(x_i, y_j)\} \in B_R; \quad \text{such that}$$
$$\min\{\rho V - \mathcal{M}V - \mathcal{L}_0 V, \mathcal{L}_1 V, \mathcal{L}_2 V\} = \rho V - \mathcal{M}V - \mathcal{L}_0 V ,$$

where V is evaluated at $(x_i, y_j; z_k)$.

In numerical experiments, we let $\Delta x = \Delta y = 0.1$, $R = 10$, $N = M = 100$ and $\epsilon = 5(10)^{-3}$. Actually from the experiments, it turns out that lower values for ϵ (for example $\epsilon = 10^{-7}$) lead to the same no shipping region. Besides, in view of the CPU time required for these computations, it does not seem reasonable to lower the values of Δx, Δy, nor to increase the number of grid points.

The scheme does not behave in a perfectly stable way, at least for the no shipping region. If one lets the scheme converge for a very long time ($\epsilon = 10^{-7}$), the cone remains globally the same, except there are a few points which oscillate around the free boundaries, that is, they appear and disappear from iteration to iteration. This phenomenon might be caused by possibly over-estimated Neumann conditions for large values of x and y. Tables I and II illustrate the monotone convergence of the scheme, showing the residual error in the L_∞ norm

$$\max(\sup_{i,j} |V^{1,n+1}_{i,j} - V^{1,n}_{i,j}|, \sup_{i,j} |V^{2,n+1}_{i,j} - V^{2,n}_{i,j}|)$$

decreasing as n increases. These tables also give the number of iterations, the value of the computed time step, and the CPU time for these two illustrative cases.

4. Discussion on Numerical Results

To illustrate some of the implications of our model for international asset pricing, we conduct some numerical experiments. We choose the following market parameter values: $\sigma_1 = 0.3$, $\sigma_2 = 0.3$, $\lambda = \mu = 0.05$, $\rho = 0.05$, $\gamma = 0.05$, $\delta = 0.05$

and $b = 0.1$. These values are chosen to be consistent with the restrictions in (6a) and (6b) needed to guarantee that V is well-defined. They were also chosen to be realistic from an economic perspective. For example, $b = 0.1$ and $\sigma_1 = 0.3$ are consistent with average real returns and volatilities observed for U.S. stocks over long horizons. Using $\sigma_2 = \sigma_1$ is consistent with a situation where the production technology is the same in both countries. Even if the production technology is the same across countries, it is likely that idiosyncratic local factors result in a correlation of less than one. Somewhat arbitrarily, we picked a medium level of correlations with $\delta = 0.5$. To preclude our results being driven primarily by large shipping costs, we set both λ and μ equal to a modest value of 0.05. For similar reasons, we used a relatively low degree of risk aversion with $\gamma = 0.05$. For the rate of time preference, we use $\rho = 0.05$ which is consistent with interest rates on 'riskless' securities such as government debt. In what follows, we frequently set $z_1 = 0.05$ and $z_2 = 0.1$. In the favorable political state, expected returns are the same across countries with $b = z_2 = 0.1$. However, the unfavorable political state represents a substantial penalty for investment in country X relative to $b = 0.01$ in country Y.

Figures 1–4 show the $(\mathcal{NS})$ regions and the equilibrium prices for the states z_1 and z_2 in the *absence of political uncertainty*. More precisely, Figures 1 and 2 correspond to $z_1 = z_2 = 0.1$ with $p_{12} = p_{21} = 0$ while Figures 3 and 4 correspond to $z_1 = z_2 = 0.08$ with $p_{12} = p_{21} = 0$. In these as well as subsequent figures, we set $K = 0.6$. Moreover, since the error due to the Neumann conditions is essentially concentrated near the boundary, we will plot the computed $(\mathcal{NS})$ regions in the domain $[0, 5] \times [0, 5]$.

Figures 1 and 3 display the 'cone of no shipping' which is a characteristic of asset pricing problems where there are shipping or transaction costs, even without political risk. Note that in Figure 3, the cone has rotated 'upward' (in a counter-clockwise direction) relative to the cone in Figure 1. This occurred because in Figure 3, the expected return for asset X is lower and hence that asset is relative less valuable. Consequently, investors are less inclined to acquire asset X by exporting for country $\mathcal{Y}$ and similarly more inclined to acquire asset Y by exporting from country $\mathcal{X}$. In both Figures 2 and 4, the level of asset X is held fixed at a value of 3. For levels of asset Y below 3, this asset is relatively scarce and asset X is relatively plentiful. Hence the price of asset X in terms of Y is relatively low. In Figure 2, the expected returns on the two assets are equal and hence their relative price is 1 when the level of Y equals 3, matching the level of X. Note that in Figure 4, the relative price does not rise to 1 until the quantity of asset Y is approximately 5 due to the lower expected return for asset X.

Then, in order to study the influence of the transition probabilities we look at values for p_{12}, p_{21} other than zero. In Figures 5–12 we represent the $(\mathcal{NS})$ regions for the states $z_1 = 0.08$ and $z_2 = 0.1$ for the following four cases

Case A: $p_{12} = p_{21} = 0.1$.
Case B: $p_{12} = 0.1$, $p_{21} = 0.9$.

Case C: $p_{12} = 0.9$, $p_{21} = 0.1$.
Case D: $p_{12} = 0.9$, $p_{21} = 0.9$.

and in Figures 13–16, we graph the equilibrium prices for the above four cases.

In our model, the size and location of the no shipping cone depends on both the current political state in the Country $\mathcal{X}$ (the politically risky country) and the probabilities for transitioning between states. Consider, for example, the Figures 5 and 6. In Figure 5, Country $\mathcal{X}$ is in the poor political state with relatively low expected returns on asset X. In Figure 6, Country $\mathcal{X}$ is currently in the favorable political state. In the favorable state, asset X is more valuable and individuals are less inclined to export from Country $\mathcal{X}$. They are also more inclined to pay the shipping cost to import from Country $\mathcal{Y}$ and, in effect, convert some of their position in asset Y into asset X. These two changes in their relative willingness to trade are manifested in the downward (clockwise) rotation of the no shipping cone between Figures 5 and 6.

To see how the transition probabilities influence the rotation and size of the no shipping cone, we can compare Figures 3 and 5. For both figures, the expected return parameter for Country $\mathcal{X}$ is 8%, which corresponds to the poor political state. However, in Figure 5 there is a 10% probability of transitioning to the better state whereas in Figure 3 that transition probability is zero. Intuitively, the increased probability of moving to a better state increases the value of asset X and alters individuals' willingness to trade. In this case, the primary effect is a reduced willingness to export X which results in a downward rotation in the lower boundary of the no shipping cone.

We also provide graphical comparisons on the relative prices of goods **X** and **Y**. Consider, for example, Figure 13. In this figure, the quantity of **X** is fixed while the quantity of **Y** is varied and the relative price of **X** (in terms of **Y**) is plotted in each of the two political states. In state z_2 (the favorable political state) asset X is relatively more valuable. However, it is interesting to note that for situations where asset Y is either quite scarce or extremely plentiful, the political state seems to have a negligible effect on relative asset pricing. Intuitively, when Y is very scarce, its value becomes extremely high and the relative value of X becomes sufficiently small that the effect of differing political states is not apparent. A symmetric argument applies when Y is extremely plentiful.

Comparing, for example, Figures 13 and 14, we can again see that the transition probabilities have a substantial effect on the relative pricing of **X** and **Y**. In Figure 13, the probability p_{21} of transitioning from the high state to the unfavorable political state is only 10%, whereas probability has increased to 90% in Figure 14. As a consequence, there is a dramatic decline in the relative price differential between high and low states, as seen in these two figures. Similar results can be seen in Figures 15 and 16. Indeed for Figure 16, the transition probabilities have both become so great that the relative price difference across political states all but disappears.

In conclusion, this paper has developed a model of international asset pricing in the presence of political risk. Although the model is simplified, it represents a substantial step towards understanding how uncertainty about future government actions can affect the prices of tradeable assets. The recent turmoil in asset prices for several Southeast Asian countries as well as Brazil serves to emphasize the importance of gaining a better understanding of the effects of political risk on asset prices. Our numerical experiments with the model provide several interesting results.

However, these results are just the beginning of an effort to better understand how political risk alters international asset pricing. In that spirit, we hope that the current paper will stimulate further research on this important issue.

Table I. Convergence in the case of no political uncertainty.

Number of iterations	Residual error	Time step	Time
100	5.84E-02	2.92E-02	
200	4.86E-02	2.93E-02	
300	3.89E-02	2.94E-02	
400	3.07E-02	2.96E-02	
500	2.51E-02	2.97E-02	
600	2.19E-02	2.97E-02	
700	1.89E-02	2.98E-02	
800	1.61E-02	2.98E-02	
900	1.37E-02	2.98E-02	
1000	1.17E-02	2.98E-02	
1100	1.01E-02	2.98E-02	
1200	8.71E-03	2.98E-02	
1300	7.52E-03	2.98E-02	
1400	6.52E-03	2.98E-02	
1500	5.66E-03	2.98E-02	
1600	4.93E-03	2.98E-02	160 minutes

Parameter values: $b = z_1 = z_2 = 0.1$, $p_{12} = p_{21} = 0, \sigma_1 = \sigma_2 = 0.3$, $\lambda = \mu = 0.05$, $\rho = 0.05$, $\gamma = 0.5$ and $\delta = 0.5$.

Table II. Convergence in a case with political risk.

Number of iterations	Residual error	Time step	Time
100	6.22E-02	3.11E-02	
200	5.11E-02	3.12E-02	
300	4.03E-02	3.14E-02	
400	3.16E-02	3.15E-02	
500	2.62E-02	3.16E-02	
600	2.27E-02	3.17E-02	
700	1.94E-02	3.17E-02	
800	1.65E-02	3.17E-02	
900	1.41E-02	3.17E-02	
1000	1.20E-02	3.17E-02	
1100	1.02E-02	3.17E-02	
1200	8.79E-03	3.17E-02	
1300	7.56E-03	3.17E-02	
1400	6.51E-03	3.17E-02	
1500	5.62E-03	3.17E-02	
1600	4.86E-03	3.17E-02	160 minutes

Parameter values: $b = 0.1$, $z_1 = 0.08$, $z_2 = 0.1$, $p_{12} = p_{21} = 0.1$, $\sigma_1 = \sigma_2 = 0.3$, $\lambda = \mu = 0.05$, $\rho = 0.05$, $\gamma = 0.5$ and $\delta = 0.5$.

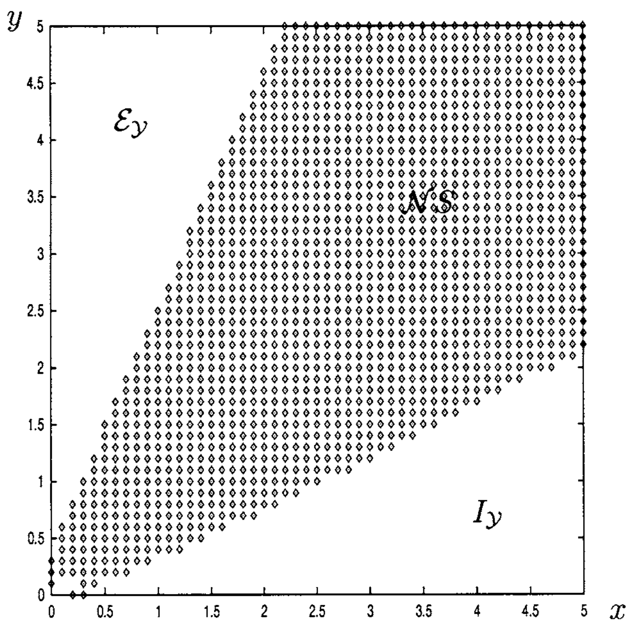

Figure 1. No political uncertainty: $z_1 = z_2 = 0.1$, $p_{12} = p_{21} = 0$, no-shipping region.

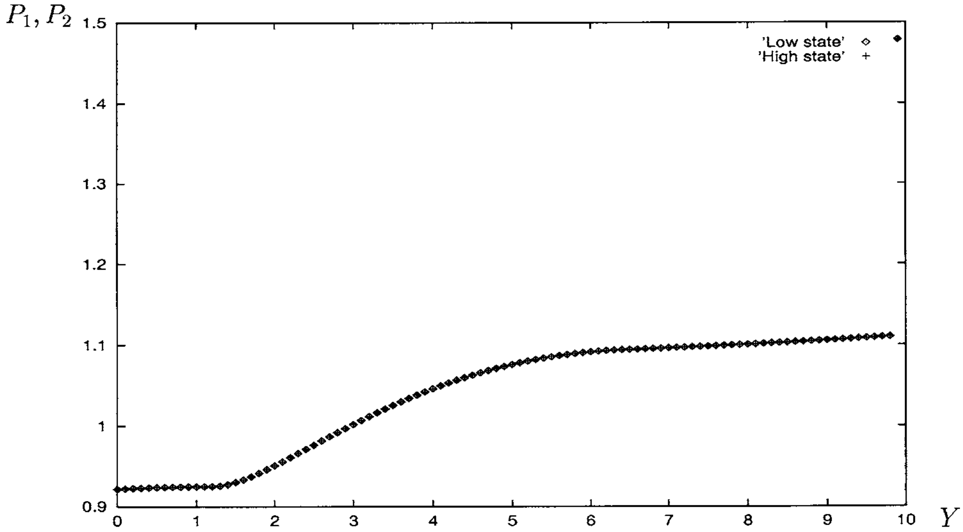

Figure 2. No political uncertainty: $z_1 = z_2 = 0.1$, $p_{12} = p_{21} = 0$, equilibrium prices for good **X** in terms of good **Y**. P_1 (low state), P_2 (high state); $X = 3$.

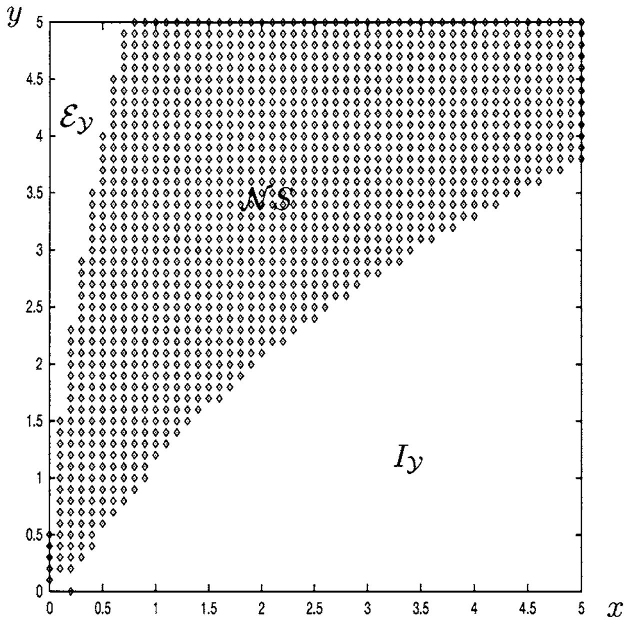

Figure 3. No political uncertainty: $z_1 = z_2 = 0.08$, $p_{12} = p_{21} = 0$, no-shipping region.

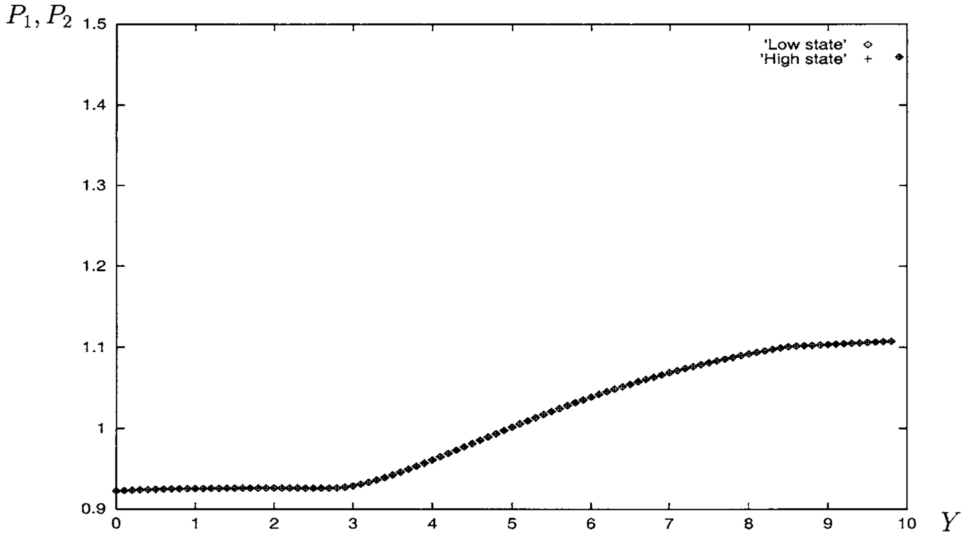

Figure 4. No political uncertainty: $z_1 = z_2 = 0.08$, $p_{12} = p_{21} = 0$, equilibrium prices for good **X** in terms of good **Y**. P_1 (low state), P_2 (high state); $X = 3$.

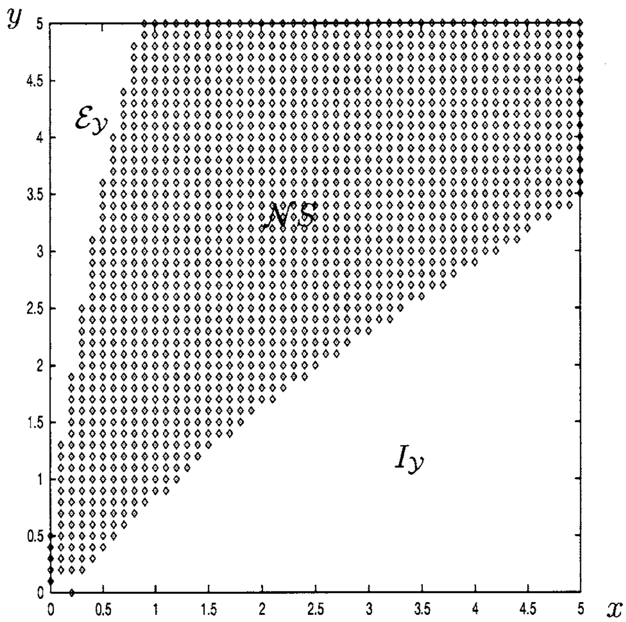

Figure 5. Case A: $z_1 = 0.08$, $z_2 = 0.1$, $p_{12} = 0.1$, $p_{21} = 0.1$; State z_1.

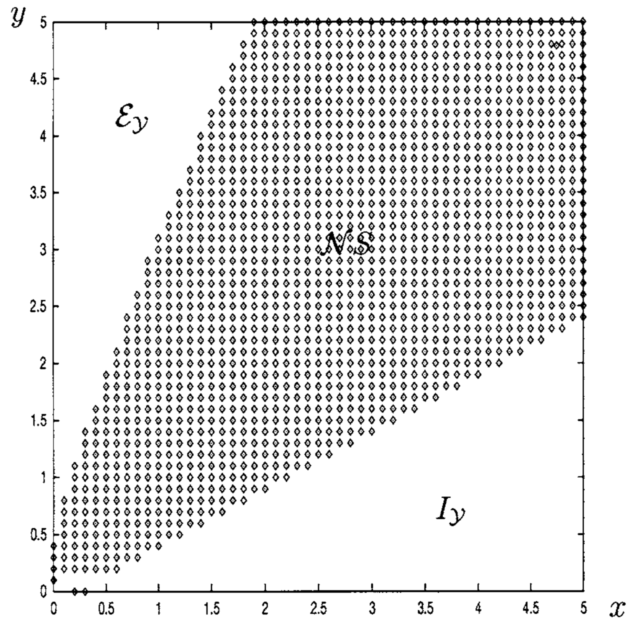

Figure 6. Case A: $z_1 = 0.08$, $z_2 = 0.1$, $p_{12} = 0.1$, $p_{21} = 0.1$; State z_2.

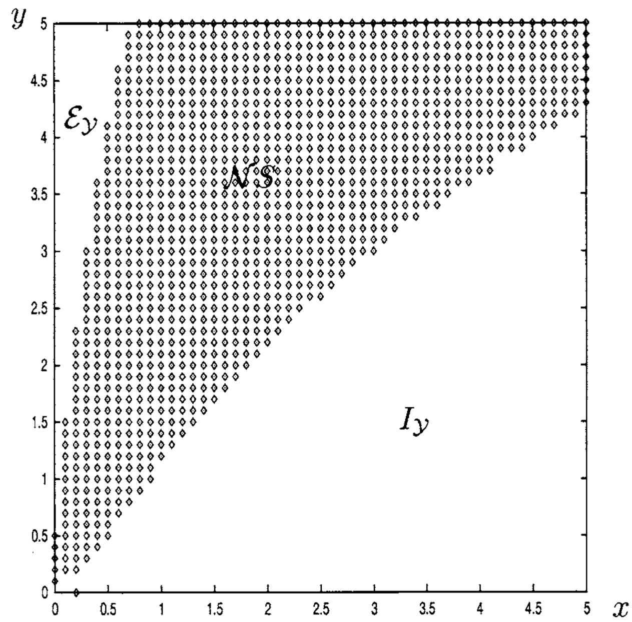

Figure 7. Case B: $z_1 = 0.08$, $z_2 = 0.1$, $p_{12} = 0.1$, $p_{21} = 0.9$; State z_1.

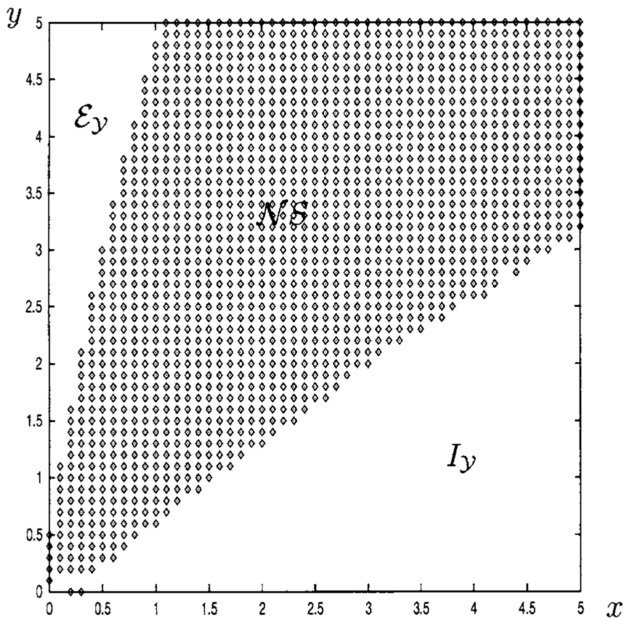

Figure 8. Case B: $z_1 = 0.08$, $z_2 = 0.1$, $p_{12} = 0.1$, $p_{21} = 0.9$; State z_2.

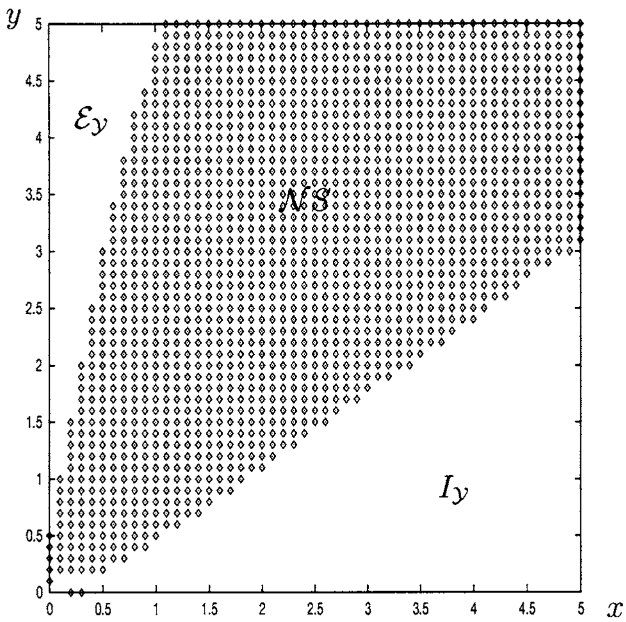

Figure 9. Case C: $z_1 = 0.08$, $z_2 = 0.1$, $p_{12} = 0.9$, $p_{21} = 0.1$; State z_1.

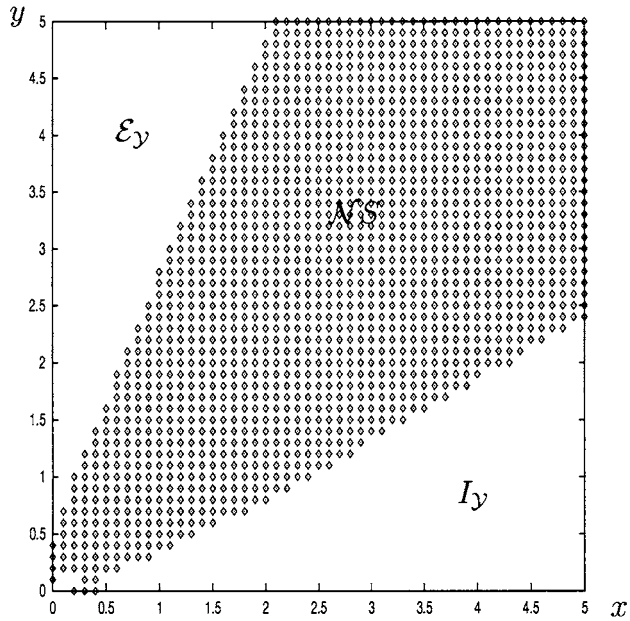

Figure 10. Case C: $z_1 = 0.08$, $z_2 = 0.1$, $p_{12} = 0.9$, $p_{21} = 0.1$; State z_2.

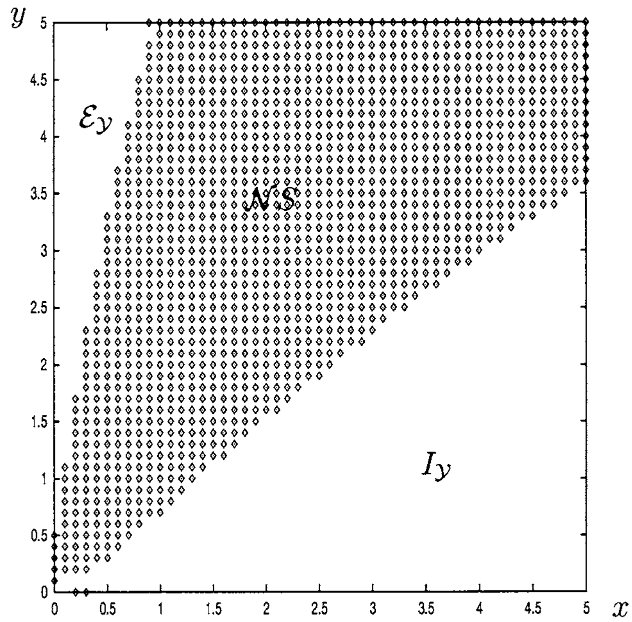

Figure 11. Case D: $z_1 = 0.08$, $z_2 = 0.1$, $p_{12} = 0.9$, $p_{21} = 0.9$; State z_1.

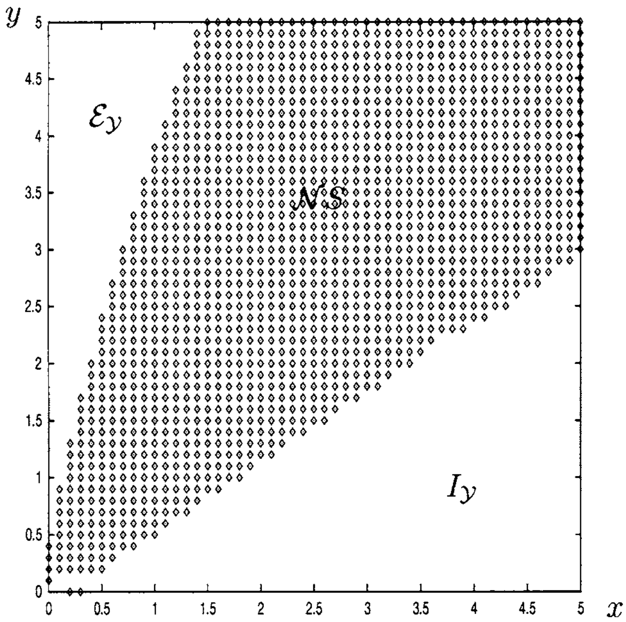

Figure 12. Case D: $z_1 = 0.08$, $z_2 = 0.1$, $p_{12} = 0.9$, $p_{21} = 0.9$; State z_2.

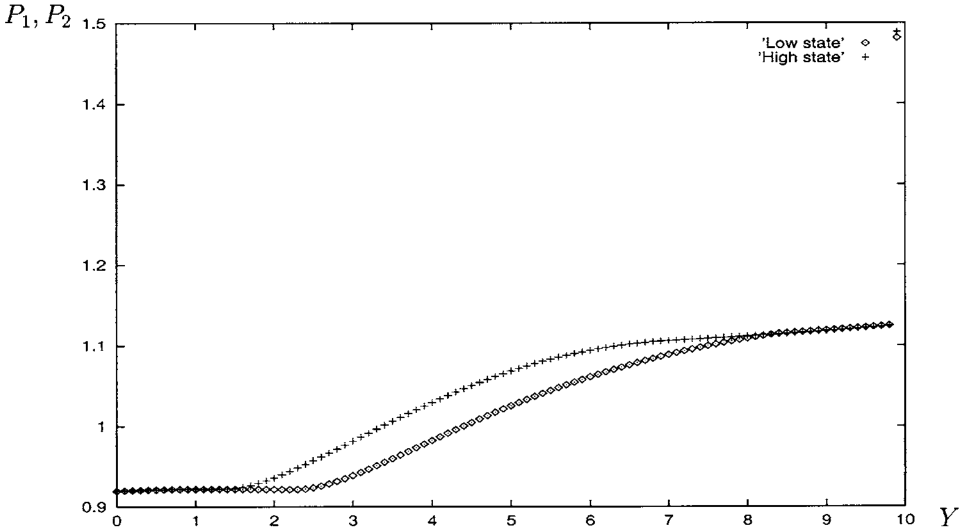

Figure 13. Case A: equilibrium prices for good **X** in terms of good **Y**. P_1 (low state), P_2 (high state); $X = 3$.

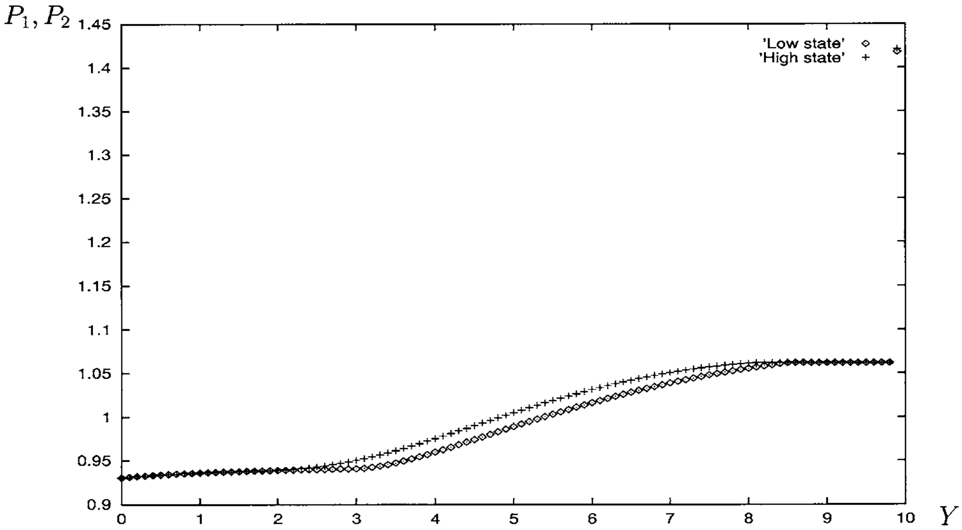

Figure 14. Case B: equilibrium prices for good **X** in terms of good **Y**. P_1 (low state), P_2 (high state); $X = 3$.

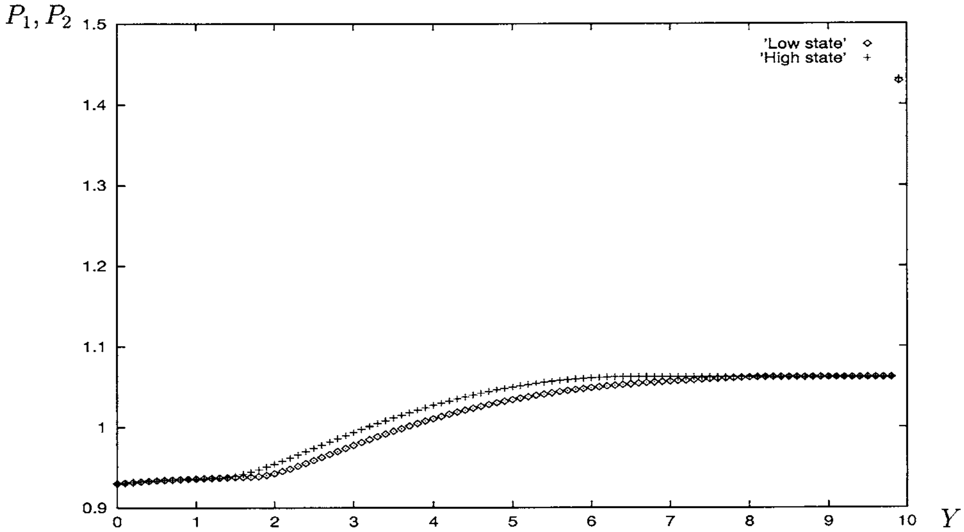

Figure 15. Case C: equilibrium prices for good **X** in terms of good **Y**. P_1 (low state), P_2 (high state); $X = 3$.

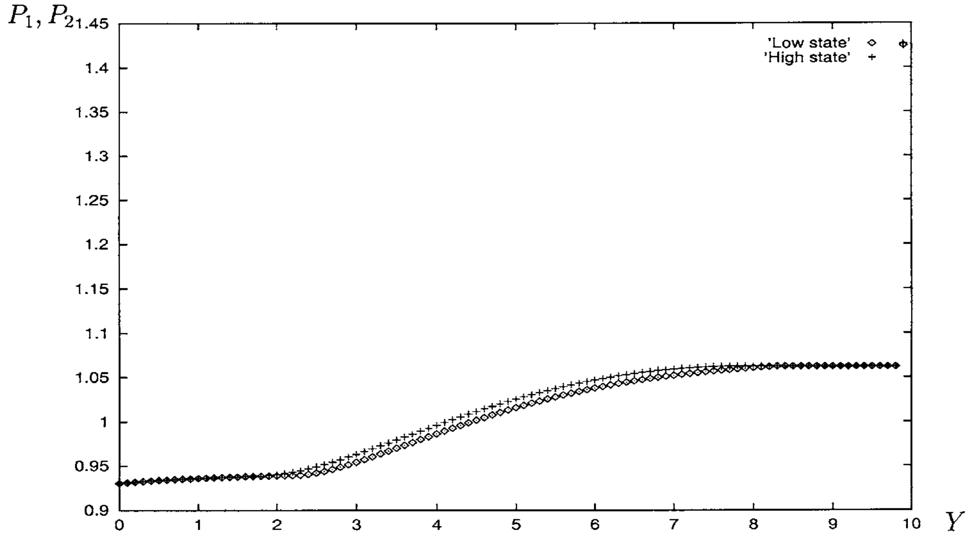

Figure 16. Case D: equilibrium prices for good **X** in terms of good **Y**. P_1 (low state), P_2 (high state); $X = 3$.

Appendix A

Consider a non-linear second order partial differential equation of the form

$$F(X, u, Du, D^2u) = 0 \quad \text{in } \Omega, \tag{A.1}$$

where Du and D^2u stand respectively for the gradient vector and the second derivative matrix of u; F is continuous in all its arguments and degenerate elliptic, meaning that

$$F(X, p, q, A + B) \leq F(X, p, q, A) \quad \text{if } B \geq 0. \tag{A.2}$$

DEFINITION A.1. *A continuous function $u : R \to R$ is a constrained viscosity solution of (A.1) if*

(i) u is a viscosity subsolution of (A.1) on $\overline{\Omega}$, that is for any $\phi \in C^2(\overline{\Omega})$ and any local maximum point $X_0 \in \overline{\Omega}$ of $u - \phi$

$$F(X_0, u(X_0), D\phi(X_0), D^2\phi(X_0)) \leq 0$$

and

(ii) u is a viscosity supersolution of (A.1) in Ω, that is for any $\phi \in C^2(\overline{\Omega})$ and any local minimum point $X_0 \in \Omega$ of $u - \phi$

$$F(X_0, u(X_0), D\phi(X_0), D^2\phi(X_0)) \geq 0.$$

Appendix B

Proof of Proposition 1. The monotonicity follows from the fact that for the point $(x+\epsilon, y)$ (respectively $(x, y+\epsilon)$) the set of admissible policies $\mathcal{A}_{z,(x+\epsilon,y)}$ satisfies $\mathcal{A}_{z,(x+\epsilon,y)} \supset \mathcal{A}_{z,(x,y)}$; the latter follows from the monotonicity and concavity of the utility function, the form of the state dynamics and the definition of the value function. These properties, together with the state constraints (4) are also used to establish the concavity of the value function. Indeed, if (C_1^x, C_1^y, L_1, M_1) and (C_2^x, C_2^y, L_2, M_2) are optimal policies for the points $(x_1, y_1; z)$ and $(x_2, y_2; z)$, then for $\lambda \in (0, 1)$, the policy $(\lambda C_1^x + (1-\lambda)C_2^x, \lambda C_1^y + (1-\lambda)C_2^y, \lambda L_1 + (1-\lambda)L_2, \lambda M_1 + (1-\lambda)M_2)$ is admissible for $(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2; z)$. For the uniform continuity of the value function on $[0, +\infty) \times [0, +\infty)$ we refer the reader to Proposition 2 in Tourin and Zariphopoulou (1994).

Proof of Proposition 2. We are only going to prove the proposition for the case $b > \hat{z}$ and $\hat{\rho}_1 > 0$, $k_1 > 0$, together with the first of inequalities (6b), since the other case can be worked out along the same lines.

We first recall Proposition 3 which states that the value function V is bounded from above by the value function $\bar{v}$ of the same stochastic control problem when there is no political risk and the process $z_t = \hat{z}$, with $\hat{z}$ being the maximal element of Z.

The rest of the proof amounts to demonstrating that $\bar{v}(x, y)$ is bounded from above by a value function which corresponds to the standard model of portfolio model with transaction costs, for which problem conditions (6a) and (6b) apply (see Davis and Norman (1990) and Shreve and Soner (1994)). To this end, observe that using the growth condition for the utility function U and a suboptimal in general policy (related to the gradient constraint $\bar{v}_y \geq (1-\mu)\bar{v}_x$), we have that

$$\bar{v}(x, y) \leq \bar{V}(x, y),$$

where $\bar{V}$ is the value function of the following stochastic control problem.

Consider the state equations

$$\begin{aligned} d\tilde{x}_t &= \hat{z}\tilde{x}_t dt + \sigma_1 \tilde{x}_t dW_t^1 - \tilde{C}_t dt - (1+\lambda) d\tilde{L}_t + (1-\mu) d\tilde{M}_t \\ d\tilde{y}_t &= b\tilde{y}_t dt + \sigma_2 \tilde{y}_t dW_t^2 + d\tilde{L}_t - d\tilde{M}_t \end{aligned}$$

and payoff

$$J(x, y, \tilde{C}, \tilde{L}, \tilde{M}) = E \int_0^{+\infty} e^{-\rho t} M \tilde{C}_t^{\gamma} \, dt \, .$$

For the rest of the arguments, we set $M = 1$ and use $\mathcal{A}$ as the generic set of admissible policies defined along the same lines as for the original problem. Then we define

$$\bar{V}(x, y) = \sup_{\mathcal{A}} E \int_0^{+\infty} e^{-\rho t} \tilde{C}_t^{\gamma} \, dt \, .$$

It is immediate to verify, using the power form of the utility and the linearity of the above state equations, that $\bar{V}$ is homogeneous of degree γ. In other words, $\bar{V}(x, y) = x^{\gamma} F(w)$ with $w = \frac{y}{x}$ and F solving the Variational Inequality

$$\begin{aligned}&\min\left\{\hat{\rho}_1 F - \tfrac{1}{2}\sigma^2 w^2 F_{ww} - k_1 w F_w - \max\nolimits_{c\geq 0}\{-cF_w + c^{\gamma}\},\right.\\ &\gamma(1+\lambda)F + F_w(1-(1+\lambda)w), -\gamma(1-\mu)F + F_w(1+(1-\mu)w)\} = 0.\end{aligned} \tag{B.1}$$

Now, consider the standard model of portfolio management in markets with transaction costs as in Davis and Norman (1990). In their model, there are two assets: a bond with riskless rate r, and a stock with mean rate of return μ and volatility σ. The utility function is of the same power type and the discount factor is β. In order to have a well-defined value function, Davis and Norman (1990) imposed the condition $\beta \geq r\gamma + (\gamma(\mu - r)^2/2\sigma^2(1-\gamma))$. For the same problem, Shreve and Soner (1994) provided a different set of conditions, in that

$$\beta > r\gamma + \frac{\gamma^2(\mu - r)^2}{2\sigma^2(1-\gamma)^2}.$$

Comparing coefficients with (B.1) and, after some tedious but otherwise straightforward calculations, we see that in order for F to be finite – and therefore the value function V – we must have either

$$\hat{\rho}_1 \geq \hat{z}(\gamma - 1) + \frac{\gamma k_1^2}{2\sigma^2(1-\gamma)}$$

or

$$\hat{\rho}_1 \geq \hat{z}(\gamma - 1) + \frac{\gamma^2 k_1^2}{2\sigma^2(1-\gamma)^2}.$$

Proof of Theorem 1. In order to simplify the presentation, we present the proof for the case that z_t is a jump Markov chain with possible states z_1 and z_2. The analysis for the general case follows along similar arguments.

The fact that the value function is a constrained viscosity solution of the system of Variational Inequalities (17a) and (17b) follows from a combination of the arguments used in Zariphopoulou (1991) and in Tourin and Zariphopoulou (1994).

In order to establish that the value function is the unique constrained viscosity solution, we need to construct a positive strict supersolution for (17a) and (17b). Once this supersolution is found, the rest of the arguments are similar to the ones used in Tourin and Zariphopoulou (1994) and they are not presented herein. We continue with the construction of the positive strict supersolution of (17a) and (17b), i.e., a function, say $G(x, y; z)$ such that

$$\begin{aligned}
&\min\Big\{ \min\{\rho G(x,y;z_1) - \mathcal{L}G(x,y;z_1) - \mathcal{H}(G_x(x,y;z_1), G_y(x,y;z_1)) \\
&\quad - p_{12}(G(x,y;z_2) - G(x,y;z_1)) - z_1 x G_x(x,y;z_1) \\
&\quad - byG_y(x,y;z_1),\ \ (1+\lambda)G_x(x,y;z_1) - G_y(x,y;z_1), \\
&\quad -(1-\mu)G_x(x,y;z_1) + G_y(x,y;z_1)\Big\}, \\
&\min\Big\{\rho G(x,y;z_2) - \mathcal{L}G(x,y;z_2) - \mathcal{H}(G_x(x,y;z_2), G_y(x,y;z_2)) \\
&\quad - p_{21}(G(x,y;z_1) - G(x,y;z_2)) - z_2 x G_x(x,y;z_2) \\
&\quad - byG_y(x,y;z_2),\ \ (1+\lambda)G_x(x,y;z_2) - G_y(x,y;z_2), \\
&\quad -(1-\mu)G_x(x,y;z_2) + G_y(x,y;z_2)\}\Big\} > \theta
\end{aligned} \tag{B.2}$$

for some positive constant θ.

To this end, we claim that there is an increasing in (x, y) and independent of z function $G(x, y)$ such that the above inequality holds. In fact, first observe that for such a function it suffices to show that

$$\begin{aligned}
&\min\Big\{\rho G - \mathcal{L}G - z_2 x G_x - byG_y - \mathcal{H}(G_x, G_y), \\
&\quad (1+\lambda)G_x - G_y, -(1-\mu)G_x + G_y\Big\} > \theta.
\end{aligned} \tag{B.3}$$

We continue with the assumption that $b > z_2$; the case $b \le z_2$ can be worked out using similar arguments. Since $\mathcal{H}$ is a decreasing function of its arguments, if G satisfies $G_y > (1-\mu)G_x$, then, in order to establish (B.3), it suffices to show

$$\begin{aligned}
&\min\Big\{\rho G - \mathcal{L}G - z_2 x G_x - byG_y - \max\nolimits_{c\ge 0}\{-cG_x + c^\gamma\}, \\
&\quad (1+\lambda)G_x - G_y, -(1-\mu)G_x + G_y\Big\} > 0.
\end{aligned} \tag{B.4}$$

The above inequality follows from the properties of the utility function and the nature of $\mathcal{H}$ as the following arguments show.

$$\begin{aligned}
\mathcal{H}(G_x, G_y) &= \max_{C^x, C^y}\Big\{-C^x G_x - C^y G_y + U(C^x, C^y)\Big\} \\
&\le \max_{C^x, C^y}\Big\{-(C^x + (1-\mu)C^y)G_x + U(C^x, C^y)\Big\} \\
&\le \max_{C^x, C^y}\Big\{-(C^x + (1-\mu)C^y)G_x + (C^x + (1-\mu)C^y)^\gamma\Big\} \\
&= \max_{c}\Big\{-cG_x + c^\gamma\Big\},
\end{aligned}$$

where $C^x \ge 0$, $C^y \ge 0$ and $c = C^x + (1-\mu)C^y \ge 0$.

Note that if $b < z_2$, it is more convenient to use that $G_x > \frac{1}{1+\lambda}G_y$ and work with the inequality

$$\mathcal{H}(G_x, G_y) \le \max_{C^x, C^y}\left\{-\left(\frac{1}{1+\lambda}C^x + C^y\right)G_x + \left(\frac{1}{1+\lambda}C^x + C^y\right)^\gamma\right\}.$$

The construction of a function that satisfies (B.4) was explicitly executed in Tourin and Zariphopoulou (1994) when the operator $\mathcal{L}$ is hypoelliptic in x i.e. when there are no higher than first order derivatives with respect to x. On the other hand, the general case we examine herein can be reduced to the degenerate case by manipulating the homogeneity properties of the value function.

To this end, we define the function G as follows. First we consider the solution g of

$$\begin{cases} (\hat{\rho}_1 + z_2)g = -\dfrac{k_1^2}{2\sigma^2}\dfrac{g_w^2}{g_{ww}} + z_2 w g_w + \max\limits_{c\geq 0}\{-cg_w + c^\gamma\} \\ g > 0,\, g_w > 0 \text{ and } g_{ww} < 0. \end{cases}$$

The reader familiar with continuous time portfolio choice problems will recognize that g is the solution to the classical Merton consumption-portfolio problem.

Now, define G by

$$G(x, y) = g(x + ky) + K + n_1 x + n_2 y\,,$$

where K, n_1, n_2 and k are positive constants and n_1, n_2 and k satisfy

$$1 - \mu < k < 1 + \lambda \text{ and } (1 + \lambda)n_1 > n_2 > \frac{\hat{\rho}_1}{\hat{\rho}_1 - (b - z_2)}(1 - \mu)n_2\,.$$

It then follows that

$$\begin{cases} (1 + \lambda)G_x(x, y) - G_y(x, y) = \\ \qquad (1 + \lambda - k)g'(x + ky) + [(1 + \lambda)n_1 - n_2] \\ -(1 - \mu)G_x(x, y) + G_y(x, y) = \\ \qquad (-1 + \mu + k)g'(x + ky) + [-(1 - \mu)n_1 + n_2]. \end{cases} \tag{B.5}$$

For the second order operator we use the choice of G, the equation that g satisfies and the homogeneity of the utility function. After tedious but straightforward calculations we get that

$$\rho G - \mathcal{L}G - z_2 x G_x - by G_y - \max_{c\geq 0}\{-cG_x + c^\gamma\} \geq (\hat{\rho}_1 + z_2)K. \tag{B.6}$$

Combining (B.5) and (B.6), we see that G satisfies (B.4) with $\theta = \min\{(\hat{\rho}_1 + z_2)K, (1 + \lambda)n_1 - n_2, n_2 - (1 - \mu)n_1\}$ and that $\theta > 0$.

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Notes

[1] A process is CADLAG if it is right-continuous with left limits.

[2] For a similar problem with general coefficients, we refer the interested reader to Scheinkman and Zariphopoulou (1996).

[3] The variational inequalities (17a) and (17b) belong to the class of equations that Barles and Souganidis (1991) examined. Our problem though is not entirely identical to theirs due to the presence of the state constraints (4). The convergence of our scheme, in the presence of the state constraints is not presented here.

References

Barles, G., Daher, C. and Romano, M. (1995). Convergence of numerical schemes for problems arising in finance theory. *Mathematical Models and Methods in Applied Sciences*, **5**, 125–143.

Barles, G. and Soner, H.M. (1998). Option pricing with transaction costs and a nonlinear Black–Scholes equation. *Finance and Stochastics*, **II** (4), 369–397.

Barles, G. and Souganidis, P.E. (1991). Convergence of approximation schemes for fully nonlinear second order equations. *Journal of Asymptotic Analysis*, **4**, 271–283.

Capuzzo-Dolcetta, I. and Lions, P.-L. (1990). Hamilton–Jacobi equations with state constraints. *Transactions of the American Mathematical Society*, **318**, 643–683.

Crandall, M., Ishii, H. and Lions, P.-L. (1992). User's guide to viscosity solutions of second order partial differential equations. *Bulletin of the American Mathematical Society*, **27**, 1–67.

Crandall, M. and Lions, P.-L. (1983). Viscosity solutions of Hamilton–Jacobi equations. *Transactions of the American Mathematical Society*, **277**, 1–42.

Davis, M.H.A. and Norman, A.R. (1990). Portfolio selection with transaction costs. *Mathematics of Operations Research*, **15**, 676–713.

Davis, M.H.A., Panas, V.G. and Zariphopoulou, T. (1993). European option pricing with transaction costs. *SIAM Journal of Control and Optimization*, **31**, 470–493.

Davis, M.H.A. and Zariphopoulou, T. (1995). American Options and Transaction Fees. *Mathematical Finance*. IMA Volumes in Mathematics and its Applications. Springer-Verlag.

Dumas, B. (1992). Dynamic equilibrium and the real exchange rate in a spatially separated world. *Review of Financial Studies*, **5**, 153–180.

Erb, C.B., Harvey, C.R. and Viskanta, T.E. (1996). Political risk, economic risk, and financial risk. *Financial Analysts Journal*, **52**, 29–45.

Fleming, W. and Soner, H.M. (1993). *Controlled Markov Processes and Viscosity Solutions*. Springer-Verlag, New York.

Ishii, H. and Lions, P.-L. (1990). Viscosity solutions of fully nonlinear second-order elliptic partial differential equations. *Journal of Differential Equations*, **83**, 26–78.

Lions, P.-L. (1983). Optimal control of diffusion processes and Hamilton–Jacobi–Bellman equations. Part 1: The dynamic programming principle and applications. Part 2: Viscosity solutions and uniqueness. *Communications in PDE*, **8**, 1101–1174, 1229–1276.

Scheinkman, J. and Zariphopoulou, T. (1996). Environmental Models with Irreversible Decisions. Fondazione Eni Enrico Mattei, Nota di Lavoro 15.

Sercu, P. and Uppal, R. (1995). *International Financial Markets and the Firm*. South-Western College Publishing, Cincinnati.

Shreve, S.E. and Soner, H.M. (1994). Optimal investment and consumption with transaction costs. *The Annals of Applied Probability*, **4**, 609–692.

Soner, H.M. (1986). Optimal control with state space constraints. *SIAM Journal on Control and Optimization*, **24**, 552–562, 1110–1122.

Tourin, A. and Zariphopoulou, T. (1994). Numerical schemes for investment models with singular transactions. *Computational Economics*, **7**, 287–307.

Tourin, A. and Zariphopoulou, T. (1995). Portfolio selection with transaction costs. *Progress in Probability*, **36**, 385–391.

Zariphopoulou, T. (1992). Investment-consumption models with transaction fees and Markov-chain parameters. *SIAM Journal on Control and Optimization*, **30**, 613–636.

Zariphopoulou, T. (1994). Investment consumption models with constraints. *SIAM Journal on Control and Optimization*, **32**, 59–84.