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CONSUMPTION–INVESTMENT MODELS WITH CONSTRAINTS

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Abstract

This paper treats a general consumption and investment problem for a single agent who consumes and distributes his wealth, dynamically, between a bond and a stock. The agent faces trading constraints: bankruptcy never occurs and the amount invested in stock must not exceed an exogeneous function of the current wealth. The objective is to maximize the expected utility of consumption. The value function is shown to be smooth solution of the associated Bellman equation and the optimal policies are determined.

1. Introduction

This paper treats a general consumption and investment problem for a single agent. The investor consumes his wealth X_t at a nonnegative rate C_t and he distributes it between two assets continuously in time. One asset is a *bond*, i.e. a riskless security with instantaneous rate of return r. The other asset is a *stock* whose value is driven by a Wiener process.

The objective is to maximize the *total expected (discounted) utility from consumption* over an infinite trading horizon. The investor faces the following *trading constraints*: his wealth must stay nonnegative, i.e. bankruptcy never occurs. Moreover, the amount π_t invested in stock must not exceed an exogeneous function $f(X_t)$ of the wealth. Finally, the agent does not pay transaction fees when he trades.

This financial model gives rise to a stochastic control problem with control variables the *consumption rate* C_t and the *portfolio vector* (π_t^0, π_t), where π_t^0 and π_t are the amount of wealth invested in bond and stock respectively. The state variable X_t is the total wealth at time t. Finally, the value function is the maximum total expected discounted utility.

The goal of this paper is to determine the value function of this control problem, to examine how smooth it is and to characterize the optimal policies. The basic tools come from the theory of partial differential equations, in particular the theory of viscosity solutions for second order partial differential equations and elliptic regularity. We first show that the value function is the unique constrained viscosity solution of the associated Hamilton–Jacobi–Bellman (HJB) equation. Then we prove that concave viscosity solutions of these equations are smooth. Finally, we obtain an explicit feedback form for the optimal policies (C^*, π^*).

Section 1

We consider a market with two assets: a *bond* and a *stock*. The price P_t^0 of the bond is given by

$$\begin{cases} dP_t^0 = rP_t^0\,dt & (t \geq 0) \\ P_0^0 = p_0 \quad , & (p_0 > 0) \end{cases}$$

where $r > 0$ is the *interest rate*. The price P_t of the stock satisfies

$$\begin{cases} dP_t = bP_t dt + \sigma P_t dW_t & (t \geq 0) \\ P_0 = p \ , & (p > 0) \end{cases}$$

where b is the *mean rate of return*, σ is the *dispersion coefficient* and the process W, which represents the source of uncertainty in the market, is a standard Brownian motion defined on the underlying probability space

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(Ω,F,P). We will denote by F_t the augmentation under P of $F_t^W = \sigma(W_s: 0 \le s \le t)$ for $0 < t < +\infty$. The interest rate r, the mean rate of return b and the dispersion coefficient σ are assumed to be constant with $\sigma \neq 0$ and $b > r > 0$.

The total current *wealth* $X_t = \pi_t^0 + \pi_t$ is the state variable; it evolves (see [20]) according to the equation

$$\begin{cases} dX_t = rX_t dt + (b-r)\pi_t dt - C_t dt + \sigma\pi_t dW_t \\ X_0 = x, \quad (x \in [0.+\infty)). \end{cases}$$

Here x is the *initial endowment* of the investor.

The *control processes* are the consumption rate C_t and the portfolio π_t. They are admissible if

(i) C_t is $\mathcal{F}_t$–progressively measurable, $C_t \ge 0$ a.e. $\forall t \ge 0$ and satisfies

$$E\int_0^\infty C_s ds < +\infty .$$

(ii) π_t is $\mathcal{F}_t$–progressively measurable and satisfies $E\int_0^\infty \pi_s^2 ds < +\infty$. Moreover $\pi_t \le f(X_t)$ a.s. $\forall t \ge 0$ where the function f: $[0, +\infty) \longrightarrow [0,+\infty)$ represents the borrowing constraints and it has the following properties:

$$(2) \qquad \begin{cases} f \text{ is increasing, concave,} \\ |f(x)-f(y)| \le K|x-y| \quad \forall x,y \ge 0 \\ f(0) \ge 0. \end{cases}$$

(iii) $X_t \ge 0$ a.s. $\forall t \ge 0$, where X_t is the trajectory given by the state equation (1) using the controls (C,π).

All the results hold for the case $f \equiv +\infty$ which was studied in [10] provided that some of the arguments are slightly modified.

We denote by $\mathcal{A}$ the set of admissible controls.

The total expected discounted utility J coming from consumption is given by

$$J(x,C,\pi) = E\int_0^{+\infty} e^{-\beta t} U(C_t) dt$$

with $(C,\pi) \in \mathcal{A}$, where U is the *utility function* and $\beta > 0$ is the *discount factor.*

The *value function* is given by

$$(3) \qquad v(x) = \sup_{\mathcal{A}} E\int_0^{+\infty} e^{-\beta t} U(C_t) dt$$

The utility function U: $[0,+\infty) \longrightarrow [0,+\infty)$ has the following properties:

$$\begin{cases} U \text{ is a strictly increasing,} \\ \text{concave, } C^2(0,+\infty) \text{ function with} \\ \lim_{c\to 0} U'(c) = +\infty, \ \lim_{c\to\infty} U'(c) = 0, \\ U(0) \ge 0, \quad U(c) \le M(1+c)^\gamma \\ \text{where } 0 < \gamma < 1 \text{ and } M > 0. \end{cases}$$

Our goal is to characterize v as classical solution of the Hamilton–Jacobi–Bellman (HJB) equation associated with this control problem and use the regularity of v to provide the optimal policies.

We now state the main results.

Theorem 1: The value function v is the unique $C^2(0,+\infty) \cap C([0,+\infty))$ solution of

$$(4) \qquad \beta v = \max_{\pi \le f(x)} [\tfrac{1}{2}\sigma^2\pi^2 v_{xx} + (b-r)\pi v_x] + \max_{c \ge 0}[-cv_x + U(c)] + rxv_x$$

in the class of concave functions.

Theorem 2: The optimal policies C_t^* and π_t^*

are given in the feedback form $C_t^* = c^*(X_t)$, $\pi_t^* = \pi^*(X_t)$ where

$$c^*(x) = (U')^{-1}(v_x(x)),$$

$$\pi^*(x) = \min\left\{f(x), -\frac{b-r}{\sigma^2}\frac{v_x(x)}{v_{xx}(x)}\right\}.$$

The single agent consumption–portfolio problem was first investigated by Merton in 1969, 1971 ([14], [15]) who considered utility functions belonging to the HARA family ("HARA" = hyperbolic absolute risk aversion). Another important contribution is the work of Karatzas, Lehoczky, Sethi and Shreve [10], and of Karatzas, Lehoczky and Shreve [11]. Pliska [17], Cox and Huang [2] and Pages [16] have used a martingale representation technology to study optimal consumption and portfolio policies.

In all the above work, trading constraints are not active. The main contribution of this paper is that it examines models with constraints (see also Zariphopoulou [20], Fleming and Zariphopoulou [7], Fitzpatrick and Fleming [5] and Vila and Zariphopoulou [19]).

Section 2

In this section we show that the value function v is the unique constrained viscosity solution of the HJB equation associated with the underlying stochastic control problem. The characterization of v as a constrained viscosity solution is natural because of the presence of the state $(X_t \geq 0)$ and control $(\pi_t \leq f(X_t))$ constraints.

The notion of *viscosity solution* was introduced by Crandall and Lions [4] for first order and by Lions [13] for second order equations. For a general overview of the theory we refer to the *user's guide* by Crandall, Ishii and Lions [3].

Next we recall the notion of *constrained viscosity solutions*, which was introduced by Soner [18] and Capuzzo–Dolcetta and Lions [1] for first order equations (see also Ishii and Lions [9]).

To this end, consider a nonlinear second order partial differential equation of the form

$$(5) \qquad F(x,u,u_x,u_{xx}) = 0 \quad \text{in } \Omega$$

where Ω is an open subset of $\mathbb{R}$ and $F: \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and (degenerate) elliptic, i.e.

$$F(x,t,P,X+Y) \leq F(x,t,P,X) \qquad \text{if } Y \geqq 0.$$

Definition 1: A continuous function $u: \overline{\Omega} \longrightarrow \mathbb{R}$ is a *constrained viscosity solution* of (5) iff

(i) u is a *viscosity subsolution* of (5) on $\overline{\Omega}$, i.e. if for any $\varphi \in C^2(\overline{\Omega})$ and any maximum point $x_0 \in \overline{\Omega}$ of $u - \varphi$

$$F(x_0,u(x_0), \varphi_x(x_0), \varphi_{xx}(x_0)) \leq 0$$

and

(ii) u is a *viscosity supersolution* of (5) in Ω, i.e. if for any $\varphi \in C^2(\overline{\Omega})$ and any minimum point $x_0 \in \Omega$ of $u - \varphi$

$$F(x_0, u(x_0), \varphi_x(x_0), \varphi_{xx}(x_0)) \geq 0.$$

Theorem 2.1: *The value function v is a constrained viscosity solution of* (4) *on* $\overline{\Omega} = [0,+\infty)$.

The fact that, in general, value functions of control problems and differential games turn out to be viscosity solutions of the associated partial differential equations is a direct consequence of the principle of dynamic programming and the definition of viscosity solutions (see for example: Lions [13]). The main difficulty, however, in the problem at hand is that the consumption rates and the portfolios are not uniformly bounded. This gives rise to some serious complications in the proofs of the results of the aforementioned papers. To overcome these difficulties we need to introduce a number of approximations of the original problem and make use repeatedly of the *stability* properties of viscosity solutions. □

We next present a comparison result for constrained viscosity solutions of (4). Comparison results for a large class of boundary problems were given by Ishii and Lions [9]. The equation on hand, however, does not satisfy some of the assumptions in [9], view of the fact that the controls are not uniformly bounded. It is therefore necessary to modify some of the arguments of Theorem II.2 of [9] to take care of these difficulties.

Theorem 2.2: *If u is an upper-semicontinuous concave viscosity subsolution of (4) on $\overline{\Omega}$ and v is a bounded from below, sublinearly growing, uniformly continuous on $\overline{\Omega}$ and locally Lipschitz in Ω viscosity supersolution of (4) in Ω, then*

$u \leq v$ on $\overline{\Omega}$. □

Section 3

In this section we show that the value function is smooth solution of the Hamilton–Jacobi–Bellman equation and we characterize the optimal policies.

Theorem 3.1: *The value function v is the unique continuous on $[0,+\infty)$ and twice continuously differentiable in $(0,+\infty)$ solution of (4) in the class of concave functions.*

The main idea of the proof is to work in intervals $(x_1,x_2) \subset [0,+\infty)$ and show that v solves a uniformly elliptic HJB equation in (x_1,x_2) with boundary conditions $v(x_1)$ and $v(x_2)$. Standard elliptic regularity theory (c.f. Krylov [12]) and the uniqueness result about viscosity solutions will yield that v is smooth in (x_1,x_2).

We next explain how we come up with the uniformly elliptic HJB equation. Formally, according to the constraints, the optimal π^* is either $f(x)$, if $-\frac{b-r}{\sigma^2}\frac{v_x(x)}{v_{xx}(x)} \geq f(x)$ or $-\frac{b-r}{\sigma^2}\frac{v_x(x)}{v_{xx}(x)}$, if $-\frac{b-r}{\sigma^2}\frac{v_x(x)}{v_{xx}(x)} < f(x)$. In the second case, we want to get a positive lower bound of π^* in $[x_1,x_2]$. Since v_x is nonincreasing and strictly positive, it is bounded from below away from zero. Therefore, it suffices to find a lower bound for v_{xx}.

Sketch of the proof: We first approximate v by a family of concave functions (v^ϵ) such that

$$v^\epsilon \longrightarrow v, \text{ locally uniformly on } \overline{\Omega}$$

and the v^ϵ's are constrained viscosity solutions on $\overline{\Omega}$ of the regularized equation

$$\beta v^\epsilon = \max_{\pi \leq f(x)} [\tfrac{1}{2}\sigma^2(\pi^2+\epsilon^2 x^2)v^\epsilon_{xx} + (b-r)\pi v^\epsilon_x] + \max_{c \geq 0}[-cv^\epsilon_x + U(c)] + rxv^\epsilon_x.$$

Moreover, it can be shown that v^ϵ is the unique smooth solution of

$$(6)\begin{cases} \beta u = \max\limits_{0 \leq \pi \leq f(x)} [\tfrac{1}{2}\sigma^2(\pi^2+\epsilon^2 x^2)u_{xx} + \\ (b-r)\pi u_x] + \max\limits_{c \geq 0}[-cu_x + U(c)] + rxu_x \\ u(x_1) = v^\epsilon(x_1), \ u(x_2) = v^\epsilon(x_2) \end{cases}$$

Next we show that there are positive constants R_1, R_2 and R, independent of ϵ, such that

$$(7) \qquad R_1 \leq v^\epsilon_x(x) \leq R_2 \text{ on } [x_1, x_2]$$

and

$$(8) \qquad |v^\epsilon_{xx}(x)| \leq R \text{ on } [x_1, x_2].$$

To get (8), we let $\zeta : \mathbb{R}^+ \longrightarrow [0,1]$ to be as follows:

i) $\zeta \in C_0^\infty$ (i.e. ζ is a smooth function with compact support),

ii) $\zeta \equiv 1$ on $[x_1,x_2]$, $\zeta \equiv 0$ on $\mathbb{R}\backslash[\bar{x}_1,\bar{x}_2]$, with $[x_1,x_2] \subset [\bar{x}_1,\bar{x}_2]$ and $\bar{x}_1 > 0$,

iii) $|\zeta_x| \leq M\zeta^p$, $|\zeta_{xx}| \leq M\zeta^p$ with $0 < p < 1$ and $M > 0$.

We consider a function z given by $z(x) = \zeta^2 v_{xx}^2 + \lambda v_x^2 - \mu v$, where λ and μ are appropriately chosen positive constants, and we look at the maximum of z on $[\bar{x}_1, \bar{x}_2]$. Using the conditions at the maximum point x_0, i.e. $z(x_0) = 0$ and $z(x_0) \le 0$, and the equation (6) we get (8).

Combining (7) and (8) we see that

$$-\frac{b-r}{\sigma^2}\frac{v_x^\epsilon}{v_{xx}^\epsilon} \ge B > 0 \quad \text{on } [x_1, x_2]$$

with $B = \frac{b-r}{\sigma^2}\frac{R_1}{R}$.

Let us now consider the equation

$$(9)\begin{cases}\beta u = \max\limits_{B \le \pi \le f(x)} [\frac{1}{2}\sigma^2(\pi^2 + \epsilon^2 x^2)u_{xx} + \\ (b-r)\pi u_x] + \max\limits_{c \ge 0}[-cu_x + U(c)] + rxu_x \\ u(x_1) = v(x_1), \ u(x_2) = v(x_2).\end{cases}$$

In view of the above analysis, we know that v^ϵ solves (9). Let $\epsilon \longrightarrow 0$. Since $v^\epsilon \longrightarrow v$, locally uniformly, v is a viscosity solution of

$$(10)\begin{cases}\beta u = \max\limits_{B \le \pi \le f(x)} [\frac{1}{2}\sigma^2\pi^2 u_{xx} + (b-r)\pi u_x] \\ + \max\limits_{c \ge 0}[-cu_x + U(c)] + rxu_x \\ u(x_1) = v(x_1), \quad u(x_2) = v(x_2).\end{cases}$$

On the other hand, (10) admits a unique smooth solution (see [12]) which is the unique viscosity solution (see [9], Theorem II.2); therefore v is smooth. □

Theorem 3.2: *The feedback optimal controls C^* and π^* are given by $c^*(x) = I(v_x(x))$ and $\pi^*(x) = \min\left\{-\frac{b-r}{\sigma^2}\frac{v_x(x)}{v_{xx}(x)}, f(x)\right\}$ for $x > 0$. The state equation (1) has a strong unique solution X_t^*, corresponding to $C_t^* = c^*(X_t^*)$ and $\pi_t^* = \pi^*(X_t^*)$ and starting at $x > 0$ at $t = 0$, which is unique in probability law up to the first time τ such that $X_\tau^* = 0$.*

Proof: The formulae for π^* and C^* follow from a standard verification theorem (see [6]) and the equation. We now show that π^* and C^* are locally Lipschitz functions of x. It is clear that v_x is locally Lipschitz because in any compact set K there exists a constant $C = C(K)$ such that $|v_{xx}| \le C(K)$, $(x \epsilon K)$. Therefore C^* is locally Lipschitz. Moreover, from the Bellman equation we have that

$$v_{xx} = H(x, v, v_x)$$

where H is a locally Lipschitz function. Since v_x is locally Lipschitz we get that v_{xx} is locally Lipschitz too. Therefore (see Gikhman and Skorohod [8]) equation (1) has a unique strong solution X_t^* in probability law up to the first time τ such that $X_\tau^* = 0$. □

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