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A solution approach to valuation with unhedgeable risks

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Abstract. We study a class of stochastic optimization models of expected utility in markets with stochastically changing investment opportunities. The prices of the primitive assets are modelled as diffusion processes whose coefficients evolve according to correlated diffusion factors. Under certain assumptions on the individual preferences, we are able to produce reduced form solutions. Employing a power transformation, we express the value function in terms of the solution of a linear parabolic equation, with the power exponent depending only on the coefficients of correlation and risk aversion. This reduction facilitates considerably the study of the value function and the characterization of the optimal hedging demand. The new results demonstrate an interesting connection with valuation techniques using stochastic differential utilities and also, with distorted measures in a dynamic setting.

Key words: Non-traded assets, stochastic factors, unhedgeable risks, portfolio management, reduced form solutions

JEL Classification: C6, D9, G1

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1 Introduction

This work is a contribution to stochastic optimization problems of expected utility. Trading takes place between a bond and a stock account with the stock price being modelled as a diffusion process. The fundamental assumption is that the coefficients of the latter depend on another process, to be referred as "stochastic factor", which is in general correlated with the underlying stock price. We model the stochastic factor as a diffusion process but we make no special assumption on the structure of its diffusion coefficients. The individual preferences are modelled in terms of a Constant Relative Risk Aversion (CRRA) function and trading takes place in a finite horizon. The objective of the agent is to maximize his expected utility from terminal wealth and to specify the optimal investment strategy.

The motivation to study this rather general class of optimal investment models comes from their wide applicability in a variety of asset pricing problems. The simplest model that fits into our framework is the one with nonlinear stock diffusion dynamics. In this setting there are no unhedgeable risks, the stochastic factor can be readily identified with the (normalized) nonlinear component and it is perfectly correlated with the underlying price process. In the past, similar models have been extensively studied using martingale theory techniques (see Karatzas 1997), but closed form solutions have been produced only for the case of logarithmic utilities (Merton 1971).

Even though the above setting applies to a number of interesting situations – among others, in models of mean reverting stock prices and in models with constant elasticity of variance – the most challenging questions arise when there is not perfect correlation between the stochastic factor and the underlying stock price. In fact, there is a lot of interest, both from the practical as well as the theoretical point of view, to explore the effects of correlation to the optimal demand for the risky asset. The effects of correlation are also important in pricing derivatives with *non-traded assets* which, even though they cannot be used in the hedging portfolio, are often closely correlated to the available for trading underlying asset. During the last years, this type of derivatives has been attracting an ever increasing interest and no unified method has been successfully developed to date. As in other cases of derivative pricing in incomplete markets – for example, in markets with transaction costs and/or trading constraints – a rather successful method has been proven to be the so-called *utility approach*. In order to apply the latter to the case of non-traded assets, one needs first to study the relevant utility optimization problems which fall into the family of problems we solve herein (see Davis 1999).

It is worth noticing that an alternative valuation approach is based on minimization criteria of the *expected hedging loss*. These criteria may either rely on mean variance hedging (see, among others, Duffie and Richardson 1991; Schweizer 1992 and 1996) or on other types of loss functionals (see Hipp and Taksar 1999). In both classes of models, the majority of the relevant stochastic minimization problems are similar to the problems we study.

From a different direction, we may view the models of optimal portfolio management with *stochastic volatility* also as valuation models with non-traded assets. Even though, the volatility is not in general observable and therefore such modelling might not be entirely realistic, one may still rely on models with diffusion volatilities after the latter are recovered through the implied volatility which is actually observable (see for example the recent work by Avellaneda and Zhu 1999 and Ledoit and Santa Clara 1999).

The main contribution of this work is a methodology to derive reduced form solutions for the value function and the optimal policies in optimal investment models when prices are affected by correlated stochastic factors. This is accomplished via the homotheticity properties of the value function combined with a novel transformation. The latter enables us to express the value function in terms of a specific power of the solution of a linear parabolic equation. This exponent depends only on the risk aversion coefficient and the correlation between the stock price and the stochastic factor. The feedback optimal policies are then expressed in a simplified way in terms of the solution of the reduced linear equation and its first derivative.

Besides the apparent reduction in constructing the value function simply by solving a linear partial differential equation, the obtained representation formulae indicate an interesting connection with two unrelated notions: “distorted” measures of risk and recursive utilities. As we demonstrate in our discussion session, the component of the value function that represents the effects of the stochastic factor can be associated to a “distorted” risk measure in terms of a new pseudo-stochastic factor. The latter turns out to be a diffusion process with dynamics similar to the ones of the original stochastic factor process but with a modified drift. As it was mentioned above, we also establish a connection between the same component of the value function and recursive utilities. By interpreting the inherent nonlinearities appropriately, we are able to write the solution in terms of a recursive utility whose aggregator and variance multiplier are explicitly specified. Even though both the above characterization results are rather preliminary, they nevertheless indicate that the reduced form solutions we derive herein, might help us to understand – in connection with the above notions – how the various market imperfections and the relevant unhedgeable risks affect the prices, the optimal demand and the hedging strategies.

Finally, our results may be applied to a different class of valuation models in imperfect markets, namely the optimal investment and consumption models with stochastic labor income. In these models, the unhedgeable risks are coming through a stream of stochastic labor income that cannot be replicated by trading the available securities (see among others, Duffie and Zariphopoulou 1993; He and Pagès 1993; Duffie et al. 1997; El Karoui and Jeanblanc-Picqué 1994; Koo 1991). Due to the special way the stochastic income affects the dynamics of the state wealth, the scaling properties and the transformation employed herein cannot be applied. On the other hand, the results of this paper might be potentially used to derive qualitative properties for the solutions with stochastic income in finite horizon for which only very general results are known up to date.

The paper is organized as follows: in Sect. 2, we introduce the investment model and we state the main results. In Sect. 3, we derive the reduced form solutions and we provide regularity and verification results for the value function and the optimal policies. Finally, in Sect. 4, we present examples, we provide an interpretation of the results, using elements from the theory of stochastic differential utility, and we conclude with future research plans.

2 The investment model and main results

We consider an optimal investment model of a single agent who manages his portfolio by investing in a bond and a stock account. The price of the bond B_t solves

$$\begin{cases} dB_t = rB_t dt \\ \\ B_0 = B \end{cases} \tag{2.1}$$

where $r > 0$ is the interest rate. The price of the stock is modelled as a diffusion process S_t solving

$$dS_t = \mu(Y_t, t)S_t dt + \sigma(Y_t, t)S_t dW_t^1 \tag{2.2}$$

with $S_0 = S \geq 0$. The process Y_t will be referred to as the "stochastic factor" and it is assumed to satisfy

$$dY_t = b(Y_t, t)dt + a(Y_t, t)dW_t^2 \tag{2.3}$$

for $Y_0 = y \in R$.

The processes W_t^1 and W_t^2 are Brownian motions defined on a probability space $(\Omega, \mathscr{F}, P)$ and they are correlated with correlation coefficient ρ with $-1 \leq \rho \leq 1$. The coefficients μ, σ, b and a are functions of the factor Y_t and time and they are assumed to satisfy all the required regularity assumptions in order to guarantee that a unique solution to (2.2) and (2.3) exists. These conditions, together with some additional growth assumptions will be introduced later; at this point, we only outlay the underlying structure of our valuation model.

The investor rebalances his portfolio dynamically by choosing at any time s, for $s \in [t, T]$ and $0 \leq t \leq T$, the amounts π_s^0 and π_s to be invested, respectively, in the bond and the stock account. His total wealth satisfies the budget constraint $X_s = \pi_s^0 + \pi_s$ and the stochastic differential equation

$$\begin{cases} dX_s = rX_s ds + (\mu(Y_s, s) - r)\pi_s ds + \sigma(Y_s, s)\pi_s dW_s^1 \\ \\ X_t = x \geq 0 \qquad 0 \leq t \leq s \leq T. \end{cases} \tag{2.4}$$

The above wealth state equation follows from the budget constraint and the dynamics in (2.1), (2.2) and (2.3). The wealth process must also satisfy the state constraint

$$X_s \geq 0 \ \text{ a.e. } \ t \leq s \leq T. \tag{2.5}$$

Remark 2.1 Even though it is assumed that the bond price is deterministic, the case of *stochastic interest rates* may be easily analyzed as long as the interest rate depends only on Y_t and time. We choose not to incorporate this in our presentation because the really interesting case is when the interest rate depends on a second stochastic factor, correlated to Y_t in general. Such a formulation gives rise to a three dimensional problem and the formulae obtained herein might not generalize in a straightforward way.

Remark 2.2 In this model, it is assumed that the investor does not have the opportunity to consume part of his wealth in the trading interval $[t, T]$. Even though the methodology introduced here can be effectively applied to models with intermediate consumption, the results are of a different nature and of independent interest. This class of optimal investment and consumption models is analyzed in Zariphopoulou (1999).

The control π_s is said to be admissible if it is $\mathscr{F}_s$-progressively measurable, where $\mathscr{F}_s = \sigma((W_u^1, W_u^2); t \leq u \leq s)$, satisfies the integrability condition $E \int_t^T \sigma(Y_s, s)^2 \pi_s^2 ds < +\infty$ and is such that the above state constraint is satisfied. We denote by $\mathscr{A}$ the set of admissible policies.

The investor's objective is to maximize his expected utility of terminal wealth

$$J(x, y, t; \pi) = E[U(X_T, Y_T)/X_t = x, Y_t = y] \tag{2.6}$$

with X_s, Y_s given in (2.4) and (2.3) respectively.

The *value function* of the investor is

$$u(x, y, t) = \sup_{\pi \in \mathscr{A}} J(x, y, t; \pi). \tag{2.7}$$

The goal herein is to analyze the value function and to determine the optimal investment strategies when the utility function is of the separable *CRRA type*

$$U(x, y) = \frac{1}{\gamma} x^\gamma h(y) \quad \text{for } \gamma < 1,\ \gamma \neq 0 \tag{2.8}$$

and $h : \mathscr{R} \to \mathscr{R}^+$ being a bounded continuous function; to simplify the exposition we assume that for some $m \in (0, 1)$, $m \leq h(y) \leq 1$, $\forall y \in R$. The classical CRRA utilities have $h(y) \equiv 1$ but we choose to incorporate the component h so that we present a general version of the model; as a matter of fact, utilities of the above form are employed in the pricing of derivatives with non-traded assets.

As it is well known, the special form of the above utilities together with the linearity of the wealth dynamics with respect to the state and control processes (see (2.4)), enables us to represent the value function in a "separable" form. In fact, the value function can be written as $u(x, y, t) = \dfrac{x^\gamma}{\gamma} V(y, t)$. To our knowledge, the component V is in general unknown except for some special cases of the correlation coefficient ρ, the risk aversion parameter $1 - \gamma$ and the components of the state dynamics. As a matter of fact, V solves a nonlinear equation for which no closed form solutions are available.

The novelty of our results lies in the fact that under a simple transformation, we manage to remove certain non-linearities that arise due to the stochastic factor. In fact, it turns out that V can be expressed as the *power* of the solution of a linear parabolic equation; this power depends only on the risk aversion coefficient and the degree of correlation between the stock price and the stochastic factor. Under certain assumptions on the coefficients of the reduced linear equation, exact solutions may be produced for the value function and the optimal policies. In general, the reduction from the original fully non-linear Hamilton-Jacobi-Bellman equation to a linear parabolic one has obvious advantages both from the analytic as well as the computational point of view. Without stating at this point the necessary technical assumptions and the regularity properties of the relevant solutions, we outline the main results below.

Proposition 2.1

i) The value function u is given by

$$u(x,y,t) = \frac{x^{\gamma}}{\gamma} v(y,t)^{\frac{1-\gamma}{1-\gamma+\rho^2\gamma}}$$

where $v : R \times [0,T] \to R^+$ solves the linear parabolic equation

$$\begin{cases} v_t + \frac{1}{2}a^2(y,t)v_{yy} + \left[b(y,t) + \rho\frac{\gamma(\mu(y,t)-r)a(y,t)}{(1-\gamma)\sigma(y,t)}\right]v_y \\ \qquad + \frac{\gamma(1-\gamma+\rho^2\gamma)}{1-\gamma}\left[r + \frac{(\mu(y,t)-r)^2}{2\sigma^2(y,t)(1-\gamma)}\right]v = 0 \\ v(y,T) = h(y). \end{cases}$$

ii) The optimal policy Π_s^* is given in the feedback form $\Pi_s^* = \pi^*(X_s^*, Y_s, s)$, $t \le s \le T$, where the function $\pi^* : [0,+\infty) \times R \times [0,T] \to R$ is defined by

$$\pi^*(x,y,t) = \left[\frac{\rho}{(1-\gamma)+\rho^2\gamma}\frac{a(y,t)v_y(y,t)}{\sigma(y,t)v(y,t)} + \frac{1}{1-\gamma}\frac{\mu(y,t)-r}{\sigma^2(y,t)}\right]x$$

with X_s^* being the state wealth process given by (2.4) when the policy Π_s^* is being used.

3 The HJB equation and reduced form solutions

In this section, we derive reduced form solutions for the value function and the optimal policies. The portfolio policies can be expressed in a feedback form in terms of the first order derivatives and the solution of a simplified linear equation whose coefficients are related to the dynamics of the risk process, the correlation and the risk aversion coefficients.

In order to demonstrate the main steps of our approach, we start with a formal analysis assuming that all the required derivatives exist in the equations we employ. The rigorous results together with necessary assumptions on the market parameters are presented in subsequent theorems.

A classical approach in stochastic control theory is to examine the Hamilton-Jacobi-Bellman (HJB) equation that the value function is expected to satisfy. This equation, which is the offspring of the Dynamic Programming Principle and stochastic analysis, turns out to be

$$u_t + \frac{1}{2}a^2(y,t)u_{yy} + b(y,t)u_y + rxu_x + \max_{\pi}\left[\frac{1}{2}\sigma^2(y,t)\pi^2 u_{xx} + \rho\pi\sigma(y,t)a(y,t)u_{xy} + (\mu(y,t)-r)\pi u_x\right] = 0. \tag{3.1}$$

Moreover u satisfies the terminal condition

$$u(x,y,T) = U(x,y) \tag{3.2}$$

for $(x,y,t) \in \overline{D} = \{(x,y,t) : x \geq 0, y \in R, 0 \leq t \leq T\}$ and U as in (2.8).

Remark 3.1 Note that the state constraint (2.5) suggests that if the initial wealth is zero, i.e. $X_t = 0$, the only admissible policy is $\pi_s = 0$ for $t \leq s \leq T$. This results in the boundary condition $u(0,y,t) = U(0,y)$. As a matter of fact, the rigorous way to recover the value at $x = 0$ is by looking at the correct solution for $x > 0$ and then pass to the limit as $x \to 0$. As we subsequently show, the correct solution is the unique constrained viscosity solution of the HJB equation, which actually coincides with the value function, and that the appropriate boundary behavior is indeed guaranteed.

Remark 3.2 The HJB equation can be simplified by introducing a new control variable $\tilde{\pi} = \sigma(y,t)\pi$ and this way absorb the variable coefficient in front of the second order derivative u_{xx}. Moreover, using the homogeneous form of the utility and the linearity properties of the state equation one could also absorb the drift term rxu_x by discounting, from the beginning, the wealth and portfolio processes by the factor $e^{-r(T-t)}$.

We continue with the construction of a candidate solution of (3.1). As it was discussed earlier, the homogeneity of the utility function together with the fact that the state X_s and the control π_s appear linearly in (2.4), suggest that the value function must be of the form

$$u(x,y,t) = \frac{x^\gamma}{\gamma}V(y,t). \tag{3.3}$$

Direct substitution in the HJB Eq. (3.1) yields that $V(y,t)$ solves

$$\frac{1}{\gamma}\left[V_t + \frac{1}{2}a^2(y,t)V_{yy} + b(y,t)V_y\right] + rV + \max_{\tilde{\pi}}\left[\frac{1}{2}(\gamma-1)\sigma^2(y,t)\tilde{\pi}^2 V + \rho\sigma(y,t)a(y,t)\tilde{\pi}V_y + (\mu(y,t)-r)\tilde{\pi}V\right] = 0 \tag{3.4}$$

together with the terminal condition

$$V(y,T) = h(y). \tag{3.5}$$

Note that the control $\tilde{\pi}$ corresponds to $\dfrac{\pi}{x}$ with π being the control variable appearing in (3.1).

We now apply, formally, the first order conditions in (3.4). Observe that the maximum (over $\tilde{\pi}$) in (3.4) is well defined because $\gamma < 1$ and one can show that $V(y,t) > 0$ (see Proposition 3.1). We have that the maximum is achieved at

$$\tilde{\pi}^*(y,t) = \frac{\rho\sigma(y,t)a(y,t)V_y(y,t) + (\mu(y,t) - r)V(y,t)}{(1-\gamma)\sigma^2(y,t)V(y,t)} \tag{3.6}$$

or, in terms of (x,y,t), at

$$\pi^*(x,y,t) = \left[\frac{\rho\sigma(y,t)a(y,t)V_y(y,t) + (\mu(y,t) - r)V(y,t)}{(1-\gamma)\sigma^2(y,t)V(y,t)}\right]x, \tag{3.7}$$

where we used (3.3). Using the form of $\tilde{\pi}^*(y,t)$ and (3.4) gives

$$\frac{1}{\gamma}\left[V_t + \frac{1}{2}a^2(y,t)V_{yy} + b(y,t)V_y\right] + rV + \frac{[(\mu(y,t) - r)V + \rho\sigma(y,t)a(y,t)V_y]^2}{2(1-\gamma)\sigma^2(y,t)V} = 0.$$

Expanding the quadratic term in the above equation yields

$$V_t + \frac{1}{2}a^2(y,t)V_{yy} + \left[b(y,t) + \rho\frac{\gamma(\mu(y,t) - r)a(y,t)}{(1-\gamma)\sigma(y,t)}\right]V_y$$
$$+\left[r\gamma + \frac{\gamma(\mu(y,t) - r)^2}{2(1-\gamma)\sigma^2(y,t)}\right]V + \rho^2\frac{\gamma a^2(y,t)}{2(1-\gamma)}\,\frac{V_y^2}{V} = 0. \tag{3.8}$$

We now make the following transformation[1]. We let

$$V(y,t) = v(y,t)^\delta \tag{3.9}$$

for a parameter δ to be determined. Differentiating yields

$$V_t = \delta v_t v^{\delta-1}, \quad V_y = \delta v_y v^{\delta-1}, \quad V_{yy} = \delta v_{yy} v^{\delta-1} + \delta(\delta-1)v_y^2 v^{\delta-2}.$$

Substituting the above derivatives in (3.8) gives

$$\delta v_t v^{\delta-1} + \frac{1}{2}a^2(y,t)\delta v_{yy} v^{\delta-1}$$
$$+\frac{1}{2}a^2(y,t)\delta(\delta-1)v_y^2 v^{\delta-2} + \left[b(y,t) + \rho\frac{\gamma(\mu(y,t) - r)a(y,t)}{(1-\gamma)\sigma(y,t)}\right]\delta v_y v^{\delta-1}$$
$$+\left[r\gamma + \frac{\gamma(\mu(y,t) - r)^2}{2(1-\gamma)\sigma^2(y,t)}\right]v^\delta + \rho^2\frac{\gamma a^2(y,t)}{2(1-\gamma)}\,\frac{\delta^2 v_y^2 v^{2(\delta-1)}}{v^\delta} = 0$$

which in turn implies that v solves the quasilinear equation

[1] To our knowledge, this is the first time this transformation is used in optimal portfolio management problems with non-traded assets.

$$v_t + \frac{1}{2}a^2(y,t)v_{yy} + \left[b(y,t) + \rho\frac{\gamma(\mu(y,t)-r)a(y,t)}{(1-\gamma)\sigma(y,t)}\right]$$
$$v_y + \frac{1}{\delta}\left[r\gamma + \frac{\gamma(\mu(y,t)-r)^2}{2(1-\gamma)\sigma^2(y,t)}\right]v + \frac{a^2(y,t)}{2}\frac{v_y^2}{v}\left[(\delta-1) + \rho^2\frac{\gamma}{1-\gamma}\delta\right] = 0.$$

The above expression indicates that if we choose the parameter δ to satisfy

$$\delta = \frac{1-\gamma}{1-\gamma+\rho^2\gamma} \tag{3.10}$$

then, Eq. (3.8) becomes a *linear parabolic differential equation*. In fact, if δ satisfies (3.10) then v solves

$$\begin{cases} v_t + \frac{1}{2}a^2(y,t)v_{yy} + [b(y,t)+c(y,t)]v_y + k(y,t)v = 0 & (3.11) \\ v(y,T) = h(y)^{1/\delta}, & (3.12) \end{cases}$$

where the coefficients $c(y,t)$ and $k(y,t)$ are given in terms of the market coefficients,

$$c(y,t) = \rho\frac{\gamma(\mu(y,t)-r)a(y,t)}{(1-\gamma)\sigma(y,t)} \tag{3.13}$$

$$k(y,t) = \frac{\gamma}{\delta}\left[r + \frac{(\mu(y,t)-r)^2}{2(1-\gamma)\sigma^2(y,t)}\right]. \tag{3.14}$$

We will refer to δ as the *distortion power*.

Remark 3.3 Observe that for all values of $\gamma < 1$ with $\gamma \neq 0$, the distortion power satisfies $\delta > 0$. As a matter of fact, $\delta < 1$ if $0 < \gamma < 1$ and $\delta > 1$ for $\gamma < 0$. Moreover, in the case of market completeness, $\rho^2 = 1$, δ becomes $1-\gamma$ and the value function is given by $u(x,y,t) = \frac{x^\gamma}{\gamma}v(y,t)^{1-\gamma}$.[2] Finally, if $\rho = 0$, δ degenerates to one and there are no non-linearities coming directly from the presence of the discount factor; in this case $u(x,y,t) = \frac{x^\gamma}{\gamma}v^0(y,t)$ with v^0 being the solution of (3.11) and (3.12) with $c(y,t) \equiv 0$.

Remark 3.4 Notice that if $0 < \gamma < 1$ (resp. $\gamma < 0$), then $k(y,t) > 0$ (resp. $k(y,t) < 0$) and under appropriate regularity and growth conditions, one expects, through the Feynman-Kac formula, the stochastic representation

$$v(y,t) = E\left[h(\widetilde{Y}_T)^{1/\delta}\exp\int_t^T \frac{\gamma}{\delta}\left[r + \frac{(\mu(\widetilde{Y}_s,s)-r)^2}{2\sigma^2(\widetilde{Y}_s,s)(1-\gamma)}\right]ds \Big/ \widetilde{Y}_t = y\right] \tag{3.15}$$

where the process $\widetilde{Y}_s$, $t \le s \le T$ solves

$$d\widetilde{Y}_s = [b(\widetilde{Y}_s,s) + c(\widetilde{Y}_s,s)]ds + a(\widetilde{Y}_s,s)dW_s \tag{3.16}$$

[2] This class of models was analyzed in Zariphopoulou (1999) in optimal portfolio management models that allow for intermediate consumption and with nonlinear stock dynamics.

with W_s being a standard Brownian motion. Observe that the above stochastic differential equation is similar to (2.3) but with a *modified drift*.

Therefore, we see that the value function u is represented as

$$u(x, y, t) = \frac{x^\gamma}{\gamma} v(y, t)^\delta \tag{3.17}$$

or, alternatively, as

$$u(x,y,t) = \frac{x^\gamma}{\gamma}\left(E\left[h(\widetilde{Y}_T)^{1/\delta}\exp\int_t^T \frac{\gamma}{\delta}\left[r + \frac{(\mu(\widetilde{Y}_s,s)-r)^2}{2(1-\gamma)\sigma^2(\widetilde{Y}_s,s)}\right]ds \Big/ \widetilde{Y}_t = y\right]\right)^\delta, \tag{3.18}$$

where $\widetilde{Y}_s$ solves (3.16) and δ is defined in (3.10).

Even though the above representation is a direct consequence of the Feynman-Kac formula, it is interesting to observe the precise structure of (3.18). Recall that in the simple framework of the Merton problem (see Merton 1969, 1971) which corresponds to constants μ and σ and $h(y) = 1$, (3.18) is replaced by $\widetilde{u}(x, t) = \frac{x^\gamma}{\gamma} e^{\lambda(T-t)}$ with $\lambda = r\gamma + \frac{\gamma(\mu - r)^2}{2(1-\gamma)\sigma^2}$. Equality (3.18) is also of the same kind but with the deterministic exponential $e^{\lambda(T-t)}$ being replaced by the expectation of a stochastic exponential. It is intriguing that the underlying process $\widetilde{Y}_s$ appearing in this expectation resembles the original stochastic factor but with a modified drift; this modified drift degenerates to the original one in the orthogonal case $\rho = 0$. We return to the interpretation of (3.18) in our discussion in Sect. 4.

Using (3.7) and the representation formula (3.17) for the value function, one obtains the following simplified expression for the *optimal feedback portfolio rule*

$$\pi^*(x,y,t) = \left[\frac{\rho}{(1-\gamma)+\rho^2\gamma}\frac{a(y,t)}{\sigma(y,t)}\frac{v_y(y,t)}{v(y,t)} + \frac{1}{1-\gamma}\frac{\mu(y,t)-r}{\sigma^2(y,t)}\right]x. \tag{3.19}$$

From classical arguments in stochastic control, one expects to recover the optimal portfolio process via $\Pi_s^* = \pi^*(X_s^*, Y_s, s)$ for $t \le s \le T$, where π^* and Y_s are as in (3.19) and (2.3) respectively, and the optimal wealth X_s^* is given by (2.4) with the optimal process Π_s^* being used.

We continue with a rigorous analysis of the above results; the main tools will come from the theories of stochastic control and viscosity solutions of the HJB equation. To this end, we start with basic assumptions on the market coefficients and we discuss some preliminary properties of the value function. To simplify the exposition, we assume that $0 < \gamma < 1$; the arguments for $\gamma < 0$ are easily modified.

Assumption i) The coefficients $\mu, \sigma, b,\ a : R \times [0,T] \to R$ satisfy the global Lipschitz and linear growth conditions

$$|f(y,t) - f(\bar{y},t)| \le K|y - \bar{y}|, \tag{3.20}$$

$$f^2(y,t) \le K^2(1 + y^2), \tag{3.21}$$

for every $t \in [0,T]$, $y, \bar{y} \in R$, K being a positive constant and f standing for μ, σ, b, and a.

ii) Uniformly in $y \in R$ and $t \in [0,T]$, the volatility coefficient $\sigma(y,t)$ satisfies $\sigma(y,t) \ge \ell > 0$ for some constant ℓ, and for some positive constant M,

$$\frac{(\mu(y,t) - r)^2}{\sigma^2(y,t)} \le M. \tag{3.22}$$

Conditions (3.20) and (3.21) are standard for the existence and uniqueness of solutions of the state stochastic differential Eqs. (2.2) and (2.3) (see Gikhman and Skorohod 1972). Condition (3.22) will be used to determine the appropriate growth conditions for the value function and to facilitate the relevant verification results.

Proposition 3.1 *i) The value function u is non-decreasing and concave with respect to the wealth variable x.*

ii) There exists a constant $\lambda > r\gamma$ such that $m\frac{x^\gamma}{\gamma}e^{r\gamma(T-t)} \le u(x,y,t) \le \frac{x^\gamma}{\gamma}e^{\lambda(T-t)}$.

Proof i) The concavity of u is an immediate consequence of the concavity of the utility function U and the fact that if $\pi^1 \in \mathscr{A}_{(x_1,y)}$, $\pi^2 \in \mathscr{A}_{(x_2,y)}$ and $\lambda \in (0,1)$ then $(\lambda\pi^1 + (1-\lambda)\pi^2) \in \mathscr{A}_{(\lambda x_1 + (1-\lambda)x_2,y)}$; the latter follows from the linear dependence of the state dynamics (2.4) with respect to the control variables and the state wealth. That u is nondecreasing in x follows from the observation that $\mathscr{A}_{(x_1,y)} \subseteq \mathscr{A}_{(x_2,y)}$ for $x_1 \le x_2$.

ii) The lower bound for the value function follows directly from the definition of u and the fact that $\pi_s = 0$, $t \le s \le T$ is an admissible policy. The upper bound follows from a standard Girsanov transformation, Hölder's inequality and the uniform bound in inequality (3.22). The relevant to the change of measure Randon-Nikodym derivative is given by $Z_T = \exp\left\{-\int_t^T \theta_s dW_s^1 - \frac{1}{2}\int_t^T \theta_s^2 dW_s^1\right\}$ with $\theta_s = \dfrac{(\mu(Y_s,s) - r)^2}{\sigma^2(Y_s,s)}$. This kind of analysis is standard in stochastic optimization problems and it is skipped for the sake of the presentation. (We refer the reader to the book of Friedlin 1985 or, in the context of relevant portfolio management problems, to the papers of Huang and Pagès 1992 and Duffie and Zariphopoulou 1993).

We continue with the study of the HJB Eq. (3.1) that the value function is expected to solve. As it is well known, for optimal decision problems a convenient class of solutions of the associated HJB equation is the class of *(constrained) viscosity solutions*. This class has been successfully used in a number of asset pricing and portfolio management problems in markets with frictions (see for example, 1992, 1994, Davis et al. 1993; Duffie and Zariphopoulou 1993; Shreve and Soner 1994; Barles and Soner 1998) and it is now routinely used in new models with similar market characteristics (among others, trading constraints, transaction costs and stochastic volatility). The analysis herein follows closely the analysis of Duffie and Zariphopoulou (1993) who provided a complete study of the value function of an optimal investment model with stochastic labor income and general utilities. Even though, as it was mentioned in the Introduction, the decomposition (3.3) and the transformation (3.9) for the value function cannot be applied, the associated HJB equation in Duffie and Zariphopoulou (1993) is of similar structure to (3.1). In fact, the analysis used by Duffie and Zariphopoulou (1993) does not require closed form solutions and it can be readily adapted for the study of Eq. (3.1). For this reason, we only state the basic results and we refer the technically motivated reader to Theorems 4.1 and 4.2 of Duffie and Zariphopoulou (1993).

The following theorem states that the value function is the unique constrained viscosity solution of (3.1) in the appropriate class. The characterization of u as a constrained solution is natural due to the presence of the state constraint (2.5). We provide the definition of (constrained) viscosity solutions and relevant references in the appendix.

Theorem 3.1 *The value function u is a constrained viscosity solution of the HJB Eq. (3.1) on $\overline{D}$ with $u(x, y, T) = \frac{x^\gamma}{\gamma} h(y)$. Moreover, u is the unique constrained viscosity solution in the class of functions that are concave, and nondecreasing in x, have sublinear growth of order γ and, for fixed x, they are of exponential growth $e^{\lambda(T-t)}$ for some constant λ, uniformly in y.*

The following theorem provides a verification result for the value function and the optimal policies.

Theorem 3.2 *The value function u is given by $u(x, y, t) = \frac{x^\gamma}{\gamma} v(y, t)^\delta$ where v is the unique viscosity solution of (3.11) and (3.12) and δ is given in (3.10).*

Proof We assume that $0 < \gamma < 1$ since the case $\gamma < 0$ follows along modified arguments. Applying the results of Ishii and Lions (1991) one easily concludes that equation (3.11) has a unique viscosity solution satisfying the boundary and terminal conditions (3.12). As a matter of fact, one can show (Lions (1993)) that v admits the stochastic representation (3.15) which in view of (3.22) yields that $v \leq e^{\lambda(T-t)}$ with $\lambda = \frac{r\gamma}{\delta} + \frac{\gamma M}{2\delta(1-\gamma)}$. Applying the definition of viscosity solutions one also gets directly that the function $F(x, y, t) = \frac{x^\gamma}{\gamma} v(y, t)^\delta$ is a viscosity solution of the HJB equation in $(0, +\infty) \times R \times [0, T]$. Moreover, at the boundary point $x = 0$, the slope of F is infinite and therefore the viscosity sub-solution property is automatically satisfied. Therefore, F is a constrained viscosity solution of the HJB equation and clearly, it also belongs to the appropriate class of solutions in which uniqueness has been established. We readily get that F coincides with the value function and therefore u is indeed given by the proposed closed form solution.

The last result of this section provides additional results for the regularity of the value function and a characterization of the optimal policies. The conditions on the coefficients can be relaxed in many ways, especially when the latter do not depend explicitly on time. We refer the reader to the book of Friedlin (1985) or Fleming and Soner (1993) for more general assumptions.

Theorem 3.3 *Assume that the functions $a(y,t)$, $b(y,t)$, and $c(y,t)$ and $k(y,t)$, given in (3.13) and (3.14), are bounded and uniformly Hölder's continuous in $R \times [0,T]$. Moreover, assume that $a(y,t)$ is uniformly elliptic, i.e. there exists a positive constant ε such that $a^2(y,t) \geq \varepsilon y^2$ for $y \in R$ and $t \in [0,T]$. Then the value function is twice continuously differentiable with respect to (x,y), for $x > 0$ and $y \in R$ and continuously differentiable with respect to t for $t \in [0,T)$. Moreover, the optimal portfolio process Π_s^* is given in the feedback form $\Pi_s^* = \pi^*(X_s^*, Y_s, s)$, $t \leq s \leq T$ where the function $\pi^* : [0,+\infty) \times R \times [0,T] \to R$ is defined by*

$$\pi^*(x,y,t) = \left[\frac{\rho}{(1-\gamma)+\rho^2\gamma} \frac{a(y,t)v_y(y,t)}{\sigma(y,t)v(y,t)} + \frac{1}{1-\gamma} \frac{\mu(y,t)-r}{\sigma^2(y,t)} \right] x \tag{3.23}$$

and X_s^* is the optimal state wealth process given by (2.4) with Π_s^* being used.

Proof We first observe that the above conditions on the coefficients guarantee that equation (3.11) (together with (3.12)) has a unique smooth solution, say $\widetilde{v}$. By the uniqueness of viscosity solutions for (3.11) we get that $\widetilde{v} \equiv v$ and therefore v is smooth. The regularity of the value function with respect to x follows then immediately from its explicit form.

Applying the first order conditions in (3.1) and using the representation formula (3.17), yields that the maximum occurs at the point $\pi^*(x,y,t)$ as given in (3.23). The fact that the optimal portfolio process is given in the feedback form $\Pi_s^* = \pi^*(X_s^*, Y_s, s)$ follows from standard verification results (see, for example, Duffie et al. 1997).

4 Examples and discussion

In this section we provide a number of applications related to the closed form solutions we derived earlier. As the examples below indicate, our results can facilitate the analysis of valuation models in portfolio management and derivative pricing in the presence of unhedgeable risks and, ultimately help us to obtain a better understanding of the effects of this kind of market frictions. Additionally, these results can also be applied to valuation models without unhedgeable risks but with non-linear stock dynamics – departing from the class of log-normal diffusion stock prices – for which no closed form solutions exist in general.

A Investment models with a single stochastic factor

This is basically the class of models extensively analyzed in previous sections and we only look briefly at the limiting cases $\rho = 0$ and $\rho = \pm 1$.

i) $\rho = 0$: When the underline stock price and the stochastic factor are not correlated, there is no distortion in the sense that $\delta = 1$. In this case,

$$u(x,y,t) = \frac{x^\gamma}{\gamma} v^0(y,t) \quad \text{and} \quad \pi^*(x,y,t) = \frac{\mu(y,t) - r}{(1-\gamma)\sigma^2(y,t)} x. \tag{4.1}$$

The Eq.(3.11) satisfied by v^0 becomes

$$\begin{cases} v_t^0 + \frac{1}{2} a^2(y,t) v_{yy}^0 + b(y,t) v_y^0 + \left[r\gamma + \dfrac{\gamma(\mu(y,t)-r)^2}{2(1-\gamma)\sigma^2(y,t)} \right] v^0 = 0 \\ v^0(y,T) = h(y) \end{cases}$$

and (3.15) or (3.17) yield

$$v^0(y,t) = E\left[h(Y_T) \exp \int_t^T \left[r\gamma + \frac{\gamma(\mu(Y_s,s)-r)^2}{2(1-\gamma)\sigma^2(Y_s,s)} \right] ds \Big/ \, Y_t = y \right]$$

with Y_s solving the original state Eq. (2.3). Notice that the drift change, represented by the coefficient $c(y,t)$ in (3.13), disappears.

The above formulae indicate the natural consequences of the absence of correlation: the component $v^0(y,t)$ is not “distorted” and also, the first component of $\pi^*(x,y,t)$ in (3.7) is eliminated. We see that, as expected, there is no hedging demand as (4.1) indicates; in fact, the ratio of wealth invested optimally in the risky asset is given by a similar expression to the so-called Merton ratio.

ii) $\rho^2 = 1$: When the underline stock price and the stochastic factor have correlation coefficient $\rho = \pm 1$, the distortion power coincides with the relative risk aversion, $\delta = 1 - \gamma$. In this case,

$$u(x,y,t) = \frac{x^\gamma}{\gamma} \widetilde{v}(y,t)^{1-\gamma}$$

and

$$\pi^*(x,y,t) = \left[(\operatorname{sgn} \rho) \frac{a(y,t)}{\sigma(y,t)} \frac{\widetilde{v}_y(y,t)}{\widetilde{v}(y,t)} + \frac{1}{1-\gamma} \frac{\mu(y,t)-r}{\sigma^2(y,t)} \right] x$$

with $\widetilde{v}$ solving

$$\begin{cases} \widetilde{v}_t + \dfrac{1}{2} a^2(y,t) \widetilde{v}_{yy} + \left[b(y,t) + (\operatorname{sgn} \rho) \dfrac{\gamma(\mu(y,t)-r) a(y,t)}{(1-\gamma)\sigma(y,t)} \right] \widetilde{v}_y + \\ \qquad\qquad + \dfrac{\gamma}{1-\gamma} \left[r + \dfrac{\mu(y,t)-r}{2(1-\gamma)\sigma^2(y,t)} \right] \widetilde{v} = 0 \\ \widetilde{v}(y,T) = h(y)^{1/(1-\gamma)}. \end{cases}$$

B Optimal investment models with non-linear price dynamics

It is well known that in the absence of correlated stochastic risk factors, the HJB Eq. (3.4) has been analyzed extensively when the coefficients μ and σ depend on time and more generally, in an adaptive way on the underlying Brownian motion (see Karatzas 1997). In most cases, the value function and the optimal policies are determined via martingale representation arguments and closed form solutions are not in general available.

The methodology of this paper can be applied specifically to models with nonlinear stock price dynamics of the form say

$$dS_s = \mu(S_s, s)S_s ds + \sigma(S_s, s)S_s dW_s.$$

The problem is easily reduced to the case of the stochastic factor $Y_s \equiv S_s$; then $\rho = 1$, $a(y,t) = \sigma(y,t)y$, $b(y,t) = \mu(y,t)y$ and $c(y,t) = \dfrac{\gamma(\mu(y,t) - r)y}{1-\gamma}$. The value function is given by

$$u(x, S, t) = \frac{x^\gamma}{\gamma} v(S, t)^{1-\gamma}$$

where v solves (3.11) modified with the above coefficients.

C Derivative pricing via utility maximization

Expected utility methods have been effectively used for the valuation of derivative securities in markets with transaction costs. This approach typically generates price bounds which are independent of the current portfolio holdings and provides a range of derivative prices advantageous for both the writer and the buyer of the securities. (See among others, Hodges and Neuberger 1989; Davis et al. 1993; Barles and Soner 1998; Constantinides and Zariphopoulou 1999). Recently, Mazaheri (1998) employed the utility maximization approach to obtain price bounds for European-type derivatives when the dynamics of the stock price are affected by a non-traded asset represented by Y_s. These bounds are determined by comparing the value function $u(x, y, t)$ to the value functions of the writer and the buyer, denoted respectively by F_W and F_B respectively. To simplify the exposition, we take $h(y) \equiv 1$.

For the case of the writer, his value function is defined as

$$F_W(x, S, y, t) = \sup_{\mathscr{A}_w} E[u(X_T - g(S_T), Y_T, T)/X_t = x, S_t = S, Y_t = y]$$

with $g(S_T)$ being the payoff of the security at expiration time T. The set $\mathscr{A}_w$ is the set of admissible policies for the investments of the writer. Because of feasibility conditions, the lowest admissible level of wealth turns out to be the value of one share of stock, in which case one gets

$$F_W(S, S, y, t) = E[u(S_T - g(S_T), Y_T, T)/S_t = S, Y_t = y]$$

or, equivalently (see (3.3) and (3.5))

$$F_W(S, S, y, t) = E\left[\frac{1}{\gamma}(S_T - g(S_T))^\gamma / S_t = S, Y_t = y\right]. \qquad (4.2)$$

The "write price", say $h_W(S, y, t)$ must satisfy

$$u(x, y, t) \leq F_W(x + h_W(S, y, t), S, y, t).$$

Similar criteria are applied for the "purchase price" but we concentrate on the "write price" only. Mazaheri (1998) established that if $H(S, y, t) : \overline{D} \to R^+$ is such that

$$u(S, y, t) = F_W(S + H(S, y, t), S, y, t)$$

or, equivalently,

$$u(S - H(S, y, t), y, t) = F_W(S, S, y, t), \tag{4.3}$$

then H provides an upper bound for the write price h_W.

The results obtained herein can be used to provide reduced form solutions for this price bound.

In fact, from (3.17), (4.2) and (4.3) we have

$$\frac{[S - H(S, y, t)]^\gamma}{\gamma} v(y, t)^\delta = E\left[\frac{1}{\gamma}(S_T - g(S_T))^\gamma / S_t = S, Y_t = y\right]$$

which in turn yields

$$H(S, y, t) = S - \left\{ \frac{E[(S_T - g(S_T))^\gamma / S_t = S, Y_t = y]}{v(y, t)^{\frac{1-\gamma}{1-\gamma+\rho^2\gamma}}} \right\}^{\frac{1}{\gamma}},$$

with v being the solution of (3.11) and (3.12).

It is worth mentioning that the above utility-based analysis can be readily generalized for other types of derivatives, namely American-type and some path-dependent ones even when they are written on multi-securities. (See Mazaheri and Zariphopoulou 1999).

D Discussion and future work

In this paper we studied an optimal investment model in markets with stochastically changing investment opportunity sets. More precisely, the underlying stock prices are modelled as diffusion processes whose coefficients depend on another correlated process, called the "stochastic factor". For the specific class of separable CRRA utilities, we derived reduced form solutions for the value function and the optimal policies in terms of a power transformation of the solution of a simple linear partial differential equation. This transformation enables us to express the value function in terms of an expected payoff that depends on a new stochastic factor whose dynamics are the same as the original ones but with a modified drift.

Even though at first sight the results herein appear rather technical, the discussion that follows indicates an interesting connection with stochastic differential utilities and, also, with distorted measures of risk. The discussion that follows is informal and it is intended only to provide some economic insights for the obtained representation formulae.

We start with an interpretation for the representation formulae (3.17) and (3.18) which involve the pseudo-stochastic factor process $\widetilde{Y}_s$, given in (3.16). Specifically, one may use ideas from the theory of stochastic differential utility to interpret the second component of the value function, namely $V(y,t) = v(y,t)^{\delta}$. The idea is to interpret the nonlinearities appearing in the reduced HJB Eq. (3.8) appropriately in order to represent V in terms of a recursive utility. The main references are the papers by El Karoui et al. (1997) and Duffie and Lions (1992).

First we recall that if Z_s^{α} is a family of solutions of the backward stochastic differential equations (BSDEs)

$$\begin{cases} -dZ_t^{\alpha} = f^{\alpha}(Z_t^{\alpha}, \widetilde{Z}_t, t)dt - \widetilde{Z}_t^{\alpha} dW_t \\ Z_T^{\alpha} = \xi\,, \end{cases}$$

with solution $(Z_t^{\alpha}, \widetilde{Z}_t^{\alpha})$ and $f(Z_t, \widetilde{Z}_t, t) \equiv \underset{\alpha}{\operatorname{essinf}} f^{\alpha}(Z_t, \widetilde{Z}_t, t)$ then the solution $(Z_t, \widetilde{Z}_t)$ of

$$\begin{cases} -dZ_t = f(Z_t, \widetilde{Z}_t, t)dt - \widetilde{Z}_t dW_t \\ Z_T = \xi, \end{cases}$$

satisfies $Z_t = \underset{\alpha}{\operatorname{essinf}} Z_t^{\alpha}$. Moreover, let g be the solution of the terminal value problem

$$\begin{cases} g_t + \mathscr{L} g + f(t, y, g(y,t), \Sigma(y,t) g_y(y,t)) = 0 \\ g(y,T) = \Psi(y) \end{cases} \tag{4.4}$$

with the differential operator being given by

$$\mathscr{L} v = \frac{1}{2}\Sigma^2(y,t) v_{yy} + B(y,t) v_y . \tag{4.5}$$

Then, under the appropriate regularity conditions, if $\widetilde{Y}_s$ solves

$$\begin{cases} d\widetilde{Y}_s = B(\widetilde{Y}_s, s)ds + \Sigma(\widetilde{Y}_s, s)dW_s \\ \widetilde{Y}_t = y \end{cases}$$

and we define

$$Z_t \equiv g(\widetilde{Y}_t, t) \;\text{ and }\; \widetilde{Z}_t \equiv \Sigma(\widetilde{Y}_t, t) g_y(\widetilde{Y}_t, t)$$

then $(Z_s, \widetilde{Z}_s)$ is the solution of the FBSDE

$$\begin{cases} -dZ_s = f(\widetilde{Y}_s, Z_s, \widetilde{Z}_s, s)ds - \widetilde{Z}_s dW_s \\ Z_T = \Psi(\widetilde{Y}_T). \end{cases}$$

Comparing (3.8) to (4.4) and (4.5) we have the identifications $g \equiv V$, $\Sigma(y,t) \equiv a(y,t)$, $B(y,t) \equiv \left[b(y,t) + \rho \frac{\gamma}{1-\gamma} \frac{(\mu(y,t) - r)}{\sigma(y,t)} a(y,t) \right]$ and

$f(t,y,V,\hat{V}) = \left[r\gamma + \frac{\gamma(\mu(y,t)-r)^2}{2(1-\gamma)\sigma^2(y,t)}\right]V + \rho^2\frac{\gamma}{2(1-\gamma)}\frac{\hat{V}^2}{V}$ with $\hat{V} \equiv a(y,t)V_y$.

We also observe that f can be written as

$$f = \max_{\xi} f^{\xi}$$

with

$$f^{\xi}(t,y,V,W) = \left[r\gamma + \frac{\gamma(\mu(y,t)-r)^2}{2(1-\gamma)\sigma^2(y,t)}\right]V + \rho^2\frac{\gamma}{2(1-\gamma)}\hat{V}^2\left(\xi - \frac{V}{4}\xi^2\right).$$

Given all the above, we can readily relate f^{ξ} to the generator of a recursive utility function. Specifically, if $\widetilde{Y}_s$ solves (3.16), rewritten here for convenience,

$$\begin{cases} d\widetilde{Y}_s = \left[b(\widetilde{Y}_s,s) + \rho\dfrac{\gamma(\mu(\widetilde{Y}_s,s)-r)a(\widetilde{Y}_s,s)}{(1-\gamma)\sigma(\widetilde{Y}_s,s)}\right]ds + a(\widetilde{Y}_s,s)dW_s \\ \widetilde{Y}_t = y \end{cases} \tag{4.6}$$

and $Z_s^{\xi_s}$ satisfies

$$\begin{cases} -dZ_s^{\xi_s} = f^{\xi_s}(\widetilde{Y}_s, Z_s^{\xi_s}, \widetilde{Z}_s^{\xi_s})ds - \widetilde{Z}_s^{\xi_s}dW_s \\ Z_T^{\xi_T} = 1 \end{cases}$$

then $Z_s^{\xi_s}$ can be written as the recursive stochastic differential utility

$$\begin{aligned} Z_s^{\xi_s} = E\Bigg[1 + \int_s^T \left\{r\gamma + \frac{\gamma(\mu(\widetilde{Y}_u,u)-r)^2}{2(1-\gamma)\sigma^2(\widetilde{Y}_u,u)}\right\}Z_u^{\xi_u}du \\ +\rho^2\frac{\gamma}{2(1-\gamma)a^2(\widetilde{Y}_u,u)}\left(\xi_u - \frac{Z_u^{\xi_s}}{4}\xi_u^2\right)d[Z_u^{\xi_u}]/\mathscr{F}_s\Bigg]. \end{aligned}$$

In terms of the recursive utility terminology, the *aggregator* is

$$\alpha(\widetilde{Y}_s, Z_s^{\xi_s}, s) \equiv \left[r\gamma + \frac{\gamma(\mu(\widetilde{Y}_s,s)-r)^2}{2(1-\gamma)\sigma^2(\widetilde{Y}_s,s)}\right]Z_s^{\xi_s}$$

and the *variance multiplier*

$$A(\widetilde{Y}_s, Z_s^{\xi_s}, s) = \frac{\rho^2\gamma}{2(1-\gamma)a^2(\widetilde{Y}_s,s)}\left(\xi_s - \frac{Z_s^{\xi_s}}{4}\xi_s^2\right).$$

Then the solution V of the non-linear Eq. (3.8) can be written as

$$V(y,t) = \max_{\xi} Z_t^{\xi}.$$

The above interpretation indicates that the stochastic utility depends on a hidden factor with coincides with the pseudo-asset $\widetilde{Y}_s$ defined in (3.13); see also (4.6). It is the process $\widetilde{Y}_s$ which affects directly the overall maximal utility of the agent and not the original one, Y_s. Of course, $\widetilde{Y}_s$ degenerates to Y_s in the orthogonal case $\rho = 0$. (A class of models with recursive utilities for affine market coefficients has been studied recently by Schroder and Skiadas 1999.)

From an entirely different point of view, one may possibly study the component V using elements from the theory of the so-called distorted probability measures. For the sake of exposition, we assume that $h(y)$, see (2.8), is given by $h(y) = \mathbb{1}_{\{y \geq K\}}$ with K a given constant, and the potential term $k(y,t)$, see (3.14), is assumed to be zero. Then one sees that v may be interpreted as the survival probability of the process $\widetilde{Y}_s$; this in turn yields that V may be written as the distorted survival probability of $\widetilde{Y}_s$, i.e.

$$V(y,t) = (Q[\widetilde{Y}_T \geq K / \widetilde{Y}_t = y])^{\delta} \tag{4.7}$$

for some probability measure Q and δ being the distortion power in (3.10).

Distorted survival probabilities have been successfully associated to non-additive measures which have been of central interest in the valuation of static insurance risks (see Wang 1996; Wang and Young 1998; Wang et al. 1997). They are proved to have desirable properties – like for example subadditivity and comonotonicity – which provide a good underlying structure for measuring risks generated by insurance claims. Due to the ever-increasing complexity of the markets, it is imperative to extend the above notions to dynamic settings in the presence of market frictions. Preliminary work in the context of dynamic insurance risks, has been recently done by Young and Zariphopoulou (1999) who provided a complete variational characterization of distorted survival probabilities of diffusion processes. In particular, they considered the survival probability $\widetilde{v}(y,t;z,T) = Q(Y_T \geq z / Y_t = y]$ of a diffusion process, say Y_s solving (2.3) and they studied its distortion, i.e. $\widetilde{V}(y,t;z,T) = g(\widetilde{v}(y,t;z,T))$ with $g : [0,1] \to [0,1]$. In Young and Zariphopoulou (1999), it is shown that under mild conditions, $\widetilde{V}$ can be written as the survival probability of a new diffusion process with a modified drift and with a combination of killing and splitting components. We note that (4.7) corresponds to the special case of concave distortion $g(z) = z^{\delta}$, $0 < \delta < 1$. An interesting question therefore arises, whether one can represent the value function, see (2.7), in terms of an integral with respect to a distorted probability measure (under a concave distortion). Such a representation would possibly facilitate the study of the effects of the unhedgeable risks – generated by the non-traded correlation factors – especially for the optimal hedging demand.

Of course, a more challenging question is whether one can represent the value function in terms of an integral with respect to a distorted measure – or in general with respect to a coherent measure – under general assumptions on the individual preferences. In the latter case, the homogeneity properties are lost and the decomposition (3.3) is not valid. Therefore, a concave distortion along the lines of (3.9) cannot hold.

Finally, the results herein may be also applied to a different category of stochastic optimization problems arising in models of imperfect hedging. Indeed, one might look into the case of a derivative security written on an underlying stock whose price depends on a non-traded asset. A hedging policy might then be specified by minimizing a loss criterion of the form

$E\left[(X_T - C)^p / X_t = x, Y_t = y\right]$ where p even and C is a constant liability. The process X_s represents the value of the hedging portfolio and its coefficients depend on another process, say Y_s, which models the non-traded asset. Such models have been studied, for particular cases of the correlation between the stock and the stochastic factor, via mean-variance criteria (see Duffie and Richardson 1991; Schweizer 1992, 1996); these criteria correspond to $p = 2$. Recently Hipp and Taksar (1999) introduced general loss criteria, for $p \neq 2$, and they produced explicit solutions for a certain range of correlation and market parameters. One may obtain their results in a somewhat more general setting, by applying the same transformation that was earlier employed herein adapted for the case of convex optimization. In fact, let the "hedging portfolio" process X_s to follow (2.4) and the stochastic factor Y_s to satisfy (2.3) with $\rho^2 = 1$. The minimal expected asymmetric loss may be viewed as the value function of the stochastic minimization problem, $u(x, y, t) = \inf_{\{\pi\}} E[(X_T - C)^p / X_t = x, Y_t = y]$. The homogeneity of degree p is then preserved and one expects to have the decomposition $u(x, y, t) = (x - C)^p V(y, t)$. Following arguments along the lines of the previous section one gets, for $\rho^2 = 1$, $u(x, y, t) = (x - C)^p v(y, t)^{p-1}$ with v solving the appropriate linear equation. We note that if $\rho^2 = 1$, the analysis of the central problem of minimizing the expected hedging loss is similar to the one of constant liability. This reduction, to a constant liability, may not be feasible for $\rho^2 \neq 1$ and the solution approach we developed herein must be extended.

Appendix A

The notion of *viscosity solutions* was introduced by Crandall and Lions (1983) for first-order equations, and by Lions (1983) for second-order equations. For a general overview of the theory we refer to the *User's Guide* by Crandall et al. (1992) and the book by Fleming and Soner (1993). Next, we recall the notion of *constrained viscosity solutions* which was introduced by Soner (1986) and Capuzzo-Dolcetta and Lions (1990) for first-order equations (see also Ishii and Lions 1990). To this end, we consider a nonlinear second order partial differential equation of the form

$$F(X, V, DV, D^2V) = 0 \quad \text{in } D \times [0, T] \tag{A.1}$$

where $D \subseteq R^2$, DV and D^2V denote the gradient vector and the second derivative matrix of V, and the function F is continuous in all its arguments and degenerate elliptic, meaning that

$$F(X, p, q, A + B) \leq F(X, p, q, A) \text{ if } B \geq 0. \tag{A.2}$$

Definition: *A continuous function $V : \overline{D} \times [0, T] \to R$ is a constrained viscosity solution of (A.1) if the following two conditions hold: i) V is a viscosity subsolution of (A.1) on $\overline{D} \times [0, T]$; that is, if for any $\phi \in C^{2,1}(\overline{D} \times [0, T])$ and any local maximum point $X_0 \in \overline{D} \times [0, T]$ of $V - \phi$,*

$$F(X_0, V(X_0), D\phi(X_0), D^2\phi(X_0)) \leq 0,$$

ii) V is a viscosity supersolution of (A.1) in $D \times [0,T]$; that is, if for any $\phi \in C^{2,1}(\overline{D} \times [0,T])$ and any local minimum point $X_0 \in D \times [0,T]$ of $V - \phi$,

$$F(X_0, V(X_0), D\phi(X_0), D^2\phi(X_0)) \geq 0.$$

References

Avellaneda, M., Zhu, Y.: A risk-neutral stochastic volatility model. Preprint (1999)

Barles G., Soner, H. M.: Option pricing with transaction costs and a nonlinear Black-Scholes equation. Finance Stochast. **2**(4), 369–398 (1998)

Capuzzo-Dolcetta, I., Lions, P.-L.: Hamilton-Jacobi equations with state constraints. Transactions Ame. Math. Soc. **318**, 543–683 (1990)

Constantinides, G., Zariphopoulou, T.: Bounds on prices of contingent claims in an intertemporal economy with proportional transaction costs and general preferences. Finance Stochast. **3** (3), 345–369 (1999).

Crandall, M., Ishii, H., Lions, P.-L.: User's guide to viscosity solutions of second order partial differential equations. Bull. Am. Math. Soc. **27**, 1–67 (1992)

Crandall, M., Lions, P.-L.: Viscosity solutions of Hamilton-Jacobi equations. Transactions Ame. Math. Soc. **277**, 1–42 (1983)

Cuoco, D.: Optimal consumption and equilibrium prices with portfolio constraints and stochastic income. J. Econ. Theory **7**(2) (1997)

Davis, M. H. A.: Option valuation and hedging with basis risk. Preprint (1999)

Davis, M. H. A., Panas, V. G., Zariphopoulou, T.: European option pricing with transaction costs. SIAM J. Control Optimization **31**, 470–493 (1993)

Duffie, D., Fleming, W. H., Soner, H. M., Zariphopoulou, T.: Hedging in incomplete markets with HARA utility. J. Econ. Dyna. Control **21**, 753–782 (1997)

Duffie, D., Lions, P.-L.: PDE solutions of stochastic differential utility. J. Math. Econ. **21** (6), 577–606 (1992)

Duffie D., Richardson, M.: Mean variance hedging in continuous time. Annals App. Prob. **1**, 1–15 (1991)

Duffie, D., Zariphopoulou, T.: Optimal investment with undiversifiable income risk. Math. Finance **3**, 135-148 (1993).

El Karoui, N., Jeanblanc-Picqué, M.: Optimization of consumption with labor income. Finance Stochast. **2**(4), 409–440 (1998)

El Karoui, N., Peng, S., Quenez, M.C.: Backward stochastic differential equations in finance. Math. Finance **7**(1), 1–71 (1997).

Fleming, W., Soner, H. M.: Controlled Markov Processes and Viscosity Solutions. New York: Springer 1993

Freidlin, M.: Functional integration and partial differential equations. Princeton: Princeton University Press 1985

Frey, R., Sin, C.: Bounds on European option prices under stochastic volatility. Math. Finance **9** (2) (1999)

Gikhman, I., Skorohod, A.: Stochastic differential equations. New York: Springer 1972

He, H. and Pagès, H.: Labor income, borrowing constraints and equilibrium asset prices: a duality approach. Journal of Economic Theory **3**, 663–696 (1993)

Hipp, C., Taksar, M.: Hedging in incomplete markets and optimal control. Preprint (1999)

Hodges, S. D., Neuberger, A.: Optimal replication of contingent claims under transaction costs. Rev. Futures Markets **8**, 222–239 (1989)

Huang, C.-F., Pagés, H.: Optimal consumption and portfolio policies with an infinite horizon: Existence and convergence. Annals Appl. Probability **2**, 36–64 (1992)

Ishii, H., Lions, P.-L.: Viscosity solutions of fully nonlinear second-order elliptic partial differential equations. J. Differential Equations **83**, 26–78 (1990)

Karatzas, I.: Lectures on the Mathematics of Finance, CRM Monograph Series. Am. Math. Soc. (1997)

Koo, H.: Consumption and portfolio selection with labor income: A continuous time approach. Math. Finance **8**(1), (1998)

Ledoit, O., Santa-Clara, P.: Relative pricing of options with stochastic volatility. Preprint (1999)

Lions, P.-L.: Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. Part 1: The dynamic programming principle and applications. Part 2: Viscosity solutions and uniqueness. Comm. PDE **8**, 1101–1174 and 1229–1276 (1983)

Mazaheri, M.: Derivative pricing with non-traded assets via utility maximization methods. Preprint (1998).

Mazaheri, M., Zariphopoulou, T.: Valuation of exotic derivatives with non-traded assets. Preprint (1999)

Merton, R. C.: Optimum consumption and portfolio rules in a continuous time model. J. Econ. Theory **3**, 373-413 (1971)

Schroder, M., Skiadas, C.: Optimal consumption and portfolio selection with stochastic differential utility. J. Econ. Theory, (to appear.)

Shreve, S., Soner, H. H.: Optimal investment and consumption with transaction costs. Annals Appl. Probability **4**(3), 609–692 (1994)

Soner, H. M.: Optimal control with state space constraints. SIAM Journal of Control and Optimization **24**, 552–562, 1110–1122 (1986)

Schweizer, M.: Mean variance hedging of general claims. Annals Appl. Probability **2**, 171–179 (1992)

Schweizer, M.: Approximation pricing and the variance optimal martingale measure. Annals Prob. **24**, 206–236 (1996)

Wang, S.: Premium calculation by transforming the layer premium density. ASTIN Bull. **25**, 71–92 (1996)

Wang, S., Young, V.: Risk-adjusted credibility premiums and distorted probabilities. Scand. Actuarial J. 143–165 (1998)

Wang, S., Young, V., Panjer, H.: Axiomatic characterization of insurance prices. Insurance: Math. Econ. **21**, 173–183 (1998)

Young, V., Zariphopoulou, T.: Computation of distorted probabilities for diffusion processes via stochastic control methods. Insurance: Math. and Econ. (In print)

Zariphopoulou, T.: Optimal investment and consumption models with non-linear stock dynamics. Math. Meth. Operations Res. **50**, 271–296 (1999)

Zariphopoulou, T.: Consumption investment models with constraints. SIAM J. Control and Optimization **32**(1), 59–84 (1994)

Zariphopoulou, T.: Investment-consumption models with transaction fees and Markov-chain parameters. SIAM J. Control and Optimization **30**, 613–636 (1992)